The Square Divisibility Lemma

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Proof. For any integer n, if n^2 is divisible by 2, then n is divisible by 2. (i.e. For any $n \in \mathbb{Z}$, if $2 \mid n^2$, then $2 \mid n$).

Let $n \in \mathbb{Z}$ be arbitrary. Suppose by way of contradiction that n^2 is divisible by 2 but n is not divisible by 2 (i.e. $2 \mid n^2$ and $2 \nmid n$). By The Division Algorithm, n is either even or odd (i.e. $n = 2q + r$ for some unique q, $r \in \mathbb{Z}$, where $0 \leq r < 2$). It will be demonstrated, in either case, that a contradiction occurs.

<u>Case 1 ($r = 0$ meaning n is even):</u>
By definition of even, $n = 2q$ for some $q \in \mathbb{Z}$. So, $2 \mid n$, which contradicts the assumption that $2 \nmid n$.

<u>Case 2 ($r = 1$ meaning n is odd):</u>
By definition of odd, $n = 2q + 1$ for some $q \in \mathbb{Z}$. By the distributive property, $n^2 = (2q+1)^2 = (2q+1)(2q+1) = 4q^2+4q+1$. Factoring out 2 from the first two terms yields $n^2 = 2(2q^2 + 2q) + 1$. Because $q \in \mathbb{Z}$, we have that $(2q^2 + 2q) \in \mathbb{Z}$. Since n^2 is odd, $2 \nmid n^2$. This contradicts the assumption that $2 \mid n^2$.

In both cases we yield a contradiction, meaning that our assumption that if $n \in \mathbb{Z}$, then $2 \mid n^2$ and $2 \nmid n$ is false. Therefore, if n^2 is divisible by 2, then n is divisible by 2 for any integer n.

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