Part -1.

① $z = f(x,y) = ax + by + c$.

$$\nabla z = \nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

② $z = f(x) = f(x_1, x_2, \dots x_N) = \sum_{i=1}^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \dots + a_Nx_N + d$.

$$\nabla z = \nabla f(x) = \cancel{\begin{bmatrix} a_1, a_2, \dots a_N \end{bmatrix}} \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix}$$

③ $z = f(x,y) = A(x-x_0)^2 + B(y-y_0)^2 + C$.

$$f_x(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)_y = 2A(x-x_0)$$

$$f_y(x,y) = \left(\frac{\partial f(x,y)}{\partial y}\right)_x = 2B(y-y_0).$$

④ $x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ $[3\times1]$ $\quad x^T = (3 \;\; 1 \;\; 4)$ $[1\times3]$.

$y = (2 \;\; 5 \;\; 1)$ $[1\times3]$ $\quad y^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ $[3\times1]$

$$B = \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix} \qquad B^T = \begin{bmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{bmatrix}.$$

$[3\times 2]$ $\qquad$ $[2\times 3]$.

$X \cdot X$ is not defined.

$x \cdot y^T = \cancel{[3\times 1]}$ is not defined.

$$x \times y = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{bmatrix}.$$

$$y \times x = \begin{bmatrix} 2 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = 15.$$

$$A \times x = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 34 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{bmatrix}$$

$B.\text{reshape}(1,6) = \begin{bmatrix} 3 & 5 & 5 & 2 & 1 & 4 \end{bmatrix}$

LLS - Single variable.

$$L(p) = L(m,b) = \sum_{i=1}^{N} (y_i - mx_i - b)^2$$

$$\frac{\partial L}{\partial m} = -2\sum_{i=1}^{N} x_i y_i + 2m\sum_{i=1}^{N} x_i^2 + 2b\sum_{i=1}^{N} x_i = 0.$$

$$\frac{\partial L}{\partial b} = -2\sum_{i=1}^{N} y_i + 2m\sum_{i=1}^{N} x_i + 2nb = 0$$

Then $\sum_{i=1}^{N} y_i = m\sum_{i=1}^{N} x_i + nb.$

$$\bar{y}_i = m\bar{x}_i + b \Rightarrow b = y_i - m\bar{x}_i$$

So. $-2\sum x_i y_i + 2m\sum x_i^2 + 2b\sum x_i = 0.$

$$\sum x_i y_i = \cancel{m\sum_{i=1}} \, m\sum x_i^2 + b\sum x_i$$

Because $b = y_i - m\bar{x}_i$

$$\sum x_i y_i = m\sum x_i^2 + (y_i - m\bar{x})\sum x_i$$

Then.

$$m = \frac{\sum(x_i y_i - \bar{y}x_i)}{\sum(x_i^2 - \bar{x}x_i)} = \frac{\sum(x_i - \bar{x})\cdot(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{Cor(x,y)}{Var(x)}.$$

LLS- Multi- variable. (Bonus)

$$x = [(x_1, y_1), (x_2 y_2), \dots, (x_n, y_n)]$$

$$y = \beta_0 + x\beta_1$$

$$SSR = \sum (y_i - \cancel{x_i\beta_1} \beta_0 - x_i\beta_1)^2$$

$$\frac{\partial SSR}{\partial \beta_0} = \sum 2(y_i - \cancel{\beta} \beta_0 - x_i\beta_1)$$

$$= 2\sum(-y_i + \beta_0 + x_i\beta_1) = 0$$

$$\beta_0 = \bar{y} - \beta_1\bar{x}$$

$$\frac{\partial SSR}{\partial \beta_1} = \sum 2(y_i - x_i\beta_1 - \beta_0)(-x_i)$$

$$= 2\sum(\beta_0 x_i + x_i^2\beta_1 - x_i y_i)$$

Set $2\sum(\beta_0 x_i + x_i^2\beta_1 - x_i y_i) = 0$.

$$\beta_1 = \frac{\sum x_i y_i - y\sum x_i}{\sum x_i^2 - \bar{x}\sum x_i} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

So $\sum (y_i - \beta_0 - x_i \beta_1)^2$

$= \sum_{i=1}^{N} \| y_i - \sum_{j=1}^{M} x_{ij} \beta_j \|^2$

$= \| y - x \beta_1^T \|^2.$

Let $y - x\beta_1^T = 0$ $\quad x\beta_1^T = y.$

Multiplying x^T on both sides of the equation.

$x^T x \beta_1^T = x^T y.$

Then $\beta^T = (x^T x)^{-1} x^T y.$