20250 HWK 6

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Question 1

Proof. Let α and β be endomorphisms of V. Let $W \subset V$ be a subspace of V that is invariant for both α and β. Then, we can construct the following short exact sequence,

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "\beta_w"] & V \arrow[r, two heads, "\pi"] \arrow[d, "\beta"] & V/W \arrow[d, dashed, "\beta_{V/W}"] \\
W \arrow[r, hook] \arrow[d, "\alpha_w"] & V \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & V/W \arrow[d, dashed, "\alpha_{V/W}"] \\
W \arrow[r, hook] & V \arrow[r, "\pi"] & V/W
\end{tikzcd}$$

Recalling that $\alpha_{V/W}$ and $\beta_{V/W}$ exist if $W \subseteq ker(\pi \circ \alpha)$ and $W \subseteq ker(\pi \circ \beta)$. We know that this is true because for $w \in W$, $\alpha(w), \beta(w) \in W$ by the invariance of alpha and beta on W. Hence, $\pi(\alpha(w)) = 0$ and $\pi(\beta(w)) = 0$. Thus, $\alpha_{V/W}$ and $\beta_{V/W}$ exist. Hence,

$$\pi \circ \alpha = \alpha_{V/W} \circ \pi$$
$$\pi \circ \beta = \beta_{V/W} \circ \pi$$

By the diagram,

$$\pi(\alpha \circ \beta) = (\alpha_{V/W} \circ \beta_{V/W}) \circ \pi$$

Consequently,

$$(\alpha \circ \beta)_{V/W} = \alpha_{V/W} \circ \beta_{V/W}$$

$\square$

Question 2

Proof. Let $\alpha : V \to V$ be an endomorphism. Let $B = (w_1, ...w_r, v_1, ...v_s)$ be a basis for V. Define $W = span(w_1, ...w_r)$

$(\Rightarrow)$. Assume that $[\alpha]_B$ is a upper triangular. Then, for each $w_1, ...w_r$,

$$\alpha(w) = \sum_{i=0}^{r} a_i w_i \tag{1}$$

Hence, α is invariant on W.

$(\Leftarrow)$ Assume that α is invariant on W. Then for each w,

$$\alpha(w) = \sum_{i=0}^{r} a_i w_i$$

Hence for each column $j = 1, ...r$, if the row $i > r$, then the element is 0 since there are no v_i in $\alpha(w)$. Consequently the matrix is upper triangular.

Define the upper right $r \times r$ matrix as A, upper left $r \times s$ matrix as B, lower right $s \times r$ matrix as C, and lower right matrix as $s \times s$ as D. Since it is an upper triangular matrix, $C = 0$. Hence $[\alpha]_B$ can be rewritten as

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

$(\Rightarrow)$ Assume that $[\alpha]_B$ is invertible. Then there exists some other matrix such that,

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \times \begin{pmatrix} E & F \\ 0 & H \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Where I is an identity matrix of the appropriate dimensions. By the rules of matrix multiplication,

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \times \begin{pmatrix} E & F \\ 0 & H \end{pmatrix} = \begin{pmatrix} AE + B0 & AF + BH \\ 0E + D0 & 0F + DH \end{pmatrix}$$

Thus,

$$AE = I$$
$$DH = I$$

Thus each corner of the upper triangular are invertible.

$(\Leftarrow)$ Using the nomenclature from the previous section, assume that A and D are invertible. Then,

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \times \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Hence $[\alpha]_B$ is invertible. $\square$

Question 3

Proof. Let $\alpha : V \to V$ be an endomorphism of a finite-dimensional vector space.

(a $\Leftrightarrow$ b)

($\Rightarrow$) Assume that α is invertible. Then it is an isomorphism and hence there exists some endomorphism $\alpha^{-1} = \beta : V \to V$. By the definition of an inverse,

$$\beta \circ \alpha = 1$$

($\Leftarrow$) Assume there exists some endomorphism, $\beta : V \to V$ such that

$$\beta \circ \alpha = 1$$

Because these functions each take $V \to V$, there are both invariant on V. Consequently,

$$\beta \circ \alpha = \alpha \circ \beta = 1$$

Hence, β is an inverse for α. Thus α is an isomorphism and hence invertible

(a $\Leftrightarrow$ c)

($\Rightarrow$) Assume that α is invertible. By the first part there exists some endomorphism $\beta : V \to V$ such that

$$\beta \circ \alpha = 1$$

Because these functions each take $V \to V$, there are both invariant on V. Consequently,

$$\beta \circ \alpha = \alpha \circ \beta = 1$$

($\Leftarrow$) Assume there exists some endomorphism, $\beta : V \to V$ such that

$$\alpha \circ \beta = 1$$

Because these functions each take $V \to V$, there are both invariant on V. Consequently,

$$\beta \circ \alpha = \alpha \circ \beta = 1$$

Hence, β is an inverse for α. Thus α is an isomorphism and hence invertible $\square$

Question 4

Proof. Let $\alpha : V \to V$ be a non-zero nilpotent endomorphism. Suppose dim(V) = 2. Take the following as a basis of V and call it B,

$$v_1 = e_1$$
$$v_2 = e_2$$

Then,

$$\alpha(B):$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\alpha(B)^2:$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

Then, α with the given matrix takes this basis to zero and hence is nilpotent. □

Question 5

Proof. Let $\alpha : V \to V$ be an endormorphism of a finite-dimensional vector space.

(a $\Rightarrow$ b)

Let α be nilpotent. Then there exists some $r \in \mathbb{N}$ such that for $v \in V$, $im(\alpha^r(v)) = 0$. Then,

$$im(\alpha) \subset V$$

Because α commutes with itself, $im(\alpha^n)$ is an invariant subspace, hence

$$im(\alpha^n) \subseteq im(\alpha^{n-1})$$

Because eventually these images become zero, there exist at least one image, namely the eventual image, such that

$$im(\alpha^\infty) \subset V$$

Then we can add in every other im that fulfills the property that

$$im(\alpha(V_i)) \subset im(V_{i-1})$$

to complete the set.

(b $\Rightarrow$ c)

Let there exist a sequence of subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$ with the property that $\alpha(V_i) \subset V_{i-1}$ for $i = 1, \ldots, r$. Then, alpha must be removing a dimension every time it is applied. Thus each vector looses its leading digit. Consequenty, for the matrix A of α, each $A_{ij} = 0$ if $i \geq j$. Thus, A is a strictly upper triangular matrix.

(c $\Rightarrow$ a)

Let the matrix of α be a strictly upper triangular matrix. Then let a basis of V be defined as the collection of vectors $(v_1, ...v_r)$. Then, multiplying this basis by the α matrix change each v_i as follows,

$$\alpha(v_i) = \sum_{k=0}^{i-1} a_k v_k$$

Intuitively this means that each vector looses one of its components each times it is multiplies by the alpha matrix. Consequently,

$$\alpha(B)^r = 0$$

Hence, α is nilpotent. $\square$

Question 6

Proof. Let $W \subset V$ be an α-invariant subspace for $\alpha : V \to V$ of a finite-dimensional vector space V. Observe that

$$ker(\alpha_W^\infty) \oplus im(\alpha_W^\infty) = W$$
$$ker(\alpha^\infty) \oplus im(\alpha^\infty) = V$$
$$ker(\alpha_{V/W}^\infty) \oplus im(\alpha_{V/W}^\infty) = V/W$$

Since,

$$W \hookrightarrow V \xrightarrow{\pi} V/W$$

is a short exact sequence, it follows that

$$ker(\alpha_W^\infty) \oplus im(\alpha_W^\infty) \xhookrightarrow{\gamma} ker(\alpha^\infty) \oplus im(\alpha^\infty) \xrightarrow{\pi} ker(\alpha_{V/W}^\infty) \oplus im(\alpha_{V/W}^\infty)$$

Hence,

$$ker(\alpha_W^\infty) \xhookrightarrow{\delta} ker(\alpha^\infty) \xrightarrow{\pi} ker(\alpha_{V/W}^\infty)$$

$$im(\alpha_W^\infty) \xhookrightarrow{\beta} im(\alpha^\infty) \xrightarrow{\pi} im(\alpha_{V/W}^\infty)$$

Finally, we must show that $im(\delta) = ker(\pi)$ and $im(\beta) = ker(\pi)$. By the first short exact sequence we know that

$$im(\gamma) = ker(\pi)$$

Hence,

$$\pi(im(\gamma)) = 0$$
$$\pi(ker(\alpha^\infty), im(\alpha^\infty)) = 0$$

Because $im(\delta) = ker(\alpha^\infty)$ and $im(\beta) = im(\alpha^\infty)$,

$$\pi(im(\delta)) = 0$$
$$\pi(im(\beta)) = 0$$

Hence, $im(\delta) = ker(\pi)$ and $im(\beta) = ker(\pi)$ □

Question 7

Proof. Let $\alpha_1, ...\alpha_n$ be pairwise commuting endomorphisms of a finite-dimensional vector space V. Proceed to decompose V into eventual kernels and images by induction on n.

Base Case
Let $n = 1$, then as shown in class we can decompose V into the eventual kernel and image of α_1 as follows,

$$V = im(\alpha_1^\infty) \oplus ker(\alpha_1^\infty)$$

Inductive Step and Hypothesis
Let $n \in N$. For any n, assume that,

$$V = \underset{I \sqcup J = \{1,\ldots,n\}}{\oplus} \Big(\big(\underset{i \in I}{\cap} \ker(\alpha_i^\infty) \big) \cap \big(\underset{j \in J}{\cap} (\alpha_j^\infty) \big) \Big)$$

Proof of Inductive Hypothesis
By what we proved in class we know that:

$$\underset{I \sqcup J = \{1,\ldots,n+1\}}{\oplus} \Big(\big(\underset{i \in I}{\cap} \ker(\alpha_i^\infty) \big) \cap \big(\underset{j \in J}{\cap} (\alpha_j^\infty) \big) \Big) = \underset{I \sqcup J = \{1,\ldots,n\}}{\oplus} \Big(\big(\underset{i \in I}{\cap} \ker(\alpha_i^\infty) \big) \cap \big(\underset{j \in J}{\cap} (\alpha_j^\infty) \big) \Big)$$

Hence, by the inductive hypothesis,

$$\underset{I \sqcup J = \{1,\ldots,n+1\}}{\oplus} \Big(\big(\underset{i \in I}{\cap} \ker(\alpha_i^\infty) \big) \cap \big(\underset{j \in J}{\cap} (\alpha_j^\infty) \big) \Big) = V$$

Thus, since they are equal, the map between the two is an isomorphism. $\square$