20250 HWK 3

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Question 1

Proof. Let $v_1, ...v_n \in V$ be a sequence of vectors. Then,

$$\langle v_1, ...v_n \rangle = \{w : w = \sum a_i \cdot v_i\}$$

Hence, swapping the place of one vector with another does not affect the span since addition is commutative. In other words,

$$a_1v_1 + ... + a_iv_i + ... + a_jv_j + ...a_nv_n = a_1v_1 + ... + a_jv_j + ... + a_iv_i + ...a_nv_n$$

Now to examine scaling. Consider the scaled series of vectors $v_1, ...bv_i, ...v_n \in V$ where $b \in k$. Then, we can construct every vector w by scaling down the ith coefficient by the multiplicative inverse of b. Hence,

$$a_1v_1 + ... + a_iv_i + ... + a_nv_n = a_1v_1 + ... + a_ib^{-1}(bv_i) + ... + a_jv_j + ...a_nv$$

Thus scaling does not impact the span of the vectors.

Finally, to examine adding one vector to another, consider the set $v_1, ...v_i + v_j, ...v_j, ...v_n \in V$. Then, if we use the same coefficients $a_1, ...a_n$ as the original set but replace a_j with $a_j - a_i$, we have

$$\begin{aligned}\sum a_iv_i &= a_1v_1 + ... + a_i(v_i + v_j) + ... + (a_j - a_i)v_j + ... + a_nv_n \\ &= a_1v_1 + ... + a_iv_i + a_iv_j + ... + a_jv_j - a_iv_j + ... + a_nv_n \\ &= a_1v_1 + ... + a_iv_i + ... + a_jv_j + ... + a_nv_n\end{aligned}$$

Hence, this set spans the same vectors since the sum of vectors are identical when using the same coefficients. ☐

Question 2

Proof. The vectors given by the question can be combined to form the following matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Reducing these vectors until we have them in echelon form will result in a basis for the span of these vectors. Doing so yields,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Which is a basis for the subspace spanned by the given vectors as by question 1 doing these operations do not change the span and by reducing to echelon form ensures that the vectors are linearly independent. □

Question 3

Proof. Let V and W be finite dimensional vector spaces. It was shown in class that $\text{Maps}(k^n, k^m)$ is isomorphic to the set of matrices with dimensions $m \times n$. Assume that $\dim V = n$ and $\dim W = m$. Hence, if we can find the dimension of the set of $m \times n$ matrices we can find the dimensionality of $\text{Maps}(V, W)$. Hence, we need a set of $m \times n$ matrices that can be combined to create any other $m \times n$ matrix. Consider the set of matrices where there is a empty $m \times n$ matrix with 1 in one element for each element of an $m \times n$ matrix. Clearly this set spans the set of all matrices and is also linearly independent. Thus this set is a basis for the set of maps. Because there is a matrix for each element, there are $m \times n$ matrices and thus $\dim Maps(k^n, k^m) = m \times n = \dim V \cdot \dim W$. □

Question 4

Proof. Consider the basis $f_1 = e_1$, $f_2 = e_1 + e_2$. We want to find $f_1^\vee$ and $f_2^\vee$ such that

$$f_1^\vee(f_1) = 1$$
$$f_2^\vee(f_2) = 1$$

Define,

$$f_1^\vee = a - b = e_1^\vee - e_2^\vee$$
$$f_2^\vee = b = e_2^\vee$$

Then,

$$f_1^\vee(f_1) = e_1^\vee(e_1) - e_2^\vee(0) = 1$$
$$f_1^\vee(f_2) = e_1^\vee(e_1) - e_2^\vee(e_2) = 1 - 1 = 0$$
$$f_2^\vee(f_1) = e_2^\vee(0) = 0$$
$$f_2^\vee(f_2) = e_2^\vee(e_2) = 1$$

Hence, $f_1^\vee$ and $f_2^\vee$ work as a dual basis. □

Question 5

Proof. Let $\alpha_1, ...\alpha_d \in k$ be distinct. To show that the deltas are linearly independent we must show that,

$$a_1\delta_1(\alpha_1) + ... + a_i\delta_i(\alpha_i) + ... + a_d\delta_d(\alpha_i) = 0$$

However, by the properties of these polynomials, for every i, this equation takes the form:

$$0 + ... + a_i + ... + 0 = 0$$

Hence, for each i, $a_i = 0$ which means that every coefficient is 0. This means that only the trivial solution remains and hence the vectors are linearly independent. Also, $\delta_1, ...\delta_d$ span $k[X]_{<d}$ as they are d independent vectors and hence their dimension is equal to that of the space. Thus they must span it. To show that $ev_{\alpha_1}, ...ev_{\alpha_d}$ is a dual basis for $\delta_1, ...\delta_n$, we must show that,

$$ev_{\alpha_j}(\delta_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

By the construction of δ,

$$\delta_i(\alpha_i) = 1$$
$$\delta_i(\alpha_j) = 0 \text{ for j} \neq i$$

Hence, $ev_{\alpha_1}, ...ev_{\alpha_d}$ is a dual basis for $\delta_1, ...\delta_n$. □

Question 6

(i)

Proof. Because $Fun_c(X,k) \subset Fun(X,k)$, there exists and isomorphism between $Fun(X,k)$ and $Fun_c(X,k)$. Similarly, there exists and isomorphism between $Fun_c(X,k)$ and $Fun_c(X,k)^\vee$. Hence, define,

$$\lambda_X : Fun(X,k) \to Fun_c(X,k) \to Fun_c(X,k)^\vee$$

Since, lambda is a composition of two isomorphic functions, lambda is an isomorphism. $\square$

(ii)

Proof. Let $\phi : X \to Y$ be a function to another set. Consider,

$$\phi^* : Fun(Y,k) \to Fun(X,k) \qquad f \mapsto f \circ \phi$$

Construct a linear map $\phi_* : Fun_c(X,k) \to Fun_c(Y,k)$ with $\phi^* = (\lambda_X)^{-1} \circ (\phi_*)^\vee \circ \lambda_Y$. Thus, breaking down the composition,

$$(\phi_*)^\vee \circ \lambda_Y = (\phi_*)^\vee (Fun_c(Y,k)^\vee) \tag{1}$$

And we need that,

$$(\lambda_X)^{-1} \circ (\phi_*)^\vee (Fun_c(Y,k)^\vee) = Fun(X,k)$$

Hence, $(\phi_*)^\vee$ should take $(Fun_c(Y,k))^\vee \to (Fun_c(X,k))^\vee$. Define ϕ_* with the following function,

$$\phi_* : Fun_c(X,k) \to Fun(X,k) \xrightarrow{\phi} Fun(Y,k) \xrightarrow{\lambda_Y} Fun_c(Y,k)^\vee \to Fun_c(Y,k)$$

Then, $(\phi_*)^\vee$ takes $(Fun_c(Y,k))^\vee \to (Fun_c(X,k))^\vee$. Consequently, the equality is satisfied. $\square$

Question 7

Proof. To prove that $\iota_{v^\vee}$ is injective, we want to show that $\ker(\iota_{v^\vee}) = 0$. In other words,

$$\begin{gathered} \iota_{v^\vee}(\xi) = 0 \\ [\iota_v(\xi)](\iota_v(v)) = 0 \\ \xi(v) = 0 \end{gathered}$$

Thus, $\xi = 0$ and hence $\ker(\iota_{v^\vee}) = 0$. Thus, $(\iota_{v^\vee}) : V^\vee \to ((V^\vee)^\vee)^\vee$ is injective. $\square$

Question 8

Proof. To show that $\dim(im(\phi)) = \dim(im(\phi^\vee))$. I will show that $im(\phi)^\vee$ is isomorphic $im(\phi^\vee)$. Define,

$$\phi^\vee : W^\vee \to im(\phi)^\vee \to V^\vee$$

Because the the second part of the map is injective, $im(\phi^\vee)$ is isomorphic to a subspace in $V^\vee$. Also, $im(\phi)^\vee$ is isomorphic to this same subspace since it is included into $V^\vee$ as well. Because V and $V^\vee$ are both finitely dimensional, they have the same dimension. Hence,

$$dim(im(\phi)) = dim(im(\phi)^\vee) = dim(im(\phi^\vee))$$

□