163 HWK 1

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Question 1

Proof. Let n be a natural number. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ so that $N \geq \frac{1}{\epsilon} - 1$. Hence, $|\frac{1}{N+1}| \leq \epsilon$. Also,

$$|\frac{1}{N+1}| = |\frac{1}{N+1}| = |\frac{N}{N+1} - 1|$$

Hence if $n \geq N$ than $|\frac{n}{n+1} - 1| < \epsilon$. Consequently, $\frac{n}{n+1}$ is convergent and $\lim_{n\to\infty} \frac{n}{n+1} = 1$ □

Question 2

Proof. Let $\epsilon > 0$. By the Archimedian Property of real numbers, let N be a real number such that $N \geq \frac{1}{\epsilon}$. Also,

$$|\frac{N^2}{N^3 + 3}| \leq |\frac{1}{N}| < \epsilon$$

Hence, for all $n > N$ then all $|\frac{n^2}{n^3+3}| < \epsilon$. Consequently, $|\frac{n^2}{n^3+3}|$ is convergent and $\lim_{n\to\infty} |\frac{n^2}{n^3+3}| = 0$ □

Question 3

Proof. Let c be a real number and let (a_n) be a convergent series such that $\lim_{n\to\infty}(a_n) = l$ and $c \neq l$. Since (a_n) is convergent, there exists an $N \in \mathbb{N}$ such that for each $n > N$, $|a_n - l| < |c - l|$. Hence, $c \neq a_n$ □

Question 4

(a)

Proof. Let (a_n) be a convergent sequence in $[0,1]$ that converges to l. Then, for each $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $n > N$, then

$$|a_n - k| < \epsilon$$

Hence,

$$|a_n - l| \leq 1$$
$$-1 \leq a_n - l \leq 1$$

Given that (a_n) is bounded below by 0 and above by 1, then

$$-1 \leq -l \leq 1 - l \leq 1$$

Hence,

$$l \leq 1$$

and

$$l \geq 0$$

Thus, $\lim_{n\to\infty}(a_n) \in [0,1]$ □

(b)

Proof. Let $n \in \mathbb{N}$. Let (a_n) be a convergent sequence defined by

$$(a_n) \stackrel{\text{def}}{=} \frac{1}{n+1}$$

Since n starts at 1, and we know that $\frac{1}{n} > 0$ then $(a_n) \in (0,1)$. Also, by the archimedean property, for each $\epsilon > 0$ we can chose an n such that $|\frac{1}{n+1}| < \epsilon$. So, $(a_n) \to 0$ as $n \to \infty$. □

Question 5

Statement: If (a_n) is a convergent sequence of integers, then there exists a natural number N such that for each $n \geq N$, $(a_N) = (a_n) = \lim_{n\to\infty}(a_n)$.

Proof. Let (a_n) be a convergent sequence of integers. Let l be the limit of said sequence. For any $\epsilon < \frac{1}{2}$ there exists only one integer that satisfies the inequality $|a_n - l| < \frac{1}{2}$. That it $a_n = l$. Hence, by the definition of a convergent sequence, there exists a N such that for each $n \geq N$, $a_n = a_N = l$. □

Question 6

Proof. ($\Rightarrow$) Let K be a natural number and let (a_n) be a convergent sequence with limit l. Then, for each $\epsilon > 0$ there exists an N such that for each $n \geq N$, $|a_n - l| < \epsilon$. Then for the sequence a_{n+K}, when $n \geq (N-K)$ then $|a_{n+K} - l| \leq |a_N - l| < \epsilon$. Hence, (a_{n+K}) is convergent and $\lim\limits_{n\to\infty}(a_n) = l = \lim\limits_{n\to\infty}(a_{n+K})$.

($\Leftarrow$) Let (a_{n+K}) be a sequence that converges to l. Then for each $\epsilon > 0$, there exists an N such that if $n > N + K$ then $|a_{n+K} - l| < \epsilon$. Consequently, for each $n > N$, $|a_n - l| < \epsilon$. Hence, $\lim\limits_{n\to\infty}(a_n) = l = \lim\limits_{n\to\infty}(a_{n+K})$. $\square$

Question 7

Proof. Assume that (a_n) is divergent. Proceeding by contradiction assume $-(a_n)$ is convergent. Hence, for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then $|-a_n - l| < \epsilon$. Thus,

$$|a_n + l| < \epsilon$$

Which implies that (a_n) is convergent, which is a contradiction. Hence, $-(a_n)$ must diverge as well. $\square$

Question 8

Proof. Assume that (b_n) is a convergent sequence. Let l be the non-zero limit of (b_n) as n goes to infinity. Then, for each $\epsilon_1 > 0$, there exists an $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, then $|b_n - l| < \epsilon_1$. We want to show that for each $\epsilon_2 > 0$ there exists some $M \in \mathbb{N}$ so that if $n \geq M$, then $|\frac{1}{b_n} - \frac{1}{l}| < \epsilon_2$.

$$\begin{aligned}|\frac{1}{b_n} - \frac{1}{l}| &= |\frac{l - b_n}{lb_n}| \\ &= \frac{|l - b_n|}{|lb_n|} \\ &< \frac{\epsilon_1}{|lb_n|}\end{aligned}$$

We know that there exists some N_2 so that $|b_n - l| < \frac{l}{2}$, and thus $|b_n| > \frac{1}{2}|l| > 0$. If we choose $N = maxN_1, N_2$ and $\epsilon_1 = \frac{\epsilon_2 |l|^2}{2}$ then,

$$|\frac{1}{b_n} - \frac{1}{l}| < \frac{\epsilon_1}{|lb_n|} < \frac{2\epsilon_2|l|^2}{2|l|^2} = \epsilon_2$$

Hence, $\frac{1}{(b_n)}$ converges to $\frac{1}{l}$ $\square$

Question 9

(a)

Proof. Let $n \in \mathbb{N}$. Define $(a_n) \stackrel{\text{def}}{=} (-1)^n$ and $(b_n) \stackrel{\text{def}}{=} (-1)^{n+1}$. It is evident that each sequence is divergent as shown in class. Then $(a_n + b_n) = 0$ and hence convergent for all n. □

(b)

Proof. Let $n \in \mathbb{N}$. Define $(b_n) \stackrel{\text{def}}{=} (-1)^n$ and $(a_n) \stackrel{\text{def}}{=} \frac{(-1)^n}{n}$. As established, (b_n) is divergent and (a_n) converges to 0. Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Suppose $l = 1$ and $n > N$. Then

$$|\frac{(-1)^n}{n} + (-1)^n - 1| = |(-1)^n(\frac{1}{n} + 1) - 1| = |\frac{1}{n}| < |\frac{1}{N}| = \epsilon$$

Hence, $(a_n + b_n)$ converges to 1. □

(c)

Proof. Let $n \in \mathbb{N}$. Define $(b_n) \stackrel{\text{def}}{=} \frac{1}{n}$. Then $\frac{1}{(b_n)} = (n)$. As established in class, n diverges. □

(d)

Proof. Let $n \in \mathbb{N}$. Assume that (a_n) is unbounded and (b_n) is convergent. Hence, for each $M_1 > 0$ there exists an $N \in \mathbb{N}$ so that $|a_N| > M_1$. Let $M_2 > 0$. If we take $M_1 = M_2 + |b_N|$, then consequently,

$$|a_N - b_N| \geq |a_N| - |b_N| > M_2$$

Hence, $(a_n - b_n)$ is unbound. □

(e)

Proof. Let $n \in \mathbb{N}$. Define $(a_n) \stackrel{\text{def}}{=} \frac{1}{n}$ and $(b_n) \stackrel{\text{def}}{=} (-1)^n$. As established in class, (a_n) is convergent and (b_n) is divergent. However, $(a_n \cdot b_n) = \frac{(-1)^n}{n}$. For each $\epsilon > 0$, if $N = \frac{1}{\epsilon}$ and $l = 0$, then for each $n > N$,

$$|\frac{(-1)^n}{n} - 0| = |\frac{1}{n}| < |\frac{1}{N}| = \epsilon$$

Hence, $(a_n \cdot b_n)$ is convergent.

□

(f)

Proof. Let $(a_n) \stackrel{\text{def}}{=} (-1)^n \cdot n$. Since $|(-1)^n \cdot n| = |n|$ we know that (a_n) similarly is unbounded. However, for each term n of (a_n), if $a_n > M$, then $a_{n+1} < -M$. Hence it is impossible for this sequence to meet the definition of diverging to either ∞ or $-\infty$. $\square$

Question 10

(a)

Proof. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Let (a_n) be a convergent sequence that converges to $l \in \mathbb{R}$. Then, for each ϵ,

$$\begin{aligned} \epsilon &> |a_n - l| \\ &> ||a_n| - |l|| \end{aligned}$$

by the triangle identity. Hence $|(a_n)|$ is convergent and $\lim_{n\to\infty} |(a_n)| = |\lim_{n\to\infty} (a_n)| = |l|$

$\square$

(b)

Proof. Let (a_n) and $(a_n - b_n)$ be convergent sequences and let $\lim_{n\to\infty} (b_n - a_n) = 0$. Let $l = \lim_{n\to\infty} (a_n)$. Let $\epsilon > 0$. By the definition of convergent, there exists some N for each (a_n) and $(a_n - b_n)$ so that if $n > N$, then

$$\begin{aligned} |a_n - l| &< \frac{\epsilon}{2} \\ |b_n - a_n| &< \frac{\epsilon}{2} \end{aligned}$$

Then,

$$\begin{aligned} \frac{\epsilon}{2} > |b_n - a_n| = |(b_n - l) - (a_n - l)| &\geq |b_n - l| - |a_n - l| \\ \frac{\epsilon}{2} + |a_n - l| &> |b_n - l| \\ \epsilon &> |b_n - l| \end{aligned}$$

Hence, (b_n) is convergent and $\lim_{n\to\infty} (b_n) = l = \lim_{n\to\infty} (a_n)$ $\square$

(c)

Proof. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Let (a_n) be a convergent sequence that converges to 0. Hence, for each ϵ there exists an $N \in \mathbb{N}$ such that if $n > N$, then

$$|a_n| < \epsilon$$

Let $b \in \mathbb{R}$ so that $|(b_n) - b| < a_n$ Consequently,

$$|b_n - b| \leq a_n \leq |a_n| < \epsilon$$

$\square$

Question 11

(a)

Proof. Let (a_n) be a convergent sequence. For each $k \in \mathbb{N}$,

$$\sigma \stackrel{\text{def}}{=} \frac{a_1 + ... + a_k}{k}$$

Define

$$l \stackrel{\text{def}}{=} \lim_{n\to\infty} a_n$$

Let $\epsilon > 0$. We want to find a $K \in \mathbb{N}$ so that for each $k \in \mathbb{N}$, if $k \geq K$, then $|\sigma_k - l| < \epsilon$. It follows that,

$$\begin{gathered} |\sigma_k - l| \\ = |\frac{a_1 + ... + a_k}{k} - l| \\ = \frac{1}{k}|(a_1 - l) + ... + (a_k - l)| \end{gathered}$$

Because (a_n) is convergent, for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq N$, then $|a_n - l| < \frac{\epsilon}{2}$. Hence,

$$\begin{gathered} \frac{1}{k}|(a_1 - l) + ... + (a_k - l)| \\ = \frac{1}{k}|(a_1 - l) + ... + (a_{N-1} - l)| + \frac{1}{k}|(a_N - l) + ... + (a_k - l)| \\ \leq \frac{1}{k}|(a_1 - l) + ... + (a_{N-1} - l)| + \frac{\epsilon}{2} \end{gathered}$$

Hence, we can choose a k such that

$$\frac{1}{k}|(a_1 - l) + ... + (a_{N-1} - l)| < \frac{\epsilon}{2}$$

Thus,

$$|\sigma_k - l| < \epsilon$$

So, $\lim_{k\to\infty} \sigma_k = l$ $\square$

(b)

Proof. Let $(a_n) \stackrel{\text{def}}{=} (-1)^n$.

Case 1:

n is even. Then $a_1 + ... + a_n = 0$. Hence $|\sigma_n| = 0 < \epsilon$.

Case 2:

n is odd. Then $a_1 + ... + a_n = -1$. Hence $|\sigma_n| = \frac{1}{n}$. Then there is a sufficiently large n given by the archimedean corollary such that $\frac{1}{n} < \epsilon$ $\square$

Wrong: 9.b, 7, 9.d(Lil bit)