163 HWK 5

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Question 1

Proof. Let (a_n) be a sequence of real numbers.. Assume that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Define,

$$s_n = \sum_{k=1}^{n} a_k$$
$$t_n = \sum_{k=1}^{n} |a_k|$$

By the definition of absolutely convergent, there exists $l \in \mathbb{R}$

$$\lim_{n \to \infty} s_n = l$$

Because, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, it is also convergent, hence there exists a $m \in \mathbb{R}$ such that,

$$\lim_{n \to \infty} t_n = m$$

Because $a_k \leq |a_k|$, $s_n \leq t_n$, and consequently, $l \leq m$. We know that $m \geq 0$ since each $a_k \geq 0$. Then,

$$|l| \leq |m| = m$$

Thus,

$$|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$$

$\square$

Question 2

(a)

Proof. Let (a_n) be a sequence of real numbers. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Take arbitrary subsequence (a_{n_k}). Then, for each $n \in \mathbb{N}$, define,

$$b_n = \begin{cases} a_{n_k} & \text{if } n = n_k \\ 0 & \text{if } n \neq n_k \end{cases}$$

Then, by construction,

$$\sum_{n=1}^{n_k} a_n = \sum_{i=1}^{k} a_{n_i}$$

Also,

$$0 \leq b_n \leq |a_n|$$

Hence, by the comparison test, since $\sum_{n=1}^{\infty} a_n$ converges, so too does $\sum_{n=1}^{\infty} b_n$. Consequently, $\sum_{i=1}^{\infty} a_{n_i}$ converges as well. $\square$

(b)

Proof. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

It has been shown in class that this series is convergent and yet $|\frac{(-1)^n}{n}| = \frac{1}{n}$ which is divergent as a series. Define

$$n_k = 2k - 1$$

Then the subsequence (a_{n_k}) contains only the negative terms of $(\frac{(-1)^n}{n})$. Define,

$$t_k = 2k$$

This subsequence contains only the positive terms. By contradiction that each of the series $\sum_{n=1}^{\infty} a_{n_k}$ and $\sum_{n=1}^{\infty} a_{t_k}$ are convergent. Consequently,

$$\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum_{k=1}^{\infty} a_{t_k} - \sum_{k=1}^{\infty} a_{n_k}$$

By the algebraic limit theorem, $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}|$ must converge. However this is a contradiction. Hence, one of these subsequences has to diverge. $\square$

Question 3

Proof. Let (a_n) be a sequence of real numbers. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Define,

$$s_n = \sum_{k=1}^{n} |a_k|$$

By the definition of absolutely convergent we know that there exists some $l \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a $N_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n > N$, then,

$$|s_n - l| < \epsilon$$

Because $\sum_{n=1}^{\infty} |a_n|$ is convergent, we know that $\lim\limits_{n \to \infty} |a_n| = 0$. Hence, there exists some $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n > N$, then,

$$|a_n| < 1$$

Consequently,

$$|a_n^2| < |a_n|$$

and

$$|a_n^3| < |a_n^2|$$

Hence,

$$|a_n^3| < |a_n|$$

Because $\sum_{n=1}^{\infty} |a_n|$ is convergent, by the stronger convergence test proved in assignment 4,$\sum_{n=1}^{\infty} |a_n^3|$ is convergent. Thus, $\sum_{n=1}^{\infty} a_n^3$ is absolutely convergent. $\square$

Question 4

Proof. Let (a_n) be sequence of real numbers such that $(\sqrt[n]{a_n})$ is convergent. Define

$$\lim_{n\to\infty} \sqrt[n]{a_n} \stackrel{\text{def}}{=} l$$

(i)

Assume that $l < 1$. Then, there exists some $r \in \mathbb{R}$ such that $l < r < 1$ by the density of real numbers. By the convergence of $(\sqrt[n]{a_n})$ to l, there exists an $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$ then

$$\sqrt[n]{a_n} - l \leq |\sqrt[n]{a_n} - l| < r - l$$

So that

$$\sqrt[n]{a_n} < r$$
$$0 \leq a_n < r^n$$

Thus, since $r < 1$, by the ratio test, $\sum_{n=1}^{\infty} r_n$ converges. Consequently, by the comparison test $\sum_{n=1}^{\infty} a_n$ converges

(ii)

Assume $l > 1$. Then there exists some $r \in \mathbb{R}$ such that $1 < r < l$ by the density of real numbers. By the convergence of $(\sqrt[n]{a_n})$ to l, there exists an $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$ then

$$l - \sqrt[n]{a_n} \leq |\sqrt[n]{a_n} - l| < l - r$$

Hence,

$$\sqrt[n]{a_n} > r$$
$$a_n > r^n$$

Then $\sum_{n=1}^{\infty} r^n$ is a geometric series with $r > 1$. By the geometric series test, this series diverges. Because $r^n < a_n$ for each $n \geq N$, by the strengthened comparison test, $\sum_{n=1}^{\infty} a_n$ diverges as well. □

Question 5

Let (s_n) and (b_n). Define $s_0 \stackrel{\text{def}}{=} 0$.

(a)

Proof. For each natural number n,

$$\sum_{k=1}^{n} b_k(s_k - s_{k-1})$$
$$= (b_1 s_1 - b_2 s_1) + (b_2 s_2 - b_3 s_2) + (b_3 s_3 - b_4 s_3) + ... + b_n s_n$$
$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + s_3(b_3 - b_4) + .. + b_n s_n$$
$$= \sum_{k=1}^{n-1} s_k(b_k - b_{k+1}) + b_n s_n$$
$$= s_n b_n - \sum_{k=1}^{n-1} s_k(b_{k+1} - b_k)$$

□

(b)

Proof. Let m and n be natural numbers such that $m > n$. Then,

$$\sum_{k=n+1}^{m} b_k(s_k - s_{k-1})$$
$$= b_{n+1}s_{n+1} - b_{n+1}s_n + b_{n+2}s_{n+2} - b_{n+2}s_{n+1} + b_{n+3}s_{n+3} - b_{n+3}s_{n+2} + ... + b_m s_m - b_m s_{m-1}$$
$$= -b_n s_n + (b_n s_n - b_{n+1}s_n) + ... + b_m s_m$$
$$= -b_n s_n - \sum_{k=n+1}^{m} s_{k-1}(b_k - b_{k-1}) + b_m s_m$$

□

Question 6

Proof. Let m and n be natural numbers such that $m > n$. Let $(a_n) = 1$ and let (b_n) be a monotone sequence of real numbers. Then,

$$\sum_{k=n+1}^{m} |b_k - b_{k-1}| = \sum_{k=n+1}^{m} a_n |b_k - b_{k-1}|$$

Assume that (b_n) is nondecreasing, then $|b_k - b_{k-1}| = |b_k| - |b_{k-1}|$. By question 5.b,

$$\sum_{k=n+1}^{m} |b_k| - |b_{k-1}| = |b_m| - b_n| - \sum_{k=n+1}^{m} |b_{k-1}|(1-1) = |b_m - b_n|$$

Assume that (b_n) is nonincreasing, then $|b_k - b_{k-1}| = -|b_k| + |b_{k-1}|$. Hence,

$$\sum_{k=n+1}^{m} |b_k - b_{k-1}| = \sum_{k=n+1}^{m} -|b_k| + |b_{k-1}|$$

$$= -|b_m| + |b_n| + \sum_{k=n+1}^{m} |b_{k-1}|(1-1) = |b_m - b_n|$$

$\square$

Question 7

Proof. Let (a_n) be a sequence of real numbers and let (b_n) be a convergent, monotone sequence of real numbers such that $\lim_{n\to\infty} b_n = 0$. Define

$$s_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k$$

Assume that (s_n) is bounded. We also know that since (b_n) is monotone and convergent to 0, it is also bounded. Also, this implies that (b_n) is nonegative. Hence, there exists and $L, M > 0$ such that

$$|b_n| < M$$
$$|s_n| < L$$

Define,

$$t_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k b_k$$

Since $a_n = s_n - s_{n-1}$, by the summation by parts formula, for $m, n \in \mathbb{N}$

$$\begin{aligned}|t_m - t_n| = |\sum_{k=n+1}^{m} b_k(s_k - s_{k-1})| &= |s_m b_m - s_n b_n - \sum_{k=n+1}^{m} s_{k-1}(b_k - b_{k-1})| \\ &\leq |b_m(s_m - s_n) - L \sum_{k=n+1}^{m} (b_k - b_{k-1})| \\ &= |b_m(s_m - s_n) - L|b_m - b_n||\end{aligned}$$

Since (b_n) is convergent, we can choose an $N \in \mathbb{N}$ sufficiently large so that $b_m(s_m - s_n) < \epsilon$ and $|b_m - b_n| < \epsilon/2$. Consequently,

$$|t_m - t_n| < |\epsilon - \frac{\epsilon}{2}| < \frac{\epsilon}{2} < \epsilon$$

Hence, $\sum_{k=1}^{\infty} a_k b_k$ is convergent. □

Question 8

Proof. Let (a_n) be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent. Let (b_n) be a bounded monotone sequence of real numbers. Then we know that there exists some $l \in \mathbb{R}$ so that for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $|b_n - l| < \epsilon$. Without loss of generality assume (b_n) is nondecreasing. Define,

$$c_n \stackrel{\text{def}}{=} l - b_n$$

Then,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} l(a_n) - \sum_{n=1}^{\infty} a_n c_n$$

The first part is convergent since the series $\sum_{n=1}^{\infty} a_n$ is convergent and the second part is convergent by question 7. Consequently,

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent. $\square$