# 163 HWK 4

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## Question 1

*Proof.* Let $(a_n)$ and $(b_n)$ be sequences of real numbers such that there exists a natural number $N$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then

$$0 \leq a_n \leq b_n$$

Then,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{N} b_n + \sum_{n=N+1}^{\infty} b_n$$

By the first iteration of the comparison test if $\sum_{n=N+1}^{\infty} b_n$ converges, then $\sum_{n=N+1}^{\infty} a_n$ converges. Also, if $\sum_{n=N+1}^{\infty} a_n$ diverges we know that $\sum_{n=N+1}^{\infty} b_n$ diverges as well. Hence, the same is true of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ since they are only offset by a constant. $\square$

## Question 2

Let p be a real number. Consider the series,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

### (a)

*Proof.* Assume that $p < 0$. Then for each $n \in \mathbb{N}$ such that $n > 2$,

$$\frac{1}{n(\ln(n))^p} > \frac{1}{n} > 0$$

Hence, by Question 1, since we know that because

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

diverges,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

does as well. $\square$

## (b)

*Proof.* Assume that $p \geq 0$. Hence both of the following series,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$
$$\sum_{k=1}^{\infty} \frac{2^k}{2^k(\ln(2^k))^p}$$

have only nonnegative terms. Let $(s_n)$ and $(t_k)$ be the respective sequences of partial sums.

($\Rightarrow$) Assume that $(s_n)$ is bounded. Then there exists $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$. Let k be a nonnegative integer. By the Archimedean property there exists an $N > 2^k$. Hence,

$$s_N = \frac{1}{2\ln(2)^p} + ... + \frac{1}{N\ln(N)^p} > \frac{1}{2\ln(2)^p} + ... + \frac{1}{2^k\ln(2^k)^p}$$

Because $(\frac{1}{n(\ln(n))^p})$ is decreasing,

$$\begin{aligned} s_N &> \frac{1}{2\ln(2)^p} + \left(\frac{1}{4\ln(4)^p} + \frac{1}{4\ln(4)^p}\right) + ... + \frac{1}{2^k\ln(2^k)^p} \\ &> \frac{1}{2\ln(2)^p} + \frac{2}{4\ln(4)^p} + ... + \frac{2^{k-1}}{2^{kp}} \\ &> \frac{1}{2}t_k \end{aligned}$$

Thus, $(t_k)$ is bounded by $2M$.

($\Leftarrow$) Assume that $(t_k)$ is bounded. Then there exists some $M > 0$ such that for each nonnegative integer k, $|t_k| < M$. Let n be a natural number. Because

$2^k$ diverges to $\infty$. Hence, there exists some nonnegative integer $K$ such that $n < 2^K$. Then,

$$\begin{aligned} s_n &= 1 + ... + \frac{1}{n^p} \\ &< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + ... + \frac{1}{2^k} = 1 + \frac{2}{2^p} + ... + \frac{2^k}{2^{kp}} \\ &< 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + ... + \frac{1}{2^{k^{p-1}}} \\ &< t_k \end{aligned}$$

Hence, $(s_n)$ is bounded by M. □

## (c)

*Proof.* ($\Rightarrow$) First assume that $p > 1$. Then,

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k(\ln(2^k))^p} = \sum_{k=1}^{\infty} \frac{1}{k^p \cdot \ln(2)^p}$$

Hence, by the p-series test, this series is convergent. By part (b), $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ is also convergent.

($\Leftarrow$) Assume that $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ is convergent. Then, by part (a), $p \geq 0$. So by part (b),

$$\sum_{k=1}^{\infty} \frac{1}{k^p \cdot \ln(2)^p}$$

is convergent. Hence, by the p-series test, $p > 1$. □

# Question 3

Let $(a_n)$ be a nonnegative sequence of real numbers. Let $(b_n)$ be a positive sequence of real numbers.

## (i)

*Proof.* Assume that $(\frac{a_n}{b_n})$ is convergent and $\lim\limits_{n\to\infty}(\frac{a_n}{b_n}) > 0$. By the definition of convergence, there exists some $l > 0$ such that for each $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\begin{aligned} |\frac{a_n}{b_n} - l| &< \epsilon \\ l - \epsilon < \frac{a_n}{b_n} &< l + \epsilon \\ (l-\epsilon)b_n < a_n &< (l+\epsilon)b_n \end{aligned}$$

Hence there exists an epsilon sufficiently small enough such that $(l-\epsilon) > 0$, then

$$b_n < \frac{a_n}{(l-\epsilon)} < a_n$$

By the comparison test, if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ does as well. Also, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ does as well. $\square$

### (ii)

*Proof.* Assume that $(\frac{a_n}{b_n})$ is convergent, $\lim\limits_{n\to\infty}(\frac{a_n}{b_n}) = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent. By the definition of convergence, there exists some $l > 0$ such that for each $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then

$$|\frac{a_n}{b_n}| < \epsilon$$
$$a_n < b_n \cdot \epsilon$$

This must work for $\epsilon < 1$, hence,

$$0 \leq a_n < b_n$$

for each natural n. Thus, by the comparison test, since $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=1}^{\infty} a_n$ is also convergent. $\square$

### (iii)

*Proof.* Assume that $(\frac{a_n}{b_n})$ diverges to $\infty$, and $\sum_{n=1}^{\infty} b_n$ is divergent. By the definition of diverging to infinity, for each $M > 0$ there exists an $N \in \mathbb{N}$ for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\frac{a_n}{b_n} > M$$

Hence,

$$a_n > M \cdot b_n$$

Hence, $(a_n)$ is unbounded and thus $\sum_{n=1}^{\infty} a_n$ is divergent. $\square$

## Question 4

*Proof.* Let $(a_n)$ be a sequence of positive real numbers such that the sequence $(a_{n+1}/a_n)$ is convergent. Assume that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$$

Hence, there exists an $M > 0$ such that for each natural n, if $n \geq M$, then

$$|\frac{a_{n+1}}{a_n}|> 1$$

Thus for each $k \geq 1$,

$$|a_{N+k}|>|a_N|> 0$$

So, for $\epsilon = a_N$, it is clear that for for $n \geq N$,

$$|a_n| \geq \epsilon$$

Hence $(a_n)$ does not converge to 0. Consequently, the series $\sum_{n=1}^{\infty} a_n$ is divergent. □

# Question 5

## (a)

*Proof.* Let $(b_n)$ be a convergent, nonincreasing sequence of nonnegative real numbers such that $\lim_{n\to\infty} b_n = 0$. Let $(s_n)$ be the sequence of partial sums of the convergent series,

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

and let S be the sum of said series. Hence,

$$\lim_{n\to\infty} s_n = S$$

It follows that

$$|S - s_n| = \sum_{k=n+1}^{\infty} b_k = b_{n+1} + b_{n+2} + ...$$

Thus, because the terms of $(b_n)$ are nonnegative,

$$|S - s_n| \leq b_{n+1}$$

for each natural number n. □

## (b)

*Proof.* We want to approximate with a magnitude of error of 1/10 so we want to get

$$|S - s_n|$$

below 1/10. By observation of the series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

It is clear that,

$$b_n = \frac{1}{n}$$

Hence, by part (a),

$$|S - s_n| \leq b_{n+1}$$

So, we need to find the n such that,

$$b_{n+1} = \frac{1}{10}$$

Because,

$$b_{n+1} = \frac{1}{n+1}$$

When $n = 9$,

$$b_{9+1} = \frac{1}{10}$$

Thus, taking $s_9$ will give us an approximation within 1/10 of S.

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} = \frac{1879}{2520}$$

□

# Question 6

## (a)

*Proof.* Consider the nonnegative series,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

Since, $n \geq 0$, for each element of the sequence,

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

Hence, because

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

converges by the ratio test, as $r = \frac{1}{3} < 1$. Hence, by the comparison test,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

is convergent. $\square$

## (b)

*Proof.* Consider the series,

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

Proceeding with the ratio test, we examine

$$\lim_{n\to\infty} \frac{\frac{1}{3^{n+1}-n-1}}{\frac{1}{3^n-n}} = \lim_{n\to\infty} \frac{3^n - n}{3^{n+1} - n - 1} = \lim_{n\to\infty} \frac{\frac{1}{3} - \frac{n}{3^{n+1}-n}}{1 - \frac{n}{3^{n+1}-n} - \frac{1}{3^{n+1}-n}}$$

Since,

$$\lim_{n\to\infty} \frac{1}{3} - \frac{n}{3^{n+1} - n} = \frac{1}{3}$$

and

$$\lim_{n\to\infty} 1 - \frac{n}{3^{n+1} - n} - \frac{1}{3^{n+1} - n} = 1$$

Then

$$\lim_{n\to\infty} \frac{\frac{1}{3^{n+1}-n-1}}{\frac{1}{3^n-n}} = \frac{1}{3} < 1$$

Hence, the series is convergent. $\square$

## (c)

*Proof.* Consider the series,

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}}$$

By the p-series test the series,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is divergent as $p = \frac{1}{2}$. Since, $\ln(1) = 0$ and ln is an increasing function, $\ln(n) \geq 0$, hence for each $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{n}} < \frac{\ln(n)}{\sqrt{n}}$$

Thus by the comparison test our series diverges. $\square$

## (d)

*Proof.* Consider the series

$$\sum_{n=1}^{\infty} \frac{n+7}{(n^2(n^3+1))^{\frac{1}{3}}} = \sum_{n=1}^{\infty} \frac{n+7}{(n^5+n^2)^{\frac{1}{3}}} > \sum_{n=1}^{\infty} \frac{n}{(2n)^{\frac{5}{3}}} = \frac{1}{2^{\frac{5}{3}}} \cdot \sum_{n=1}^{\infty} \frac{1}{(n)^{\frac{2}{3}}}$$

By the p-series test, because $\frac{2}{3} < 1$, the second series diverges. Because these terms are all positive, both series are nonnegative. Hence, by the comparison test,

$$\sum_{n=1}^{\infty} \frac{n+7}{(n^2(n^3+1))^{\frac{1}{3}}}$$

diverges. $\square$

## (e)

*Proof.* Consider the series,

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

Expanding the terms of $\frac{n^n}{n!}$ gives us,

$$\frac{n^n}{n!} = \frac{n \cdot n \cdot ... \cdot n}{1 \cdot 2 \cdot ... \cdot n}$$

Hence,

$$\frac{(n+1)^{n+1}}{(n+1)!} > \frac{n^n}{n!}$$

Thus this sequence is increasing. Also,

$$\frac{1^1}{1!} = 1$$

Consequently $\frac{n^n}{n!} \geq 1$ and thus cannot converge to 0. Hence the series is divergent. $\square$