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Question 1

Proof. Consider the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$f(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } 0 \leq x < 1/2; \\ 1, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Define $P \stackrel{\text{def}}{=} (P_n, P_1)$ where P_n is defined as,

$$P_n \stackrel{\text{def}}{=} \{x/n : x = 1, ...n\}$$

and

$$P_1 \stackrel{\text{def}}{=} \{0, 1\}$$

This partition of the domain results in subrectangles $S \in \mathbb{S}$. Now we can defin for these subrectangles,

$$m(f, S) = \inf\{f(x, y) : x, y \in S\}$$
$$M(f, S) = \sup\{f(x, y) : x, y \in S\}$$

Then,

$$L(f, P) \stackrel{\text{def}}{=} \sum_{S \in \mathbb{S}} m(f, S)\text{vol}(S)$$
$$U(f, P) \stackrel{\text{def}}{=} \sum_{S \in \mathbb{S}} M(f, S)\text{vol}(S)$$

By the definition of the partitions,

$$\text{vol}(S) \stackrel{\text{def}}{=} \frac{1}{n}$$

Take $\frac{1}{2}$ as a candidate for $\sup\{L(f,P) : P \in \mathbb{P}\}$. Then, for each $\epsilon > 0$, take the partition $P = (P_{\frac{1}{\epsilon}}, P_1)$. Then,

$$L(f,P) = \sum_{S \in \mathbb{S}} m(f,S)\text{vol}(S) = \sum_{S \in \mathbb{S}} \epsilon \cdot m(f,S)$$

Only one subrectangle can contain $x = \frac{1}{2}$. The supremum for that subrectangle is 1 only if, $x \geq 1/2$ for the entire subrectangle. In that case,

$$L(f,P) = 0(1/2) + 1(1/2)$$

The other option is the subrectangle containing some $x < 1/2$, in which case

$$L(f,P) > 1/2 - \epsilon$$

Which satisfies the least upper bound approximation property. Hence the lower sum is 1/2. Similarly for the upper sum,

$$U(f,P) < 1/2 + \epsilon$$

which satisfies the greatest lower bound approximation property. Since, the upper and lower integrals are both 1/2, then the function is integrable and,

$$\int_{[0,1]\times[0,1]} f = \frac{1}{2}$$

$\square$

Question 2

Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by $f(x,y) \overset{\text{def}}{=} x + y^2$.

(a)

Proof. For each natural number n, define a partition Q_n of $[0,1]$ by

$$Q_n \overset{\text{def}}{=} (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1)$$

And define a partition P_n of $[0,1] \times [0,1]$ by $P_n \overset{\text{def}}{=} (Q_n, Q_n)$. The subrectangles of P_n are the rectangles,

$$S_{j,k} \overset{\text{def}}{=} \left[\frac{j-1}{n}, \frac{j}{n}\right] \times \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, n$. Then,

$$\text{vol}(S_{j,k}) = \left(\frac{1}{n}\right)^2 \tag{1}$$

And for each rectangle,

$$M(f, S_{j,k}) = f\left(\frac{j}{n}, \frac{k}{n}\right) = \frac{j}{n} + \frac{k^2}{n^2} = \frac{jn + k^2}{n^2}$$
$$m(f, S_{j,k}) = f\left(\frac{j-1}{n}, \frac{k-1}{n}\right) = \frac{j-1}{n} + \frac{(k-1)^2}{n^2} = \frac{jn - n + (k-1)^2}{n^2}$$

$\square$

(b)

Proof. Then,

$$\begin{aligned}
U(f,P_n) &= \sum_{j=1}^{n}\sum_{k=1}^{n} M(f,S_{j,k})\mathrm{vol}(S_{j,k}) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{jn+k^2}{n^2}\cdot\left(\frac{1}{n}\right)^2 \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{jn+k^2}{n^4} \\
&= \frac{1}{n^4}\left(\sum_{j=1}^{n}(jn^2)+\sum_{k=1}^{n}(k^2n)\right) \\
&= \frac{1}{n^4}\left(\frac{n^3(n+1)}{2}+\frac{n^2(n+1)(2n+1)}{6}\right) \\
&= \frac{1}{2}+\frac{1}{2n}+\frac{2}{6}+\frac{1}{6n}+\frac{2}{6n}+\frac{1}{6n^2} \\
&= \frac{5}{6}+\frac{1}{n}+\frac{1}{6n^2}
\end{aligned}$$

and,

$$\begin{aligned}
L(f,P_n) &= \sum_{j=1}^{n}\sum_{k=1}^{n} m(f,S_{j,k})\mathrm{vol}(S_{j,k}) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{jn-n+(k-1)^2}{n^2}\cdot\left(\frac{1}{n}\right)^2 \\
&= \frac{1}{n^4}\left(n^2\sum_{j=1}^{n}(j-1)+n\sum_{k=1}^{n}(k-1)^2\right) \\
&= \frac{1}{n^4}\left(\frac{n^3(n+1)}{2}-n^3+n\sum_{k=1}^{n}(k^2-2k+1)\right) \\
&= \frac{1}{n^4}\left(\frac{n^3(n+1)}{2}-n^3+\frac{n^2(n+1)(2n+1)}{6}-n^2(n+1)+n^2\right) \\
&= \frac{1}{n^4}\left(\frac{n^3(n+1)}{2}-2n^3+\frac{n^3(n+1)(2n+1)}{6}\right) \\
&= \frac{n^4}{2n^4}+\frac{n^3}{2n^4}-\frac{2n^3}{n^4}+\frac{2n^4+n^3+2n^3+n^2}{6n^4} \\
&= \frac{1}{2}+\frac{1}{2n}-\frac{2}{n}+\frac{2}{6}+\frac{3}{6n}+\frac{1}{6n^2} \\
&= \frac{5}{6}-\frac{1}{n}+\frac{1}{6n^2}
\end{aligned}$$

$\square$

(c)

Proof. It is clear that the function is bounded on its domain, so take $\epsilon > 0$. Then,

$$\begin{aligned} &U(f, P_n) - L(f, P_n) \\ &= \frac{5}{6} + \frac{1}{n} + \frac{1}{6n^2} - \left(\frac{5}{6} - \frac{1}{n} + \frac{1}{6n^2}\right) \\ &= \frac{2}{n^2} \end{aligned}$$

So, we by the archimedean corollary we can choose an $n \in \mathbb{N}$ sufficiently large such that,

$$\frac{2}{n^2} < \epsilon$$

Hence, by the criterion for integrability, f is integrable. $\square$

(d)

Proof. Suppose $\frac{5}{6}$ is the value of the integral of f. Take $\epsilon > 0$. For $n \in \mathbb{N}$,

$$\frac{5}{6} < \frac{5}{6} + \frac{1}{n} + \frac{1}{6n^2} < \frac{5}{6} + \frac{2}{n}$$

Thus, we by the archimedean property, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\frac{5}{6} + \frac{2}{n} < \frac{5}{6} + \epsilon$$

Hence, $\frac{5}{6}$ is the infimum of the set of $U(f, P)$. Similarly, there exists another $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\frac{5}{6} - \frac{1}{n} < \frac{5}{6} - \epsilon$$

Then, because $\frac{1}{n} > \frac{1}{6n^2}$ for $n \in \mathbb{N}$,

$$\frac{5}{6} - \epsilon < \frac{5}{6} - \frac{1}{n} < \frac{5}{6} - \frac{1}{n} + \frac{1}{6n^2} < \frac{5}{6}$$

Hence, $\frac{5}{6}$ is also the supremum for the set of all $L(f, P)$. Thus,

$$\int_{[0,1]\times[0,1]} f = \frac{5}{6}$$

$\square$

Question 3

Let R be a nonempty, closed, bounded rectangle in $\mathbb{R}^n$ such that $\mathrm{vol}(R) > 0$. Let $f : R \to \mathbb{R}$. Assume that there exists a nonempty, finite subset A of R such that for each $\mathbf{x}$ in R, if $\mathbf{x} \notin A$ then $f(\mathbf{x}) = 0$.

(a)

Proof. Because there are a finite amount of points in A, there exists a maximum and minimum value. Hence, the function is bounded on A. Thus there exists an $M > 0$ such that for each $\mathbf{x} \in A$,

$$|f(\mathbf{x})| \leq M$$

Then, by assumption, for every $\mathbf{x} \in R$,

$$|f(\mathbf{x})| \leq M$$

As $f(\mathbf{x}) = 0$ at all other points. □

(b)

Let $\epsilon > 0$. Let $I_1, \ldots, I_n$ be intervals such that $R = I_1 \times \ldots \times I_n$. Let N be the number of elements in A. For each $k = 1, \ldots n$

- Let P_k' be the partition of I_k formed by the endpoints of I_k and the kth coordinate of points in A.
- Let P_k'' be a partition of I_k with subintervals of length less than

$$\frac{1}{2N}\left(\frac{\epsilon}{2M}\right)^{1/n}$$

- Let P_k be the common refinement of P_k' and P_k''.

Define $P \stackrel{\text{def}}{=} (P_1, \ldots P_n)$ and let $\mathbb{S}$ denote the set of subrectangles of P. Let S_1 denote the set of subrectangles that contain a point in A and let S_2 be the set of subrectangles that do not contain a point in A.

(i)

Proof. There are N points in A. Examining a single dimension of $\mathbb{R}^n$ like the number line, it is clear by the construction of our intervals that these points can be a part of a maximum of 2 subrectangles if they are at the intersection between two subrectangles. Hence, there are a maximum of $2N$ subrectangles with elements of A in one dimension. Expanding this to $\mathbb{R}^n$ requires only taking,

$$(2N)^n$$

as the same is true for each dimension up to n. □

(ii)

Proof. Proceed considering two cases:

Case 1:

Assume that there are no points for A in S. Then $f(\mathbf{x}) = 0$ and so

$$m(f, S) = 0 = M(f, S)$$

Hence,

$$-M \leq m(f, S) \leq 0 \leq M(f, S) \leq M$$

Case 2:

Assume that there are a finite amount of points from A in the subrectangle. Then we know that there must be some points not from A in the subrectangle so for $\mathbf{a} \in A$

$$m(f, S) = \min\{0, f(\mathbf{a})\}$$
$$M(f, S) = \max\{0, f(\mathbf{a})\}$$

Hence,

$$m(f, S) \leq 0 \leq M(f, S)$$

Because f is bounded, it also follows that,

$$-M \leq m(f, S) \leq 0 \leq M(f, S) \leq M$$

□

(iii)

Proof. We know that

$$\text{vol}(S) \leq \left(\frac{1}{2N}\left(\frac{\epsilon}{2M}\right)^{1/n}\right)^n = \frac{1}{(2N)^n}\frac{\epsilon}{2M}$$

Hence,

$$-M \leq m(f, S) \leq 0 \leq M(f, S) \leq M$$
$$\frac{-\epsilon}{2(2N)^n} \leq \frac{m(f, S)\epsilon}{2M(2N)^n} \leq m(f, S)\text{vol}(S) \leq 0 \leq M(f, S)\text{vol}(S) \leq \frac{M(f, S)\epsilon}{2M(2N)^n} \leq \frac{\epsilon}{2(2N)^n}$$

Then, summing over each $S \in \mathbb{S}$

$$\sum_{S \in \mathbb{S}} \frac{-\epsilon}{2(2N)^n} \leq \sum_{S \in \mathbb{S}} m(f, S)\text{vol}(S) \leq 0 \leq \sum_{S \in \mathbb{S}} M(f, S)\text{vol}(S) \leq \frac{\epsilon}{2(2N)^n}$$
$$\frac{-\epsilon}{2}\sum_{S \in \mathbb{S}} \frac{1}{(2N)^n} \leq L(f, P) \leq 0 \leq U(f, P) \leq \frac{\epsilon}{2}\sum_{S \in \mathbb{S}} \frac{1}{(2N)^n}$$

By part (i), we know that there are a maximum of $(2N)^n$ subrectangles so,

$$\sum_{S \in \mathbb{S}} \frac{1}{(2N)^n} \leq 1$$

Consequently

$$\frac{-\epsilon}{2} \leq \sum_{S \in \mathbb{S}} \frac{-\epsilon}{2(2N)^n}$$
$$\frac{\epsilon}{2} \sum_{S \in \mathbb{S}} \frac{1}{(2N)^n} \leq \frac{\epsilon}{2}$$

So,

$$\frac{-\epsilon}{2} \leq L(f,P) \leq 0 \leq U(f,P) \leq \frac{\epsilon}{2}$$

□

(c)

Proof. By part(iii),

$$\frac{-\epsilon}{2} \leq L(f,P) \leq 0 \leq U(f,P) \leq \frac{\epsilon}{2}$$

Thus,

$$\inf\{U(f,P) : P \in \mathbb{P}\} = 0$$
$$\sup\{L(f,P) : P \in \mathbb{P}\} = 0$$

As by the approximation property for least upper and greatest lower bounds, for each $\epsilon > 0$

$$0 - \epsilon < \frac{-\epsilon}{2} \leq L(f,P) \leq 0$$
$$0 \leq U(f,P) \leq \frac{\epsilon}{2} < 0 + \epsilon$$

Because they upper and lower integrals are equal, f is integrable and

$$\int_R f = 0$$

□

Question 4

Proof. Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) \overset{\text{def}}{=} \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

First take the following integral,

$$\begin{aligned} &\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx \\ &= \int_0^1 \left(\frac{x}{2x^4} - 0 \right) dx \\ &= \int_0^1 \frac{1}{2x^3} dx \\ &= \left[\frac{-1}{4x^2} \right]_0^1 \\ &= \frac{-1}{4} \end{aligned}$$

Then the other direction,

$$\begin{aligned} &\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy \\ &= \int_0^1 \left(\frac{-y}{2y^4} - 0 \right) dy \\ &= -\int_0^1 \frac{1}{2y^3} dy \\ &= -\left[\frac{-1}{4y^2} \right]_0^1 \\ &= \frac{1}{4} \end{aligned}$$

Clearly these two values are not equal. I think that that the reason that Fubini's theorem does not apply is because the function is not continuous at 0. □