20250 HWK 2

James Gillbrand

April 2024

Question 1

(i)

Proof. We will attempt to construct an inverse for the following function to show that it is bijective

$$\begin{gathered} F : Maps(V_1 \oplus V_2, W) \to Maps(V_1, W) \times Maps(V_2, W) \\ \phi \mapsto (\phi \circ \iota_1, \phi \circ \iota_2) \end{gathered}$$

So if we define a function as follows:

$$\begin{gathered} G : Maps(V_1, W) \times Maps(V_2, W) \to Maps(V_1 \oplus V_2, W) \\ (\phi_1, \phi_2) \mapsto \phi_1 \circ \pi_1 + \phi_2 \circ \pi_2 \end{gathered}$$

Then,

$$\begin{gathered} F \cdot G(\phi_1, \phi_2) = F(\phi_1 \circ \pi_1 + \phi_2 \circ \pi_2) \\ = (\phi_1 \circ \pi_1 \circ \iota_1 + \phi_2 \circ \pi_2 \circ \iota_1, \phi_1 \circ \pi_1 \circ \iota_2 + \phi_2 \circ \pi_2 \circ \iota_2) \\ = (\phi_1, \phi_2) \end{gathered}$$

Also let $v_1 \in V_1$ and $v_2 \in V_2$, then

$$\begin{gathered} G \cdot F(\phi)(v_1, v_2) = G(\phi \circ \iota_1(v_1), \phi \circ \iota_2(v_2)) \\ = (\phi \circ \iota_1 \circ \pi_1(v_1) + \phi \circ \iota_2 \circ \pi_2(v_2)) \\ = \phi(v_1, 0) + \phi(0, v_2) \\ = \phi(v_1, v_2) \end{gathered}$$

Hence, both compositions are the identity so thus G is an inverse for F. Consequently, a bijection has been constructed. $\square$

(ii)

Proof. We will attempt to construct an inverse for the following function to show that it is bijective,

$$F : (W, V_1 \oplus V_2) \to (W, V_1) \times (W, V_2)$$
$$\phi \mapsto (\pi_1 \circ \phi, \pi_2 \circ \phi)$$

Consider the function

$$G : (W, V_1) \times (W, V_2) \to (W, V_1 \oplus V_2)$$
$$(\phi_1, \phi_2) \mapsto (\iota_1 \circ \phi_1 + \iota_2 \circ \phi_2)$$

Thus,

$$F \circ G(\phi_1, \phi_2) = F(\iota_1 \circ \phi_1 + \iota_2 \circ \phi_2)$$
$$= (\pi_1(\iota_1 \circ \phi_1 + \iota_2 \circ \phi_2), \pi_2(\iota_1 \circ \phi_1 + \iota_2 \circ \phi_2))$$
$$= (\phi_1, \phi_2)$$

Also,

$$G \circ F(\phi) = G(\pi_1 \circ \phi, \pi_2 \circ \phi)$$
$$= (\iota_1 \circ \pi_1 \circ \phi + \iota_2 \circ \pi_2 \circ \phi)$$
$$= (\phi, \phi) = \phi$$

□

Question 2

Proof. Want to show that

$$Mat(\phi \circ \psi) = Mat(\phi) \cdot Mat(\psi)$$

It is given that the ijth element of the product of two matrices is

$$(Mat(A) \cdot Mat(B))_{ij} = \sum_j Mat(A)_{ij} Mat(B)_{jk}$$

Also,

$$(\phi \circ \psi)(e_i) = \phi(\sum_j Mat(\psi)_{ij} e_j)$$
$$= \sum_j Mat(\psi)_{ij} \phi(e_j)$$
$$= \sum_j (Mat(\psi)_{ij} \sum_k (Mat(\phi)_{jk} e_k))$$
$$= \sum_k (\sum_j (Mat(\psi)_{ij} Mat(\phi)_{jk}) e_k)$$

Since this is the formula for the ith column, the ij element is

$$(\sum_j (Mat(\psi)_{ij} Mat(\phi)_{jk})$$

which agrees with the formula. $\square$

Question 3

Did it

Question 4

Proof. Let $W \subset V$ be a vector subspace equipped with addition and scalar multiplication inherited from V. Because W is a subspace of V, it contains the 0 vector, and is closed under addition and scalar multiplication, inherited from V. Hence, W is a vector space. To show that $i : W \to V$ is a linear map, we need to show that for each $a, b \in W$, and $c \in k$

$$i(a+b) = i(a) + i(b)$$
$$and$$
$$c \cdot i(a) = i(c \cdot a)$$

So, because i is an inclusion,

$$i(a+b) = a + b = i(a) + i(b)$$

Also,

$$c \cdot i(a) = c \cdot a = i(c \cdot a)$$

Hence, i is a linear map. $\square$

Question 5

Proof. Let V_1, V_2 be subspaces of the vector space V. Then, let $a, b \in V_1 \cap V_2$. So,

$$a, b \in V_1$$
$$a, b \in V_1$$

Consequently for any $c \in k$,

$$a + b \in V_1$$
$$a + b \in V_2$$
$$c \cdot a \in V_1$$
$$c \cdot a \in V_2$$

So, $V_1 \cap V_2$ is closed under addition and scalar multiplication. Also, $0 \in V_1 \cap V_2$ because each is a subspace and thus contains 0. Hence, $V_1 \cap V_2$ is a subspace of V.

Let $V = \mathbb{R}^2$. Define $V_1 \stackrel{\text{def}}{=} \{(x,0) : x \in \mathbb{R}\}, V_2 \stackrel{\text{def}}{=} \{(y,y) : y \in \mathbb{R}\}$. Then, if we take $v_1 = (1,0) \in V_1$ and $v_2 = (1,1) \in V_2$. Then

$$v_1 + v_2 = (2,1) \notin V_1 \cup V_2$$

Hence, $V_1 \cup V_2$ is not closed under addition and thus isn't always subspace. $\square$

Question 6

Proof. Let $\phi : V \mapsto V'$ be a surjective linear map. Let $A : \{W \subset V\}, B : \{W' \subset V'\}$. Also, let $ker(\phi) \in W$ for each $w \in A$. Define $F : A \mapsto B$. Given $W' \in A$,

$$F(W') \stackrel{\text{def}}{=} \phi^{-1}(W')$$

Because W' is a subspace, $0 \in W'$. Then $\ker(\phi) \in \phi^{-1}(W')$. Hence $\phi^{-1}(W') \in B$.

Given $W_1', W_2' \in A$. Suppose $F(W_1') = F(W_2')$. Then, take $v \in W_1'$. So

$$\phi^{-1}(v) \subseteq \phi^{-1}(W_2')$$

Hence, $v \in W_2'$ because ϕ is surjective, and so $W_1' \subseteq W_2'$. Without loss of generality $W_1' = W_2'$. Hence, F is injective. Now for surjectivity.

Given $W \in B$, want to find W' such that $F(W') = W$. Let $w' = \phi(w)$. Then

$$W \subseteq \phi^{-1} \circ \phi(w)$$

Also for $v \in \phi^{-1} \circ \phi(v)$, $\phi(v) \in \phi(W)$. So there exists a $w \in W$ so that $\phi(v) = \phi(w)$. Then

$$\phi(v) - \phi(w) = 0$$
$$\phi(v - w) = 0$$

So $v - w \in ker(\phi) \in W$. Because $w \in W$ and $v - w \in W$. Then because W is a subspace, $v \in W$. Hence $F(W') = \phi^{-1} \circ \phi(W) = W$. Hence F is injective. Thus F is a bijection.

$\square$

Question 7

Proof. Since $\pi^2 = \pi$, for any $v \in V$

$$\pi^2(v) = \pi(v)$$
$$\pi^2(v) - \pi(v) = 0$$
$$\pi(v) \cdot (\pi(v) - 1(v)) = 0$$

Since we are considering all $v \in V$, $\pi(v) = im(\pi)$. Then we know that $ker(\pi - 1)$ is the set of all v such that $\pi(v) - v = 0$. Hence it is all v such that $v = \pi(v)$. For each, $m \in im(\pi)$. Then $\pi(m) = m$. Hence $ker(\pi - 1) = im(\pi)$

$ker(\pi) = \{v : \pi(v) = 0\}$. Also $im(\pi) = \{v : \pi(v) = v\}$. Hence the only shared point is $v = 0$ and thus these subspaces are complementary.

□

Question 8

Proof. Consider V/W. We want to show that it is a vector space when equipped with the operations of addition and scalar multiplication. Define $\pi : V \to V/W$. Let $v, v', v'' \in V$. Let$c, d \in k$.

Addition:

- $\pi(v + v') = \pi(v) + \pi(v')$ because π is a linear function. Hence, $\pi(v + v') \in V/W$
- $\pi(v + v') = \pi(v) + \pi(v') = \pi(v' + v)$.
- $\pi(v + v') + \pi(v'') = \pi(v) + \pi(v') + \pi(v'') = \pi(v) + \pi(v' + v'')$
- Because W is a subset, $0 \in W$. So $\pi(0) = 0$. Also, $\pi(v+0) = \pi(v)+\pi(0) = \pi(v)$. Hence an additive identity exists.
- Because v is in a vector space, there exists some additive inverse, $-v$. Hence, $\pi(v) + \pi(-v) = \pi(v - v) = \pi(0) = 0$. Thus V/W is equipped with an additive inverse.

Hence it fulfills the requirements of addition.

Scalar Multiplication:

- $\pi(c \cdot v) = c \cdot \pi(v)$ because π is a linear map. Hence, $c \cdot \pi(v) \in V/W$.
- $c \cdot (\pi(v + v')) = c \cdot \pi(v) + c \cdot \pi(v') = \pi(c \cdot v) + \pi(c \cdot v')$
- $d \cdot \pi(c \cdot v) = c \cdot \pi(d \cdot v)$
- $1 \cdot \pi(v) = \pi(1 \cdot v) = \pi(v)$

Hence V/W is a vector space. □