162 HWK 6

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Question 1

(1.a)

Proof. Define $f : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ as

$$f(x) \stackrel{\text{def}}{=} \tan(x)$$

By Proposition 48 in the course notes,

$$f'(x) = \sec^2(x)$$

Thus f is differentiable and hence continuous $\square$

(1.b)

Proof. Let $a, b \in (\frac{-\pi}{2}, \frac{\pi}{2})$ such that $f(a) = f(b)$. Thus,

$$\tan(a) = \tan(b)$$
$$\frac{\sin(a)}{\cos(a)} = \frac{\sin(b)}{\cos(b)}$$
$$\sin(a)\cos(b) = \sin(b)\cos(a)$$
$$\sin(a)\cos(b) - \sin(b)\cos(a) = \sin(a - b) = 0$$

Because $a, b \in (\frac{-\pi}{2}, \frac{\pi}{2})$, $(a - b) \in (-\pi, \pi)$. For $n \in \mathbb{Z}$, $\sin(x) = 0$ when $x = 2\pi n$. The only such point in $(-\pi, \pi)$ is at $x = 0$. Thus

$$a - b = 0$$
$$a = b$$

So, by the definition of injectivity, tan(x) is injective on $(\frac{-\pi}{2}, \frac{\pi}{2})$ $\square$

(1.c)

Part 1

Proof. Let M be a positive real number. Because $\lim_{x \to \frac{\pi}{2}^-} \cos(x) = 0$. For each $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < \frac{\pi}{2} - x < min\{\delta, \frac{\pi}{2}\}$$

then

$$\sin(x) > 0$$
$$|\cos(x)| < \epsilon$$

Since we are approaching from the left,

$$|\cos(x)| = \cos(x)$$

Thus, there exists a δ such that

$$\cos(x) < M \sin(x)$$
$$\frac{\cos(x)}{\sin(x)} < M$$
$$\frac{\sin(x)}{\cos(x)} > M$$
$$\tan(x) > M$$

Hence, for any $M > 0$, there exists an x close enough to $\frac{\pi}{2}$ such that $\tan(x) > M$. Hence tan approaches infinity as $x \to \frac{\pi}{2}$ □

Part 2

Proof. Let M be a negative real number. Because $\lim_{x \to -\frac{\pi}{2}^+} \cos(x) = 0$. For each $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < \frac{\pi}{2} + x < min\{\delta, \frac{\pi}{2}\}$$

then

$$\sin(x) < 0$$
$$|\cos(x)| < \epsilon$$

Since we are approaching from the right,

$$|\cos(x)| = \cos(x)$$

Thus, there exists a δ such that

$$\begin{aligned}
\cos(x) &< M\sin(x) \\
\frac{\cos(x)}{\sin(x)} &> M \\
\frac{\sin(x)}{\cos(x)} &< M \\
\tan(x) &< M
\end{aligned}$$

Hence, for any $M < 0$, there exists an x close enough to $-\frac{\pi}{2}$ such that $\tan(x) < M$. Hence tan approaches infinity as $x \to -\frac{\pi}{2}$ □

(1.d)

Proof. Let y be any real number. By part c, for any y there exists $a, b \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$f(a) < y < f(b)$$

Thus because $\tan(x)$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$, it is also continuous on $[a, b]$. Thus by the intermediate value theorem, there exists a $c \in [a, b]$ such that

$$\tan(c) = y$$

□

Question 2

Proof. Let y be a real number. Define $f : (0, \infty) \to \mathbb{R}$ as

$$f(x) \stackrel{\text{def}}{=} \ln(x) = \int_1^x \frac{1}{t} dt$$

By the First Fundamental Theorem of Calculus,

$$f'(x) = \frac{1}{x}$$

Hence f is differentiable and thus continuous on its domain. As established in class, the image of $\ln(x)$ is all real numbers. As such, for each y there exists some $a, b \in (0, \infty)$ such that $f(a) < y < f(b)$. Hence, by the Intermediate Value Theorem, there exists a $c \in (a, b)$ such that $f(c) = y$.

□

Question 3

(3.a)

Proof. Let a and b be real numbers such that $a < b$. Let $f : [a,b] \to \mathbb{R}$ be continuous on [a,b], differentiable on (a,b), and $f(x) > 0$ for each x in (a,b). Thus, both $f(x)$ and $ln(x)$ are differentiable on (a,b), so by the Chain Rule, $\ln \circ f(x)$ is differentiable on (a,b).

By the Chain Rule,

$$(\ln \circ f(x))' = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$$

□

(3.b)

Proof. By part a, $(\ln \circ f(x))$ is an antiderivative for $\frac{f'(x)}{f(x)}$. Thus by the Second Fundamental Theorem of Calculus,

$$\int_a^b \frac{f'(x)}{f(x)} dx = (\ln \circ f(b)) - (\ln \circ f(a))$$

□

Question 4

(4.a)

Proof. Define $g : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ by

$$g(x) \stackrel{\text{def}}{=} \ln(|x|)$$

$\ln(x)$ is differentiable and $|x|$ is differentiable for $x \neq 0$. Because 0 isn't in the domain of g, g is differentiable by the Chain Rule.

Case 1:

$x > 0$

$$|x| = x$$
$$\ln(|x|) = \ln(x)$$

Thus,

$$g'(x) = (\ln(x))' = \frac{1}{x}$$

for $x > 0$.

Case 2:

$x < 0$

$$|x| = -x$$
$$\ln(|x|) = \ln(-x)$$

By the Chain Rule,

$$g'(x) = (\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$$

for $x < 0$.

Hence, for $x \neq 0$, $g'(x) = \frac{1}{x}$ $\square$

(4.b)

Proof. Let a and b be real number such that $a < b$. Let $f : (a, b) \to \mathbb{R}$ be differentiable such that $f(x) \neq 0$ for each x in (a,b).

By part a, $\ln(|x|)$ is differentiable when its domain doesn't include 0. It is given that f is differentiable and that it's image does not include 0. As such by the Chain Rule, $\ln(|f(x)|)$ is differentiable and

$$(\ln \circ |f|)' = \frac{1}{f}f' = \frac{f'}{f}$$

$\square$

Question 5

Proof. Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function. Define $F : [1, \infty) \to \mathbb{R}$ by

$$F(x) = \int_1^x \frac{f(t)}{t} dt$$

Assume f is bounded. Thus, there exist some $M \in \mathbb{N}$ such that for each $x \in (1, \infty)$,

$$|f(x)| < M$$

Hence,

$$|F(x)| = |\int_1^x \frac{f(t)}{t} dt| \leq \int_1^x |\frac{f(t)}{t}| dt$$
$$\leq \int_1^x \frac{M}{t} dt = M \int_1^x \frac{1}{t} dt = M \ln(x)$$

Because $x > 1$, $\ln(1) = 0$, and ln is strictly increasing.

$$\ln(x) > 0$$

Thus,

$$|\ln(x)| = \ln(x)$$

and so

$$\frac{|F(x)|}{|\ln(x)|} = |\frac{F(x)}{\ln(x)}| \leq M$$

Hence, F(x)/ln(x) is bounded.

$\square$