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Question 1

Proof. For each natural number n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) \stackrel{\text{def}}{=} \frac{1}{n(1+x^2)}$$

For any $x \in \mathbb{R}$,

$$x^2 \geq 0$$

Consequently, for each x,

$$|\frac{1}{n(1+x^2)}| \leq \frac{1}{n}$$

Then, for each $\epsilon > 0$, by the archimedean property, there exists an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Hence, for each $n \geq N$,

$$|\frac{1}{n(1+x^2)}| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

Hence, $(f_n(x))$ is uniformly convergent. $\square$

Question 2

Proof. For each natural number n, define $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(x) \overset{\text{def}}{=} \begin{cases} \min\{n, 1/x\} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Let $x \in (0,1]$ and $\epsilon > 0$. Then, by the archimedean property there exists an $N \in \mathbb{N}$ such that $N > \frac{1}{x}$. Hence, for each $n \geq N$, $f_n(x) = \frac{1}{x}$. Consequently,

$$|f_n(x) - \frac{1}{x}| = |\frac{1}{x} - \frac{1}{x}| = 0 < \epsilon$$

If $x = 0$, then

$$|0 - 0| < \epsilon$$

Now to show that each function is bounded, fixing n, each $f_n(x) \leq n$. However, as shown previously,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Because $\frac{1}{x}$ on the domain $(0,1)$ is unbounded, $\lim\limits_{n\to\infty} f_n(x)$ is unbounded. $\square$

Question 3

Proof. Assume that $f_n(x)$ is a uniformly convergent sequence of bounded real-valued functions. Then there exists an $M_n > 0$ for each f_n such that for all $x \in \mathbb{R}$,

$$M_n > |f_n(x)|$$

Also, there exists a real number l and there exists a natural number N such that for each natural number n, if $n \geq N$, then

$$|M_n - l| < 1$$

By the triangle inequality,

$$\begin{gathered} |M_n| - |l| < |M_n - l| < 1 \\ M_n < 1 + |l| \end{gathered}$$

Take,

$$M = \max\{M_1, ...M_{N-1}, 1 + |l|\}$$

Then, $M \geq 1 + |l| > 0$ and $M \geq M_n \geq |f_n(x)|$. Hence, the sequence is bounded. □

Question 4

Proof. Let (f_n) be a pointwise convergent sequence of bounded real-valued functions on a nonempty set A of real numbers and let f be the pointwise limit of (f_n).

$(\Rightarrow)$ Assume that $(f_n(x))$ is uniformly convergent. Then, for all $x \in A$ and each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then,

$$|f_n - f| < \epsilon$$

Hence, no matter what $x \in A$,

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0$$

Consequently,

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A\} = 0$$

$(\Leftarrow)$ Assume that,

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A\} = 0$$

Let $\epsilon > 0$. There exists some $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq N$,

$$|\sup\{|f_n(x) - f(x)| : x \in A\} - 0| < \epsilon$$

By the definition of a supremum,

$$|f_n(x) - f(x)| \leq \sup\{|f_n(x) - f(x)| : x \in A\}$$

Hence,

$$|f_n(x) - f(x)| < |\sup\{|f_n(x) - f(x)| : x \in A\}| < \epsilon$$

Consequently, (f_n) is uniformly convergent. $\square$

Question 5

For each natural number n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) \stackrel{\text{def}}{=} \sqrt{x^2 + \frac{1}{n}}$$

(a)

Proof. Let $a, b \geq 0$. Then we want to show the following.

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$$
$$|\sqrt{a} - \sqrt{b}|^2 \leq \sqrt{|a-b|}^2$$
$$a - 2\sqrt{a}\sqrt{b} - b \leq |a-b|$$

Taking the left side,

$$(a - 2\sqrt{a}\sqrt{b} - b) \leq a - b \leq |a-b|$$

Hence the inequality holds. □

(b)

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Then for each $n \geq N$,

$$|f_n(x) - \sqrt{x^2}| \leq \sqrt{|f_n(x) - \sqrt{x^2}|} = \sqrt{|\frac{1}{n}|} = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon$$

Hence (f_n) is uniformly convergent. Each f_n is differentiable as they differ by a constant from x. Then $\lim_{n\to\infty} f_n(x) = \sqrt{x^2}$. By observation, $\sqrt{x^2} = |x|$ which we have proven is not differentiable at $x = 0$. □

Question 6

Proof. For each natural number n, define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) \overset{\text{def}}{=} \frac{x}{1+n^2x^2}$$

Let $\epsilon > 0$ and fix $x \in \mathbb{R}$. Then

$$|\frac{x}{1+n^2x^2}| \leq |\frac{x}{n^2x^2}| = |\frac{1}{n^2x}|$$

Choose $N \in \mathbb{N}$ such that $N > \sqrt{\frac{1}{\epsilon|x|}}$. Then if $n \geq N$,

$$|\frac{1}{n^2x}| \leq \frac{1}{N^2|x|} = \epsilon$$

Hence (f_n) is pointwise convergent. Since x and $(1+n^2x^2)$ are differentiable, by the combination of differentiable functions, each f_n is differentiable. Then it is clear that,

$$f_n'(x) = \frac{(1+n^2x^2) - 2x^2n^2}{(1+n^2x^2)^2} = \frac{1-x^2n^2}{1+2n^2x^2+n^4x^4} = \frac{\frac{1}{n^4} - \frac{x^2}{n^2}}{\frac{1}{n^4} - \frac{2x^2}{n^2} + x^4}$$

Now to show that (f_n') is pointwise convergent. By the algebraic limit theorem, since each term is a pointwise convergent sequence, the entire term is pointwise convergent. Evaluating

$$\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} \frac{1}{1} = 1$$

However,

$$(\lim_{n\to\infty} f_n(0))' = (0)' = 0$$

Hence, the two expressions are not equal. $\square$

Question 7

Define $\phi : \mathbb{R} \to \mathbb{R}$ such that for each $x \in [-1, 1]$,

$$\phi(x) = |x|$$

For each $x \in \mathbb{R}$,

$$\phi(x + 2) = \phi(x)$$

(a)

Proof. For each $k \in \mathbb{Z}$ if $x \neq 2k - 1$ then we know that ϕ is continuous there as it is exactly $|x|$ which is a continuous function. Consequently we must show that

$$\lim_{x \to (2k-1)^-} \phi(x) = 1 = \lim_{x \to (2k-1)^+} \phi(x) = \phi(2k - 1)$$

First, by the properties of ϕ,

$$\phi(2k - 1) = \phi(-1) = 1$$

Also for $k = 1$,

$$\lim_{x \to (2k-1)^-} \phi(x) = \lim_{x \to (-1)^-} \phi(x) = 1$$

And for $k = 0$,

$$\lim_{x \to (2k-1)^+} \phi(x) = \lim_{x \to (-1)^+} \phi(x) = 1$$

Hence, ϕ is continuous. $\square$

(b)

Proof. Take the series of functions

$$\sum_{n=1}^{\infty} \frac{3^n \phi(4^n x)}{4^n}$$

We know that $|\phi| \leq 1$. Hence,

$$|\frac{3^n \phi(4^n x)}{4^n}| \leq \frac{|3^n|}{|4^n|}$$

So, for

$$M_n = \frac{|3^n|}{|4^n|}$$

$$|\frac{3^n \phi(4^n x)}{4^n}| \leq M_n$$

Because $3/4 < 1$ by the geometric series test, $\sum_{n=1}^{\infty} M_n$ is convergent. Thus, by the Weierstrauss M-test, $\sum_{n=1}^{\infty} \frac{3^n \phi(4^n x)}{4^n}$ is uniformly convergent. $\square$

(c)

Proof. Define,

$$f(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{3^n \phi(4^n x)}{4^n}$$

For each $n \in \ltimes$,

$$\frac{3^n \phi(4^n x)}{4^n}$$

is a constant multiple of ϕ. Because ϕ is a continuous function, each f_n is a continuous function. By the continuous uniform limit theorem for series, f is continuous. $\square$