162 HWK 8

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Question 1

Proof. Let n be a nonegative integer and let $a_0, ..., a_n$ be real numbers. Define $P : \mathbb{R} \to \mathbb{R}$ as

$$P(x) \stackrel{\text{def}}{=} \sum_{j=0}^{n} a_j x^j$$

To show that $P_n(x)$ is differentiable of order k for $k \in \mathbb{N}$, we can observe that it is a polynomial and hence differentiable of any order k. To establish the derivative, proceed by induction.

Base Case:

k = 1

Case 1: $k > n$

$$\begin{gathered} 1 > n \\ n = 0 \\ P(x) = 0 \\ P^1(x) = 0 \end{gathered}$$

Case 2: $k \leq n$

$$1 \leq n$$

$$P(x) = \sum_{j=0}^{n} a_j x^j$$

$$P(x) = a_0 + \sum_{j=1}^{n} a_j x^j$$

$$P^1(x) = \sum_{j=1}^{n} j a_j x^{j-1}$$

$$P^1(x) = \sum_{j=1}^{n} \frac{j!}{(j-1)!} a_j x^{j-1}$$

Inductive Step:

Let k be any natural number

Inductive Hypothesis:

Assume that

$$P^{(k)}(x) = \begin{cases} \sum_{j=k}^{n} \frac{j!}{(j-k)!} a_j x^{j-k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Proof of Induction

Case 1: $k > n$

$$P^{(k+1)}(x) = (P^{(k)}(x))' = (0)' = 0$$

Case 2: $k \leq n$

$$P^{(k+1)}(x) = (P^{(k)}(x))' = (\sum_{j=k}^{n} \frac{j!}{(j-k)!} a_j x^{j-k})'$$

$$P^{(k+1)}(x) = (k!(a_k) + \sum_{j=k+1}^{n} \frac{j!}{(j-k)!} a_j x^{j-k})'$$

$$P^{(k+1)}(x) = (k!(a_k))' + \sum_{j=k+1}^{n} \frac{j!(j-k)}{(j-k)!} a_j x^{j-k-1}$$

$$P^{(k+1)}(x) = \sum_{j=k+1}^{n} \frac{j!}{(j-k-1)!} a_j x^{j-k-1}$$

$$P^{(k+1)}(x) = \sum_{j=k+1}^{n} \frac{j!}{(j-(k+1))!} a_j x^{j-(k+1)}$$

Thus, via induction, for each natural k,

$$P^{(k)}(x) = \begin{cases} \sum_{j=k}^{n} \frac{j!}{(j-k)!} a_j x^{j-k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

$\square$

Question 2

Proof. Let $x \in \mathbb{R}$ such that $x \leq 0$ and $n \in \mathbb{N}$ such that $n \geq 0$. Define the open interval I as $(x, 0)$ and $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \exp(x)$$

Define R_n as the nth order remainder of f at 0. Since the derivative of exp is itself, f is differentiable any amount of times. Hence by Langrange's version of the Taylor Theorem, there exists a $c \in I$ such that

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x)^{n+1}$$

Because the derivative of exp is itself, and exp > 0, then $R_n(x) \geq 0$. Also, because f is strictly increasing and since $c < 0$

$$f(c) \leq f(0) = 1$$

Likewise, since $f(x) = f'(x)$

$$f^n(c) \leq f^n(0) = 1$$

Hence,

$$R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$
$$R_n(x) \leq \frac{|x|^{n+1}}{(n+1)!}$$
$$-R_n(x) \geq -\frac{|x|^{n+1}}{(n+1)!}$$
$$-\frac{|x|^{n+1}}{(n+1)!} \leq -R_n(x) \leq 0 \leq R_n(x) \leq \frac{|x|^{n+1}}{(n+1)!}$$
$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

□

Question 3

(a)

Proof. By the definition of a Taylor Polynomial, a polynomial of order n for the function f at point a is

$$P_n = \sum_{k=0}^{n} \frac{f^n(a)}{n!}(x-a)^n$$

Given by the problem, $a = 0$, and $f \stackrel{\text{def}}{=} \ln(x)$.
Hence,

$$\begin{aligned}
f^0(a) &= \ln(x) = \ln(1) = 0 \\
f^1(a) &= \frac{1}{x} = \frac{1}{1} = 1 \\
f^2(a) &= \frac{-1}{x^2} = \frac{-1}{1} = -1 \\
f^3(a) &= \frac{2}{x^3} = \frac{2}{1} = 2
\end{aligned}$$

Thus,

$$\begin{aligned}
P_0 &= 0 \\
P_1 &= \frac{1}{1}(x-1)^1 = x - 1 \\
P_2 &= (x-1) + \frac{-1}{(2)!}(x-1)^2 = (x-1) - \frac{(x-1)^2}{2} \\
P_3 &= (x-1) - \frac{(x-1)^2}{2} + \frac{2}{(3)!}(x-1)^3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}
\end{aligned}$$

□

(b)

Proof. Because $\ln(x)$ exists on $(0, \infty)$ and is differentiable 4 times on this interval. By Lagrange's version of the Taylor Theorem, there exists a $c \in (\frac{1}{2}, \frac{3}{2})$ such that

$$R_3(x) = \frac{f^4(c)}{(4)!}(x-1)^4$$

Also,

$$f^4(x) = \frac{-6}{x^4}$$

Thus,

$$R_3(x) = \frac{-6}{(4)!c^4}(x-1)^4$$

Hence, for c and x in $(\frac{1}{2}, \frac{3}{2})$, $R_3(x)$ is maximised for $c = \frac{1}{2}$ and $x = \frac{1}{2}, \frac{3}{2}$
Consequently for $x \in (\frac{1}{2}, \frac{3}{2})$,

$$R_3(x) \leq R_3(1/2) = \frac{-6}{(4)!(1/2)^4}(1/2-1)^4$$
$$R_3(x) \leq -\frac{1}{4}$$

for each $x \in (\frac{1}{2}, \frac{3}{2})$. The maginitude of error is 1/4. ☐

(c)

Proof. Let n be a positive integer such that the remainder of the Taylor Polynomial of order n, for the function ln, located at 1 when approximating $\frac{5}{4}$ is less than or equal to 0.01. Hence by Taylor's Theorem, there exists a $c \in (1, \frac{5}{4})$ such that,

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-1)^{n+1} \leq 0.01$$

By examining the derivatives of ln, we can construct an expression for derivatives 1 through n.

$$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

Hence,

$$R_n(x) = \frac{(-1)^{n+1}(n)!}{(n+1)!c^{n+1}}(x-1)^{n+1}$$

Because $c \in (1, \frac{5}{4})$, we know that

$$\frac{1}{c^{n+1}} \leq 1$$

And since we are evaluating at $\frac{5}{4}$, let $x = \frac{5}{4}$.
So,

$$|R_n(\frac{5}{4})| \leq |\frac{(n)!}{(n+1)!}(\frac{1}{4})^{n+1}| = \frac{(\frac{1}{4})^{n+1}}{(n+1)} \leq 0.01$$

Hence,

$$n \geq 1.626$$

Consequently, the smallest non-negative n that works is n=2.

$$P_2(\frac{5}{4}) = (\frac{5}{4} - 1) - \frac{(\frac{5}{4} - 1)^2}{2} = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$$

To prove that n=2 works,

$$\frac{1}{4^3(3)} = \frac{1}{192} \leq 0.01$$

□

(d)

Proof. Let r be a real number in $(0, 1)$. By Taylor's Theorem, there exists a $c \in (1 - r, 1 + r)$ such that for each $x \in (1 - r, 1 + r)$

$$|R_2(x)| = |\frac{f^3(c)}{3!}(x - 1)^3|$$

We know that $f^3(x) = \frac{2}{x^3}$ so,

$$|R_2(x)| = |\frac{2}{3!c^3}(x - 1)^3|$$

Hence, for x and c in $(1 - r, 1 + r)$, $R_2(x)$ is maximised for $c = 1 - r$ and $x = 1 - r, 1 + r$. Hence for $x \in (1 - r, 1 + r)$,

$$|R_2(x)| \leq |\frac{2}{3!(1 - r)^3}((1 - r) - 1)^3|$$

The solution to the following inequality provides the maximum r to keep $R_2(x) <$ 0.003 on (1-r,1+r)

$$|\frac{2}{3!(1 - r)^3}((1 - r) - 1)^3| \leq 0.003$$

So,

$$r = 0.17219$$

Then the maximum value of $R_2(x)$ on (0.82781, 1.17219) is:

$$|R_2(x) = |\frac{2}{3!(0.82781)^3}((0.82781) - 1)^3|$$
$$\frac{(0.17219)^3}{3(0.82781)^3} \leq 0.002999$$

Which checks out. □

Question 4

Proof. Evaluate the following using Taylor Polynomials:

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x - \sin(x)}$$

The Taylor Polynomial formula for e^x and $\sin(x)$ at 0 are as follows:

$$\sin(x) : P_n(0) = \sum_{k=0}^{n} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$e^x : P_n(0) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Hence by Langrange's form of Taylor's Theorem, there exists a c,d in (0,x) such that

$$e^x - 1 - x - x^2/2 = R_2(x) = \frac{e^c}{3!}(x^3)$$
$$x - \sin(x) = -R_2(x) = \frac{\cos(d)}{3!}(x^3)$$

Thus,

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x - \sin(x)} = \lim_{x \to 0} \frac{e^c}{\cos(d)}$$

Since c and d are selected from (0,x), as x approaches 0, c and d approach 0. Consequently,

$$\lim_{x \to 0} \frac{e^c}{\cos(d)} = \lim_{x \to 0} \frac{e^x}{\cos(x)}$$

Because both of these functions are continuous, and that $cos(0) \neq 0$, we know that the combination is also continuous. It follows then that

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x - \sin(x)} = \frac{e^0}{\cos 0} = 1$$

□

Question 5

(a)

Proof. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) \overset{\text{def}}{=} \begin{cases} \frac{e^x-1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then, we know that e^x, -1, and $\frac{1}{x}$ are continuous for $x \neq 0$. Hence, by combinations of continuous functions, $\frac{e^x-1}{x}$ is continuous for $x \neq 0$.

Additionally, since both

$$\lim_{x\to 0} e^x - 1 = 0 \tag{1}$$

$$\lim_{x\to 0} x = 0 \tag{2}$$

L'Hopital's rule can be used, yielding

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} e^x = 1$$

Hence, $f(x)$ is continuous for all x. Thus $f(x)$ is integrable on [0,1] $\square$

(b)

Proof. As proven in class, the taylor polynomial for e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!}$$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x)$$

Hence,

$$\frac{e^x-1}{x} = \frac{\sum_{k=1}^{n} \frac{x^k}{k!} + R_n(x)}{x} = \sum_{k=1}^{n} \frac{x^{k-1}}{k!} + \frac{R_n(x)}{x} = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{R_n(x)}{x}$$

Hence, the error in the integral can be represented in the following way:

$$\int_0^1 (f(x) - \sum_{k=1}^{n} \frac{x^{k-1}}{k!})dx = \int_0^1 f(x)dx - \int_0^1 \sum_{k=1}^{n} \frac{x^{k-1}}{k!}dx = \int_0^1 \frac{R_n(x)}{x}dx$$

Suppose n=6, then, by Langrange's form of Taylor's Theorem, we know that there exists a $c \in (0,1)$ such that

$$R_6(x) = \frac{e^c}{7!}(x)^7$$

$$R_6(x) \leq \frac{ex^7}{7!} \leq \frac{3x^7}{7!}$$

So,

$$\int_0^1 \frac{R_7(x)}{x} dx \leq \int_0^1 \frac{3x^6}{7!}$$
$$\int_0^1 \frac{R_7(x)}{x} dx \leq \frac{3(1)^7}{7!(7)} \leq 0.0001 = 10^{-4}$$

Hence, n provides an estimate with error less than 10^{-4}
To estimate the value of the integral,

$$\int_0^1 \sum_{k=0} 56 \frac{x^k}{(k+1)!} dx = \int_0^1 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720}$$
$$\int_0^1 \sum_{k=1}^{4} \frac{x^{k-1}}{k!} dx = [x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \frac{x^6}{4320}]_0^1$$
$$\int_0^1 \sum_{k=1}^{4} \frac{x^{k-1}}{k!} dx = 1 + \frac{1}{4} + \frac{1}{18} + \frac{1}{96} + \frac{1}{600} + \frac{1}{4320} = 1.3179$$

$\square$

Question 6

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that f, f', and f'' are bounded. Define

$$M_0 \stackrel{\text{def}}{=} \sup\{|f(x)| : x \in \mathbb{R}\}$$
$$M_1 \stackrel{\text{def}}{=} \sup\{|f'(x)| : x \in \mathbb{R}\}$$
$$M_2 \stackrel{\text{def}}{=} \sup\{|f''(x)| : x \in \mathbb{R}\}$$

Let $x \in \mathbb{R}$ and $h > 0$.

By the definition of the Remainder,

$$R_n(x) = f(x) - P_n(x)$$

Finding the second order remainder of $f(x)$ at x+h yields,

$$R_1(x+h) = f(x+h) - \sum_{n=0}^{1} \frac{f^n(x)}{n!}(x+h-x)^n$$
$$R_1(x+h) = f(x+h) - f(x) - f'(x)(h)$$

By Lagrange's Taylor Theorem, there exists a $c \in (x, x+h)$ such that

$$R_1(x) = \frac{f''(c)}{2}(x-(x+h))^2 = \frac{f''(c)}{2}(h)^2$$

Hence,

$$\frac{f''(c)}{2}(h)^2 = f(x+h) - f(x) - f'(x)(h)$$
$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{f''(c)}{2}(h)$$

By definition of M_0, and M_2

$$|f(x+h) - f(x)| \leq 2M_0$$
$$|f''(x)| \leq M_2$$

Then,

$$|f'(x)| = |\frac{f(x+h)-f(x)}{h} - \frac{f''(c)}{2}(h)|$$
$$|f'(x)| \leq \frac{|f(x+h)-f(x)|}{h} + \frac{|f''(c)|}{2}(h)$$
$$|f'(x)| \leq \frac{2M_0}{h} + \frac{M_2 h}{2}$$

Define

$$g(h) \stackrel{\text{def}}{=} \frac{2M_0}{h} + \frac{M_2 h}{2}$$

Then, minimizing this function,

$$g'(h) = -\frac{2M_0}{h^2} + \frac{M_2}{2}$$
$$0 = -\frac{2M_0}{h^2} + \frac{M_2}{2}$$
$$h = 2\sqrt{\frac{M_0}{M_2}}$$

Thus,

$$g(2\sqrt{\frac{M_0}{M_2}}) = \frac{2M_0}{2\sqrt{\frac{M_0}{M_2}}} + \frac{M_2 2\sqrt{\frac{M_0}{M_2}}}{2}$$
$$g(2\sqrt{\frac{M_0}{M_2}}) = \frac{M_0}{\sqrt{\frac{M_0}{M_2}}} + M_2\sqrt{\frac{M_0}{M_2}}$$
$$g(2\sqrt{\frac{M_0}{M_2}}) = M_2\sqrt{\frac{M_0}{M_2}} + M_2\sqrt{\frac{M_0}{M_2}}$$
$$g(2\sqrt{\frac{M_0}{M_2}}) = 2M_2\sqrt{\frac{M_0}{M_2}}$$

Hence, because M_1 is the least upper bound for $|f'(x)|$, $M_1 \geq 0$ and,

$$M_1 \leq 2M_2\sqrt{\frac{M_0}{M_2}}$$

Consequently,

$$M_1^2 \leq 4M_2^2\frac{M_0}{M_2} = 4M_2M_0$$

$\square$