20250 HWK 5

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Question 1

Proof. Let $v_1, ...v_r \in V$ be a set of vectors. You can put them into echelon form so that they are linearly independent. Then, for $i = 1, ...m$, if there is no v such that $LC(v) = i$, add e_i to the set.

Implementation: Take the following as the matrix of the set of vectors,

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Reduce it to echelon form as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, by the algorithim we are missing only $i = 5$. Hence, the complement is e_5. $\square$

Question 2

Proof. Let $v_1, ...v_n$ be the reduced echelon basis for subspace W in k^m. $LP(v_r) = n$ iff $e_n \in W$. $LP(v_r) = l$ iff $\langle e_{l+1}, ...e_n \rangle \notin W$. Also, $\langle e_l, ...e_n \rangle \cap W = \langle v_r \rangle$. Then, begin with $\langle e_n \rangle$, if $\dim(\langle e_n \rangle \cap W) = 1$ there exists a unique v_n in reduced echelon form in the basis of W. Continue for each $e_n, e_{n-1}, ...e_2$. Then, if $dim(\langle e_1, ...e_n \rangle \cap W) = dim(W)$ there exists some unique v_1. Hence we have a unqiue reduced echelon form set of vectors that make up the basis. $\square$

Question 3

Proof. For d=0, there is 1 subspace since dim = 0.

For d=1, there are p^5 possible vectors to choose from. Ignoring the zero vector $p^5 - 1$. However, we must ignore vectors that are scalar multiples of our first vector as they produce the same subspace. This amounts to removing $p-1$ such vectors. Hence, there are $\frac{p^5-1}{p-1} = p^4 + p^3 + p^2 + p + 1$ subspaces for d=1.

For d=2, adding up all of the possible vectors yields that there are a total of $p^6 + p^5 + 2p^4 + 2p^3 + 2p^2 + p + 1$ subspaces.

For d=3, the number of subspaces if given by,

$$n = p^6 + p^5 + 2p^4 + 2p^3 + 2p^2 + p + 1$$

For d=4, the number of subspaces if given by,

$$n = p^4 + p^3 + p^2 + p + 1$$

For d=5, the number of subspaces if given by,

$$n = 1$$

□

Question 4

Proof. Let e_1 = the price of Avasa horses, e_2 = the price of Haya horses, and e_3 = the price of camels. Let K be value of one person.. After the exchange, $5e_1 + e_2 + e_3 = e_1 + 7e_2 + e_3 = e_1 + e_2 + 8e_3 = K$. Hence,

$$4e_1 = 6e_2 = 7e_3 = K - e_1 - e_2 - e_3$$

In order for these values to be integers, $K - e_1 - e_2 - e_3$ must be a multiple $4, 6$ and 7. The lowest common multiple is 84. Given this,

$$\begin{aligned} 4e_1 = 6e_2 = 7e_3 &= 84 \\ e_1 &= 21 \\ e_2 &= 14 \\ e_3 &= 12 \end{aligned}$$

Consequently, $K = 84 + 21 + 14 + 12 = 131$. The total value of the animals is $3 * K = 393$. $\square$

Question 5

Proof. We want to find a vector that can be expressed in terms of $v_1, ...v_r = V$ and $w_1, ...w_r = W$. Hence, we want to find an X such that,

$$(V|-W)X = 0$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} X = 0$$

Hence,

$$X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Consequently, $(v_1, v_2, v_3) * \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (w_1, w_2, w_3) * \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ □

Question 6

Proof. Let $[\phi]_{B,C}$ be the matrix that represents the linear map ϕ. Then, the rows of $[\phi]_{B,C}$ are covectors for ϕ. Hence when we take the transpose of $[\phi]_{B,C}$ we swap rows and columns, hence the new matrix takes in elements of $C^{\vee}$ and outputs elements of $B^{\vee}$. Thus,

$$[\phi^{\vee}]_{C^{\vee}B^{\vee}} = ([\phi]_{B,C})^T$$

□

Question 7

Proof. Let $v_1, ...v_n$ be a basis for k^n. Write A as the matrix composed of the column vectors $v_1, ...v_n$. Then take the inverse of A as,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Hence, by the properties of an inverse,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Also,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} = \begin{bmatrix} a_1(v_1) & a_1(v_2) & \dots & a_1(v_n) \\ a_2(v_1) & a_2(v_2) & \dots & a_2(v_n) \\ \vdots & \vdots & \ddots & \vdots \\ a_n(v_1) & a_n(v_2) & \dots & a_n(v_n) \end{bmatrix}$$

Hence for each $i \neq j$, $a_i v_j = 0$ and otherwise, $a_i v_j = 1$. Thus by definition each $a_i = v_i^\vee$. $\square$