20250 HWK 3

James Gillbrand

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Question 1

Proof. Let W and W' be complementary subspaces of a vector space V. Show that the composite function defined by $A : W' \to V \xrightarrow{\pi} V/W$ is an isomorphism.

Since W and W' are complementary subspaces, $W \cup W' = 0$. Hence, $ker(A) = 0$ and thus A is injective.

By the surjectivity of π, for any $x \in V/W$ and there exists $v \in V$ so that $\pi(v) = x$. However, since W and W' are complementary subspaces, v can be written as $v = w + w'$ where $w \in W$ and $w' \in W'$. So $x = \pi(w) + \pi(w')$ because π is a linear map. However, $\pi(w) = 0$ since $w \in W$ so $x = \pi(w)$. Hence for each $x \in V/W$, there exists a $w \in W$ so that $A(w) = 0$. So A is surjective.

Because A is both injective and surjective, it is an isomorphism. $\square$

Question 2

(a) $\Leftrightarrow$ (d)

Proof. ($\Rightarrow$) Let ϕ be a splitting of $V \to V_1 \bigoplus V_2$. Then, by the short exact sequence, ϕ is an isomorphism. Hence,

$$\phi^{-1}(im(\iota_2))$$

where ι_2 is a function $V_2 \to V_1 \bigoplus V_2$, is a function for a complementary subspace to $im(\alpha)$ in V.

($\Leftarrow$) Let $W \subset V$ be a complementary subspace to $im(\alpha)$. Then, we can construct a splitting function $\phi : V \to V_1 \bigoplus V_2$ where

$$\phi(W) \to V_2$$
$$\phi(im(\alpha)) \to V_1$$

$\square$

(b) $\Leftrightarrow$ (d)

Proof. ($\Rightarrow$) Let $\sigma : V_2 \to V$ be a linear map with the property that $\beta \circ \sigma = 1$. Then $f : \sigma \to im(\sigma)$ creates a complementary subspace to $im(\alpha)$. Hence, given σ we can produce a complementary subspace, $im(\sigma)$.
($\Leftarrow$) Let $W \subset V$ be a complementary subspace to $im(\alpha)$. The function $g : V_2 \to W \to V$ is one such linear map that fulfills the requirements of σ. □

(c) $\Leftrightarrow$ (d)

Proof. ($\Rightarrow$) Let $\rho : V \to V_1$ be a linear map with the property that $\rho \circ \alpha = 1$. Then $ker(\rho)$ is a complementary subspace to $im(\alpha)$. This is because

$$\rho(im(\alpha)) = V_1$$

and

$$\rho(ker(\rho) = 0.$$

Hence, they are complements
($\Leftarrow$) Let $W \subset V$ be a complementary subspace to $im(\alpha)$. Then we can construct $\rho : V \to V_1$ with the property that $\rho \circ \alpha = 1$ if the $ker(\rho) = W$. Then every element in V is sent to an elements of V_1 and $\rho \circ \alpha(V_1) = V_1$ □

Question 3

Proof. Let $\phi : V \to W$ be a linear map and let $v_1, ..., v_n \in V$. Then

$$\phi(\langle v_1, ..., v_n \rangle) = \{\phi(a_1 \cdot v_1 + ... + a_n \cdot v_n) : a_i \in k\}$$

where k is the field of V. Hence, by the definition of a linear map,

$$\phi(a_1 \cdot v_1 + ... + a_n \cdot v_n) = a_1 \cdot \phi(v_1) + ... + a_n \cdot \phi(v_n)$$

Also,

$$\{a_1 \cdot \phi(v_1) + ... + a_n \cdot \phi(v_n) : a_i \in k\} = \langle \phi(v_1), ..., \phi(v_n) \rangle$$

Hence,

$$\phi(\langle v_1, ..., v_n \rangle) = \langle \phi(v_1), ..., \phi(v_n) \rangle$$

□

Question 4

Proof. Let V be a finite dimension vector space. Let $v_1, ...v_n \in V$ be a sequence of linearly independent vectors. Proceed with the following operation:

Start with the 0 vector. If this spans, it is the basis. If this does not span V, there exists some $v_1 \notin span\{0\}$. Add this v_1 to the vector sequence as it is linearly independent from 0.

If this spans, it is the basis of V. If this sequence of vectors is not a basis, continue the process by adding $v_n \notin span\{v_1, ...v_{n-1}\}$ to the sequence until the sequence spans V. At that point the sequence will remain linearly independent while spanning V and hence be a basis for V. □

Question 5

Proof. Let $\alpha : V \to V$ be linear map from a finite-dimensional vector space to itself.

Case 1:

Assume that α is an isomorphism. Then it is both injective and surjective by definition

Case 2:

Assume that α is injective. Then $ker(\alpha) = 0$. Because V is finitely dimensional, we know that

$$\dim(V) = \dim(ker(\alpha)) + \dim(im(\alpha))$$

Hence,

$$\dim(V) = \dim(0) + \dim(im(\alpha))$$
$$\dim(V) = \dim(im(\alpha))$$

Thus because

$$\dim(V/W) = \dim(V) - \dim(W)$$
$$\dim(V/im(\alpha)) = 0$$

and so $V = im(\alpha)$ so α is surjective and hence, an isomorphism.

Case 3:

Assume that α is surjective. Hence, $im(\alpha) = V$. So,

$$\dim(V/im(\alpha)) = 0$$

Hence,

$$\dim(V) = \dim(im(\alpha))$$

Thus,

$$\dim(ker(\alpha)) = 0$$

Hence, $ker(\alpha) = 0$. So, α is injective and thus isomorphic. □

Question 6

Proof. Let $k = \mathbb{F}_p$ where

$$\mathbb{F}_p \stackrel{\text{def}}{=} \mathbb{Z}/p$$

Then, by the definition of the this kind of field, there are p terms in k. Hence, there are p^2 terms in k^2. We know that the basis must have two vectors since it is of a k^2 space. Given this, we can deduce that there are $p^2 - 1$ choices for the first vector since we must remove 0. Having selected one vector, we must remove all scalar multiples of said vector for the next selection in order to obtain a linearly independent set. Thus, there are $p^2 - p$ options for the second vector of the basis. Thus the total number of possible bases is given by,

$$(p^2 - 1)(p^2 - p) = p^4 - p^3 - p^2 + p$$

□

Question 7

Proof. Let V_1 and V_2 be subspaces of a finite-dimensional vector space V. We want to show that

$$dim(V_1 + V_2) = dim(V_1) + dim(V_2) - dim(V_1 \cap V_2)$$

To show this, let $v_1, ..., v_n$ be the bases of $V_1 \cap V_2$. So $dim(V_1 \cap V_2) = n$. We can extend this into a basis of V_1 by adding in linearly independent vectors as detailed by question 4. This is possible since it is already a basis of $V_1 \cap V_2$ and hence linearly independent for V_1 and V_2. We can do the same for V_2. Let $v_1, ..., v_n, w_1, ..., w_j$ be the basis of V_1 and $v_1, ..., v_n, u_1, ..., u_m$ be the basis of V_2. So,

$$dim(V_1) = n + j$$
$$dim(V_2) = n + m$$

Thus,

$$dim(V_1) + dim(V_2) - dim(V_1 \cap V_2) = n + j + n + m - n = n + j + m$$

If $v_1, ..., v_n, w_1, ..., w_j, u_1, ..., u_m$ is linearly independent then it is the basis of $dim(V_1 + V_2)$ and hence $n + j + m = dim(V_1 + V_2)$, proving the initial equality. To do this it must be shown that for any scalars x, y, z if

$$x_1 \cdot v_1 + ... + x_n \cdot v_n + y_1 \cdot w_1 + ... + y_j \cdot w_j + z_1 \cdot u_1 + ... + z_m \cdot u_m = 0$$

then, $x, y, z = 0$. In other words,

$$z_1 \cdot u_1 + ... + z_m \cdot u_m = -x_1 \cdot v_1 - ... - x_n \cdot v_n - y_1 \cdot w_1 - ... - y_j \cdot w_j$$

Hence, the LHS is an element of V_1, but also since the u vectors are only a part of the basis of V_2, the LHS is in $V_1 \cap V_2$. Thus, we can rewrite the LHS with new scalars g as:

$$g_1 \cdot v_1 + ... + g_n \cdot v_n = -x_1 \cdot v_1 - ... - x_n \cdot v_n - y_1 \cdot w_1 - ... - y_j \cdot w_j$$

This can only be true if all the g's are 0 since $v_1, ..., v_n, w_1, ..., w_j$ is linearly independent. So the equation becomes,

$$x_1 \cdot v_1 + ... + x_n \cdot v_n + y_1 \cdot w_1 = 0$$

We already know that this sequence of vectors is linearly independent since it is the basis of $V_1 \cap V_2$ and hence this equation has only the trivial solution. Thus $v_1, ..., v_n, w_1, ..., w_j, u_1, ..., u_m$, is linearly independent as well. Hence,

$$\begin{aligned} dim(V_1 + V_2) &= dim(v_1, ..., v_n, w_1, ..., w_j, u_1, ..., u_m) = n + j + m \\ &= n + j + n + m - n = dim(V_1) + dim(V_2) - dim(V_1 \cap V_2) \end{aligned}$$

$\square$