Question 1

(i)

Proof. ($\Rightarrow$) Assume f is surjective. Hence, for each element $y \in Y$ there exists a $z \in X/\sim$ such that $f(z) = y$.
Since $\tilde{f} = f \circ \pi$, $\pi : X \to X/\sim$, and π is surjective. Thus,

$$\tilde{f}(X) = f(X/\sim) = Y$$

Hence $\tilde{f}$ is surjective.

($\Leftarrow$) Assume $\tilde{f}$ is surjective. Hence, for each $y \in Y$, there exists a $x \in X$ such that $y = \tilde{f}(x)$. Then, because $f \circ \pi = \tilde{f}$, for each $y \in Y$, there exists $\pi(x)$ so that $f(\pi(x)) = y$. Hence, f is surjective.

□

(ii)

Proof. ($\Rightarrow$) Assume f is injective. Then, if $\pi(x) = \pi(x')$ it follows that $f(\pi(x)) = f(\pi(x'))$. Consequently, if $x \sim x'$, then $\pi(x) = \pi(x')$ and $\tilde{f}(x) = \tilde{f}(x')$ by characteristics of a quotient set function.

($\Leftarrow$) Assume that $x \sim x' \Leftrightarrow \tilde{f}(x) = \tilde{f}(x')$. Since $x \sim x'$, then $\pi(x) = \pi(x')$. Hence, if $\pi(x) = \pi(x')$, then $f(\pi(x)) = f(\pi(x'))$. Thus, f is injective. □

Question 2

Proof. Let $p, q \in \mathbb{Z}$. Let $a \sim a'$ so that

$$a - a' = np$$

Define

$$\{0\} \stackrel{\text{def}}{=} \{b : b = nq\}$$

Then,

$$a + \{0\} = a' + n(p + q) = a$$

Hence, $\{0\}$ is the additive identity for $\mathbb{Z}/n\mathbb{Z}$. Similarly, define

$$\{1\} \stackrel{\text{def}}{=} \{c : c = 1 + nq\}$$

Then,

$$\begin{gathered}
a \cdot \{1\} \\
= (a' + np) \cdot (1 + nq) \\
= a' + np + a'nq + n^2pq \\
= a' + n(p + a'q + npq) \\
= a
\end{gathered}$$

Hence, $\{1\}$ is the multiplicative identity for $\mathbb{Z}/n\mathbb{Z}$. $\square$

Question 3

Proof. Let V be a vector space and let $v \in V$. We know that $0 + 0 = 0$ since it is the additive identity. Hence,

$$\begin{gathered}
0 \cdot v = (0 + 0) \cdot v \\
0 \cdot v = 0 \cdot v + 0 \cdot v
\end{gathered}$$

Thus by the uniqueness of the additive identity,

$$0 \cdot v = 0$$

Additionally, we want to show that $(-1) \cdot v$ is the additive inverse of v. So, given the multiplicative identity and the distributive property of vector spaces,

$$v + (-1) \cdot v = (1) \cdot v + (-1) \cdot v = v \cdot (1 - 1) = 0$$

Consequently,

$$(-1) \cdot v = -v$$

$\square$

Question 4

Proof. Let X be a set and V a vector space. Assume the Fun(X,V) is equipped with addition and scalar multiplication via:

$$(f + g)(x) = f(x) + g(x) \quad (a \cdot f)(x) = a \cdot f(x)$$

Hence,

$$a \cdot (f + g)(x) = a \cdot (f(x) + g(x)) = a \cdot f(x) + a \cdot g(x)$$

Also,

$$(a + b) \cdot f(x) = (a \cdot f)(x) + (b \cdot f)(x) = a \cdot f(x) + b \cdot f(x)$$

Thus, Fun(X,V) is a vector space. $\square$

Question 5

Let $\phi, \psi : V \to W$ be linear maps.

(i)

Let $v, v' \in V$. Hence,

$$(\phi + \psi)(v + v') = \phi(v + v') + \psi(v + v')$$

Because each of these functions are linear maps,

$$\begin{gathered} \phi(v + v') + \psi(v + v') \\ = \phi(v) + \phi(v') + \psi(v) + \psi(v') \\ = (\phi + \psi)(v) + (\phi + \psi)(v') \end{gathered}$$

Hence the first requirement of a linear map is met. Next, take $a \in k$. Then, because each function is a linear map,

$$\begin{gathered} (\phi + \psi)(v \cdot a) \\ = \phi(v \cdot a) + \psi(v \cdot a) \\ = a \cdot \phi(v) + a \cdot \psi(v) \\ = a \cdot (\phi + \psi)(v) \end{gathered}$$

Thus, $(\phi + \psi)$ is a linear map.

(ii)

Let $v, v' \in V$. Hence,

$$\begin{gathered} a \cdot \phi(v + v') \\ = (a \cdot \phi)(v + v') \\ = a \cdot (\phi(v) + \phi(v')) \\ = a \cdot \phi(v) + a \cdot \phi(v') \end{gathered}$$

Also,

$$\begin{gathered} (a \cdot \phi)(b \cdot v) \\ = a \cdot (\phi(b \cdot v)) \\ = a \cdot (b \cdot (\phi(v))) \\ = b \cdot (a \cdot \phi)(v) \end{gathered}$$

Thus, $a \cdot \phi$ is a linear map.

Question 6

Proof. Let V be a vector space. Take the linear maps, $\psi : k^n \to V$ to be defined as:

$$k^n \to V$$

Given these linear maps, point in k^n will deliver a set of vectors $(v_1, v_2, ..., v_n)$. In order to construct a bijection, we want to construct an inverse of that takes

$$(v_1, v_2, ...v_n) \to k^n$$

and gives us a linear map. We know that

$$\psi((k^n)) = (v_1, v_2, ..., v_n)$$

If we take, $\psi(e_1)$, then $v_1 = \psi_1$. Hence, $v_i = \psi_i$. So, $\psi = (\psi_1, \psi_2, ..., \psi_n) = (\psi(e_1), \psi(e_2), ..., \psi(e_n)) = (v_1, v_2, ..., v_n)$. Hence we can find the linear map given just the vectors and thus a bijection is created between the two pieces of data. □