163 HWK 2

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Question 1

Proof. Let $(a_n), (b_n), (c_n)$ be sequences of real numbers. Assume that (a_n) and (c_n) are convergent, $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n) \stackrel{\text{def}}{=} l$, and that there exists a natural number N such that for each natural number n, if $n > N$, then $a_n \leq b_n \leq c_n$. Let $\epsilon > 0$. There exist a N_1, N_2 so that if $n > N$,

$$|a_n - l| < \epsilon$$
$$|c_n - l| < \epsilon$$

Also,

$$a_n - l \leq b_n - l \leq c_n - l$$

Hence

$$-\epsilon < a_n - l \leq b_n - l \leq c_n - l < \epsilon$$

So,

$$|b_n - l| < \epsilon$$

Thus, b_n is convergent and $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n) = \lim_{n\to\infty}(b_n)$ □

Question 2

Proof. Let (a_n) be an increasing sequence of natural numbers. Then for each $n \in \mathbb{N}$, $a_n < a_{n+1}$. Hence, by the properties of natural numbers, $a_n + 1 = a_{n+1}$ and so $a_n = a_0 + n$. Thus, by the archimedean property, for any $M > 0$, we can choose a $N \in \mathbb{N}$ so that $a_0 + N \geq M$. So, if $n > N$, then $a_n > M$. Consequently, (a_n) diverges to ∞. □

Question 3

Proof. Let (a_n) be a non-decreasing sequence of real numbers and let (a_{n_k}) be a convergent subsequence of (a_n) that converges to l. Hence, for each $\epsilon > 0$, there exists a $K \in \mathbb{K}$ so that if $k > K$,

$$|a_{n_k} - l| < \epsilon$$

For each $n > n_k$, since (a_n) is nondecreasing,

$$a_{n_k} \leq a_n \leq a_{n_{k+1}}$$
$$a_{n_k} - l \leq a_n - l \leq a_{n_{k+1}} - l$$

Also,

$$|a_{n_k} - l| < \epsilon$$
$$|a_{n_{k+1}} - l| < \epsilon$$
$$-\epsilon < a_{n_k} - l$$
$$a_{n_{k+1}} - l < \epsilon$$
$$-\epsilon < a_n - l < \epsilon$$
$$|a_n - l| < \epsilon$$

Hence, (a_n) is a a convergent sequence. □

Question 4

Proof. Let $n^{\frac{1}{n}}$ be a sequence of real numbers for $n \in \mathbb{N}$. Let $\epsilon > 0$. Define $a_n \overset{\text{def}}{=} n^{\frac{1}{n}} - 1$. Then,

$$n^{\frac{1}{n}} = a_n + 1$$
$$n = (a_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k$$

Because n is natural number

$$n \geq 1$$
$$n^{\frac{1}{n}} \geq 1$$

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} a_n^k > \binom{n}{2} a_n^2$$

Thus,

$$(a_n + 1)^n > \frac{n(n-1)a_n^2}{2}$$

Hence,

$$0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$$

Then, by the archimedean property, we can choose an N sufficiently large so that for each epsilon, if $n > N$ then,

$$|a_n| = |n^{\frac{1}{n}} - 1| < \epsilon$$

Thus, $n^{\frac{1}{n}}$ converges to 1. □

Question 5

Proof. Let (a_n) be a bounded, non-increasing sequence of real numbers. Define

$$A \stackrel{\text{def}}{=} \{a_n : n \in \mathbb{N}\}$$

Because (a_n) is a sequence, $A \neq 0$. Since, (a_n) is bounded then A is bounded as well. So, A is bounded below. By the greatest lower bound axiom, A has a greatest lower bound. Define,

$$l \stackrel{\text{def}}{=} \inf[A]$$

Let $\epsilon > 0$. By the approximation property of greatest lower bounds, there exists an $N \in \mathbb{N}$ such that

$$l \leq a_n < l + \epsilon$$

Because (a_n) is non-increasing, for each $n > N$,

$$l - \epsilon < l \leq a_n < l + \epsilon$$
$$|a_n - l| < \epsilon$$

Hence, (a_n) is convergent and

$$l = \lim_{n \to \infty} (a_n) = \inf[A]$$

□

Question 6

Proof of arithmetic mean-geometric mean inequality

Proof. Let $a, b \geq 0$. Then,

$$\begin{aligned} 0 &\leq (a-b)^2 \\ &= a^2 - 2ab + b^2 \\ &= a^2 + 2ab + b^2 - 4ab \\ &= (a+b)^2 - 4ab \end{aligned}$$

Hence,

$$4ab \leq (a+b)^2$$
$$\sqrt{ab} \leq \frac{a+b}{2}$$

$\square$

Proof that $x_n > 0$

Proof. Proceeding by induction,

Base Case: n=1

Then, it is given that $x_1 > 0$

Inductive Step:

Let $n \in \mathbb{N}$

Inductive Hypothesis:

Assume that $x_n > 0$

Proof of Inductive Step:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n}) = \frac{x_n}{2} + \frac{c}{2x_n}$$

Because c, x_n are positive real numbers,

$$x_{n+1} > 0$$

Hence, by induction, $x_n > 0$

$\square$

Assumptions:

Let c be a positive real number, and let x_1 be any positive real number such that $x_1^2 \geq c$. For each natural number n, define

$$x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n})$$

(a)

Proof. Proceeding by induction to show that $x_n^2 \geq c$:

Base Case: n=1

By the assumptions of the question,

$$x_1^2 \geq c$$

Inductive Step:

Let $n \in \mathbb{N}$

Inductive Hypothesis:

Assume that

$$x_n^2 \geq c$$

Proof of Inductive Step:

Need to show that

$$x_{n+1}^2 \geq c$$

It's given that,

$$x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n})$$

So, by the arithmetic mean-geometric mean inequality,

$$\frac{1}{2}(x_n + \frac{c}{x_n}) \geq \sqrt{x_n \cdot \frac{c}{x_n}} = \sqrt{c}$$

Hence,

$$x_{n+1}^2 \geq c$$

So, by means of induction, $x_n^2 \geq c$ for any natural n.

$\square$

(b)

Proof. In order to show that (x_n) is nonincreasing, we need to show that $x_{n+1} \leq x_n$. We know that

$$x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n}) = x_{n+1} = \frac{1}{2}(x_n + x_n \frac{c}{x_n^2})$$

By part a,

$$x_n^2 \geq c$$

So

$$x_n + x_n \frac{c}{x_n^2} \leq 2x_n$$

Hence,

$$x_{n+1} \leq \frac{1}{2}(2x_n) = x_n$$

Thus, (x_n) is nonincreasing. $\square$

(c)

Proof. Since $x_n > 0$ and because (x_n) is nonincreasing, it is bounded above by x_1 so $0 < x_n \leq x_1$. Hence, (x_n) is bounded and monotone, so it is convergent. Define

$$l \stackrel{\text{def}}{=} \lim_{n\to\infty} x_n$$

So by the shift property of sequences and limit laws,

$$\begin{aligned} l = \lim_{n\to\infty} x_{n+1} &= \frac{1}{2}(\lim_{n\to\infty} x_n + \lim_{n\to\infty} \frac{c}{x_n}) \\ &= \frac{1}{2}(l + \frac{c}{l}) \end{aligned}$$

Hence,

$$\begin{aligned} 2l &= l + \frac{c}{l} \\ l^2 &= c \\ l &= \sqrt{c} \end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n = l = \sqrt{c}$$

$\square$

Lemma 1

Proof. Let (a_n) be a bounded sequence in $\mathbb{R}$ that does not converge to $l \in \mathbb{R}$. Hence, there exists an $\epsilon > 0$ such that for each $N \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n \geq N$, and $|a_n - l| \geq \epsilon$. Thus, there exists $n_1 \in \mathbb{N}$ so that $n_1 \geq 1$ and $|a_{n_1} - l| \geq \epsilon$. There also exists $n_2 \in \mathbb{N}$ so that $n_2 \geq n_1 + 1 > n_1$ and $|a_{n_2} - l| \geq \epsilon$.

For each $k \in \mathbb{N}$, there exists $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} \geq n_k + 1 > n_k$ and $|a_{n_k} - l| \geq \epsilon$.

Then (n_k) is an increasing sequence of natural numbers. So, (a_{n_k}) is a subsequence of (a_n). So, because (a_n) is bounded, by the Bolzano-Weierstrass theorem, (a_{n_k}) has a convergent subsequence $(a_{n_{k_j}})$ and $\lim\limits_{j \to \infty} a_{n_{k_j}} \neq l$. Hence, there is a convergent subsequence of (a_n) that doesn't converge to l. □

Question 7

Proof. Assume that (a_n) is a bounded and divergent sequence of real numbers. By Bolzano-Weierstrass,, there exists a convergent subsequence, (a_k) of (a_n) that is convergent. Define

$$l \stackrel{\text{def}}{=} \lim_{k \to \infty} a_k$$

Applying lemma 1, there exists a convergent subsequence, (a_m) of (a_n) that does not converge to l. Define

$$j \stackrel{\text{def}}{=} \lim_{m \to \infty} a_m$$

Hence, there are two subsequence which both converge to unique real numbers. □

Question 8

Proof. $(\Rightarrow)$ Assume that (a_n) is a convergent sequence of real numbers. Define,

$$l \stackrel{\text{def}}{=} \lim_{n \to \infty} a_n$$

Let $k \in \mathbb{N}$. Define

$$(n_k) \stackrel{\text{def}}{=} 2k$$

Because this sequence is increasing, and $a_{n_k} = a_{2k}$, then a_{2k} is a subsequence of (a_n). Because it is a subsequence of a convergent sequence, a_{2k} is convergent itself and also converges to l. So, by the shift property of sequences, a_{2k+1} is convergent and

$$l = \lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1}$$

($\Leftarrow$) Assume that a_{2k} and a_{2k+1} are convergent sequences so that

$$\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1} \stackrel{\text{def}}{=} l$$

Thus, for each $\epsilon > 0$ there exists a $K_1, K_2 \in \mathbb{N}$ so that if $k \geq \max\{K_1, K_2\}$, then

$$|a_{2k} - l| < \epsilon$$
$$and$$
$$|a_{2k+1} - l| < \epsilon$$

Hence, if $n \geq 2 \cdot \max\{K_1, K_2\}$, then

$$|a_n - l| < \epsilon$$

□

Question 9

Let (a_n) be a bounded sequence of real numbers such that each convergent subsequence of (a_n) converges to l.

(a)

Proof. Assume that (a_n) is either divergent or converges to a value that isnt l. In either case, we know that the subsequence (a_{n_k}) of (a_n) isn't convergent to l when $(n_k) = k$. Negating the definition of convergence to l gives us that for the subsequence (a_k) of (a_n) there exists an $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists a natural k such that $k \geq K$ and $|a_k - l| \geq \epsilon$. □

(b)

Proof. Because (a_n) is a bounded sequence, (a_{n_k}) is also bounded. Thus, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $(a_{n_{k_j}})$. However, because for (a_{n_k}) there exists and $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $k \geq K$ and $|a_{n_k} - l| \geq \epsilon$, since $(a_{n_{k_j}})$ is a subsequence of (a_{n_k}), the same is true of $(a_{n_{k_j}})$. Hence, we know that

$$\lim_{j\to\infty} (a_{n_{k_j}}) \neq 0.$$

□

(c)

Proof. Proceeding by contradiction, assume that (a_n) is either convergent to something other than l or divergent. Then, by part(a) and part(b), there exists a convergent subsequence, $(a_{n_{k_j}})$ that doesn't converge to l, which is a contradiction. Hence, (a_n) must converge to l. $\square$