20250 Supp HWK 1

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April 2024

Question 1

(a)

(i)

Proof. An example of a function $f : X \to \{0, 1\}$ is the function that sends integers to 0 and fruits to 1. In this case,

$$\{0, 1, \text{Fun}(\{\text{pomelo}\}, \{3, 4\})\} \to \{0\}$$
$$\{\text{citron, mandarin}\} \to \{1\}$$

$\square$

(ii)

Proof. Let $g : X \to \{0, 1\}$ be the function that works as follows,

$$\{0, 1\} \to 0$$
$$\{\text{citron, mandarin}, \text{Fun}(\{\text{pomelo}\}, \{3, 4\})\} \to 1$$

Then the kernel of g is $\{0, 1\}$ or Y. This means

$$g^{-1}(\{0\}) = Y$$

$\square$

(b)

Proof. Let X be an arbitrary set. I will proceed to construct a bijection between

$$\text{Fun}(X, \{0, 1\}) \qquad \text{and} \qquad \{\text{S : S is a subset of X}\}$$

If we create a function

$$\phi : \text{Fun}(X, \{0, 1\}) \to \ker(\text{Fun})$$

. Then we construct a subsect from a given function. To show that this is a bijection I will construct an inverse. Define

$$\phi^{-1} : S \to \text{Fun}(X, \{0,1\})$$
$$\text{Fun}(S) \mapsto 0, \text{Fun}(!S) \mapsto 1$$

Then,

$$\phi \circ \phi^{-1}(S) = \phi(\text{Fun}(X, \{0,1\}) = \ker(\text{Fun}) = S$$

Also,

$$\phi^{-1} \circ \phi(\text{Fun}(X, \{0,1\})) = \phi^{-1}(\ker(\text{Fun}) = (\text{Fun}(X, \{0,1\}))$$

Hence, both compositions yield the identity, so they are inverses. Hence, ϕ is bijective. □

Question 2

(a)

Consider the sets

$$X = \{0,1\} \qquad Y = \{\text{blue}, \text{red}\} \qquad Z = \{a, b, c\}$$

(i)

Proof. An example function $f : X \times Y \to Z$ is the function that sends

$$(0, y) \to a$$
$$(1, \text{blue}) \to b$$
$$(1, \text{red}) \to c$$

where $y \in Y$. □

(ii)

Proof. An example function $g : X \to \text{Fun}(Y, Z)$ is the function that sends

$$0 \to \phi : Y \to Z$$
$$1 \to \psi : Y \to Z$$

Where ϕ and ψ are distinct functions from $Y \to Z$. □

(b)

Proof. Let X,Y, and Z be arbitrary sets. ($\Rightarrow$) Define

$$F : X \to \text{Fun}(Y, Z)$$

Then define $\phi : F \to G$ where we construct a function $G : X \times Y, \to Z$ that takes

$$(x, y) \to F(x)(y)$$

($\Leftarrow$) Define

$$G : X \times Y \to Z$$

Then define $\phi^{-1} : G \to F$ where we construct a function $F : X \to \text{Fun}(Y, Z)$ that takes

$$x \to G(x, -)$$

Hence,

$$\phi \circ \phi^{-1}(G) = \phi(F) = G$$
$$\phi^{-1} \circ \phi(F) = \phi^{-1}(G) = F$$

Hence, there exists a bijection between the two sets. $\square$