20250 HWB

James Gillbrand

May 2024

Question 1

See figure 1 at the bottom of the pdf.

Question 2

Take $\mathbb{F}_7$ to be the field.

(i)

Proof. First imagine the one dimensional example in k. There is exactly one unique line, with can be constructed by taking a vector from the origin to any of the 6 other points in the space. Hence, there are 6 vector choices and one unique line subspace..

Now in k^2, there are a total of $7^2 - 1$ vectors to choose from as we can take the 0 vector to create a line. However, we need to account for vectors that create the same subspace, as such we need to remove all other vectors that lie in that line. As shown above, there are $7 - 1$ vectors that share a subspace, hence the total amount of line subspaces in k^2 is given by,

$$\frac{7^2 - 1}{7 - 1} = 8$$

Also there is exactly one plan in k^2 as $dim(plane) = dim(k^2) = 2$

Extending into k^3, there are $7^3 - 1$ vectors to choose from, without the 0 vector. Hence, there are $\frac{7^3-1}{7-1}$ lines in k^3. Now to construct a plane, we select another vector. It can't be a multiple of the initial vector, so we have to exclude 7 vectors right off the bat including the 0 vectors. Hence we have $7^3 - 7$ choices. Now to eliminate those that create the same plane. Clearly, linear multiples will create the same plane so that leaves, $\frac{7^3-7}{7-1}$ vectors. However, vectors that are linear combinations of both of these vectors must also be removed. Hence, the total amount unique planes is given by, $\frac{7^3-7}{7^2-7}$. Consequently, the total number of planes is

$$\frac{7^3-7}{7^2-7} = 400$$

Knowing this, we can extend the line formula from previous into k^3. As such the total number of lines is given by

$$\frac{7^3-1}{7-1} = 57$$

$\square$

(ii)

Proof. Because any plane in k^3 is isomorphic to k^2, there are 8 lines in each plane as shown in part(i). $\square$

(iii)

Proof. Let H and H' be distinct planes. We know that they share the 0 vector. Hence, if we they share another point such that they also share linear multiples of this point, there is exactly one shared line. Let $a_1v_1 + a_2v_2 \in H$ and $b_1w_1 + b_2w_2 \in H'$. Then $dim(H) = dim(H') = 2$, however $dim(H \cup H') = 3$ as we are operating in k^3. Thus, one of v_1, v_2, w_1, w_2 must be a linear combination of the other three. Without loss of generality, let $w_1 = c_1v_1 + c_2v_2 + c_3w_1$. Consequently, the vector $-c_3w_1 + w_2 = c_1v_1 + c_2v_2 + c_3w_1 - c_3w_1 \in H \cap H'$. Consequently, this produces a line contained in both planes. $\square$

Question 3

The game spot it depends on every card sharing one of its 8 images with each of the other cards in the deck. Take the 8 images as the 8 lines that make up a plane. Then, when you take take another card, as long as the planes are distinct, you are guaranteed that one of the images/lines will match the other plane.

Figure 1: Enter Caption