163 HWK 3

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Question 1

($\Rightarrow$) Let f be a continuous function. Let (x_n) be a convergent sequence of real numbers. Define

$$\lim_{n\to\infty}(x_n) \stackrel{\text{def}}{=} l$$

To show that $\lim_{n\to\infty} f(x_n) = f(l)$ we must show that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $_1 > N_1$, then

$$|f(x_n) - f(l)| < \epsilon$$

Since f is continuous there exists a $\delta > 0$ such that if $|x_n - l| < \delta$ then the inequality above holds. Hence, because (x_n) converges to l, we can choose a $N_1 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n > N_1$ then

$$|x_n - l| < \delta$$

for any delta. Hence, if we take $N = N_1$ then

$$|f(x_n) - f(l)| < \epsilon$$

So, $\lim_{n\to\infty} f(x_n) = f(l) = f(\lim_{n\to\infty}(x_n))$

($\Leftarrow$) Let f be a function of real numbers. Let (x_n) be a convergent sequence of real numbers. Proceeding by contraposition assume that f is discontinuous. Then for some $l \in \mathbb{R}$ there exists some $\epsilon > 0$ such that for each $\delta > 0$, $|x - l| < \delta$ and $|f(x) - f(l)| \geq \epsilon$. Hence, if (x_n) converges to l, then there exists a $N \in \mathbb{N}$ sufficiently large enough that $|x_n - l| < \delta$. However, this implies that

$$|f(x_n) - l| \geq \epsilon$$

Which means that f does not converge and consequently is divergent. Thus, by contraposition, if f is convergent and

$$\lim_{n\to\infty} f(x_n) = f(l) = f(\lim_{n\to\infty}(x_n))$$

then f is continuous.

Question 2

(a)

Proof. Let $(\sqrt{n})$ be a sequence of real numbers. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $N > \frac{1}{4\epsilon^2}$. Hence,

$$\frac{1}{2\sqrt{N}} < \epsilon$$

Then let $n \geq N$ so that

$$\begin{aligned} \frac{1}{2\sqrt{N}} &\geq \frac{1}{2\sqrt{n}} \\ \geq \frac{1}{\sqrt{n+1}+\sqrt{n}} &= (\sqrt{n+1}-\sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \sqrt{n+1}-\sqrt{n} = |\sqrt{n+1}-\sqrt{n}| \end{aligned}$$

Hence,

$$|\sqrt{n+1}-\sqrt{n}| < \epsilon$$

□

(b)

Proof. To show that $(\sqrt{n})$ is divergent I will show that it is unbounded. So, let $M > 0$. Let $N = (M+1)^2$. Then,

$$|\sqrt{N}| = |M+1| > M$$

Hence, there exists a natural number such that the magnitude of the sequence $(\sqrt{n})$ exceeds any real number. Thus, it is unbounded and consequently divergent. □

Question 3

(a)

Proof.

$$s_n = \sum_{k=1}^{n} \frac{3}{k^2+3k+2} = 3\sum_{k=1}^{n}(\frac{1}{k+1} - \frac{1}{k+2}) = 3(\frac{1}{2} - \frac{1}{n+2})$$

So, we know that this sequence is convergent by the algebraic limit theorem and shift theorem for sequences. Hence, the series converges as well. Using the shift theorem for sequences we can solve for the limit as follows:

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty} 3(\frac{1}{2} - \frac{1}{n+2}) = \frac{3}{2} - 3\lim_{n\to\infty}\frac{1}{n+2} \\ &= \frac{3}{2} - 3\lim_{n\to\infty}\frac{1}{n} = \frac{3}{2} - 3(0) = \frac{3}{2}\end{aligned}$$

Hence, the series converges to $\frac{3}{2}$ □

(b)

Proof.

$$s_n = \sum_{k=1}^{n} \frac{3^{k+2} + (-1)^{k-1}2^k(k+1)(k+2)}{3^{k+1}(k+1)(k+2)} = \sum_{k=1}^{n} \frac{3}{(k+1)(k+2)} + \sum_{k=1}^{n} \frac{(-1)^{k-1}2^k}{3^{k+1}}$$

We know that the first term converges to $\frac{3}{2}$ by part a. Now to examine the second part,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}2^k}{3^{k+1}} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{3}(\frac{2}{3})^k$$

Because $(-1)\cdot(-1) = 1$,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{3}(\frac{2}{3})^k = \sum_{k=1}^{k} \frac{(-1)^{k+1}}{3}(\frac{2}{3})^k = \sum_{k=1}^{n} \frac{-1}{3}(\frac{-2}{3})^k$$

This is clearly a geometric series so we know that it converges since $|r| = \frac{2}{3} < 1$ and specifically that it converges to

$$\frac{\frac{-1}{3}}{(1-\frac{-2}{3})} = -\frac{1}{3(\frac{5}{3})} = -\frac{1}{5}$$

Hence, the entire series converges and

$$\lim_{n\to\infty} s_n = \frac{3}{2} - \frac{1}{5} = \frac{13}{10}$$

by the algebraic limit theorem for series. □

(c)

Proof. To show that the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$

is divergent, I will show that the sequence,

$$\left(1+\frac{1}{n}\right)^n$$

is convergent but not to 0 and by the nth term test, the series will diverge. First, we want to show that the sequence is non-decreasing. So we can show that

$$\begin{gathered} a_{n+1} \geq a_n \\ \frac{a_{n+1}}{a_n} \geq 1 \end{gathered}$$

So,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} \\ &= \left(1+\frac{1}{n}\right)\cdot\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \\ &= \left(1+\frac{1}{n}\right)\cdot\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1} \\ &= \left(1+\frac{1}{n}\right)\cdot\left(1-\frac{1}{n^2+2n+1}\right)^{n+1} \\ &\geq \left(1+\frac{1}{n}\right)\cdot\left(1-\frac{n+1}{n^2+2n+1}\right) \\ &= \left(1+\frac{1}{n}\right)\cdot\left(1-\frac{1}{n+1}\right) \\ &= 1-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n^2+n} \\ &= 1+\frac{(n+1)-n-1}{n^2+n} = 1 \end{aligned}$$

Hence, the sequence is nondecreasing. Now to show that it is bounded. By the binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 2+\sum_{k=2}^{n}\binom{n}{k}\cdot\left(\frac{1}{n}\right)^k \\ &= 2+\sum_{k=2}^{n}\frac{n!}{k!(n-k)!n^k} \\ &< 2+\sum_{k=2}^{n}\frac{1}{k(k-1)} \\ &= 2+\sum_{k=2}^{n}\frac{1}{k-1}-\frac{1}{k} \\ &= 2+1-\frac{1}{n}<3\end{aligned}$$

Hence, the series is bounded by 3. Thus, we know that it is convergent. Because $n \in \mathbb{N}$ we know that each term in

$$\left(1+\frac{1}{n}\right)^n > 1.$$

Hence, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq 1$. Therefore by the nth term test the series

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$$

must diverge. □

Question 4

(a)

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of real numbers. Let $c \neq 0$. By contradiction assume that

$$\sum_{n=1}^{\infty} ca_n$$

is convergent. Then by the algebraic limit theorem for series, $c\sum_{n=1}^{\infty} a_n$ would be convergent as well. However this is a contradiction since we assume that $\sum_{n=1}^{\infty} a_n$ is divergent. □

(b)

Proof. Let (a_n) be a sequence of nonzero real numbers that diverges to ∞. As a counterexample, take

$$(a_n) \stackrel{\text{def}}{=} (n)$$

This sequence clearly diverges to infinity as for any $M > 0$, if we take $N = M+1$, for any $n > N$

$$a_n = n > N = M + 1 > M$$

However,

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

diverges as shown in class, contradcicting the statement. □

(c)

Proof. By let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n + b_n$ be convergent series. Then

$$-\sum_{n=1}^{\infty} a_n$$

is also convergent, so

$$\sum_{n=1}^{\infty} a_n + b_n - \sum_{n=1}^{\infty} a_n$$

is convergent. So,

$$\sum_{n=1}^{\infty} a_n + b_n - \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$$

by the algebraic limit theorem for series. This implies that $\sum_{n=1}^{\infty} b_n$ is convergent by the same theorem. □

(d)

Proof. Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} a_n b_n$ be convergent series. In order to prove that

$$\sum_{n=1}^{\infty} a_n b_n \neq \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n$$

let

$$(a_n) = \frac{1}{2^n}$$
$$(b_n) = \frac{1}{3^n}$$

Each of these are convergent by the geometric series test since $\frac{1}{2}, \frac{1}{3} < 1$. However,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{(2 \cdot 3)^n}$$

By the geometric series test,

$$\sum_{n=1}^{\infty} \frac{1}{(2 \cdot 3)^n} = \frac{1}{1 - \frac{1}{6}} = \frac{6}{5}$$

On the other hand, again using the geometric series test,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} = 2 \cdot \frac{3}{2} = 3$$

Since, these values are not equal, the statement is false. □

Question 5

(a)

Proof. Let the number of sides for the nth be defined as follows:

$$sides_n = 3^{n+1}$$

□

(b)

Proof. Let the length of each side of each triangle for the nth term of the sequence be defined as follows:

$$length_n = \frac{1}{2^n}$$

□

(c)

Proof. Since by observation triangles do not share sides,

$$P_n = sides_n \cdot length_n = \frac{3^{n+1}}{2^n}$$

□

(d)

Proof. Let P_n be defined as follows,

$$P_n = \frac{3^{n+1}}{2^n} = 3\left(\frac{3}{2}\right)^n$$

Let $M > 0$. Choose $N \in \mathbb{N}$ such that $N = 2(M-1)-1$. Then for each $n \in \mathbb{N}$ if $n \geq N$, then

$$n > 2(\frac{M}{3} - 1) - 1$$

$$3\left(1 + \frac{n+1}{2}\right) > M$$

By Bernoulli's inequality,

$$3\left(\frac{3}{2}\right)^n \geq 3\left(1 + \frac{n+1}{2}\right)$$

Hence, for any M, if $n > N$ then

$$3\left(\frac{3}{2}\right)^n > M$$

Consequently, $3\left(\frac{3}{2}\right)^n$ diverges to infinity. □

(e)

Proof. The area removed from S_{n-1} to create S_n is defined as

$$A_n \stackrel{\text{def}}{=} \frac{\sqrt{3}}{16} \cdot \frac{3^{n-1}}{4^{n-1}}$$

□

(f)

Proof. Then, the total area removed from S_0 to create S_n is

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{\sqrt{3}}{16} \cdot \frac{3^{k-1}}{4^{k-1}} \tag{1}$$

where 1 is the length of one side of the triangle □

(g)

Proof. Because the series is of the form $\sum_{n=1}^{\infty} ar^n$ where $a = \frac{\sqrt{3}}{16}$ and $r = \frac{3}{4}$, we know that it is a geometric series. By the geometric series test, this series converges to

$$\frac{a}{1-r}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\sqrt{3}}{16} \cdot \frac{3^{n-1}}{4^{n-1}} = \frac{\frac{\sqrt{3}}{16}}{1-\frac{3}{4}} = \frac{\sqrt{3}}{4}$$

□

(h)

Proof. The area of a triangle is given by

$$A = \frac{1}{2}bh$$

Assuming each side is length 1, we need only to find h. Using trigonometry,

$$h = \frac{\sqrt{3}}{2}$$

Hence,

$$A = \frac{\sqrt{3}}{4}$$

Hence the area minus the subtracted area is as follows,

$$T_n = \frac{\sqrt{3}}{4} - \sum_{n=1}^{\infty} \frac{\sqrt{3}}{16} \cdot \frac{3^{n-1}}{4^{n-1}}$$

As n tends to infinity,

$$T_n = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

□