162 HWK 7

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Question 1

(a)

Proof. Let M be any positive real number. Define N as

$$\ln(M) \stackrel{\text{def}}{=} N$$

Because we know that the $\exp x$ is strictly increasing and defined on all real numbers, for $x > N$,

$$\exp x > \exp N$$
$$\exp x > \exp(\ln M)$$
$$\exp x > M$$

Hence for any positive M, and $x > N$, $\exp x > M$. Thus, because the exponential function is strictly increasing, as $x \to \infty$, $\exp x \to \infty$. □

(b)

Proof. Let $\epsilon > 0$ and $M > 0$. Define

$$M \stackrel{\text{def}}{=} \ln(\epsilon)$$

We know that exp > 0 because exp is strictly increasing and it is its own derivative. Choose an $x < M$ and

$$|\exp(x) - 0| = \exp(x)$$

Because $x < M$ and exp is strictly increasing,

$$\exp(x) < \exp(M)$$

Also,

$$M = \ln(\epsilon)$$
$$\exp(M) = \epsilon$$

Thus,

$$|\exp(x) - 0| < \epsilon$$

Hence, as $x \to -\infty$, $\exp(x) \to 0$ $\square$

Question 2

(a)

Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(r) = 0$ for each rational number r. By contradiction, assume that there exists some a such $f(a) \neq 0$. Then, by the definition of continuity for each $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$|x - a| < \delta$$

then

$$|f(x) - f(a)| < \epsilon$$

Hence if we take $\epsilon < f(a)$, then on $(a - \delta, a + \delta)$, $f(x) > 0$. By the density of rationals, we know that there has to be a rational on $(a - \delta, a + \delta)$ and hence there exists some r such that

$$f(r) > 0$$

which is a contradiction. Thus, $f(x) = 0$ for any real x $\square$

(b)

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f(r) = g(r)$ for each rational r. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = f(x) - g(x)$$

Thus for any rational r, h is continuous because it is a combination of continuous functions and

$$h(r) = 0$$

Thus, by part a, $h(x) = 0$ for any real x. Hence,

$$f(r) = g(r)$$

Question 3

(a)

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+y) = f(x)f(y)$ for all real numbers x and y. Let $n \in \mathbb{N}$.

$$f(n) = f(n(1)) = f(1)^n$$

It follows that

$$f(n) * f(-n) = f(n-n) = f(0)$$

Thus, $f(-n) = f(n)^{-1}$ by the unique multiplicative inverse axiom. Hence the property holds for all integers. Let $m \in \mathbb{Z}$ so that $r \in \mathbb{Q}$ when $r = \frac{m}{n}$. So,

$$f(r)^n = f(\frac{m}{n})^n = f(n * \frac{m}{n}) = f(m) = f(1)^m$$

Then,

$$f(\frac{m}{n}) = f(m)^{\frac{1}{n}} = f(1)^{m^{\frac{1}{n}}} = f(1)^{\frac{m}{n}}$$

□

(b)

Case 1

$f(0) = 0$
Then because $f(x - x)$ by the zero product property, $f(x) = 0$

Case 2

For any real number x and rational number r. Because

$$f(r) = f(1)^r \tag{1}$$

and both of these functions are continuous. By question 2,

$$f(x) = f(1)^x$$

Thus, let $f(1) = a$ so $f(x) = a^x$ □

Question 4

Proof. Proof. Let $a \in \mathbb{R}$, define $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) \stackrel{\text{def}}{=} x^a$$

Define $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \exp(x)$$
$$h(x) = a \ln(x)$$

Each of these functions are differentiable as we know the exponential and natural log functions are differentiable. Hence, their composition is differentiable.

$$g(h(x)) = \exp(a \ln(x)) = x^a = f(x)$$

Thus f(x) is differentiable. By the chain rule,

$$(g(h(x)))' = g'(h(x))h'(x) = \exp(a \ln(x))(\frac{a}{x}) = \frac{ax^a}{x} = ax^{a-1}$$

$\square$

Question 5

(a)

Proof. Let a and b be real numbers such that $a < b$. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions such that $g(x) > 0$ for each $x \in [a, b]$. Assume there exists a real number C such that for each x in [a,b]

$$f(x) \leq C + \int_a^x fg$$

Define $h : [a, b] \to \mathbb{R}$ as

$$h(x) = \frac{C + \int_a^x fg}{\exp(\int_a^x g)}$$

Then h is differentiable since f,g, are continuous. Hence, the integral of f and g are differentiable. Also, the exponential function is differentiable so by combination of differentiable functions, h(x) is differentiable. So,

$$\begin{aligned} h'(x) &= \frac{fg(\exp(\int_a^x g)) - (C + \int_a^x fg)(\exp(\int_a^x g)g}{(\exp(\int_a^x g)^2} \\ &= \frac{g(f - C - \int_a^x fg)}{\exp(\int_a^x g)} \end{aligned}$$

Since

$$f(x) \leq C + \int_a^x fg f(x) - C - \int_a^x fg \leq 0$$

So,

$$h'(x) \leq 0$$

Thus, for any x in [a,b]

$$\begin{gathered} h(x) \leq h(a) \\ \frac{f(x)}{\exp(\int_a^x g)} \leq \frac{C + (\int_a^x fg)}{\exp(\int_a^x g)} \leq C \end{gathered}$$

Hence,

$$f(x) \leq C \exp(\int_a^x g)$$

$\square$

(b)

Proof. Assume that $f(x) \geq 0$ for each x in [a,b], $f(0) = 0$, f is differentiable on (a,b), and $f'(x) = g(x)f(x)$ for each x in (a,b).

Because $g(x)$ and $f(x)$ are greater than 0 and $f(x)g(x) > 0$, f is increasing. Since 0 is in [a,b], $a \leq 0$. Hence since f is increasing, $f(0) = 0$, and $f(x) \geq 0$, then for any $x \leq 0$, $f(x) = 0$. Thus, $f(a) = 0$.

Since f is differentiable, by the Second Fundamental Theorem of Calculus,

$$f(x) = f(x) - f(0) = \int_0^x f'(x)dx = \int_0^x f(x)g(x)dx$$

Since we know that $f(x) = 0$ for $x \in [a, 0]$, then

$$\int_a^0 f(x)g(x)dx = 0$$

Hence,

$$f(x) = \int_0^x f(x)g(x)dx + \int_a^0 f(x)g(x)dx = \int_a^x f(x)g(x)$$

In other words

$$f(x) \leq C + \int_a^x f(x)g(x)$$

Where C=0. Thus, by part a,

$$0 \leq f(x) \leq C \exp\left(\int_a^x g\right) = 0$$

So,

$$f(x) = 0$$

for $x \in [a, b]$ □

Question 6

(a)

Proof. Because we know that

$$\frac{e^x}{x}$$

goes to ∞ as x goes to ∞. It follows that

$$e^{x^2} \geq e^x$$

since $e^{a^2} = e^1$ and $(e^{x^2})' > (e^x)'$. Thus for each $M > 0$, there is a $K > 0$ such that if $x > max\{K, 1\}$

$$M < \frac{e^x}{x} \leq \frac{e^{x^2}}{x}$$

Hence, $\frac{e^{x^2}}{x} \to \infty$ as $x \to \infty$. $\square$

(b)

Proof. By part a, for each $\epsilon > 0$ there exists a $K > 0$ such that if $x > max\{K, 1\}$, then

$$\frac{e^{x^2}}{x} > \frac{1}{\epsilon}$$

Hence by taking the reciprocal, for all $x > max\{K, 1\}$,

$$\frac{x}{e^{x^2}} < \epsilon$$

Hence by the definition of a limit at infinity, $\frac{x}{e^{x^2}} \to 0$ as $x \to \infty$ $\square$

(c)

Proof. By part b, for each $\epsilon > 0$,there exists a $K > 0$ such that if $x > K$ then

$$|\frac{x}{e^{x^2}}| < \epsilon$$

Let $\delta = \frac{1}{K}$. Hence if $h < \delta$, then $\frac{1}{h} > K$, so

$$|\frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}| < \epsilon$$

Hence, as $x \to 0$, $\frac{\frac{1}{h}}{e^{\frac{1}{h^2}}} \to 0$ $\square$

(d)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \stackrel{\text{def}}{=} \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In order for this function be differentiable, the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

must exist. For the limit at 0, $a = 0$. Hence, the limit becomes

$$\lim_{h \to 0} \frac{f(h)}{h}$$

Because h never is equal to 0,

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = \lim_{h \to 0} \frac{1/h}{e^{1/h^2}}$$

By part c, this limit is equal to 0. Hence f'(0) exists and

$$f'(0) = 0$$

For $x \neq 0$, we know that the exponential function is differentiable. So by the chain rule,

$$(e^{-1/x^2})' = \frac{2e^{-1/x^2}}{x^3}$$

Thus,

$$f'(x) = \begin{cases} \frac{2e^{-1/x^2}}{x^3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Question 7

Proof. The first use of the L'Hopital's rule is correct as

$$\lim_{x \to 1} x^3 + x - 2 = 0$$
$$\lim_{x \to 1} x^2 - 3x - 2 = 0$$

Hence L'Hopital's rule is applicable. The issue comes with the second application as now

$$\lim_{x \to 1} 3x^2 + 1 = 4$$
$$\lim_{x \to 1} 2x + -3 = -1$$

Hence the top and bottom do not equal 0 and so L'Hopital's rule does not apply. $\square$