163 HWK 7

James Gillbrand

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Question 1

Proof. Let (a_n) be a sequence of nonzero real numbers.

Case 1

Assume that $(|\frac{a_n+1}{a_n}|)$ diverges to ∞. Then, for any $M > 0$ there exists an $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq N$, then

$$|\frac{a_{n+1}}{a_n}| > M$$

Hence,

$$|a_{n+1}| > |a_n|M$$

Consequently, we can choose an M such for each $n \geq N$,

$$|a_n| > \epsilon$$

for any epsilon. Hence, (a_n) does not converge to 0 and thus,

$$\sum_{n=1}^{\infty} a_n$$

diverges.

Case 2

Assume $(|\frac{a_n+1}{a_n}|)$ is convergent and that $\lim_{n\to\infty} |\frac{a_n+1}{a_n}| > 1$. Then

$$\lim_{n\to\infty} |a_n| \neq 0$$

Hence, the sum does not converge by the nth term test. □

Question 2

Proof. Consider the power series,

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Assume that $0 \leq x \leq 1$, then

$$\frac{x^n}{n^2} \leq \frac{1}{n^2}$$

Because the sum of $\frac{1}{n^2}$, is convergent, by the comparison test, so too does

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

for $0 \leq x \leq 1$. If $x = -1$, it is also convergent by the alternating series test. If $x > 1$, then

$$\lim_{n\to\infty} \frac{x(n^2)}{(n+1)^2} = x > 1$$

Hence it doesn't converge. Thus the interval of convergence is $[-1, 1]$ and the radius is 1. $\square$

Question 3

Proof. Let a be a nonzero number. Let $f : \{x \in \mathbb{R} : x \neq a\} \to \mathbb{R}$ be defined as

$$f(x) \overset{\text{def}}{=} \frac{1}{x - a_0}$$

By the ratio test we know that

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Let $r = \frac{x}{a_0}$ and $a = -\frac{1}{a_0}$. Then,

$$\sum_{n=0}^{\infty} ar^n = \frac{1}{-a_0(1 - \frac{x}{a_0})} = \frac{1}{x - a_0} = f(x)$$

Hence, by the geometric series test, $f(x)$ is convergent exactly when $|\frac{x}{a_0}| < 1$. So for $|x| < a_0$. Thus the radius of convergence is $(-|a_0|, |a_0|)$ $\square$

Question 4

(a)

Proof. The following series, obtained from the geometric series test will allow us to obtain the desired series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Taking the derivative,

$$(\sum_{n=0}^{\infty} x^n)' = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$$

For $x = \frac{1}{2}$, because $x < 1$ we know that this series converges and that

$$\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

$\square$

(b)

Proof. Proceeding by the same reasoning as above,

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$$

$$(\sum_{n=0}^{\infty} nx^n)' = \sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{-x-1}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{-x^2-x}{(1-x)^3}$$

So, for $x = \frac{1}{2}$

$$\sum_{n=0}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \frac{-1/4-1/2}{(1-1/2)^3}$$

$$= \frac{-3/4}{1/8} = \frac{-24}{4} = -6$$

$\square$

Question 5

(a)

Proof. Define,

$$f(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n (x-c)^n$$

Let k be a natural number. Proceed with induction on the derivative of f.

Base Case

Let k = 1, then,

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} n a_n (x-c)^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{n!}{(n-1)!} a_n (x-c)^{n-1} \end{aligned}$$

Inductive Step and Hypothesis

Let $k \in \mathbb{N}$ and assume that,

$$f^k(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n (x-c)^{n-k}$$

Proof of Inductive Step

Take

$$\begin{aligned} f^{k+1}(x) &= (\sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n (x-c)^{n-k})' \\ &= \sum_{n=0}^{\infty} (n-k) \frac{n!}{(n-k)!} a_n (x-c)^{n-k-1} \\ &= \sum_{n=0}^{\infty} \frac{n!}{(n-(k+1))!} a_n (x-c)^{n-(k+1)} \end{aligned}$$

Hence, by induction

$$f^k(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n (x-c)^{n-k}$$

□

(b)

Proof. Let k be a nonnegative integer. Then,

$$f^k(c) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n (c-c)^{n-k}$$

This term is only nonzero when $n = k$, hence,

$$f^k(c) = k! a_k$$

$\square$

(c)

Proof. Assume that $f(x) = 0$ for each x in $(c - R, c + R)$. Then, per a problem from quarter 1, $f'(x) = 0$. Inductively this can be extended to all nonnegative derivatives $f^k(x)$. Hence,

$$f^k(x) = 0 = k! a_k$$

Thus,

$$a_k = 0$$

$\square$

Question 6

Proof. Define,

$$g(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n(x-c)^n$$

Assume that there exists some convergent subsequence $(x_k) \in (c-R, c+R)$ such that $x_k \neq c$, $\lim_{n\to\infty} x_k = c$ and $g(x_k) = 0$. Since, $g(x)$ is power series, it is continuous on $(c-R, c+R)$. Hence by assignment 3 problem 1,

$$\lim_{k\to\infty} g(x_k) = g(\lim_{n\to\infty} x_k) = g(c) = 0$$

Also,

$$\lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} \sum_{n=0}^{\infty} a_n(x_k-c)^n = a_0 + \lim_{k\to\infty} \sum_{n=1}^{\infty} a_n(x_k-c)^n$$

Define,

$$f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n(x-c)^n$$

Then, similarly,

$$\lim_{k\to\infty} f(x_k) = f(\lim_{n\to\infty} x_k) = f(c) = 0$$

Hence,

$$a_0 + \lim_{k\to\infty} \sum_{n=1}^{\infty} a_n(x_k-c)^n = a_0 + 0 = 0$$

□

Question 7

Let c be a real number and (a_n) be a sequence of real numbers. Assume that $R > 0$ is the radius of convergence for the following power series. Define $f : (c - R, c + R) \to \mathbb{R}$ by,

$$f(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n (x-c)^n$$

Assume that there exists a convergent subsequence (x_k) in $(c - R, c + R)$ such that

- $x_k \neq c$
- $\lim_{n\to\infty} x_k = c$
- $f(x_k) = 0$ for each natural k

For each nonnegative integer k, define $g_k : (c - R, c + R) \to \mathbb{R}$ as,

$$g_k(x) = \sum_{n=k}^{\infty} a_n (x-c)^{n-k}$$

(a/b)

Proof. Proceed by induction,

Base Case:
Let $k = 0$. There is nothing to prove as

$$(x-c)^0 g_0(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = f(x)$$

Also,

$$0 = f(x_j) = (x_j 0 c)^0 g_0(x_j) = \sum_{n=0}^{\infty} a_n (x_j - c)^n$$

By question 6, $a_0 = 0$

Inductive Step and Hypothesis
Let $k \geq 0$. Assume that

$$f(x) = (x-c)^k g_k(x) = (x-c)^k \sum_{n=k}^{\infty} a_n (x-c)^{n-k}$$

and that $a_k = 0$

Proof of Inductive step
Take the case of $k+1$. Then,

$$f(x) = (x-c)^k \sum_{n=k}^{\infty} a_n (x-c)^{n-k}$$
$$= (x-c)^k (a_k (x-c)^0) + (x-c)^k \sum_{n=k+1}^{\infty} a_n (x-c)^{n-k}$$
$$= (x-c)^k (a_k) + (x-c)^{k+1} \sum_{n=k+1}^{\infty} a_n (x-c)^{n-k-1}$$
$$= (x-c)^{k+1} \sum_{n=k+1}^{\infty} a_n (x-c)^{n-k-1}$$

This expression is the same as,

$$f(x) = (x-c)^{k+1} g_{k+1}(x)$$

since,

$$g_{k+1}(x) = \sum_{n=k+1}^{\infty} a_n (x-c)^{n-(k+1)}$$

Also, by question 6, $a_{k+1} = 0$.

Thus by induction the proof is complete. $\square$

Question 8

Proof. Let c be a real number. Let (a_n) and (b_n) be sequences of real numbers. Assume that r is a positive real number such that the power series,

$$\sum_{n=0}^{\infty} a_n(x-c)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(x-c)^n$$

are pointwise convergent on $(c-r, c+r)$. Assume that

$$\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} b_n(x-c)^n$$

for each $x \in (c-r, c+r)$. Define $f : (c-r, c+r) \to \mathbb{R}$ as

$$f(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n(x-c)^n - \sum_{n=0}^{\infty} b_n(x-c)^n$$

Define (c_n) as the sequence where the nth term is given by $c_n = a_n - b_n$. Then,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-c)^n$$

Also, $f(x) = 0$ for each $x \in (c-r, c+r)$ by assumption. Consequently by question 5, part c, each $c_n = 0$. Hence,

$$a_n - b_n = 0$$
$$a_n = b_n$$

□

Question 9

Let the sequence (a_n) be the Fibonnaci Sequence, defined by $a_1 \overset{\text{def}}{=} 1$, $a_2 \overset{\text{def}}{=} 1$, and

$$a_{n+1} \overset{\text{def}}{=} a_n + a_{n-1}$$

for each $n = 2, 3, 4....$ Assume that $(\frac{a_{n+1}}{a_n})$ is convergent.

(a)

Proof.

$$\frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_{n-1} + a_{n-2}}$$

Because each a_n is a positive term,

$$\frac{a_{n-1}}{a_{n-1} + a_{n-2}} < 1$$

Hence

$$1 + \frac{a_{n-1}}{a_{n-1} + a_{n-2}} < 2$$

□

(b)

Proof. Consider the following power series

$$\sum_{n=1}^{\infty} a_n x^{n-1}$$

Since the center is 0, proving that the series is convergent at $x = \frac{1}{2}$ is enough to show that the radius, $R \geq \frac{1}{2}$. So we must consider the following series,

$$\sum_{n=1}^{\infty} a_n \frac{1}{2^{n-1}}$$

Proceeding with the ratio test we will examine the following limit:

$$\lim_{n\to\infty} \frac{a_{n+1}x^n}{a_n x^{n-1}} = \lim_{n\to\infty} \frac{a_{n+1}x}{a_n} = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

By part 1, this limit is less than or equal to 2. Hence,

$$_{n\to\infty} \frac{a_{n+1}x^n}{a_n x^{n-1}} < 1$$

Consequently the power series converges for $|x| \leq \frac{1}{2}$. Thus, $R \geq \frac{1}{2}$. □

(c)

Proof. Define $f : (-R, R) \to \mathbb{R}$ as,

$$f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n x^{n-1}$$

Then,

$$\begin{aligned} f(x) - xf(x) - x^2 f(x) &= \sum_{n=1}^{\infty} a_n x^{n-1} - \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n x^{n+1} \\ &= a_1 x^0 + \sum_{n=2}^{\infty} a_n x^{n-1} - \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n x^{n+1} \\ &= 1 + \sum_{n=1}^{\infty} a_{n+1} x^n - \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n x^{n+1} \\ &= 1 + \sum_{n=1}^{\infty} (a_n + a_{n-1}) x^n - \sum_{n=1}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n \\ &= 1 + \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-1} x^n \\ &= 1 + 0 + \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-1} x^n = 1 \end{aligned}$$

Thus,

$$\begin{aligned} f(x)(1 - x - x^2) &= 1 \\ f(x) &= \frac{-1}{x^2 + x - 1} \end{aligned}$$

□

(d)

Proof. Define,

$$\alpha \stackrel{\text{def}}{=} \frac{-1 - \sqrt{5}}{2} \quad \text{and} \quad \beta \stackrel{\text{def}}{=} \frac{-1 + \sqrt{5}}{2}$$

Let $x \in \mathbb{R}$ and assume that $x \neq \alpha$, $x \neq \beta$. Then,

$$\begin{aligned}
\frac{1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{x-\beta} &= \frac{1/\sqrt{5}((x-\beta)-(x-\alpha))}{x^2 - x\alpha - x\beta + \alpha\beta} \\
&= \frac{\alpha-\beta}{\sqrt{5}(x^2+2x/2+1/4(1-5)} \\
&= \frac{-\sqrt{5}}{\sqrt{5}(x^2+x-1)} \\
&= \frac{-1}{x^2+x-1}
\end{aligned}$$

□

(e)

Proof. Take the following equation,

$$\frac{1/\sqrt{5}}{x-\alpha} - \frac{1/\sqrt{5}}{x-\beta}$$

By question 3, these can be approximated by the following power series:

$$\frac{1}{\sqrt{5}}(\sum_{n=1}^{\infty} -\frac{1}{\alpha}(\frac{x}{\alpha})^n - \sum_{n=1}^{\infty} -\frac{1}{\beta}(\frac{x}{\beta})^n)$$

□

(f)

Proof.

$$\begin{aligned}
\frac{1}{\sqrt{5}}(\sum_{n=1}^{\infty} -\frac{1}{\alpha}(\frac{x}{\alpha})^n - \sum_{n=1}^{\infty} -\frac{1}{\beta}(\frac{x}{\beta})^n) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{5}}(-\frac{1}{\alpha^{n+1}} + \frac{1}{\beta^{n+1}})x^n \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{5}}(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^{n+1}\beta^{n+1}})x^n \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{5}(-1)^{n+1}}(\alpha^{n+1}-\beta^{n+1})x^n \\
&= \sum_{n=1}^{\infty} \frac{(\alpha^n-\beta^n)}{\sqrt{5}(-1)^n}x^{n-1} \\
&= \sum_{n=1}^{\infty} a_n x^{n-1}
\end{aligned}$$

Hence, $a_n = \frac{(\alpha^n - \beta^n)}{\sqrt{5}}$. □