20250 HWK 7

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Question 1

Proof. Let $\alpha : V \to V$ be an endomorphism and let W_{nilp}, W_{inv} be αinvariant complementary subspaces of V and let $a_{W_{nilp}}, a_{W_{inv}}$ be nilpotent and invertible restrictions respectively. We know that,

$$W_{nilp} \oplus W_{inv} = V = ker(\alpha^{\infty}) \oplus im(\alpha^{\infty})$$

Also, it is clear that $W_{nilp} \subseteq ker(\alpha^{\infty})$, as by definition there exists some d such that $\alpha(W_{nilp})^d = 0$. Also we know that since $a_{W_{inv}}$ is invertible,

$$W_{inv} = a_{W_{inv}}(W_{inv}) im(\alpha^{\infty})$$

Hence,

$$W_{nilp} \oplus W_{inv} \subseteq ker(\alpha^{\infty}) \oplus im(\alpha^{\infty})$$

Which suffices to show that,

$$W_{nilp} = ker(\alpha^{\infty})$$
$$W_{inv} = im(\alpha^{\infty})$$

$\square$

Question 2

Proof. Let β be a commuting endomorphism with α. Let $v \in V_\lambda$. That means that $\alpha(v) = \lambda \cdot v \in V_\lambda$. We want to show that $\beta(v) \in V_\lambda$. Then,

$$\beta(v) = \beta(\alpha(v)) = \alpha(\beta(v))$$
$$\beta(\alpha(v)) = \beta(\lambda \cdot v) = \lambda\beta(v)$$

Thus,

$$\alpha(\beta(v)) = \lambda\beta(v)$$

Which means that $\beta(v) \in V_\lambda$. Thus the eigenspace is beta-invariant.

Let $v \in V_{(\lambda)}$. That means that there exists some number d such that, $\alpha(v)^d = \lambda v$. Then,

$$\beta(v) = \alpha(\beta(v))^d = (\alpha(v))$$

which proves that $\beta(v) \in V_{(\lambda)}$ and hence this subspace is beta-invariant.

Let $v \in V^{n.s.}$. Then $\alpha^d(v) \neq \lambda v$ for any $\lambda \in k$ and any d. Hence, $\alpha(v)$ is also in $V^{n.s.}$. Thus,

$$\beta(v) = \beta(\alpha(v)) = \alpha(\beta(v)) \neq \lambda(\beta(v))$$

Hence, $\beta(v) \in V^{n.s.}$. Consequently, $V^{n.s}$ is beta invariant $\square$

Question 3

Proof. ($\Rightarrow$) Assume that $\alpha_1, ..., \alpha_n$ are simultaneously diagonalizable. Hence, there exists a basis where $[\alpha_1], ..., [\alpha_n]$ are diagonal. Hence, for any $\alpha_i, \alpha_j \in \alpha_1, ..., \alpha_n$,

$$\alpha_i \circ \alpha_j = [\alpha_i][\alpha_j]$$

Multiplying two diagonal matrixes is the same as multiplying each diagonal element by its partner in the other matrix. As such,

$$[\alpha_i][\alpha_j] = [\alpha_j][\alpha_i]$$

Thus, these endormorphisms commute pairwise and are clearly diagonal.

($\Leftarrow$) Assume that $\alpha_1, ..., \alpha_n$ pairwise commute and are diagonalizable. Take an arbitrary pair of these endomorphisms α_i, α_j. We know that we can decompose V into a sum of V_λ eigenspaces for α_i. If we can show that α_j is diagonalizable on its restriction to each of these subspaces, this will imply that we can write α_j as a diagonal matrix with the same basis as α_i. By question 2, we know that α_j is invariant for V_λ, hence we can decompose

$$\alpha_j : V \to V$$

into

$$\oplus \alpha_{j_\lambda} : \underset{\lambda \in spec(\alpha_i)}{\oplus V_\lambda} \to \underset{\lambda \in spec(\alpha_i)}{\oplus V_\lambda}$$

Now, suppose for each λ that $v_1 + ... + v_n$ is the sum of the eigenvectors for α_j in a given α_i eigenspace. Then each $v_i \in V_\lambda$. Consequently we can write a_{j_λ} in diagonal form with these vectors. Also, because these vectors are in V_λ they can be written in the same basis as the basis that makes α_i diagonalizable. Hence, α_i and α_j are simultaneously diagonalizable. This is true for any pairing so can be extended to all $\alpha_1, ..., \alpha_n$. $\square$

Question 4

Proof. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vectors space and let $W \subset V$ be an α-invariant subspace. There exists the general short exact sequence,

$$W \hookrightarrow V \twoheadrightarrow V/W$$

Which we can decompose as follows,

$$\begin{tikzcd}
W \arrow[r, hook, "\iota"] \arrow[d] & V \arrow[r, two heads, "\pi"] \arrow[d] & V/W \arrow[d] \\
\bigoplus\limits_{\lambda \in spec(\alpha)} W_{(\lambda)} \oplus W^{n.s.} \arrow[r, hook, "\oplus\iota_\lambda"] & \bigoplus\limits_{\lambda \in spec(\alpha)} V_{(\lambda)} \oplus V^{n.s.} \arrow[r, two heads, "\oplus\pi_\lambda"] & \bigoplus\limits_{\lambda \in spec(\alpha)} (V/W)_{(\lambda)} \oplus (V/W)^{n.s.}
\end{tikzcd}$$

With the downward arrows representing isomorphisms. Hence, each ι_i is injective and π_i is surjective. For $\lambda \in spec(\alpha)$, $im(\iota_\lambda) = W_{(\lambda)}$. Hence, $\pi_\lambda(im(\iota_\lambda)) = \pi_\lambda(W) = 0$. Thus the following short exact sequence exists

$$W_{(\lambda)} \xhookrightarrow{\iota_\lambda} V_{(\lambda)} \xtwoheadrightarrow{\pi_\lambda} (V/W)_{(\lambda)}$$

The same is true for the nonspectral spaces and as such the following is also a short exact sequence.

$$W^{n.s} \xhookrightarrow{\iota_\lambda} V^{n.s} \xtwoheadrightarrow{\pi_\lambda} (V/W)^{n.s}$$

$\square$

Question 5

Proof. Take $k = \mathbb{R}$. Then the transformation defined by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has no eigenvectors as it sends $e_1 \to e_2$ and $e_2 \to -e_1$. Neither of these are linear scalings. Thus $V_{(\lambda)} = 0$ and so $V^{n.s.} = \mathbb{R}^2$

Take $k = \mathbb{F}_2$. In this case,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This transformation keeps switching the e_1 and e_2 vectors, which implies that the only eigen vector is $(1, 1)$ where it gets sent to itself. Thus $V_{(\lambda)} =< 1, 1 >$. Consequently, $V^{n.s}$ composes the rest of the vector space.

Take $k = \mathbb{C}$. Then the transformation defined by,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has the eigenvalues $i, -i$. Hence, our eigenvectors given by the equation for $\lambda = i$, where $a, b \in \mathbb{C}$.

$$-ai - b = 0$$
$$a - bi = 0$$

Thus,

$$a = bi$$
$$-(bi^2) - b = 0$$
$$b = b$$
$$a = bi$$

so (bi, b) is a eigenvector for α. For $\lambda = -i$

$$ai - b = 0$$
$$a + bi = 0$$
$$a = a$$
$$b = ai$$

Hence (a, ai) is an eigenvector for α. Hence, a basis for the eigenspace is the set of vectors $(i, 1), (-i, 1)$. Since these are two linearly independent vectors in $\mathbb{C}^2$, they space $\mathbb{C}^2$ and hence $V_{(\lambda)} = \mathbb{C}$ and $V^{n.s} = 0$

$\square$

Question 6

Proof. Let α be an endomorphism of a finite-dimensional vector space V. Assume that α is invertible. Then for any element $v \in V_{(\lambda)}$ for α, then $\alpha^{-1}(v) = \frac{v}{\lambda}$. Hence the generalized eigenspace decomposition for α^{-1} has the same exact susbspaces as α only each $V_{(\lambda)}$ becomes $V_{(\frac{1}{\lambda})}$ and $V^{n.s}$ remains the same.

Now to consider $\alpha^{\vee}$. By definition $\alpha^{\vee} : V^{\vee} \to V^{\vee}$ such that

$$\xi \to \xi \circ \alpha$$

Hence for $v \in V_{\lambda}$,

$$\xi \circ \alpha(v) = \xi(\lambda \cdot v) = \lambda \cdot \xi(v)$$

Hence $\alpha^{\vee}$ is decomposable into the same subspaces as α, those being,

$$\bigoplus_{\lambda \in spec(\alpha)} V_{(\lambda)} \oplus V^{n.s.}$$

$\square$

Question 7

Proof. Let α be an endomorphism of a finite-dimensional vector space V.

($\Rightarrow$) Assume that α is diagonalizable. Hence, there exists $\lambda_1, ...\lambda_r \in k$ unique eigenvectors of α. This means that for each λ_i, there exists a non trivial

$$ker(\alpha - \lambda \cdot \mathbb{I})$$

Also, because α is diagonalizable, it can be decomposed into only eigenspaces. Hence, every $v \in V$ is also a $v \in V_\lambda$ for some lambda. This means that there exists some $i = 1, ...r$ such that,

$$\alpha(v) - \lambda_i \cdot \mathbb{I} = 0$$

For $f(X)$ defined by,

$$f(X) \stackrel{\text{def}}{=} \prod(X - \lambda_i)$$

Then,

$$f(\alpha) = \prod(\alpha(v) - \lambda_i)$$

As show above, no matter what v we choose, one of these terms will be 0 and hence because it is a product,

$$f(\alpha) = 0$$

($\Leftarrow$) Assume that $f(\alpha) = 0$ for some polynomial of the form $f(X) \stackrel{\text{def}}{=} \prod(X - \lambda_i)$ with $\lambda_1, ...\lambda_r \in k$ distinct. That means that for any $v \in V$, there exists at least one term in the product that equals 0. Without loss of generality,

$$(\alpha(v) - \lambda_i) = 0$$
$$(\alpha(v) - \lambda_i \cdot \mathbb{I}) = 0$$

This means that each $v \in V_{\lambda_i}$. Hence, α must be diagonalizable as clearly there exists a basis of V composed of it's eigenvectors. $\square$