Finite Difference of the 1D Explicit Heat Equation

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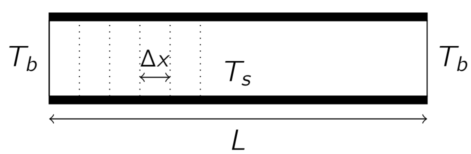

Figure 1: 1D Heat Model

The one-dimensional transient heat conduction equation without heat generating sources is given by:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \tag{1}$$

where ρ is the density, c_p heat capacity, k thermal conductivity, T temperature, x distance, and t time.

If ρ, c_p and k are constant then the equation can be simplified to:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{2}$$

where:

$$\kappa = \frac{k}{\rho c_p} \tag{3}$$

An example of the heat equation in practice is where a thin body with thermal conductivity κ (e.g. rod or laminate) is at a starting temperature T_s and is heated at both ends at a constant temperature T_b while being insulated along its length L (Figure 1). In this situation, one might be interested in how long it will take for the body to reach T_b.

We cen solve this numerically by discretising the continuous derivatives $(\partial t, \partial x)$ in Equation (2) using finite difference methods. Lets first look at the time derivative $\frac{\partial T}{\partial t}$, which can be approximated using the forward finite difference approximation.

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{T^{n+1} - t^n} = \frac{T_i^{n+1} - T_i^n}{\Delta t} \tag{4}$$

where n is a step in time and i is a point along the discretised domain.

Second, we turn our attention to the spatial derivative $\frac{\partial^2 T}{\partial x^2}$, which can be approximated using the central difference approximation.

$$\begin{aligned} \frac{\partial T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) &\approx \frac{\frac{T_{i+1}^n - T_i^n}{\Delta x} - \frac{T_i^n - T_{i-1}^n}{\Delta x}}{\Delta x} \\ &\approx \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \end{aligned} \tag{5}$$

Substituting Equations (4) and (5) into Equation (2) gives:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \kappa \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\right) \tag{6}$$

Rearranging for T_i^{n+1} gives:

$$T_i^{n+1} = T_i^n + \kappa \Delta t \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\right) \tag{7}$$