

Finite Difference of the 1D Explicit Heat Equation

Dr J. A. Gopsill

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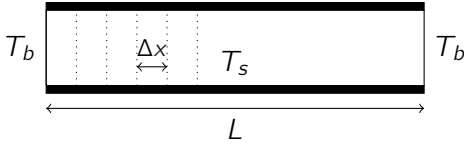


Figure 1: 1D Heat Model

The one-dimensional transient heat conduction equation without heat generating sources is given by:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \quad (1)$$

where ρ is the density, c_p heat capacity, k thermal conductivity, T temperature, x distance, and t time.

If ρ , c_p and k are constant then the equation can be simplified to:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad (2)$$

where:

$$\kappa = \frac{k}{\rho c_p} \quad (3)$$

Examples of the heat equation being used in practice is where a thin body with thermal conductivity κ (e.g. rod or laminate) is at a starting temperature of T_s , is heated at both ends at a constant temperature T_b and insulated along its length L so there is no heating along the orthogonal axes (Figure 1). In this situation, one might be interested in how long it will take for the body to reach T_b .

We can solve this numerically by discretising the continuous derivatives ($\partial t, \partial x$) in Equation (2) using finite difference methods. Let's first look at the time derivative $\frac{\partial T}{\partial t}$, which can be approximated using the forward finite difference approximation.

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{T_i^{n+1} - t^n} = \frac{T_i^{n+1} - T_i^n}{\Delta t} \quad (4)$$

where n is a step in time and i is a point along the discretised domain.

Second, we turn our attention to the spatial derivative $\frac{\partial^2 T}{\partial x^2}$, which can be approximated using central difference approximation.

$$\begin{aligned} \frac{\partial T}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) \approx \frac{\frac{T_{i+1}^n - T_i^n}{\Delta x} - \frac{T_i^n - T_{i-1}^n}{\Delta x}}{\Delta x} \\ &\approx \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \end{aligned} \quad (5)$$

Substituting Equations (4) and (5) into Equation (2) gives:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \kappa \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \right) \quad (6)$$

Rearranging for T_i^{n+1} gives:

$$T_i^{n+1} = T_i^n + \kappa \Delta t \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \right) \quad (7)$$