Notes by J.H.Davenport

29 May 2024

1 The case of the twisted "Similarly"

[Mig74] is a key piece of the jigsaw. It uses $||\cdot||$ for the L_2 norm of a polynomial.

Lemma 1 *Let $P(X)$ be a polynomial with complex coefficients and α be a nonzero complex number. Then*

$$||(X+\alpha)P(X)|| = |\alpha|||(X+\overline{\alpha}^{-1})P(X)||.$$

The proof goes as follows. [Mig74] writes

$$\begin{aligned} P(X) &= \sum_{k=0}^{m} a_k X^k, \\ Q(X) &= (X+\alpha)P(X) = \sum_{k=0}^{m+1} (a_{k-1}+\alpha a_k) X^k \\ R(X) &= (X+\overline{\alpha}^{-1})P(x) = \sum_{k=0}^{m+1} (a_{k-1}+\overline{\alpha}^{-1} a_k) X^k \end{aligned}$$

with the notation $a_{-1} = a_{m+1} = 0$.

Figure 1: Mignotte Lemma 1a in Lean

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lemma Mignotte1974L1a (p :  ℂ[X])  (α: ℂ) :
    (L2normSq ((X + C α) * p):ℂ) = ∑ᶠ k : ℕ,
      ((‖(X * p).coeff k‖^2:ℝ) +
        α*p.coeff k * conj ((X * p).coeff k) +
        conj α * (X * p).coeff k * conj (p.coeff k) +
        (‖α * p.coeff k‖^2:ℝ)).re := by
  simp only [L2normSq_finsum]
  congr; ext k
  refine (ofReal_re _).symm.trans ?_; congr
  simp [add_mul]
  cases k <;> simp [mul_pow, normSq_eq_conj_mul_self, ← normSq_eq_abs, ← ofReal_pow]
  ring
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Then

$$||Q||^2 = \sum_{k=0}^{m+1} |a_{k-1} + \alpha a_k|^2 = \sum_{k=0}^{m+1} (a_{k-1} + \alpha a_k)\overline{(a_{k-1} + \alpha a_k)}$$

which expands to

$$\sum_{k=0}^{m+1} \left(|a_{k-1}|^2 + \alpha a_k \overline{a_{k-1}} + \overline{\alpha} a_{k-1} \overline{a_k} + |\alpha^2||a_k|^2\right). \tag{1}$$

This is accomplished in Lean by the code in Figure 1.

[Mig74] then says "Expanding $|a|^2||R||^2$ yields the same sum".

However, if we expand $|a|^2||R||^2$ naively as above, we actually get

$$\sum_{k=0}^{m+1} \left(|\alpha^2||a_{k-1}|^2 + \alpha a_k \overline{a_{k-1}} + \overline{\alpha} a_{k-1} \overline{a_k} + |a_k|^2\right). \tag{2}$$

In general (1) and (2) are different: the $|\alpha|^2$ multiplies different terms. And indeed, for any k the index-k summands in (1) and (2) do differ. However, it is legitimate to re-express (1) as (3):

$$\sum_{k=0}^{m+1} \left(|a_{k-1}|^2\right) + \sum_{k=0}^{m+1} (\alpha a_k \overline{a_{k-1}} + \overline{\alpha} a_{k-1} \overline{a_k}) + \sum_{k=0}^{m+1} \left(|\alpha^2||a_k|^2\right)$$

and then as

$$||P||^2 + \sum_{k=0}^{m+1} (\alpha a_k \overline{a_{k-1}} + \overline{\alpha} a_{k-1} \overline{a_k}) + |\alpha^2|||P||^2. \tag{3}$$

A similar operation on (2) gives (4) :

$$|\alpha^2|||P||^2 + \sum_{k=0}^{m+1} (\alpha a_k \overline{a_{k-1}} + \overline{\alpha} a_{k-1} \overline{a_k}) + ||P||^2, \tag{4}$$

and now the equality between (3) and (4) is obvious,

References

[Mig74] M. Mignotte. An Inequality about Factors of Polynomials. *Math. Comp.*, 28:1153–1157, 1974.