February 15, 2017

In the following I use bold-face for the divergent integrals and normal type for the finite pieces, as defined in the TSIL manual. I use `TAI` to refer to TARCER notation and A for TSIL notation as there is a different normalisation between such integrals. We define $\kappa = 1/16\pi^2$.

The field theory we will deal with here is that with Lagrangian

$$\mathcal{L} = -\frac{1}{2}m^2 s^2 - \frac{g}{3!}s^3 - \frac{\lambda}{4!}s^4 \tag{1}$$
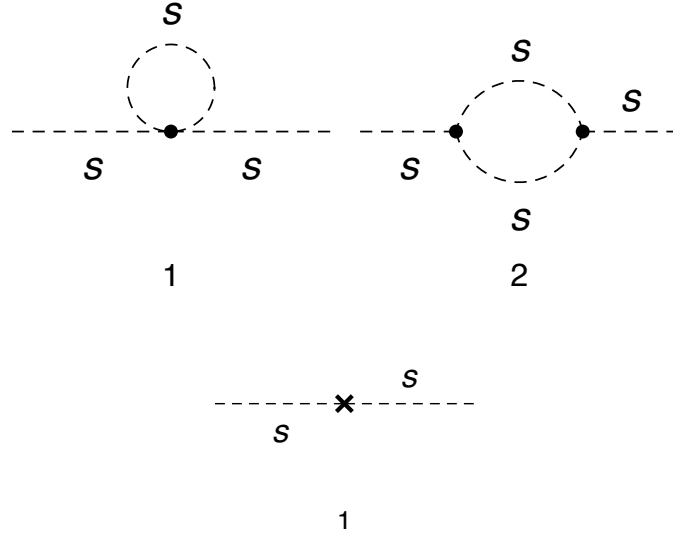
1 One-loop self energy

Figure 1: The one-loop order loop (*top*) and counter-term (*bottom*) corrections to the scalar propagator.

The scalar propagator has two loop corrections and one counter-term correction at the one-loop order which are represented in Figure **??** top and bottom respectively. Although these are easily written down by hand we use `Mass Builder` to carry out the calculation and demonstrate where we need to account for sign differences between the computational tools we interface with.

The amplitudes from the loop corrections as output from `TARCER` are

$$\Pi_1^{(1)} = -i\kappa\frac{\lambda}{2}\mathbf{TAI}(m^2) \tag{2}$$

and

$$\Pi_2^{(1)} = -i\kappa \frac{g^2}{2} \mathbf{TBI}(m^2, m^2) \tag{3}$$

first we must account for an overall negative sign in the amplitude, and then use the relationship $i\mathbf{TAI} = \mathbf{A}$ and $-i\mathbf{TBI} = \mathbf{B}$ to convert to TSIL notation

$$\Pi_1^{(1)} = \kappa \frac{\lambda}{2} \mathbf{A}(m^2) \tag{4}$$

and

$$\Pi_2^{(1)} = -\kappa \frac{g^2}{2} \mathbf{B}(m^2, m^2). \tag{5}$$

The counter-term diagram is trivial to calculate by hand, yet again we will take the result from TARCER which is

$$\Pi_1^{(1c)} = -d_1 \tag{6}$$

so again we need to make a sign change and obtain

$$\Pi_1^{(1c)} = d_1. \tag{7}$$

We can now determine this counter-term coupling from a simple calculation at one-loop, which we must do by hand. The full one-loop self energy, including counter-terms, is

$$\Pi_1 = \frac{1}{2}\kappa\lambda \left(A(m^2) + \epsilon A_\epsilon(m^2) - \frac{m^2}{\epsilon} \right) - \frac{1}{2}\kappa g^2 \left(B(m^2) + \epsilon B_\epsilon(m^2) + \frac{1}{\epsilon} \right) + d_1 \tag{8}$$

so the divergent part is

$$\Pi_1^{1/\epsilon} = -\frac{1}{2\epsilon}\kappa \left(\lambda m^2 + g^2 \right) + d_1 \tag{9}$$

so we find

$$d_1 = \frac{\kappa}{2\epsilon} \left(g^2 + \lambda m^2 \right) + \mathcal{O}(\kappa^2) \tag{10}$$

where we allow for high order terms which are required to cancel divergences from amplitudes above the one-loop order. This counter-term is all we will need to compute the finite piece of the self energy at two-loop order under certain assumptions.

So at the one-loop level we make one sign change to the TARCER output to be able to compute the self energy correctly via the TSIL integrals.

2 Two-loop self energy

There are 9 loop corrections and 5 counter-term corrections to the scalar propagator at the two-loop level. We present the results as output by TARCER and the required conversions to TSIL integrals on a diagram by diagram basis in this section.

Diagram 1
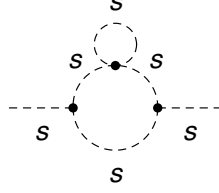
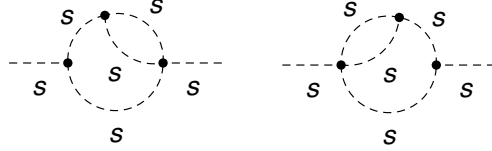This digram is given by TARCER to be

$$\Pi_1^{(2)} = \kappa^2 \frac{g^2 \lambda}{4} \frac{\mathbf{TAI}(m^2) \left[2(D-3)m^2 \mathbf{TBI}(m^2, m^2) - (D-2)\mathbf{TAI}(m^2) \right]}{m^2(4m^2 - p^2)} \tag{11}$$

so after accounting for the negative sign, and converting to TSIL notation (which also introduces a negative sign, due to the two factors of i) we obtain

$$\Pi_1^{(2)} = \kappa^2 \frac{g^2 \lambda}{4} \frac{\mathbf{A}(m^2) \left[-2(D-3)m^2 \mathbf{B}(m^2, m^2) - (D-2)\mathbf{A}(m^2) \right]}{m^2(4m^2 - p^2)} \tag{12}$$
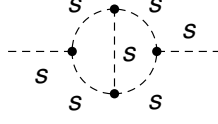
Diagram 2 and 3

Diagrams 2 and 3 are identical and are given in TARCER to be

$$\Pi_2 = \kappa^2 \frac{g^2 \lambda}{2} \mathbf{TVI}(m^2, m^2, m^2, m^2) \tag{13}$$

so combining these together and using the result $\mathbf{TVI} = -\mathbf{U}$ we obtain, after including the sign change, the same result as we could get by simply looking at the topology of the diagram

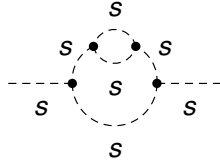$$\Pi_2 = \kappa^2 g^2 \lambda \mathbf{U}(m^2, m^2, m^2, m^2). \tag{14}$$

Diagram 4

$$\Pi_4 = \kappa^2 \frac{g^4}{2} \mathbf{TFI}(m^2, m^2, m^2, m^2, m^2) \tag{15}$$

Accounting for the negative sign and using the relationship $\mathbf{TFI} = \mathbf{M}$ we obtain the same result as the notes

$$\Pi_4 = -\kappa^2 \frac{g^4}{2} \mathbf{M}(m^2, m^2, m^2, m^2, m^2) \tag{16}$$
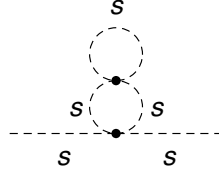
Diagram 5

This diagram is given by TARCER to be after full reduction

$$
\begin{aligned}
\Pi_5 = & \kappa^2 \frac{g^4}{12m^4} \left(\frac{3(3D-8)m^2(2m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{TJI}(m^2, m^2, m^2) \right. \\
& + \frac{4m^2(m^2-p^2)(9m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{TJI}_2(m^2, m^2, m^2) \\
& + \frac{3(D-2)m^2(2m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{TKI}(m^2, m^2, m^2) \\
& + \frac{m^2(Dp^2+2(D-9)m^2)}{4m^2-p^2} \mathbf{TVI}(m^2, m^2, m^2, m^2) \\
& \left. + 2(D-2)\mathbf{TAI}(m^2)\mathbf{TBI}(m^2, m^2) \right)
\end{aligned} \tag{17}
$$

accounting for the negative sign and converting to TSIL notation we obtain

4

$$\Pi_5 = -\kappa^2 \frac{g^4}{12m^4} \left(\frac{3(3D-8)m^2(2m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{S}(m^2,m^2,m^2) \right.$$

$$- \frac{4m^2(m^2-p^2)(9m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{T}(m^2,m^2,m^2)$$

$$+ \frac{3(D-2)m^2(2m^2-p^2)}{p^2(4m^2-p^2)} \mathbf{I}(m^2,m^2,m^2) \tag{18}$$

$$- \frac{m^2(Dp^2+2(D-9)m^2)}{4m^2-p^2} \mathbf{U}(m^2,m^2,m^2,m^2)$$

$$\left. + 2(D-2)\mathbf{A}(m^2)\mathbf{B}(m^2,m^2) \right)$$

Diagram 6

This diagram is given by TARCER to be

$$\Pi_6 = \kappa^2 \frac{\lambda^2(D-2)}{8} \frac{\left(\mathbf{TAI}(m^2)\right)^2}{m^2} \tag{19}$$

converting to TSIL notation and accounting for the negative sign we obtain

$$\Pi_6 = \kappa^2 \frac{\lambda^2(D-2)}{8} \frac{\left(\mathbf{A}(m^2)\right)^2}{m^2} \tag{20}$$

now we must set $D = 4 - r\epsilon$ and account for finite contributions which come from the divergent parts of the basis integrals. I leave the factor of r in for now. The basis integrals can be expressed as a finite piece plus a divergent piece, so we have

$$\mathbf{TAI}(x) = \frac{ix}{\epsilon} + \texttt{TAI}(x) \tag{21}$$

where we have converted (1.28) of the TSIL manual into the TARCER definition using

5

$\mathbf{A} = ia\mathbf{TAI}$ and taking $a = 1$ for small ϵ [1]. So the amplitude becomes

$$\Pi_1 = \frac{\lambda^2(2 - r\epsilon)}{8m^2}\left(\texttt{TAI}(m^2) + \frac{im^2}{\epsilon}\right)^2$$

$$= \frac{\lambda^2}{4m^2}\texttt{TAI}(m^2)\left(\frac{\texttt{TAI}(m^2)}{m^2} - ir\right)$$

where we ignore any divergent terms (assuming these will be cancelled by the counterterms) and take the limit of $\epsilon \to 0$. Finally we need to evaluate this in terms of TSIL integrals, which have a different normalization. The relationship is $\mathbf{A} = ia\mathbf{TAI}$ where $a = (4\pi\mu^2)^{2-D/2} \approx 1 + \frac{r\epsilon}{2}\log(4\pi\mu^2) + \mathcal{O}(\epsilon^2)$. Since we have already removed the divergent parts and are dealing with finite integrals, I take $\epsilon = 0$ and we simply have $\mathbf{A} = i\mathbf{TAI}$. Therefore we end up with, after setting $r = 2$ and taking account for the negative sign as with all other integrals,

$$\Pi_1 = \kappa^2\frac{\lambda^2}{4}A(m^2)\left(\frac{A(m^2)}{m^2} + 2\right) \tag{22}$$
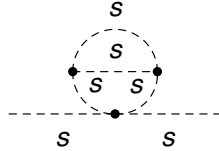
Diagram 7
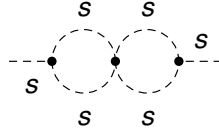
This diagram is given by TARCER to be

$$\Pi_7 = \kappa^2\frac{g^2\lambda}{4}\left(\mathbf{TBI}(m^2, m^2)\right)^2 \tag{23}$$

accounting for negative sign and using $-i\mathbf{TBI} = \mathbf{B}$ we obtain

$$\Pi_7 = \kappa^2\frac{g^2\lambda}{4}\left(\mathbf{B}(m^2, m^2)\right)^2 \tag{24}$$

Diagram 8

---

[1] I have tried keeping these additional ϵ terms in here coming from the a factor, but I have found it leads to only more problems and additional terms which don't make sense to me. Although I'm still a little uncertain, I have convinced myself taking $\epsilon \to 0$ for converting between these different normalisations is okay in this context.

This diagram is given by TARCER to be

$$\Pi_8 = -\kappa^2 \frac{g^2\lambda}{12m^2}(D-3)\mathbf{TKI}(m^2, m^2, m^2) \tag{25}$$
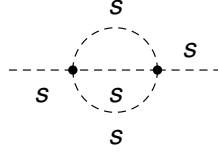
converting to TSIL notation and accounting for the negative sign

$$\Pi_8 = \kappa^2 \frac{g^2\lambda}{12m^2}(D-3)\mathbf{I}(m^2, m^2, m^2) \tag{26}$$

but we need another negative sign for the definition of K (apparently, check this)

$$\Pi_8 = -\kappa^2 \frac{g^2\lambda}{12m^2}(D-3)\mathbf{I}(m^2, m^2, m^2) \tag{27}$$

Diagram 9

This diagram is given by TARCER to be

$$\Pi_9 = \kappa^2 \frac{\lambda^2 \mathbf{TJI}(m^2, m^2, m^2)}{6} \tag{28}$$

where $\mathbf{TJI} = \mathbf{TJI}\{1, 1, 1\}$ which is equivalent to the TSIL integral $S(x, y, z)$ for $a = 1$, accounting for the negative sign we obtain

$$\Pi_9 = -\kappa^2 \frac{\lambda^2}{6}\mathbf{S}(m^2, m^2, m^2)$$

Counter-terms

The two-loop order counter-terms cancel divergences from the loop corrections and provide finite contributions to the total self energy. In the MS bar scheme the counter-term couplings are of at least first order in $\mathcal{O}(1/\epsilon)$, so we can identify which counter-term diagrams will contribute finite contributions.

Counter-diagram 1
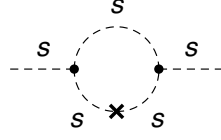
$$\Pi_1 = \kappa \frac{i d_1 g^2}{2m^2(4m^2 - p^2)} \left((D-2)\mathbf{TAI}(m^2) - 2(D-3)\mathbf{TBI}(m^2, m^2) \right) \tag{29}$$
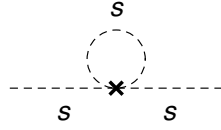
converting to TSIL notation, accounting for the negative sign and multiplying by $1/\epsilon$ we obtain

$$\Pi_1 = \kappa \frac{d_1}{\epsilon} \frac{g^2}{2m^2(4m^2 - p^2)} \left((2-2\epsilon)\mathbf{A}(m^2) + 2(1-2\epsilon)\mathbf{B}(m^2, m^2) \right) \tag{30}$$

from which we obtain the finite contributions

$$\Pi_1 = \kappa d_1 \frac{g^2}{2m^2(4m^2 - p^2)} \left(-2A(m^2) + 2A_\epsilon(m^2) - 4m^2 B(m^2, m^2) + 2Bm_\epsilon^2(m^2, m^2) \right) \tag{31}$$
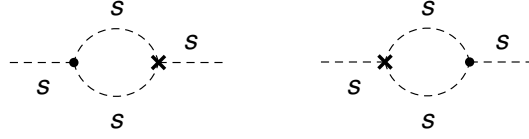
Counter-diagram 2

$$\Pi_2 = -\kappa i \frac{d_\lambda}{2} \mathbf{TAI}(m^2) \tag{32}$$

which after accounting for the negative sign and converting from TARCER to TSIL we obtain

$$\Pi_2 = -\kappa \frac{d_\lambda}{2} \mathbf{A}(m^2) \tag{33}$$

Counter-diagram 3 and 5

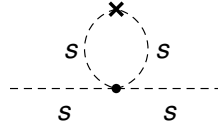$$\Pi_{3,5} = -\kappa i \frac{g d_g}{2} \mathbf{TBI}(m^2, m^2) \tag{34}$$

Combining these together and accounting for the negative sign we obtain

$$\Pi_{3,5} = i\kappa g d_g \mathbf{TBI}(m^2, m^2) \tag{35}$$

converting to TSIL notation and inserting the counter-term coupling (each introduces a negative sign)

$$\Pi_{3,5} = \kappa^2 g \frac{c_{1,1}^g}{\epsilon} \mathbf{B}(m^2, m^2) \tag{36}$$

Counter-diagram 4

$$\Pi_4 = \kappa \frac{-i\lambda d_1(D-2)}{4m^2} \mathbf{TAI}(m^2) \tag{37}$$

accounting for the negative sign and changing notation we obtain

$$\Pi_4 = \kappa \frac{\lambda d_1(D-2)}{4m^2} \mathbf{A}(m^2) \tag{38}$$

Comments

In Stephen's working the $A_\epsilon(x')$ terms cancel between the diagrams and the counter-term diagrams. However,

$$A_\epsilon(x') = \frac{A_\epsilon(x)}{x} - \frac{A(x)}{x}$$

so there is actually a $A(x)$ term here that is cancelling out between the normal diagrams and the finite contribution from the counter-terms, but it's kind of hidden. In our approach, we do not invoke this derivative at all (and don't carry around non-expanded things like $A_\epsilon(x')$) , and thus our diagram (diagram 5) explicitly contains this $A(x)$ term, so it appears wrong until we actually consider the counter-term. In Stephen's approach we could actually just ignore the $A_\epsilon(x')$ term from the beginning knowing it will cancel. All this only becomes apparent when we expand out $A_\epsilon(x')$ and look at the counter-term and normal diagrams separately.