SCHÄFFER'S CONJECTURE FOR $k = 14$

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ABSTRACT. We extend the results of Bennett-Győry-Pintér to prove Schäffer's conjecture for $k = 14$.

CONTENTS

1. INTRODUCTION

Schäffer's conjecture [8] concerns the following diophantine problem.

Conjecture 1.1. Let $S_k(x) = 1^k + 2^k + \cdots + x^k$. Then the only solutions to the equation

$$S_k(x) = y^n \tag{1}$$

in $x, y, k, n \in \mathbb{N}$ and $n \geq 2$ is the trivial solution $(x, y, k, n) = (1, 1, k, n)$ unless

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\},$$

in which case there are infinitely many solutions in $x, y \in \mathbb{N}$, or

$$(k, n, x, y) = (2, 2, 24, 70).$$

The conjecture has been proven in the following cases.

Theorem 1.2. *(Bennett-Győry-Pintér [3])* For $1 \leq k \leq 11$ and $(k, n) \notin \{(1, 2), (3, 2), (3, 4), (5, 2)\}$, equation (2.1) has only the trivial solution, unless $k = 2$, in which case there is the additional solution $(n, x, y) = (2, 24, 70)$.

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Bennett et al. showed this using a wide variety of methods. We extend this result with the following theorem.

Theorem 1.3. When $k = 14$, there are no solutions to (1).

2. Descent step

For even k, we may write

$$S_k(x) = \frac{1}{C_k} x(x+1)(2x+1)T_k(x) \tag{2}$$

where $C_k \in \mathbb{Z}$. Let $D = \gcd(x(x+1), T_k(x))$ and $d = \gcd(2x+1, T_k(x))$ where we consider the gcd over $\mathbb{Z}$ for a particular $x \in \mathbb{Z}$. For a fixed k, we will use the following result to get a finite list of possible values for D and d.

Lemma 2.1. Let $f = \alpha x + \beta, g = \gamma x^e + \ldots \in \mathbb{Z}[x]$, $\alpha, \delta \neq 0$ be such that the resultant $\mathrm{Res}(f, g, x) = \delta$ where δ is squarefree. Then for all $x_0 \in \mathbb{Z}$, $(f(x_0), g(x_0)) \mid \delta$.

Proof. Suppose $x_0 \in \mathbb{Z}$. We have that $p \mid (f(x_0), g(x_0))$ if and only if $p \mid \mathrm{Res}(f, g, x)$ so the primes dividing $(f(x_0), g(x_0))$ must be among the primes dividing $\mathrm{Res}(f, g, x) = \delta$. Recall that

$$\mathrm{Res}(f, g, x) = \alpha^e g(x_0). \tag{3}$$

Let p be a prime dividing δ. Set $x = x_0 + px_1$ and note

$$f_1 = f(x_0 + px_1)/p, g_1 = g(x_0 + px_1)/p \in \mathbb{Z}[x_1]. \tag{4}$$

Then

$$\mathrm{Res}(f_1, g_1, x_1) = \alpha^e g(x_0)/p \tag{5}$$

is coprime to p. Hence, $p \| g(x_0)$.

□

Corollary 2.2. For $2 \le k \le 14$ even, we have that

(i) D and d are both square-free,
(ii) if a prime $p \mid D$ or $p \mid d$, then $p \| T_k(x)$.

Proof. We apply Lemma 2.1 to each linear factor where we note the three linear factors are pairwise coprime. □

The values for D and d are listed in Table 2. We will also need the following lemmas.

Lemma 2.3. Suppose

$$ab = dy^n$$

for $a, b, d, y, n \in \mathbb{N}$ where $(a, b) = 1$. Then there exist $d_1, d_2, y_1, y_2 \in \mathbb{N}$ such that

$$d = d_1 d_2, \quad y = y_1 y_2$$
$$(d_1, d_2) = (y_1, y_2) = 1$$
$$a/d_1 = y_1^n, \quad b/d_2 = y_2^n.$$

k	D divides	d divides	C_k
2	1	1	6
4	1	7	$2 \cdot 3 \cdot 5$
6	1	31	$2 \cdot 3 \cdot 7$
8	3	$3 \cdot 127$	$2 \cdot 3^2 \cdot 5$
10	5	$5 \cdot 7 \cdot 73$	$2 \cdot 3 \cdot 11$
12	691	$23 \cdot 89 \cdot 691$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$
14	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7 \cdot 8191$	$2 \cdot 3^2 \cdot 5$

TABLE 1. Summary of constants

Proof. This is an application of unique factorization in $\mathbb{Z}$. $\square$

Note that since $D \mid T_k(x)$, we have $D \mid C_k y^n$. Next, define $m := (D, C_k)$ and $m' := \frac{D}{m}$ so that $(m', C_k) = 1$. Since $m' \mid D \mid C_k y^n$ and $(m', C_k) = 1$, we have $m' \mid y^n$. Additionally, m' is squarefree, thus $m' \mid y$. Let $y_m := \frac{y}{m'}$.

Multiplying (2) by $\frac{C_k}{D}$, we have

$$\begin{aligned} x(x+1)(2x+1)\frac{T_k(x)}{D} &= \frac{C_k y^n}{D} \\ &= \frac{C_k y^n}{mm'} \\ &= \frac{C_k}{m}\frac{y^n}{m'} \\ &= \frac{C_k}{m}(m')^{n-1} y_m^n. \end{aligned}$$

Since x, $x+1$, and $2x+1$ are pairwise coprime, and by Corollary 2.2 that if p is a prime dividing D, then $p \| T_k(x)$, then

$$\left(x(x+1), (2x+1)\frac{T_k(x)}{D} \right) = 1. \tag{6}$$

Next, we know $(m')^{n-1} \mid x(x+1)(2x+1)\frac{T_k(x)}{D}$. Since we have $m' \mid D \mid x(x+1)$, if $m' \mid (2x+1)\frac{T_k(x)}{D}$, then this would contradict (6). Hence $\gcd(m', (2x+1)\frac{T_k(x)}{D}) = 1$ so $(m')^{n-1} \mid x(x+1)$. We can write

$$y_m^n = u^n v^n, \tag{7}$$

where

$$u^n := \frac{1}{2(m')^{n-1}\alpha} x(x+1), \qquad v^n := \frac{1}{\beta}(2x+1)\frac{T_k(x)}{D} \tag{8}$$

as the right hand sides of (8) are coprime. Here $2(m')^{n-1}\alpha\beta = \frac{C_k}{m}(m')^{n-1}$ or more simply, $\alpha\beta = \frac{C_k}{2m}$. Further, we can write

$$u^n = u_0^n u_1^n \tag{9}$$

where either

$$u_0^n := \frac{1}{\alpha_0 (m')^{n-1}} x, \quad u_1^n := \frac{1}{\alpha_1}(x+1) \tag{a}$$

or

$$u_0^n := \frac{1}{\alpha_0} x, \quad u_1^n := \frac{1}{\alpha_1 (m')^{n-1}}(x+1) \tag{b}$$

as the right hand sides of (a) and (b) are coprime. Here $\alpha_0\alpha_1 = 2\alpha$.

Note that since $\gcd(x, x+1) = 1$, we must have either $\gcd(m', x) = 1$ or $\gcd(m', x+1) = 1$ and hence when $m' \neq 1$, we have (exclusively) $(m')^{n-1} \mid x$ or $(m')^{n-1} \mid x+1$.

In the rest of the set up, we will assume that we are in case (a) as the case of (b) is similar. Rearranging and substituting the expressions for u_0^n and u_1^n, we get

$$\alpha_0 (m')^{n-1} u_0^n + 1 = \alpha_1 u_1^n. \tag{10}$$

Remark 2.4. Note that if $(D, d) \neq 1$, then $\exists p$ such that $p \mid (x(x+1), 2x+1)$, which is a contradiction. Hence, for the rest of this discussion, we may assume that $(d, D) = 1$.

Next, we factor $v^n = \frac{1}{\beta}(2x+1)\frac{T_{14}(x)}{D}$ in a similar manner to how we split y_m^n into $u^n v^n$. Define $d_0 = (d, \beta)$, $d_1 = \frac{d}{d_0}$. We have $(d_1, \beta) = 1$ and $d_1 \mid \beta v^n$ so $d_1 \mid v^n$. Since d_1 is squarefree, $d_1 \mid v$. Let $v_d := \frac{v}{d_1}$. Then

$$\begin{aligned}
(2x+1)\frac{T_k(x)}{dD} &= \frac{\beta v^n}{d} \\
&= \frac{\beta}{d_0}\frac{v^n}{d_1} \\
(2x+1)\frac{T_k(x)}{dD} &= \frac{\beta}{d_0} d_1^{n-1} v_d^n
\end{aligned}$$

Also we need to assume that if p is a prime dividing d, then $p \| T_k(x)$ (for $k = 14$, it is easy to verify that $p = 3, 5, 7, 8191$ and if $p \mid 2x+1$ then $p^2 \nmid T_{14}(x)$.) So

$$\gcd\left(2x+1, \frac{T_k(x)}{dD}\right) = 1.$$

Since $d_1 \mid 2x+1$, so $\gcd\left(d_1, \frac{T_k(x)}{dD}\right) = 1$. Hence $d_1^{n-1} \mid 2x+1$.

Then we can write $v_d^n = v_0^n v_1^n$ where

$$v_0^n := \frac{1}{d_1^{n-1}\beta_0}(2x+1), \qquad v_1^n := \frac{1}{\beta_1}\frac{T_k(x)}{dD} \tag{11}$$

as again the right hand sides of (11) are coprime, where $\beta_0\beta_1 = \frac{\beta}{d_0}$. Substituting $u_0^n = \frac{1}{\alpha_0(m')^{n-1}}x$ into v_0^n yields

$$2\alpha_0(m')^{n-1}u_0^n + 1 = \beta_0 d_1^{n-1}v_0^n \tag{12}$$

Finally, doing (12) - 2(10) gives

$$2\alpha_1 u_1^n - 1 = \beta_0 d_1^{n-1}v_0^n \tag{13}$$

Remark 2.5. Each of $\alpha_i, \beta_i, m', d_1$ are pairwise coprime.

To consider both the (a) and (b) case simultaneously, we may define

$$w_0 = \begin{cases} (m')^{n-1}u_0^n & \text{case a} \\ u_0^n & \text{case b} \end{cases}, \qquad w_1 = \begin{cases} u_1^n & \text{case a} \\ (m')^{n-1}u_1^n & \text{case b} \end{cases}, \qquad w_2 = d_1^{n-1}v_0^n$$

and instead consider the equivalent system of equations

$$\alpha_1 w_1 - \alpha_0 w_0 - 1 = 0 \tag{14}$$

$$\beta_0 w_2 - 2\alpha_0 w_0 - 1 = 0 \tag{15}$$

$$2\alpha_1 w_1 - \beta_0 w_2 - 1 = 0 \tag{16}$$

We have proved the following.

Proposition 2.6. For even k, suppose

$$S_k = \frac{1}{C_k}x(x+1)(2x+1)T_k(x) = y^n$$

has solution x, y in $\mathbb{N}$. Let $D = \gcd(x(x+1), T_k(x))$, $d = \gcd(2x+1, \frac{T_k(x)}{D})$ and $m' = \frac{D}{\gcd(D, C_k)}$. Assume

(i) D and d are both square-free,
(ii) if a prime $p \mid D$ or $p \mid d$, then $p \| T_k(x)$.

Then for some $\alpha, \beta, \alpha_0, \alpha_1, \beta_0 \in \mathbb{N}$ satisfying

$$\alpha\beta = \frac{C_k}{2\gcd(D, C_k)}, \qquad \alpha_0\alpha_1 = 2\alpha, \qquad \beta_0 \mid \frac{\beta}{\gcd(d, \beta)},$$

the system of equations

$$\alpha_1 w_1 - \alpha_0 w_0 - 1 = 0 \tag{17}$$

$$\beta_0 w_2 - 2\alpha_0 w_0 - 1 = 0 \tag{18}$$

$$2\alpha_1 w_1 - \beta_0 w_2 - 1 = 0 \tag{19}$$

is solvable in $u_0, u_1, v_0 \in \mathbb{N}$ where

$$w_0 = \begin{cases} (m')^{n-1}u_0^n & \text{if } (m')^{n-1} \mid x \\ u_0^n & \text{if } (m')^{n-1} \mid x+1 \end{cases}, \qquad w_1 = \begin{cases} u_1^n & \text{if } (m')^{n-1} \mid x \\ (m')^{n-1}u_1^n & \text{if } (m')^{n-1} \mid x+1 \end{cases}, \qquad w_2 = \left(\frac{d}{\gcd(d, \beta)}\right)^{n-1} v_0^n.$$

3. The modular method

Theorem 3.1. *(Darmon-Merel [4])* The only non-zero primitive solutions (a, b, c) to the equation $x^n + y^n = 2z^n$ satisfies $abc = \pm 1$.

If one of the equations from the descent are of the form of Darmon-Merel, then the case leads to either the trivial solution or no solution.

Indeed, if equation (2.10) satisfies Darmon-Merel, then we have $\alpha_0 = 1, \alpha_1 = 2$ or vice versa. In either case, by Darmon-Merel, the only solution satisfies $|u_0 u_1| = 1$. If $\alpha_0 = 1$, then $u_0 = u_1 = 1$ so $x = \alpha_0 u_0^n = 1$, which is the trivial solution. If $\alpha_1 = 1$, then $u_0 = u_1 = -1$ so $x = \alpha_0 u_0^n = -2 \notin \mathbb{N}$.

If either (2.11) or (2.12) satisfy Darmon-Merel, then $\beta_0 = 1 = d_1$ and either $\alpha_0 = 1$ or $\alpha_1 = 1$ respectively. In the case of (2.11), we get $u_0 = -1 = v_0$ so again, $x = -2 \notin \mathbb{N}$. In (2.12), $u_1 = 1 = v_0$ so $x = 0 \notin \mathbb{N}$.

If Darmon-Merel doesn't apply, then to each of (17), (18), and (19), we may respectively attach the elliptic curves,

$$E_0 : y^2 = \begin{cases} x(x+1)(x - \alpha_0 w_0) & \alpha_0 \equiv 0 \pmod{2} \\ x(x+1)(x + \alpha_1 w_1) & \alpha_1 \equiv 0 \pmod{2} \end{cases} \tag{20}$$

$$E_1 : y^2 = x(x+1)(x - 2\alpha_0 w_0) \tag{21}$$

$$E_2 : y^2 = x(x+1)(x + 2\alpha_1 w_1) \tag{22}$$

By Kraus-Halberstadt [5], we can compute the level of the mod p Galois representation associated to each of the curves after level-lowering by

$$N_i = \begin{cases} 2\mathrm{rad}'(R_i) & v_2(R_i) = 0 \text{ or } v_2(R) \geq 5 \\ 2\mathrm{rad}'(R_i) & 1 \leq v_2(R_i) \leq 3, xyz \text{ even} \\ \mathrm{rad}'(R_i) & v_2(R_i) = 4 \\ 32\mathrm{rad}'(R_i) & v_2(R_i) = 1, xyz \text{ odd} \\ 8\mathrm{rad}'(R_i) & v_2(R_i) \in \{2, 3\}, xyz \text{ odd} \end{cases}$$

where

$$R_0 = \alpha_0 \alpha_1 m', \quad R_1 = \begin{cases} 2\alpha_0 m' \beta_0 d_1 & \text{case a} \\ 2\alpha_0 \beta_0 d_1 & \text{case b} \end{cases}, \quad R_2 = \begin{cases} 2\alpha_1 \beta_0 d_1 & \text{case a} \\ 2\alpha_1 m' \beta_0 d_1 & \text{case b} \end{cases}$$

To any of the mod p Galois representations, we may attach a modular form f that arises mod p. Let $K = \mathbb{Q}(a_n(f))$. Then there exists a prime ideal $\mathcal{B}$ over p such that for all primes $\ell \neq p$,

(1) If E_i is non-singular mod p, then $a_\ell(E) \equiv a_\ell(f) \pmod{\mathcal{B}}$
(2) If E_i is singular mod p, then $\ell + 1 \equiv \pm a_\ell(f) \pmod{\mathcal{B}}$

In particular, define

$$B_i(f) := \begin{cases} \text{Norm}(a_\ell(E_i) - a_\ell(f)) & E_i \text{ non-singular mod } \ell \\ \text{Norm}((a_\ell(f))^2 - (\ell+1)^2) & E_i \text{ singular mod } \ell \end{cases}$$

Thus any solution to eqi must have exponent $p \mid B_i(f)$, or $p = \ell$. Note that since this holds for all primes $\ell \neq p$, we can consider $B_i(f)$ for various ℓ and take the intersection of possible primes.

4. Local constraints

While the modular method can eliminate most primes, we are left with a finite list of primes that are still possible as exponents.

First, we try the method of Kraus [7]. Fix a prime p. Let $\ell \equiv 1 \pmod{p}$ and $\ell \nmid N_i$. Further, suppose that there exists a solution to the system in u_0, u_1, v_0 mod ℓ. Then from the previous section, we know that the only possible exponents must be a prime dividing $B_i(f)$. Thus, if $p \nmid B_i(f)$, then we p doesn't solve the system. Our implementation of Kraus is outlined in Algorithm 1.

Algorithm 1: The method of Kraus for a fixed exponent p

```
for prime ℓ ≡ 1 (mod p), ℓ ∤ N_i do
    for u_0^p ∈ {a^p (mod ℓ) : a ∈ Z/ℓZ} do
        Solve for u_1^p and v_0^p.
        if No solution exists in u_1^p, v_0^p then
            Continue u_0^p
        end
        Compute E_i reduced mod ℓ.
        Compute A := {p : p | B_i(f) for some f of level N_i, ∀i = 0, 1, 2}
        if 0 ∈ A then
            Continue ℓ
        end
        else
            Continue u_0^p
        end
    end
end
```

If the method of Kraus doesn't eliminate p from being an exponent, then we instead consider the system mod p^2 and show by brute force that there are no solutions to the system in u_0, u_1, v_0 with fixed p.

5. Linear forms in logarithms

We will make use of two powerful theorems to deal with any cases not handled by the modular and local methods.

Theorem 5.1. *(Bennett [2])* If $A, B, n \in \mathbb{Z}$, $AB \neq 0$, $n \geq 3$, then

$$|Ax^n - By^n| = 1$$

has at most one solution in positive integers (x, y).

In particular, if $A = B + 1$, then the only solution is $(x, y) = (1, 1)$.

Theorem 5.2. *(BBGyP [1])* If $1 < B \leq 400$, then all integer solutions (x, y, n) of

$$|x^n - By^n| = 1$$

with $|xy| > 1, n \geq 3$ and with $(B, n) \notin \{(235, 23), (282, 23), (295, 29), (329, 23), (354, 29)\}$ are given by

$$\begin{aligned}
n = 3,\ & (B, x, y) = (7, \pm(2, 1)), (9 \pm (2, 1)), (17, \pm(18, 7)), (19, \pm(8, 3)),\\
& (20 \pm (19, 7)), (26, \pm(3, 1), (63, \pm(4, 1)), (91, \pm(9, 2)), (124, \pm(5, 1)),\\
& (126, \pm(5, 1)), (182, \pm(17, 3)), (215, \pm(6, 1)), (217, \pm(6, 1)),\\
& (254, \pm(19, 3)), (342, \pm(7, 1)), (344, \pm(7, 1))\\
n = 4,\ & (B, x, y) = (5, \pm 3, \pm 2), (15, \pm 2, \pm 1), (17, \pm 2, \pm 1), (39, \pm 5, \pm 2),\\
& (80, \pm 3, \pm 1), (150, \pm 7, \pm 2), (255, \pm 4, \pm 1).\\
n = 5,\ & (B, x, y) = (31, \pm(2, 1)), (242, \pm(3, 1)), (244, \pm(3, 1))\\
n = 6,\ & (B, x, y) = (63, \pm 2, \pm 1)\\
n = 7,\ & (B, x, y) = (127, \pm(2, 1)), (129, \pm(2, 1)),\\
n = 8,\ & (B, x, y) = (255, \pm 2, \pm 1)
\end{aligned}$$

6. The even cases $k = 4, 6, 8, 14$ and n prime

We take a computational approach to solving the titled cases. We use the following notation:

- f_i is a newform of level N_i.
- ℓ is a prime
- $n, w_0 \in \mathbb{Z}/\ell\mathbb{Z}$ are fixed
- $E_i(w_0, n)$ is the elliptic curve E_i calculated using n and w_0.
- $B_i(f, w_0, n) := \begin{cases} \mathrm{Norm}(a_\ell(E_i) - a_\ell(f)) & E_i(w_0, n) \text{ non-singular mod } \ell \\ \mathrm{Norm}((a_\ell(f))^2 - (\ell + 1)^2) & E_i(w_0, n) \text{ singular mod } \ell \end{cases}$
- $A_i(w_0, n) := \{p \text{ prime} : p \mid B_i(f), f \text{ arises from } E_i(w_0, n) \bmod p\}$
- $A_\ell(f_0, w_0, n)$ are the prime exponents possible as a solution for a given newform f_0, parameters w_0, n, and auxillary prime ℓ.
- $A_\ell(f_0)$ are the prime exponents possible as a solution for a given newform f_0 and auxiliary prime ℓ.
- $A(f_0)$ are the prime exponents possible as a solution for a given newform f_0 independent of any prime ℓ.
- A are the prime exponents possible as a solution independent of a newform.

After fixing $k \in \{4, 6, 8, 14\}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1, d_1, m'$, Algorithm 2 finds at least one prime $\ell < 1000$ such that $A_\ell(f_0, w_0, n)$ is finite and then one of the local methods eliminates each $p \in A_\ell(f_0, w_0, n) \smallsetminus \{2\}$, except possibly for the cases listed the following table.

Algorithm 2: Show that no solutions to (1) exist for fixed $\alpha_0, \alpha_1, \beta_0, \beta_1, m', d_1$

```
for each newform f_0 of level N_0 do
    for prime ℓ ∤ N_0 do
        for n ∈ Z/ℓZ do
            if ℓ | β_0 d_1 then
                Compute E_0 reduced mod ℓ
                Compute A_ℓ(f_0, w_0, n) := {p : p | B_0(f_0, w_0, n)} ∪ {ℓ}
            end
            if ℓ ∤ N_1 N_2 then
                for w_0 ∈ Z/ℓZ do
                    Compute E_i reduced mod ℓ
                    Compute
                     A_ℓ(f_0, w_0, n) := {p : p | B_0(f_0, w_0, n)} ∩ A_1(w_0, n) ∩ A_2(w_0, n) ∪ {ℓ}
                end
            end
            if A_ℓ(f_0, w_0, n) is finite then
                Apply local methods to each p ∈ A_ℓ(f_0)
            end
            else
                try next ℓ
            end
            Replace A_ℓ(f_0) with A_ℓ(f_0) ∪ A_ℓ(f_0, w_0, n).
        end
        Replace A(f_0) with A(f_0) ∩ A_ℓ(f_0).
        if A(f_0) ⊆ {2, 3, 5} then
            Continue f_0.
        end
    end
    Replace A with A ∪ A(f_0)
end
```

k	n	Case
$k = 4, 8, 14$	$n \in \mathbb{N}$	$\alpha_0 = 2, \alpha_1 = 3, \beta_0 = 5, \beta_1 = 1 = m' = d_1$
$k = 8, 14$	$n \in \mathbb{N}$	$\alpha_0 = 2, \alpha_1 = 9, \beta_0 = 5, \beta_1 = 1 = m' = d_1$
$k = 14$	$n = 3, 5$	$\alpha_0\alpha_1\beta_1 \mid 90,\ m' = 7,\ \beta_0 = 1 = d_1$

The first exceptional case has $(u_0, u_1, v_0) = (1, 1, 1)$ $\forall n \in \mathbb{N}$ as a solution, so we can't expect the algorithm to apply here. In the second exceptional case, for all primes $\ell < 1000$, there exists a modular form f_0 and $w_0, n \in \mathbb{Z}/\ell\mathbb{Z}$ such that $A_\ell(f_0, w_0, n)$ is infinite. While it's possible that some $\ell > 1000$ exists to make this set finite, we instead appeal to Bennett.

In both of the first two exceptional cases, (12) leads to

$$4u_0^n + 1 = 5v_0^n$$

and by Bennett, the only solution is $(u_0, v_0) = (1, 1)$. Then $x = \alpha_0 u_0^n = 2$. Also, $S_4(2) = 1 + 2^4 = 17$, $S_8(2) = 1 + 2^8 = 257$, and $S_{14}(2) = 16385 = 5 \cdot 29 \cdot 113$, all of which are squarefree so we must have $n = 1$. But we assumed n is prime, so there are no solutions in these cases.

In the last exceptional case, for at least one modular form f_0, Algorithm 2 shows that $\{2\} \subsetneq A(f_0) \subseteq \{2, 3, 5\}$ so we appeal instead to BBGyP to handle these cases. Since $\beta_0 = 1 = d_1$, in the case where $m' \mid x$, (16) leads to

$$2\alpha_1 u_1^n - v_0^n = 1$$

and in the case where $m' \mid x + 1$, (15) leads to

$$v_0^n - 2\alpha_0 u_0^n = 1$$

In either case, BBGyP shows that there are no solutions when $n = 3, 5$ and $2\alpha_i \mid 90$.

Finally, Jacobson, Pintér, and Walsh [6] have verified Schäffer's conjecture when $k \leq 58$ and $n = 2$. Thus, since we have shown that there are no solutions to the system (14), (15), (16), by Proposition 2.6 there are no solutions to $S_k(x) = y^n$ for $k = 4, 6, 8, 14$.

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