ON THE GROUND STATE OF NONLINEAR SCHRÖDINGER EQUATIONS WITH SATURABLE NONLINEARITY: A NON-EXISTENCE RESULT FOR THE CRITICAL CONSTANT AND AN ALMOST EVERYWHERE UNIQUENESS RESULT

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Abstract. Nonlinear Schrödinger equations (NLS) with saturable nonlinearity is a model for beam propagation in photorefractive crystals which prompts many intriguing nonlinear phenomena such as the formation of spatial solitons, self-trapping and self-focusing. This work is concerned with the ground state (an energy minimizer under the L^2-normalization condition) of nonlinear Schrödinger equations with saturable nonlinearity which includes a constant refractive index c in $\mathbb{R}^2$. From [Lin et al. 2014 *J. Math. Phys.* **55** 011505], it is known that: (i) there exists a ground state as the coefficient Γ of saturated nonlinearity is strictly large than T_c, where T_c is the threshold constant depending mainly on c; (ii) there is no ground state when $\Gamma < T_c$. The purpose of this paper is threefold. Firstly, based on investigating the corresponding Nehari and Pohozaev identities carefully, we prove the non-existence of ground state when $\Gamma = T_c$, and the uniqueness of the ground state for almost every $\Gamma \in (T_c, \infty)$. Moreover, regarding $\Gamma \in (T_c, \infty)$ as a parameter, we employ a compactness argument to show that, for $p > 2$, ground states converge to zero in L^p as $\Gamma \searrow T_c$. The last result can be viewed as a formation of soliton when Γ increases to T_c and across it Combining our results with [Lin et al. 2014 *J. Math. Phys.* **55** 011505], the necessary and sufficient condition for the ground state existence is established. Moreover, we mathematically confirm the vanishing behavior of the ground state as Γ approaches T_c (but their L^2 norm are preserved at the same time).

Keywords. Schrödinger equations, saturable nonlinearity, ground states, best constant, uniqueness

1. Introduction

Nonlinear Schrödinger equation with saturable nonlinearity and refractive index function is a mathematical model for the study of nonlinear optics [3, 4, 5, 6, 10, 19, 20]. Such a model can be read as:

$$-i\frac{\partial\psi}{\partial z} = -\Delta\psi - \Gamma\frac{I(x)+|\psi|^2}{1+I(x)+|\psi|^2}\psi, \text{ for } x\in\mathbb{R}^2, z>0, \tag{1.1}$$

where $I : \mathbb{R}^2 \to (-1,\infty)$, Γ is the coupling constant, $\psi = \psi(x,z) : \mathbb{R}^2\times\mathbb{R}\to\mathbb{C}$. This equation describes wave propagation in nonlinear optics. For example, in electromagnetic wavepackets propagation through a photorefractive waveguide, ψ is the renormalization of its slowly varying amplitude, z is the propagation direction along the waveguide and the refractive index (distribution) function $I = I(x)$ describes the unperturbed photorefractive material. $x = (x_1, x_2) \in \mathbb{R}^2$ is the transverse coordinate and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the transverse Laplacian.

In this paper, we consider the existence and uniqueness problems for ground state (an energy minimizer under the L^2-normalization condition) of (1.1). More precisely, a *ground state* of (1.1) is defined as a minimizer for the following minimization problem

$$\mathscr{E}_\Gamma = \inf\left\{\mathbb{E}_\Gamma[w] : w\in H^1(\mathbb{R}^2), \|w\|_{L^2}=1\right\}, \tag{1.2}$$

where

$$\mathbb{E}_\Gamma[w] := \int_{\mathbb{R}^2} |\nabla w|^2 - \Gamma\left[w^2 - \ln\left(1+\frac{w^2}{1+I(x)}\right)\right].$$

The L^2-normalization condition in (1.2) comes from the L^2-norm conservation of equation (1.1), which states $\int_{\mathbb{R}^2}|\psi|^2 = \int_{\mathbb{R}^2}|\psi_0|^2$. Here ψ_0 is the initial condition of (1.1). Therefore, without loss of generality, we may assume $\int_{\mathbb{R}^2} |\psi_0|^2 = 1$ and get this L^2-normalization condition. Note that problem (1.2) is *not* a minimization problem which searches an energy minimizer among all solutions of equation (1.9) introduce below with an *a-priori* given $\lambda > 0$ (cf. [16, 17, 14, 1]).

For problem (1.3) without refractive index function (that is, $I(x) \equiv 0$ in (1.2)), which is motivated from the experiments for counterpropagating two-wave mixing in a photorefractive crystal, Lin et al. [11] observed a threshold constant $T_c > 0$, defined in (1.5) below, such that a ground state exists for $\Gamma > T_c$ and does not exist for $\Gamma < T_c$. The proof can be extended to the case $I(x) \equiv c > -1$ with some minor modifications (cf. Remark 1.1(iii) below). On the other hand, for several different classes of non-constant functions $I(x)$, Lin, Wang and Wang [13] used a different argument to prove the existence of a ground state of (1.1) for $\Gamma > \Gamma_0(I)$, where $\Gamma_0(I) > 0$ is some constant depending mainly on $I(x)$ (also see [23, 24]). However, they obtain no result for $\Gamma \leqslant \Gamma_0(I)$, that is, $\Gamma_0(I)$ may not be a threshold constant as T_c is.

Therefore, in this paper, we consider the constant refractive index case $I(x) \equiv c > -1$ and prove the non-existence result for problem (1.2) at the critical case $\Gamma = T_c$. We further study the limiting behavior of minimizers as $\Gamma \searrow T_c$. Moreover, we obtain the uniqueness theorem (up to translations) for problem (1.2) at almost every $\Gamma > T_c$.

To state our main results, we reformulate the minimization problem (1.2) as the following form

$$e_\Gamma := \inf \left\{ E_\Gamma[w] : w \in H^1(\mathbb{R}^2), \|w\|_{L^2} = 1 \right\}, \tag{1.3}$$

where

$$E_\Gamma[w] := \int_{\mathbb{R}^2} |\nabla w|^2 - \Gamma \left[\frac{w^2}{1+c} - \ln\left(1 + \frac{w^2}{1+c}\right)\right]. \tag{1.4}$$

(Here we use both the assumption $I(x) \equiv c > -1$ and the L^2-normalization condition for the above reformulation.) We next recall the following theorem from [11, Theorem 1.1] and [13, Theorem 2.1 and Remark 2.1]:

Theorem A. *Let $c > -1$ be a constant and e_Γ be defined in* (1.3). *Define*

$$T_c := \inf \left\{ \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} \frac{w^2}{1+c} - \ln\left(1 + \frac{w^2}{1+c}\right)} : w \in H^1(\mathbb{R}^2), \|w\|_{L^2} = 1 \right\} > 0. \tag{1.5}$$

(i) *If $\Gamma < T_c$, then $e_\Gamma = 0$. In this situation, e_Γ cannot be attained by a minimizer, i.e.* (1.3) *has no ground state.*
(ii) *If $\Gamma > T_c$, then $e_\Gamma < 0$ has a positive radial minimizer $u_\Gamma(x) = u_\Gamma(|x|)$ that is monotone decreasing in $|x|$.*

Remark 1.1. (i) *Similar to the proof of Theorem A(i), we have $e_{T_c} = 0$.*
(ii) *From Theorem A, T_c is a threshold constant for the ground state existence.*
(iii) *In* [13, Remark 2.1], *it is mentioned that one can use the method of* [11] *to prove Theorem A. However, we provide an alternative proof in Appendix A to show all the cases $c > -1$ can be deduced from the case $c = 0$, which is proved in* [11]. *The proof is based on a scaling argument.*

1.1. **Main Results.** Our first result in this paper is the non-existence of the energy minimizer of e_Γ when $\Gamma = T_c$ and the L^p-vanishing behavior of u_Γ for $p > 2$ as $\Gamma \searrow T_c$ (but, at the same time, $\|u_\Gamma\|_{L^2} = 1$ for $\Gamma > T_c$).

Theorem 1.2. *Let $c > -1$ be a constant and e_Γ be defined in* (1.3). *Then*

(i) *e_{T_c} cannot be attained by a minimizer, i.e., problem* (1.3) *has no ground state when $\Gamma = T_c$.*
(ii) *When $\Gamma \searrow T_c$, we have, for any minimizer u_Γ of e_Γ,*

$$u_\Gamma \to 0 \quad \text{in } L^p(\mathbb{R}^2) \tag{1.6}$$

for any $p \in (2, \infty]$ and

$$\|u_\Gamma\|_{C^{1,\alpha}(\overline{B_R})} \to 0 \tag{1.7}$$

for any $R > 0$ and $\alpha \in (0,1)$. Here B_R is the ball centered at the origin with radius R.

Remark 1.3. *By Remark 1.1(i), searching for an energy minimizer for e_{T_c} is equivalent to searching for an extremal function for T_c. Theorem 1.2 shows that it is impossible to exist. On the other hand, minimization problem* (1.5) *for T_c is reminiscent of those problems in searching extremal functions for the best constants in Sobolev-type inequalities (cf.* [2, 22, 25] *and references therein) since, by the fact $0 \leqslant s^2 - \ln(1+s^2) \leqslant \frac{1}{2}s^4$, we have $T_c \geqslant 2(1+c)^2 S_{2,4} > 0$, where $S_{2,4}$, defined by*

$$S_{2,4} = \inf\left\{ \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} w^4} : w \in H^1(\mathbb{R}^2), \|w\|_{L^2} = 1 \right\},$$

is the largest constant C such that the following Soboloev inequality holds for all $w \in H^1(\mathbb{R}^2)$

$$C \int_{\mathbb{R}^2} w^4 \leqslant \int_{\mathbb{R}^2} |\nabla w|^2 \int_{\mathbb{R}^2} w^2.$$

Since it is well-known that $S_{2,4}$ has a minimizer (cf.[25]*) and is related to the cubic NLS in $\mathbb{R}^2$, Theorem 1.2 supports an essential difference between the cubic NLS equation and NLS equation with saturable nonlinearity in $\mathbb{R}^2$.*

From Theorem A and 1.2(i), we see $\Gamma > T_0$ is the sufficient and necessary condition for the ground state existence. It is natural to ask whether a ground state is unique (up to translations). Our second result in this paper is the uniqueness result for problem (1.3) at almost every $\Gamma \in (T_c, \infty)$:

Theorem 1.4. *Let*

$$\Lambda = \{\Gamma \in (T_c, \infty) : e_\Gamma \text{ is attained by a unique energy minimizer (up to translations)}\}. \tag{1.8}$$

Then $(T_c, \infty) \setminus \Lambda$ has Lebesgue measure zero.

That is, for almost every $\Gamma \in (T_c, \infty)$, the minimizer for (1.3) *is unique (up to translations).*

Remark 1.5. *From this uniqueness result, one might predict a stronger result: the uniqueness is true for every Γ. However, this might be wrong. There are minimization problems that depend on a parameter τ continuously, but the uniqueness for their minimizers fails at a single $\tau = \tau_0$ while the uniqueness holds for every point in a punctured neighborhood of τ_0 :*

Example 1.6. *Let $\tau \in \mathbb{R}$ and $E_\tau(x) = |\tau|\, x^2$ for $x \in \mathbb{R}$. Consider the minimization problems with a parameter τ : $e_\tau := \min_{x \in \mathbb{R}} E_\tau(x)$. For every $\tau \neq 0, e_\tau = 0$ and it has a unique minimizer. On the other hand, for $\tau = 0, e_tau = 0$, but its set of minimizers is $\mathbb{R}$ and hence the uniqueness fails.*

For the future work, we may want to answer whether Theorem 1.4 can be extended from almost everywhere to everywhere.

The proofs of Theorem 1.2 and 1.4 will be given in Sections 2 and 3, respectively. The main difficulty for proving Theorem 1.2 and 1.4 is that the energy functional $\mathbb{E}_\Gamma[u]$ itself does not provide enough information for the present situations. Hence we cannot apply the methods in [11, 13] directly. To overcome this difficulty, we use the corresponding Pohozaev and Nehari identities (see (2.1) and (2.3) below) to extract more information about a minimizer and its Lagrange multiplier. More precisely, we recall that an energy minimizer $u = u_\Gamma$ of e_Γ satisfies the following Euler-Lagrange equation

$$\Delta u + \Gamma \left(1 - \frac{1}{1 + \frac{u^2}{1+c}}\right) \frac{u}{1+c} = \lambda u \quad \text{in } \mathbb{R}^2 \tag{1.9}$$

with the Lagrange multiplier $\lambda = \lambda_{\Gamma, u_\Gamma}$ (corresponding to the L^2-norm constraint), which is *not a-priori* given and depends on Γ and u_Γ. The dependence of λ on energy minimizers is the main difficulty for the uniqueness problem since different minimizers may solve different elliptic equations, which means the classical uniqueness theorem about positive solution to (1.9) [18, 21] cannot be applied directly here. We will exclude such dependency: inspired by the work [9], we show that the Lagrange multiplier λ, through the Pohozaev identity, can be determined by the derivative of $\Gamma \mapsto e_\Gamma$ (see (3.9)), which exists for almost every Γ and is independent of the choice of the energy minimizer for e_Γ.

On the other hand, to prove the non-existence result in Theorem 1.2(i), we assume on the contrary that there is an energy minimizer u for e_{T_c}. We carefully investigate the relations between kinetic energy term $\|\nabla u\|_{L^2}$ and the nonlinear potential terms of in the following three identities: $e_{T_c} = 0$, Pohozaev and Nehari identities. A contradiction is then drawn from comparing the nonlinear potential terms in the Nehari and Pohozaev identities of (1.9) (see (2.7)).

Finally, concerning the vanishing behavior in Theorem 1.2(ii) and the aforementioned problem of computing the derivative of the function $\Gamma \mapsto e_\Gamma$ in Theorem 1.4, we need to answer the following question: Suppose $\{u_{\Gamma+\epsilon}\}_{\epsilon>0}$ is a sequence of energy minimizers for $\{e_{\Gamma+\epsilon}\}_{\epsilon>0}$. Does a subsequence of $\{u_{\Gamma+\epsilon}\}$ converge to an energy minimizer U of e_Γ as $\epsilon \to 0$?

To answer this question, the main task is to ensure the limit function U preserves the L^2 normalization condition $\|U\|_{L^2} = 1$. Note that this is not a direct consequence of the compact embedding $H^1_r(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ for $p > 2$. In the proofs of [11, Theorem 1.1(ii)] and [12, Theorem B], the authors encounter the same question and use the maximum principle to equation (1.9) to solve it. Although this strategy still works here, we prove it by a new and simpler proof which exploits the strict decreasing of $\Gamma \mapsto E_\Gamma(U)$ and the scaling balance for functions in $\mathbb{R}^2$ (see (2.15)). (It can also be applied to simplify the aforementioned proofs in [11, 12]).

The rest of this paper is organized as follows. In Sections 2 and 3, we give the proofs for Theorem 1.2 and 1.4, respectively. Finally, some conclusions of our results are given in Section 4.

Notations. Throughout this paper, all integrals are taken over $\mathbb{R}^2$ and all dx in the integrals are omitted; $L^p(\mathbb{R}^2)$ is the usual Lebesgue space with norm $\|u\|^p_{L^p} = \int_{\mathbb{R}^2} |u|^p$; For $k \in \mathbb{N}$, $H^k(\mathbb{R}^2)$ denotes the usual Sobolev space and $H^k_r(\mathbb{R}^2) := \{u \in H^k(\mathbb{R}^2) : u \text{ is radial}\}$. $\langle \cdot, \cdot \rangle$ stands for the standard inner product on $L^2(\mathbb{R}^2)$.

$W^{2,p}(B_R)$ and $C^{1,\alpha}(\overline{B_R})$ are usual Sobolev space and uniform Hölder space on the ball B_R centered at the origin in $\mathbb{R}^2$ with radius R.

2. Proof of Theorem 1.2

2.1. **Proof for Theorem 1.2(i).** We first recall the corresponding Pohozaev and Nehari identities for a minimizer $u \in H^1(\mathbb{R}^2) \cap \{w : \|w\|_{L^2} = 1\}$ of (1.3). By Lagrange multiplier principle, u satisfies the Euler-Lagrange equation (1.9) for some $\lambda \in \mathbb{R}$. Multipling (1.9) by $x \cdot \nabla u$ and integrating the result over $\mathbb{R}^2$, we obtain the Pohozaev identity

$$\Gamma \int_{\mathbb{R}^2} \frac{u^2}{1+c} - \ln\left(1 + \frac{u^2}{1+c}\right) = \lambda \int_{\mathbb{R}^2} u^2 = \lambda. \tag{2.1}$$

From (2.1) and the fact $0 < s^2 - \ln(1+s^2) < s^2$ for $s > 0$, we have

$$0 < \lambda < \frac{\Gamma}{1+c}. \tag{2.2}$$

On the other hand, multiplying (1.9) by u and integrating the result over $\mathbb{R}^2$, we obtain the Nehari identity

$$\int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma \int_{\mathbb{R}^2} \frac{u^2}{1+c} - \frac{\frac{u^2}{1+c}}{1 + \frac{u^2}{1+c}} = -\lambda. \tag{2.3}$$

To prove Theorem 1.2(i), we let $\Gamma = T_c$ in (2.1) and (2.3). Suppose, on the contrary, that e_{T_c} defined in (1.3) is attained by a minimizer $u \in H^1(\mathbb{R}^2) \cap \{w : \|w\|_{L^2} = 1\}$. By Remark 1.1(i), we have

$$0 = e_{T_c} = \int_{\mathbb{R}^2} |\nabla u|^2 - T_c \int_{\mathbb{R}^2} \frac{u^2}{1+c} - \ln\left(1 + \frac{u^2}{1+c}\right). \tag{2.4}$$

Along with (2.1), we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 = \lambda. \tag{2.5}$$

Combining (2.5), (2.3) and (2.1), we see

$$\begin{aligned} 2\lambda = T_c \int_{\mathbb{R}^2} \frac{u^2}{1+c} - \frac{\frac{u^2}{1+c}}{1+\frac{u^2}{1+c}} &= T_c \int_{\mathbb{R}^2} 2\left[\frac{u^2}{1+c} - \ln\left(1+\frac{u^2}{1+c}\right)\right] + h(\frac{u^2}{1+c}) \\ &= 2\lambda + T_c \int_{\mathbb{R}^2} h(\frac{u^2}{1+c}), \end{aligned} \tag{2.6}$$

where

$$h(s) := s - \frac{s}{1+s} - 2(s - \ln(1+s)).$$

Hence

$$\int_{\mathbb{R}^2} h(\frac{u^2}{1+c}) = 0. \tag{2.7}$$

On the other hand, it is easy to verify $h(s) < h(0) = 0$ for $s > 0$ since $h'(s) = -\frac{s^2}{(1+s)^2} < 0$. Hence (2.7) is violated by the fact $u \not\equiv 0$ (since $\|u\|_{L^2} = 1$). This contradiction implies that e_{T_c} cannot be attained. Therefore, the proof for Theorem 1.2(i) is complete.

2.2. **Proof for Theorem 1.2(ii).** First, to emphasize the possibility of non-uniqueness for energy minimizers of e_Γ in current stage, we denote the set of minimizer for (1.3) by

$$\Lambda_\Gamma := \{u_\Gamma : u_\Gamma\text{'s are energy minimizers for } e_\Gamma\}, \tag{2.8}$$

which is non-empty if and only if $\Gamma > T_c$. The proofs for (1.6) and (1.7) rely on the compact embedding of $H^1_r(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ for $p > 2$ (cf. [15]). Due to this reason, we first prove every minimizer in Λ_Γ is positive, radially symmetric (up to translations), decreasing in $r = |x|$ by verifying the hypotheses of a classical theorem of Gidas, Ni and Nirenberg [7].

Lemma 2.1. *Let $\Gamma \in (T_c, \infty)$ and $w \in \Lambda_\Gamma$. Then w is positive, radially symmetric about the origin (up to translations) and strictly decreasing in $r = |x|$.*

Proof. Let $w \in \Lambda_\Gamma$. Then w is nonnegative and satisfies the Euler–Lagrange equation (1.9) for some $\lambda > 0$ (by (2.2)). Using the standard regularity theory and strong maximum principle, we know $w \in C^2(\mathbb{R}^2)$ and $w > 0$. Moreover, applying De Giorgi's local boundedness theorem for (1.9) (cf. [8, Theorem 8.17]), we have $w(x) \to 0$ as $|x| \to \infty$. Then by the classical theorem of Gidas–Ni–Nirenberg (cf. [7, Theorem 2]), one immediately obtain the radial symmetry and decreasing property of w. Hence, we complete the proof of Lemma 2.1. $\square$

From Lemma 2.1, we may assume $u_\Gamma \in \Lambda_\Gamma$ is radial and decreasing in $r = |x|$ for Γ is closed to T_c. Now we prove Theorem 1.2(ii) in three steps:

Step 1. Prove L^p convergence (1.6) **for the case** $2 < p < \infty$. For $\Gamma \in (T_c, T_c+1)$, we have

$$0 \leqslant \int_{\mathbb{R}^2} |\nabla u_\Gamma|^2 = e_\Gamma + \Gamma \int_{\mathbb{R}^2} \frac{u_\Gamma^2}{1+c} - \ln(1+\frac{u_\Gamma^2}{1+c}) < \frac{T_c+1}{1+c}.$$

Therefore, $\{u_\Gamma\}_{\Gamma\in(T_c,T_c+1)}$ is bounded in $H^1_r(\mathbb{R}^2)$. By the compact embedding of $H^1_r(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ (see [15]), there is a function $U \in H^1(\mathbb{R}^2)$ such that

$$u_\Gamma \rightharpoonup U \quad \text{weakly in } H^1_r(\mathbb{R}^2) \tag{2.9}$$

and

$$u_\Gamma \to U \quad \text{in } L^p(\mathbb{R}^2) \tag{2.10}$$

as $\Gamma \searrow T_c$ (up to a subsequence) for $2 < p < \infty$.

We claim that $U \equiv 0$. Suppose, on the contrary, $U \not\equiv 0$, then we show that U is an energy minimizer for e_{T_c} by the following compactness argument: By the fact $0 \leqslant s - \ln(1+s) \leqslant \frac{1}{2}s^2$ for $s \geqslant 0$, we have

$$(2.11) \qquad 0 \leqslant \frac{u_\Gamma^2}{1+c} - \ln\left(1+\frac{u_\Gamma^2}{1+c}\right) \leqslant \frac{u_\Gamma^4}{2(1+c)^2}$$

Applying generalized dominated convergence theorem to (2.11) with (2.10) (for $p = 4$), we get

$$(2.12) \qquad \int_{\mathbb{R}^2} \frac{u_\Gamma^2}{1+c} - \ln\left(1+\frac{u_\Gamma^2}{1+c}\right) \to \int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right).$$

as $\Gamma \searrow T_c$ (up to a subsequence). By (2.9) and (2.12), we can apply Fatou's lemma to $E_\Gamma[U]$. As a consequence, we see

$$(2.13) \qquad \begin{aligned} E_\Gamma[U] &= \int_{\mathbb{R}^2} |\nabla U|^2 - T_c \int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right) \\ &\leqslant \liminf_{\Gamma\searrow T_c} \int_{\mathbb{R}^2} |\nabla u_\Gamma|^2 - \Gamma \int_{\mathbb{R}^2} \frac{u_\Gamma^2}{1+c} - \ln\left(1+\frac{u_\Gamma^2}{1+c}\right) \\ &= \liminf_{\Gamma\searrow T_c} e_\Gamma \\ &= e_{T_c}. \end{aligned}$$

On the other hand, since $\|u_\Gamma\|_{L^2} = 1$, we apply Fatou's lemma to $\|U\|_{L^2}$ to get $0 < \alpha \leqslant 1$, where

$$\alpha := \|U\|_{L^2}.$$

Now we shall prove $\alpha = 1$. Suppose, on the contrary, $\alpha < 1$. Let $U_\alpha(x) := U(\alpha x)$, then we obtain

$$(2.14) \qquad \|\nabla U\|_{L^2} = \|\nabla U_\alpha\|_{L^2} \quad \text{and} \quad \|U_\alpha\|_{L^2} = 1.$$

Moreover, by (2.13) and (2.14), we see

$$(2.15) \qquad \begin{aligned} e_{T_c} &\leqslant \int_{\mathbb{R}^2} |\nabla U_\alpha|^2 - T_c\left[\frac{U_\alpha^2}{1+c} - \ln\left(1+\frac{U_\alpha^2}{1+c}\right)\right] \\ &= \int_{\mathbb{R}^2} |\nabla U|^2 - T_c\alpha^{-2}\left[\frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right)\right] \\ &< \int_{\mathbb{R}^2} |\nabla U|^2 - T_c\left[\frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right)\right] \\ &\leqslant e_{T_c}, \end{aligned}$$

which is a contradiction. Hence $\|U\|_{L^2} = \alpha = 1$ and then U is an energy minimizer for e_{T_c} by (2.13). However, Theorem 1.2(i) implies e_{T_c} has no energy minimizer. Therefore, we prove the claim that the limit function $U \equiv 0$.

Furthermore, we observe the limit function $U \equiv 0$ of any convergent subsequence of $\{u_\Gamma\}$ is independent of the choice of subsequence. Consequently, $u_\Gamma \to 0$ in L^p as $\Gamma \searrow T_c$ (not only up to a subsequence), for $p \in (2, \infty)$. The proof for Step 1 is complete.

Step 2. Prove local Hölder convergence (1.7) **by standard $W^{2,q}$ estimate of linear elliptic equation.** To prove (1.7), we rewrite (1.9) as

$$\Delta u_\Gamma = \left(\lambda - \frac{\Gamma}{1+c} + \frac{\Gamma}{1+c+u_\Gamma^2}\right) u_\Gamma =: f(x),$$

Note that for $R > 0$ and $q \in (2, \infty)$, we can use standard regularity theory to ensure $\Delta u_\Gamma = f(x)$ a.e. in B_{2R} and $u_\Gamma \in W^{2,q}(B_{2R})$. Furthermore, by the standard interior $W^{2,q}$ estimate (see [8, Theorem 9.11]), we have

$$
\begin{aligned}
\|u_\Gamma\|_{W^{2,q}(B_R)} &\leqslant C(q,R)\left(\|u_\Gamma\|_{L^q(B_{2R})}+\|f\|_{L^q(B_{2R})}\right) \\
&\leqslant C(q,R)\left(1+\frac{\Gamma}{1+c}\right)\|u_\Gamma\|_{L^q(B_{2R})}
\end{aligned}
\tag{2.16}
$$

for some constant $C(q,R)$ depending on q and R only, where the second inequality comes from $-\frac{\Gamma}{1+c}<\lambda-\frac{\Gamma}{1+c}+\frac{\Gamma}{1+c+u_\Gamma^2}<\frac{\Gamma}{1+c}$ (by (2.2)). Using (2.16) and (1.6), we get

$$
\lim_{\Gamma\searrow T_c}\|u_\Gamma\|_{W^{2,q}(B_{2R})}=0,
$$

for $q\in(2,\infty)$. Moreover, combining with Sobolev inequality, we complete the proof for (1.7).

Step 3. Prove (1.6) **for the case** $p=\infty$. This step immediately follows from Step 2 and the decreasing property of u_Γ.

Combining Step 1-3, the proof for Theorem 1.2 is now completed.

3. Almost Everywhere Uniqueness: Proof of Theorem 1.4

In this section, we use an idea from [9, Corollary 1.1] to prove Theorem 1.4. We compute the right and left derivatives of $\Gamma\mapsto e_\Gamma$. Upon its derivative and Pohozaev identity (2.1), we prove that the Lagrange multiplier λ in (1.9) is independent of the choice of energy minimizer for almost every $\Gamma\in(T_c,\infty)$. Based on this result and Lemma 2.1, we can apply McLeod's uniqueness theorem [18] for (1.9) to show the uniqueness of minimizer for (1.3).

First, we calculate the right and left derivatives of e_Γ.

Lemma 3.1. *For each* $\Gamma\in(T_c,\infty)$, *we have*

$$
\lim_{\epsilon\downarrow 0}\frac{e_{\Gamma+\epsilon}-e_\Gamma}{\epsilon}=-s_\Gamma \tag{3.1}
$$

$$
\lim_{\epsilon\downarrow 0}\frac{e_{\Gamma-\epsilon}-e_\Gamma}{-\epsilon}=-i_\Gamma, \tag{3.2}
$$

where

$$
s_\Gamma:=\sup\left\{\int_{\mathbb{R}^2}\frac{w^2}{1+c}-\ln\left(1+\frac{w^2}{1+c}\right):w\in\Lambda_\Gamma\right\},\quad i_\Gamma:=\inf\left\{\int_{\mathbb{R}^2}\frac{w^2}{1+c}-\ln\left(1+\frac{w^2}{1+c}\right):w\in\Lambda_\Gamma\right\}, \tag{3.3}
$$

and Λ_Γ *is defined in* (2.8).

Proof. We only prove (3.1) since (3.2) can be proved similarly.

For each $\Gamma>T_c$, by Lemma 2.1, we pick an arbitrary positive radial minimizer $u_\Gamma\in\Lambda_\Gamma$ for e_Γ. Then it is easy to see for $\epsilon>0$, we have

$$
\begin{aligned}
e_{\Gamma+\epsilon}&\geqslant e_\Gamma-\epsilon\int_{\mathbb{R}^2}\frac{u_{\Gamma+\epsilon}^2}{1+c}-\ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right),\\
e_\Gamma&\geqslant e_{\Gamma+\epsilon}+\epsilon\int_{\mathbb{R}^2}\frac{u_\Gamma^2}{1+c}-\ln\left(1+\frac{u_\Gamma^2}{1+c}\right).
\end{aligned}
$$

Therefore we obtain

$$
-\int_{\mathbb{R}^2}\frac{u_{\Gamma+\epsilon}^2}{1+c}-\ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right)\leqslant\frac{e_{\Gamma+\epsilon}-e_\Gamma}{\epsilon}\leqslant-\int_{\mathbb{R}^2}\frac{u_\Gamma^2}{1+c}-\ln\left(1+\frac{u_\Gamma^2}{1+c}\right).
$$

Taking supremum over $u_\Gamma\in\Lambda_\Gamma$ for the above expression, we get

$$
-\int_{\mathbb{R}^2}\frac{u_{\Gamma+\epsilon}^2}{1+c}-\ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right)\leqslant\frac{e_{\Gamma+\epsilon}-e_\Gamma}{\epsilon}\leqslant-s_\Gamma. \tag{3.4}
$$

On the other hand, since

$$\int_{\mathbb{R}^2} |\nabla u_{\Gamma+\epsilon}|^2 = e_{\Gamma+\epsilon} + (\Gamma+\epsilon)\int_{\mathbb{R}^2} \frac{u_{\Gamma+\epsilon}^2}{1+c} - \ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right)$$
$$\leqslant e_\Gamma + (\Gamma+1)(1+c)^{-1},$$

$\{u_{\Gamma+\epsilon}\}_{\epsilon\in(0,1)}$ is uniformly bounded in $H_r^1(\mathbb{R}^2)$ and hence converges to some $U \in H_r^1(\mathbb{R}^2)$ weakly as $\epsilon \to 0$ (up to a subsequence). Furthermore, similar to (2.11)–(2.13), we use the compact embedding of $H_r^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$, generalized Lebesgue dominated convergence theorem and the weak convergence of $u_{\Gamma+\epsilon}$ to get

$$\int_{\mathbb{R}^2} \frac{u_{\Gamma+\epsilon}^2}{1+c} - \ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right) \to \int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right) \tag{3.5}$$

as $\epsilon \to 0$ (up to a subsequence),

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla U|^2 - \Gamma\int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right) \\
&\leqslant \liminf_{\epsilon\to 0} \int_{\mathbb{R}^2} |\nabla u_{\Gamma+\epsilon}|^2 - (\Gamma+\epsilon)\int_{\mathbb{R}^2} \frac{u_{\Gamma+\epsilon}^2}{1+c} - \ln\left(1+\frac{u_{\Gamma+\epsilon}^2}{1+c}\right) \\
&= \liminf_{\epsilon\to 0} e_{\Gamma+\epsilon} \\
&= e_\Gamma.
\end{aligned} \tag{3.6}$$

On the other hand, since $\|u_{\Gamma+\epsilon}\|_{L^2} = 1$, we apply Fatou's lemma to $\|U\|_{L^2}$ to get $0 \leqslant \alpha \leqslant 1$, where

$$\alpha := \|U\|_{L^2}.$$

Moreover, by (3.6) and $e_\Gamma < 0$, we have $\alpha > 0$. By the same argument as (2.15), we have $\alpha = 1$ (with the constant T_c is replaced by Γ). Therefore, U is a minimizer of e_Γ and hence, by (3.3), we have

$$-s_\Gamma \leqslant -\int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right). \tag{3.7}$$

Combining (3.4), (3.5) and (3.7), we obtain

$$-s_\Gamma \leqslant -\int_{\mathbb{R}^2} \frac{U^2}{1+c} - \ln\left(1+\frac{U^2}{1+c}\right) = \liminf_{\epsilon\to 0} \frac{e_{\Gamma+\epsilon}-e_\Gamma}{\epsilon} \leqslant \limsup_{\epsilon\to 0} \frac{e_{\Gamma+\epsilon}-e_\Gamma}{\epsilon} \leqslant -s_\Gamma,$$

which is exactly (3.1). □

Proof for Theorem 1.4. It is obvious that the function $\Gamma \mapsto e_\Gamma$ is decreasing and hence is almost everywhere differentiable. Then by Lemma 3.1, we have, for almost every $\Gamma > T_c$,

$$\frac{d}{d\Gamma} e_\Gamma = -\int_{\mathbb{R}^2} \frac{u^2}{1+c} - \ln\left(1+\frac{u^2}{1+c}\right) \quad \text{for all } u \in \Lambda_\Gamma. \tag{3.8}$$

Combining (3.8) and (2.1), we see, for almost every $\Gamma > T_c$,

$$\lambda = -\Gamma\frac{de_\Gamma}{d\Gamma}. \tag{3.9}$$

Since the right-hand side of the last formula is independent of the choice of minimizers, so does the left-hand side. That is, $\lambda = \lambda_\Gamma$ is independent of the choice of energy minimizers of e_Γ for almost every $\Gamma > T_c$. Combining with Lemma 2.1 and (1.9), we see for almost every $\Gamma > T_c$, all energy minimizers of e_Γ satisfy the following ODE with the same Lagrange multiplier $\lambda_\Gamma \in (0, \frac{\Gamma}{1+c})$

$$\begin{cases} u'' + \frac{1}{r}u' + \Gamma\left(1 - \frac{1}{1+\frac{u^2}{1+c}}\right)\frac{u}{1+c} = \lambda_\Gamma u \quad \text{for } r > 0, \\ u'(0) = 0, \ \lim_{r\to\infty} u(r) = 0, \\ u(r) > 0 \quad \text{for } r \geqslant 0. \end{cases} \tag{3.10}$$

Applying McLeod's uniqueness theorem [18, Theorem 1], we know that the solution of (3.10) must be unique. Hence, for almost every $\Gamma \in (T_c, \infty)$, the energy minimizer for e_Γ is unique (up to translations). The proof for Theorem 1.4 is complete. ☐

4. Concluding Remarks

In this article, we show the non-existence of ground state for NLS with saturable nonlinearity and constant refractive index at the critical case $\Gamma = T_c$. Thus, not only is $\Gamma > T_c$ a sufficient condition for ground state existence (Theorem A(ii)), but it is also a necessary condition (see Theorem A(i) and 1.2). We further, regarding $\Gamma \in (T_c, \infty)$ as a parameter, prove the L^p-vanishing behavior of the ground state for $p > 2$. Lastly, we obtain the uniqueness of ground states (up to translations) for almost every $\Gamma \in (T_c, \infty)$. For the future work, we may want to investigate that: (1) whether the uniqueness of minimizer holds for *every* $\Gamma \in (T_c, \infty)$ and (2) study the corresponding threshold constants for the cases of non-constant refractive index functions $I(x)$ and then extend the results in this paper.

Acknowledgements

The author would like to thank Prof. Tai-Chia Lin for the following two things: (1) sending his papers [11, 12] to him on April 28, 2017, so that the author was aware of the two aforementioned problems about uniqueness and best constant were unsolved at that time; (2) some discussions before Oct. 2018. He also wants to thank Prof. Chiun-Chang Lee for proofreading of the manuscript in May 2020 and Prof. Xiaoyu Zeng for pointing out the use of De Giorgi's theorem to show the decay of w in the proof of Lemma 2.1.

Appendix A. An Alternative Proof of Theorem A

As mentioned in Remark 1.1(iii), we give a proof for Theorem A [13, Theorem 2.1] based on its special $c = 0$, that is, [11, Theorem 1.1]. To distinguish this special case, we introduce

$$\widehat{e}_\Omega = \inf\left\{\widehat{E}_\Omega[w] : w \in H^1(\mathbb{R}^2), \|w\|_{L^2} = 1\right\}, \tag{A.1}$$

where $\Omega \in \mathbb{R}$ and

$$\widehat{E}_\Omega[w] = \int_{\mathbb{R}^2} |\nabla w|^2 - \Omega \int_{\mathbb{R}^2} w^2 - \ln\left(1 + w^2\right). \tag{A.2}$$

(A.1) and (A.2) are exactly the case $c = 0$ in the ground state energy (1.3) and its energy functional (1.4). We recall the following theorem from [11, Theorem 1.1] (that is, Theorem A in the special case $c = 0$):

Theorem A.1. *Define*

$$T_0 := \inf\left\{\frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} w^2 - \ln(1 + w^2)} : w \in H^1(\mathbb{R}^2), \|w\|_{L^2} = 1\right\} > 0. \tag{A.3}$$

(i) *If $\Omega < T_0$, then $\widehat{e}_\Omega = 0$. In this situation, $\widehat{e}_\Omega$ cannot be attained by a minimizer, i.e.* (A.1) *has no ground state.*
(ii) *If $\Omega > T_0$, then $\widehat{e}_\Omega < 0$ has a positive radial minimizer $\widehat{u}_\Omega(x) = \widehat{u}_\Omega(|x|)$ that is monotone decreasing in $|x|$.*

Next, we investigate the relationship between (1.3) and (A.1):

Lemma A.2. *Let $c > -1$ and e_Γ be defined in* (1.3)*. Then $e_\Gamma = (1+c)\widehat{e}_{\frac{\Gamma}{(1+c)^2}}$, where the right hand side is defined in* (A.1) *with $\Omega = \frac{\Gamma}{(1+c)^2}$. On the other hand, let T_c be defined in* (1.5)*, then we have $T_c = (1+c)^2 T_0$, where T_0 is defined in* (A.3)*.*

Proof. For each $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{L^2} = 1$, we define $w(x) = \frac{1}{\sqrt{1+c}} u(\frac{x}{\sqrt{1+c}})$. Then we have $\|w\|_{L^2} = 1$, $\int_{\mathbb{R}^2} |\nabla w|^2 = \frac{1}{1+c} \int_{\mathbb{R}^2} |\nabla u|^2$ and $\int_{\mathbb{R}^2} w^2 - \ln(1+w^2) = (1+c) \int_{\mathbb{R}^2} \frac{u^2}{1+c} - \ln\left(1 + \frac{u^2}{1+c}\right)$. Hence we have

$$E_\Gamma[u] = (1+c)\widehat{E}_{\frac{\Gamma}{(1+c)^2}}[w] \geqslant (1+c)\widehat{e}_{\frac{\Gamma}{(1+c)^2}}, \tag{A.4}$$

where the left hand side of the above equality is defined in (1.4) and the right hand side of the above equality is defined in (A.2) with $\Omega = \frac{\Gamma}{(1+c)^2}$. Taking the infimum of (A.4) over u, we have $e_\Gamma \geqslant (1+c)\widehat{e}_{\frac{\Gamma}{(1+c)^2}}$. Similarly, we have $e_\Gamma \leqslant (1+c)\widehat{e}_{\frac{\Gamma}{(1+c)^2}}$. Thus, we complete the proof of first assertion in Lemma A.2. Since the second assertion in Lemma A.2 can also be proved in the same manner, we omit the details. □

Finally, combining Lemma A.2 and Theorem A.1, we complete the proof for Theorem A.

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