Boundary Singularity of Macroscopic Variables for Linearized Boltzmann Equation with Cutoff Soft Potential

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The boundary singularity for stationary solutions of the linearized Boltzmann equation with cutoff soft potential in a slab is studied. An asymptotic formula for the gradient of the moments is established, which reveals the logarithmic singularity near the planar boundary. Similar results for cutoff hard-sphere and hard potential were proved in Chen [J. Stat. Phys. **153**(1), 93–118 (2013)] and Chen and Hsia [SIAM J. Math. Anal. **47**(6), 4332–4349 (2015)]. We extend their results to the case of soft potential $-\frac{3}{2} < \gamma < 0$. Since the solution space from the known existence theory is equipped with a weighted L^2 integrability for the velocity variables that behaves differently from the solution space for hard potential case, we cannot apply their arguments directly. To overcome this crux, we employed a different version of smoothing property for weighted L^2 space in Golse and Poupaud [Math. Methods Appl. Sci. **11**(4), 483–502 (1989)] to carry out the idea of Chen and Hsia. We then successfully extend the boundary singularity result to the soft potential case $-\frac{3}{2} < \gamma < 0$.

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I. INTRODUCTION

Among various models of gases, the kinetic theory is suitable for modelling boundary phenomena. In the studies of the effects of the boundary, problems related to the steady behavior of a rarefied gas bounded by a pair of planar walls are simplest yet indispensable. It has attracted attention of many authors[3–5,8,13,17,18,24,25,27,28]. The main topic of this article is to study the boundary singularity for the macroscopic variables that are calculated as the moments of the velocity distribution function. More precisely, we investigate that, in the model of stationary linearized Boltzmann equation with cutoff soft potential in a slab, the gradients of these macroscopic variables diverge, which resemble $\ln x$ as a function of the distance x from a planar boundary.

To the best of our knowledge, this kind of singular boundary behavior was first examined for the linearized Boltzmann–Krook–Welander equation by Sone[19,20] and Sone and Onishi[21,22]. For the linearized Boltzmann equation, Chen, Liu and Takata[12] considered the thermal transpiration problem for a highly rarefied gas with hard-sphere potential and proved an asymptotic formula for the derivative of flow velocity. From their formula, we know the derivative of flow velocity diverges to infinity logarithmically at the boundary and the shear stress is a constant. This seems to be contradictory from the fluid dynamics viewpoints and the notion of viscosity coefficient is unclear in this situation. But it is reasonable to be true since the region, a highly rarefied gas near a boundary, is inside the kinetic layer. See Refs. 11 and 25 for related results.

Besides the aforementioned cases, the phenomenon of boundary singularity appears in some other situations. Regarding linearized Boltzmann equation with the hard-sphere potential and hard potential case, Chen[9] and Chen and Hsia[10] established an asymptotic formula for the gradient of the moments of solutions to the stationary linearized Boltzmann equation, which shows the logarithmic singularity near the boundary. Here, the solution space is inherited from the existence theory of Bardos, Caflisch, and Nicolaenko[6] and Golse and Poupaud[13] for the Milne and Kramers problems.

The main interest of this article is to investigate whether Chen–Hsia's asymptotic formula holds for soft potential case. The linear and nonlinear Boltzmann equations with cutoff soft potential have been studied extensively[2,7,13,15,16,23,26]. The solution space for stationary solutions is also inherited from the aforementioned work of Golse and Poupaud[13]. This solution space has a weighted L^2-integrability for the microscopic velocity variables. For the soft potential case, the properties of this weighted space are different from those of solution space for hard potential case. As a result, the consequence that L^2 norm is controlled by this weighted L^2 norm, which is used frequently in Ref. 10 for hard potential case, is no longer true for soft potential case. Due to this reason, we cannot apply Chen–Hsia's arguments directly. To overcome this crux, we employed a different version of smoothing property for weighted L^2 space in Ref. 13 to carry out the idea of Chen–Hsia. It turns out that we can extend the boundary singularity result to the soft potential case $-\frac{3}{2} < \gamma < 0$.

The Boltzmann equation reads

$$\partial_t \mathbf{F} + \xi \cdot \nabla_{\mathbf{x}} \mathbf{F} = Q(\mathbf{F}, \mathbf{F}), \tag{1}$$

where $\mathbf{F} = \mathbf{F}(t, \mathbf{x}, \xi)$ denotes the density distribution function of the particle gas at time $t \geqslant 0$, position $\mathbf{x} = (x, y, z) \in U$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where U is a subset of $\mathbb{R}^3$.

The Boltzmann collision operator $Q(\mathbf{F}, \mathbf{G})$, which omits the dependence on t and x, is given in the following non-symmetric form

$$Q(\mathbf{F}, \mathbf{G})(\xi) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} [\mathbf{F}(\xi_*')\mathbf{G}(\xi') - \mathbf{F}(\xi_*)\mathbf{G}(\xi)] \, |\xi_* - \xi|^\gamma B(|\cos\theta|) d\Omega d\xi_*, \tag{2}$$

where $\Omega \in \mathbb{S}^2$ is parametrized by $\theta \in [0, \pi]$ and $\epsilon \in [0, 2\pi]$ such that $\cos\theta = \frac{(\xi_* - \xi)\cdot\Omega}{|\xi_* - \xi|}$. Here, (ξ_*, ξ) and (ξ_*', ξ') are the velocities of the particles before and after the collision, respectively, which satisfy

$$\xi' = \xi + (\Omega \cdot (\xi_* - \xi))\Omega, \quad \xi_*' = \xi_* - (\Omega \cdot (\xi_* - \xi))\Omega. \tag{3}$$

In the related physical models, if the force law between particles follows the inverse power law, i.e., the interparticle force is proportional to r^{-s} (here r is the interparticle distance and $s > 2$), then the collision kernel is in the form $|\xi_* - \xi|^\gamma B(|\cos\theta|)$ in (2), where

$$\gamma = \frac{s-5}{s-1}. \tag{4}$$

The case $\gamma = 1$ (i.e., $s \to \infty$) is called the hard-sphere model, the case $0 < \gamma < 1$ ($s > 5$) is called the hard potential model, the case $\gamma = 0$ ($s = 5$) is called the Maxwellian model, and the case $-3 < \gamma < 0$ ($2 < s < 5$) is called soft potential model. Furthermore, we follow Grad's idea (cf. Ref. 14) to make the following angular cutoff assumption

$$0 < B(|\cos\theta|) \leqslant C|\cos\theta|, \tag{5}$$

for some constant $C > 0$ and for all $\theta \in [0, \pi]$ so that the function B is integrable over $\mathbb{S}^2$ to make us able to separate the collision operator (2) into gain term and loss term.

In a nondimensionalized setting, we consider the linearized Boltzmann equation around the following standard Maxwellian

$$M(\xi) = \pi^{-\frac{3}{2}} e^{-|\xi|^2}. \tag{6}$$

That is, we plug $\mathbf{F} = M + M^{\frac{1}{2}} f$ into (1) and neglect the higher order term of f. This leads to the following linearized Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_{\mathbf{x}} f = L(f), \tag{7}$$

where

$$L(f) = M^{-\frac{1}{2}} \left(Q(M^{\frac{1}{2}} f, M) + Q(M, M^{\frac{1}{2}} f) \right). \tag{8}$$

Under the assumptions (2) and (5), L is the sum of a multiplicative operator $-\nu$ and a smoothing integral operator K :

$$L(h)(\xi) = -\nu(\xi) h(\xi) + K(h)(\xi). \tag{9}$$

Before we state the formulation of our problem in details, we give a brief account of some important properties of ν and K that we need[7,14]. First, it is well-known that the collision frequency function ν satisfies

$$\nu(\xi) = \nu(|\xi|) \tag{10}$$

and

$$\nu_0(1 + |\xi|)^\gamma \leqslant \nu(\xi) \leqslant \nu_1(1 + |\xi|)^\gamma \tag{11}$$

for some $0 < \nu_0 < \nu_1$.

To state the properties of integral operator K, we define a weighted L^∞ norm and a weighted L^2 space in the following: For $a \geqslant 0$,

$$\|h\|_{L^{\infty,a}} := \sup_{\xi} (1 + |\xi|)^a |h(\xi)|, \tag{12}$$

and

$$L^*_\xi(\mathbb{R}^3) := \{ f : \|f\|_{L^*} < \infty \}, \tag{13}$$

where

$$\|h\|_{L^*} := \|h\|_{L^2(\nu(\xi) d\xi)} = \left(\int_{\mathbb{R}^3} h(\xi)^2 \nu(\xi) d\xi \right)^{\frac{1}{2}}. \tag{14}$$

It is known that (cf. Refs. 7 and 13), applying the integral operator K raises up the decay rates of functions in the following sense:

Lemma I.1. *Let $-3<\gamma\leqslant 1$ and $a\geqslant 0$. Then there exists a positive constant $C_{a,\gamma}$ depending on a and γ such that*

$$\|K(h)\|_{L^{\infty,a+2-\gamma}}\leqslant C_{a,\gamma}\|h\|_{L^{\infty,a}}. \tag{15}$$

Moreover, if $-\frac{3}{2}<\gamma\leqslant 1$, then there is a positive constant C_γ depending on γ such that

$$\|K(h)\|_{L^{\infty,\frac{3}{2}-\gamma}}\leqslant C_\gamma\|h\|_{L^2}, \tag{16}$$

$$\|K(h)\|_{L^{\infty,\frac{3}{2}-\frac{\gamma}{2}}}\leqslant C_\gamma\|h\|_{L^*}. \tag{17}$$

We shall also use the following smoothing effect of K (cf. Ref. 10):

Lemma I.2. *Let $-2<\gamma\leqslant 1$ and $q\in[1,\infty]$. Then there exists a constant $C_{q,\gamma}>0$ depending on q and γ such that*

$$\|\nabla_\xi K(h)\|_{L^q}\leqslant C_{q,\gamma}\|h\|_{L^q}, \tag{18}$$

for all $h\in L^q$.

We shall discuss the range of γ in the above two lemmas here. Lemma I.1 was discovered for the case $-1<\gamma\leqslant 1$ in Caflisch[7] (Proposition 6.1) and Golse and Poupaud[13] (Lemma 2.1). On the other hand, Lemma I.2 appeared in Ref. 10 for the case $\gamma\in[0,1]$. The crux in the proofs of Lemmas I.1 and I.2 is the pointwise estimates of the kernels $k(\xi,\xi_*)$, $\partial_\xi k(\xi,\xi_*)$ of the integral operator K and $\nabla_\xi K$. These pointwise estimates have a similar version for the case $-3<\gamma\leqslant -1$, which can be proved by modifying Caflish's proof. To see the optimality of the range of γ in the above two lemmas, we look into the formulas of the kernel of K, which is decomposed into two functions k_1 and k_2 that come from the loss term and gain term of the collision operator Q in (8), respectively, and are to be estimated separately. In a simpler formula for $k_1(\xi,\xi_*)$, it is clear that, for each fixed ξ, $k_1(\xi,\xi_*)$ and $\partial_\xi k_1(\xi,\xi_*)$, as functions of ξ_*, have singularities in form of $|\xi-\xi_*|^\gamma$ and $|\xi-\xi_*|^{\gamma-1}$, respectively, as γ is negative. On the other hand, roughly speaking, we shall fix ξ and apply Young's inequality to estimate the ξ_*-integrals in the left hand sides of (16) and (17). Thus we require $k_1(\xi,\xi_*)$, as a function of ξ_*, to be $L^2_{\xi_*}$-integrable. As a result, it is natural to request $2\gamma>-3$ for (16) and (17). Similarly, for deriving (18), we require $\partial_\xi k_1$ to be $L^1_{\xi_*}$-integrable so that we can apply the Young's ineqaulity here (more precisely, the Schur's test.) Therefore we need $\gamma - 1 > -3$ for (18).

In this article, we consider the special slab domain $U = \{\mathbf{x} = (x, y, z) : 0 < x < l, (y, z) \in \mathbb{R}^2\}$ and study the stationary solutions to the linearized Boltzmann equation that are uniform on any plane paralleled to the boundaries $x = 0, l$ of U. In this situation, (7) is reduced to the following stationary equation

$$\xi_1 \partial_x f(x, \xi) = L(f)(x, \xi), \quad \text{for } 0 < x < l,\ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \tag{19}$$

We consider the following incoming boundary data:

$$f(0, \xi), \quad \text{given for } \xi_1 > 0 \tag{20}$$

and

$$f(l, \xi), \quad \text{given for } \xi_1 < 0. \tag{21}$$

Definition I.3. *We call $f \in L^\infty_x([0, l], L^*_\xi(\mathbb{R}^3))$ a solution to (19) if it satisfies the following integral equation:*

$$f(x, \xi) = \begin{cases} e^{-\frac{\nu(\xi)}{|\xi_1|}x} f(0, \xi) + \int_0^x \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}(x-s)} K(f)(s, \xi) ds & \text{for} \quad \xi_1 > 0 \\ e^{-\frac{\nu(\xi)}{|\xi_1|}(l-x)} f(l, \xi) + \int_x^l \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}(s-x)} K(f)(s, \xi) ds & \text{for} \quad \xi_1 < 0 \end{cases} \tag{22}$$

Remark I.4. *Both the solution spaces considered in Ref. 13 for Milne and Kramers problems are included in $L^\infty_x([0, l], L^*_\xi(\mathbb{R}^3))$ if x is restricted to $[0, l]$.*

We define the moments as follows.

Definition I.5. *The α-moment is defined as*

$$\sigma_\alpha(x) = \int_{\mathbb{R}^3} f(x, \xi) \phi_\alpha(\xi) d\xi,$$

where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_i' s \textit{ are nonnegative integers,}$$

and

$$\phi_\alpha(\xi) := \xi^\alpha M^{\frac{1}{2}}(\xi) = \pi^{-\frac{3}{4}} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3} e^{-\frac{|\xi|^2}{2}}.$$

The macroscopic variables are defined through the moments. For example, $\sigma_{(0,0,0)}$ is the density, $\sigma_{(1,0,0)}$ is the flow velocity in the x_1 direction, and $\frac{2}{3}(\sigma_{(2,0,0)}+\sigma_{(0,2,0)}+\sigma_{(0,0,2)})-\sigma_{(0,0,0)}$ is the temperature. We will use the following inequality frequently without special mentions:

$$|\phi_\alpha| \leqslant C_\alpha e^{-\frac{|\xi|^2}{3}}. \tag{23}$$

We also adopt the convention $0^0 = 1$ and use the notations

$$\mathbb{R}^{3+} := \{\xi \in \mathbb{R}^3 : \xi_1 > 0\} \quad \text{and} \quad \mathbb{R}^{3-} := \{\xi \in \mathbb{R}^3 : \xi_1 < 0\}.$$

The main theorem of this article is a generalization of Ref. 10 to the case of soft potentials $-\frac{3}{2} < \gamma < 0$.

Theorem I.6. *Let $\gamma \in (-\frac{3}{2}, 1]$. Assume (2) and (5) hold. Suppose $f \in L^\infty_x([0,l], L^*_\xi))$ is a solution to (19) such that $\nabla f(0,\cdot) \in L^p_\xi(\mathbb{R}^{3+})$ for some $p \in (1,\infty)$, $f(0,\xi) \in L^\infty_\xi(\mathbb{R}^{3+})$, and $f(l,\xi) \in L^\infty_\xi(\mathbb{R}^{3-})$. Then for small $x > 0$, we have*

$$\partial_x \sigma_\alpha(x) = -\ln x \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\alpha(0,\xi_2,\xi_3) L(f)(0,0^+,\xi_2,\xi_3)\, d\xi_2 d\xi_3 + O(\langle f \rangle), \tag{24}$$

where

$$L(f)(0,0^+,\xi_2,\xi_3) := \lim_{\xi_1 \to 0^+} L(f)(0,\xi_1,\xi_2,\xi_3), \tag{25}$$

$$\langle f \rangle := 1 + |||f||| + \|f(0,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3+})} + \|f(l,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3-})} + \|\nabla f(0,\cdot)\|_{L^p_\xi(\mathbb{R}^{3+})}, \tag{26}$$

$$|||f||| := \sup_{x\in[0,l]} \|f(x,\cdot)\|_{L^*}, \tag{27}$$

and "O" means the big O notation with a generic constant that may depend on α, γ and p.

Remark I.7. *The hard potential case $0 \leqslant \gamma \leqslant 1$ in Theorem I.6 was proved in Ref. 10. In this article, we shall focus on the proof for the case $-\frac{3}{2} < \gamma < 0$.*

To prove Theorem I.6, we formally differentiate (22) with respect to the space variable x. For the case of $\xi_1 > 0$, we get the following formula

$$\frac{\partial}{\partial x} f(x,\xi) = -\frac{\nu(\xi)}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}x} f(0,\xi) + \frac{1}{|\xi_1|} K(f)(x,\xi) - \int_0^x \frac{\nu(\xi)}{|\xi_1|^2} e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|} K(f)(s,\xi) ds \tag{28}$$

As mentioned in Ref. 10, in view of integrability of the product of ϕ_α and $\partial_x f$ in ξ, the first term of right hand side of (28) has a singularity at $x = 0$. On the other hand, although the second and third terms of (28) have the factors $|\xi_1|^{-1}$ and $|\xi_1|^{-2}$ respectively, they also involve the integral operator K which is well-known to improve the regularity in velocity variables. To overcome the subtlety caused by the factors $|\xi_1|^{-1}$ and $|\xi_1|^{-2}$, we use the strategy of Ref. 10 which anticipates some cancellations between the last two terms of (28). More precisely, we use the identity

$$\int_0^x e^{-\frac{\nu(\xi)}{|\xi_1|}(x-s)} ds = \frac{|\xi_1|}{\nu(\xi)} \left(1 - e^{-\frac{\nu(\xi)}{|\xi_1|}x}\right) \tag{29}$$

to rewrite (28) as

$$\begin{aligned} \frac{\partial}{\partial x} f(x,\xi) =& \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}x} Lf(0,\xi) + \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}x} [K(f)(x,\xi) - K(f)(0,\xi)] \\ &+ \int_0^x \frac{\nu(\xi)}{|\xi_1|^2} e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|} [K(f)(x,\xi) - K(f)(s,\xi)] ds, \end{aligned} \tag{30}$$

if $\xi_1 > 0$. Similarly, if $\xi_1 < 0$,

$$\begin{aligned} \frac{\partial}{\partial x} f(x,\xi) =& \frac{-1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}(l-x)} Lf(l,\xi) + \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}(l-x)} [K(f)(l,\xi) - K(f)(x,\xi)] \\ &+ \int_x^l \frac{\nu(\xi)}{|\xi_1|^2} e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|} [K(f)(s,\xi) - K(f)(x,\xi)] ds. \end{aligned} \tag{31}$$

Here, we outline the strategy of the proof of Theorem I.6 as follows. First, the desired logarithmic singularity will be extracted from the first term of (30) in Section II under an additional regularity assumption on the boundary data. On the other hand, the contribution from the second and third terms of (30) will be proved to be uniformly bounded in Section III via the Lipschitz-Type continuity of the integral operator K. Regarding the main difference between the proofs of our results and Ref. 10, we address in Remarks II.4 and III.6. For the future works, we may want to extend this logarithmic formula for the case $-3 < \gamma \leqslant -\frac{3}{2}$ and the full Boltzmann equation.

Notations

For functions g and h, we use $g \lesssim h$ to mean $g \leqslant Ch$ pointwisely for some generic constant $C > 0$. We also use $g \lesssim_\gamma h$ to mean $g \leqslant C_\gamma h$ for some generic constant $C = C_\gamma > 0$ depending on γ.

II. ASYMPTOTIC FORMULA ASSOCIATED WITH THE LOGARITHMIC SINGULARITY

As mentioned in the end of introduction, the source of singularity for the derivatives of the moments is the contribution from the first term on the right hand side of (30). More precisely, from Definition I.5 for the α-moment and the reorganized formula (30) for $\partial_x f(x,\xi)$, we shall prove the following lemma in this chapter.

Lemma II.1. *Let $f(x,\xi)$ be the solution of* (19) *under the angular cutoff assumption* (2) *and* (5) *with soft potential consideration $-\frac{3}{2}<\gamma<0$. Assume that $f(0,\xi)\in L^{\infty}(\mathbb{R}^{3+})$, $f(l,\xi)\in L^{\infty}_{\xi}(\mathbb{R}^{3-})$ and $\nabla_{\xi}f(0,\xi)\in L^{p}_{\xi}(\mathbb{R}^{3+})$ for some $p\in(1,\infty)$. Then*

$$\begin{aligned}\partial_x\sigma^{+}_{\alpha 1}(x) &:= \int_{\xi_1>0}\phi_{\alpha}(\xi)\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}L(f)(0,\xi)d\xi \\ &= -\ln(x)\int_{\mathbb{R}}\int_{\mathbb{R}}\phi_{\alpha}(0,\xi_2,\xi_3)L(f)(0,0^{+},\xi_2,\xi_3)d\xi_2 d\xi_3+O(\langle f\rangle),\end{aligned}\tag{32}$$

as $x\to 0$. Here

$$L(f)(0,0^{+},\xi_2,\xi_3):=\lim_{\xi_1\to 0^{+}}L(f)(0,\xi_1,\xi_2,\xi_3)\tag{33}$$

and $\langle f\rangle$ is defined in (26)*.*

The hard potential case $0\leqslant\gamma\leqslant 1$ of Lemma II.1 was proved in Ref. 10. Here we adapt their arguments to prove Lemma II.1 with soft potential case $-\frac{3}{2}<\gamma<0$. We summary these modifications in the end of this section. First, we apply the spherical coordinates

$$\xi=(\xi_1,\xi_2,\xi_3)=(\rho\cos\theta,\rho\sin\theta\cos\phi,\rho\sin\theta\sin\phi)\tag{34}$$

to (32) to get

$$\partial_x\sigma^{+}_{\alpha 1}(x)=\int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{1}{\rho\cos\theta}e^{-\frac{\nu(\rho)}{\rho\cos\theta}x}L(f)(0,\xi_1,\xi_2,\xi_3)\phi_{\alpha}(\xi)\rho^2\sin\theta d\theta d\phi d\rho.$$

We further set $z=\cos\theta$ to obtain

$$\partial_x\sigma^{+}_{\alpha 1}(x)=\int_0^{\infty}\left(\int_0^{2\pi}\int_0^{1}\frac{1}{z}e^{-\frac{\nu(\rho)}{\rho z}x}F(\rho,z,\phi)dz\,d\phi\right)e^{-\frac{\rho^2}{2}}\rho d\rho,\tag{35}$$

where

$$F(\rho,z,\phi):=L(f)(0,\xi_1,\xi_2,\xi_3)\xi^{\alpha}\pi^{-\frac{3}{4}}.\tag{36}$$

Now, we recall the definition of exponential integral

$$E_1(x):=\int_0^1\frac{1}{z}e^{-\frac{x}{z}}dz,\quad(x>0)\tag{37}$$

and present its properties which is essential in our analysis (cf. Ref. 1):

Lemma II.2. *As $x \to 0^+$, we have*

$$E_1(x) = -\Upsilon - \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k \cdot k!} = -\ln x + O(1), \tag{38}$$

where Υ is the Euler-Mascheroni constant. On the other hand, if $x \geqslant \delta > 0$, then $E_1(x) \leqslant C(\delta)$, where $C(\delta)$ is a constant depending on δ only.

Now we apply integration by parts to the innermost integral of (35) with

$$H(z,x) := -\int_z^1 \frac{1}{u} e^{-\frac{\nu(\rho)}{\rho u} x} du \tag{39}$$

to get

$$\begin{aligned} \int_0^1 \frac{1}{z} e^{-\frac{\nu(\rho)}{\rho z} x} F(\rho, z, \phi) dz &= \int_0^1 \left(\frac{\partial}{\partial z} H(z,x) \right) F(\rho, z, \phi) dz \\ &= E_1 \left(\frac{\nu(\rho)}{\rho} x \right) F(\rho, 0, \phi) - \int_0^1 H(z,x) \left(\frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz. \end{aligned} \tag{40}$$

Let

$$I := \int_0^\infty \int_0^{2\pi} E_1 \left(\frac{\nu(\rho)}{\rho} x \right) F(\rho, 0, \phi) e^{-\frac{\rho^2}{2}} \rho d\phi d\rho, \tag{41}$$

$$II := \int_0^\infty \int_0^{2\pi} \left[\int_0^1 H(z,x) \left(\frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz \right] e^{-\frac{\rho^2}{2}} \rho d\phi d\rho. \tag{42}$$

Then from (35) and (40)–(42), we see that

$$\partial_x \sigma_{\alpha 1}^+(x) = I + II. \tag{43}$$

Now, to extract the logarithmic singularity from I, we decompose the domain of ρ into $(0, \rho_0)$ and $[\rho_0, \infty)$ and denote the integral as I_s and I_l, respectively, where

$$\rho_0 = \rho_0(x) := \sup \Big\{ \rho : \frac{\nu(\rho)}{\rho} x > 1 \Big\}. \tag{44}$$

Note that $\frac{x}{\rho_0(x)} = \frac{1}{\nu(\rho_0(x))}$. Hence for $\rho < \rho_0$, we have

$$\frac{\nu(\rho)}{\rho} x > \frac{\nu(\rho)}{\rho_0(x)} x = \frac{\nu(\rho)}{\nu(\rho_0)} \geqslant \frac{\nu_0 (1+\rho)^\gamma}{\nu_1 (1+\rho_0)^\gamma} \geqslant \frac{\nu_0}{\nu_1}, \tag{45}$$

where we use the inequality (11) for frequency function ν and the fact $\gamma < 0$ for the last two inequalities. Note that the above lower bound of $\frac{\nu(\rho)}{\rho} x$ is independent of x.

On the other hand, for $\rho \geqslant \rho_0(x)$, we infer from the definition (44) that

$$\frac{\nu(\rho)}{\rho} x \leqslant 1, \tag{46}$$

which hints that the logarithmic singularity comes from I_l. To prove it, we note that the definition (36) of F and the properties (11) and (17) of the collision frequency function ν and the integral operator K imply the following lemma directly.

Lemma II.3. *Let F be defined in* (36)*. Then*

$$e^{-\frac{\rho^2}{2}}\rho|F(\rho,0,\phi)| \lesssim_{\alpha,\gamma} \langle f\rangle e^{-\frac{\rho^2}{4}} \tag{47}$$

for all $\rho>0$ and $\phi\in[0,2\pi]$. Here $\langle f\rangle$ is defined in (26)*.*

Using the asymptotic formula (38) of $E_1(x)$ and the upper bound estimate (47) for $F(\rho,0,\phi)$, we obtain

$$\begin{aligned}
I_l :&= \int_{\rho_0}^{\infty}\int_0^{2\pi} E_1\left(\frac{\nu(\rho)}{\rho}x\right)F(\rho,0,\phi)e^{-\frac{\rho^2}{2}}\rho d\phi d\rho \\
&= -\ln x\int_{\rho_0}^{\infty}\int_0^{2\pi} F(\rho,0,\phi)e^{-\frac{\rho^2}{2}}\rho d\phi d\rho + O(\langle f\rangle) \\
&= -\ln x\int_{0}^{\infty}\int_0^{2\pi} F(\rho,0,\phi)e^{-\frac{\rho^2}{2}}\rho d\phi d\rho + O(\langle f\rangle(1+\rho_0(x)|\ln x|)) \\
&= -\ln x\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\alpha(0,\xi_2,\xi_3)L(f)(0,0^+,\xi_2,\xi_3)d\xi_2 d\xi_3 + O(\langle f\rangle(1+\rho_0(x)|\ln x|)).
\end{aligned} \tag{48}$$

Note that

$$\frac{x}{\rho_0(x)} = \frac{1}{\nu(\rho_0(x))} \geqslant \frac{1}{\nu_1(1+\rho_0(x))^\gamma} \geqslant \frac{1}{\nu_1}, \tag{49}$$

where we use $\gamma<0$ for the last ineqaulity. Therefore we have $\rho_0(x)\leqslant \nu_1 x$, and hence

$$|\rho_0(x)|\ln x|| \lesssim |x|\ln x|| \lesssim 1 \tag{50}$$

as $x\to 0^+$. Combining (48) and (50), we see

$$I_l = -\ln x\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_\alpha(0,\xi_2,\xi_3)L(f)(0,0^+,\xi_2,\xi_3)d\xi_2 d\xi_3 + O(\langle f\rangle) \tag{51}$$

as $x\to 0^+$. On the other hand, by Lemmas II.2, II.3 and the lower bound (45) of $\frac{\nu(\rho)}{\rho}x$, we have

$$\begin{aligned}
I_s :&= \int_0^{\rho_0}\int_0^{2\pi} E_1\left(\frac{\nu(\rho)}{\rho}x\right)F(\rho,0,\phi)e^{-\frac{\rho^2}{2}}\rho d\phi d\rho \\
&\lesssim \int_0^{\rho_0}\int_0^{2\pi} |F(\rho,0,\phi)|e^{-\frac{\rho^2}{2}}\rho d\phi d\rho \\
&\lesssim \langle f\rangle.
\end{aligned} \tag{52}$$

Combining (51) and (52), we see

$$I = -\ln x \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\alpha(0, \xi_2, \xi_3) L(f)(0, 0^+, \xi_2, \xi_3) d\xi_2 d\xi_3 + O(\langle f \rangle) \tag{53}$$

as $x \to 0^+$.

For the term II defined in (42), we use a similar method as Ref. 10 to see

$$|II| \lesssim_{p,\alpha} \langle f \rangle, \tag{54}$$

and we refer to it for the details. The main difference here is that, for the estimates related to the integral operator K, we use (17) instead of (16) due to the existence theorem in Ref. 13 and the fact that L^2 norm is not controlled by L^* norm when $\gamma < 0$. Finally, by (43), (53) and (54), we complete the proof for Lemma II.1.

Remark II.4. *We summary the main difference between the proofs for hard and soft potential case:*

(i) The proof of a uniform lower bound for $\frac{\nu(\rho)}{\rho} x$ when $\rho < \rho_0(x)$, see (45).

(ii) The proof for $\rho_0(x) = O(x)$ as $x \to 0$, see (49).

(iii) On the estimates of the integral operator K, we use (17) instead of (16) due to the solution space and the fact that L^2 norm is not controlled by L^* norm when $\gamma < 0$.

III. UPPER BOUND ESTIMATES VIA LIPSCHITZ-TYPE CONTINUITY OF THE INTEGRAL OPERATOR

To complete the proof of Theorem I.6, we shall prove several upper bound estimates concerning the terms in the right hand sides of (30) and (31) in this chapter.

Lemma III.1. *Let $-\frac{3}{2} < \gamma < 0$. Let $f \in L^\infty_x([0, l], L^*_\xi(\mathbb{R}^3))$ be a solution to (19) for the case that L, which is defined in (8), satisfies (9), (11), and (16)–(18) such that*

$$\begin{cases} f(0, \xi) \in L^\infty_\xi(\mathbb{R}^{3+}), \\ f(l, \xi) \in L^\infty_\xi(\mathbb{R}^{3-}). \end{cases}$$

Then we have

$$\left|\int_{\xi_1>0}\phi_\alpha\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}L(f)(0,\xi)d\xi\right| \lesssim_{\alpha,\gamma} (|\ln x|+1)\langle f\rangle', \tag{55}$$

$$\left|\int_{\xi_1>0}\phi_\alpha\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}(K(f)(x,\xi)-K(f)(0,\xi)d\xi\right| \lesssim_{\alpha,\gamma} \langle f\rangle', \tag{56}$$

$$\left|\int_{\xi_1>0}\phi_\alpha\int_0^x\frac{\nu(\xi)}{|\xi_1|^2}e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|}(K(f)(x,\xi)-K(f)(s,\xi))dsd\xi\right| \lesssim_{\alpha,\gamma} \langle f\rangle', \tag{57}$$

where

$$\langle f\rangle' := |||f||| + \|f(0,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3+})} + \|f(l,\cdot)\|_{L^\infty_\xi(\mathbb{R}^{3-})}. \tag{58}$$

Remark III.2. *We only assume that the linearized collision operator L defined in* (8) *satisfies* (9)*,* (11)*, and* (16)*–*(18)*. This may include potentials whose form are more general than* (2) *and* (5)*.*

Estimates (55)–(57) have similar versions for the case $\xi_1 < 0$ (i.e., for (31)) and their proofs can be derived in a similar way as the one for (55)–(57). To sum up, we conclude the following lemma.

Lemma III.3. *Suppose $f \in L^\infty([0,l], L^*_\xi(\mathbb{R}^3))$ is a solution to* (19) *with Grad's angular cutoff potential with $-\frac{3}{2} < \gamma < 0$ and $f(0,\cdot) \in L^\infty_\xi(\mathbb{R}^{3+})$ and $f(l,\cdot) \in L^\infty_\xi(\mathbb{R}^{3-})$. Then*

$$|\partial_x\sigma^+_\alpha(x)| \lesssim_{\alpha,\gamma} (|\ln(x)|+1)\langle f\rangle', \tag{59}$$

$$|\partial_x\sigma^-_\alpha(x)| \lesssim_{\alpha,\gamma} (|\ln(l-x)|+1)\langle f\rangle', \tag{60}$$

where

$$\sigma^+_\alpha(x) = \int_{\xi_1>0}\phi_\alpha(\xi)f(x,\xi)d\xi, \quad \sigma^-_\alpha(x) = \int_{\xi_1<0}\phi_\alpha(\xi)f(x,\xi)d\xi.$$

The hard potential case $0 \leqslant \gamma \leqslant 1$ of Lemmas III.1 and III.3 were proved in Ref. 10. Here we adapt their arguments to prove Lemma III.1 with soft potential case $-\frac{3}{2} < \gamma < 0$.

Proof of Lemma III.1. The proof falls naturally into three steps that we give the proofs of (55), (57) and (56), respectively.

Step 1. The strategy to prove (55) is the same as the proof of Lemma II.1. Since we are only pursuing an upper bound here instead of the precise formula (32), we get, with the help of (9), (11), and (17),

$$\begin{aligned}&\left|\int_{\xi_1>0}\phi_\alpha\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}L(f)(0,\xi)d\xi\right|\\&\quad\lesssim_\alpha\int_{\xi_1>0}e^{-\frac{|\xi|^2}{4}}\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}\left((1+|\xi|)^\gamma\|f(0,\cdot)\|_{L^\infty(\mathbb{R}^{3+})}+(1+|\xi|)^{-\frac{3}{2}+\frac{\gamma}{2}}\|f(0,\cdot)\|_{L^*_\xi}\right)d\xi\\&\quad\lesssim_\alpha\langle f\rangle'\int_{\xi_1>0}e^{-\frac{|\xi|^2}{4}}\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}d\xi.\end{aligned}\tag{61}$$

Note that the integral in the right hand side of (61) can be estimated in the same fashion as the one for (32)–(51), where the term $F(\rho,0,\phi)$ in (35)–(51) is replaced by the simpler constant function 1 here. Thus we have

$$\int_{\xi_1>0}e^{-\frac{|\xi|^2}{4}}\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}d\xi\lesssim|\ln(x)|+1.\tag{62}$$

Combining (61) and (62), we obtain (55).

Step 2. We shall give a proof for (57) in this step. Replacing $0,l$ by s,x in (22), we see

$$\begin{aligned}K(f)(x,\xi)-K(f)(s,\xi)=&\int_{\xi_{*1}>0}k(\xi,\xi_*)\left(e^{-\frac{\nu(\xi_*)}{|\xi_{*1}|}|x-s|}-1\right)f(s,\xi_*)d\xi_*\\&+\int_{\xi_{*1}>0}k(\xi,\xi_*)\int_s^x\frac{1}{|\xi_{*1}|}e^{-\frac{\nu(\xi_*)}{|\xi_{*1}|}|x-t|}K(f)(t,\xi_*)dtd\xi_*\\&+\int_{\xi_{*1}<0}k(\xi,\xi_*)\left(1-e^{-\frac{\nu(\xi_*)}{|\xi_{*1}|}|x-s|}\right)f(x,\xi_*)d\xi_*\\&-\int_{\xi_{*1}<0}k(\xi,\xi_*)\int_s^x\frac{1}{|\xi_{*1}|}e^{-\frac{\nu(\xi_*)}{|\xi_{*1}|}|s-t|}K(f)(t,\xi_*)dtd\xi_*\\=:&\ H_1+H_2+H_3+H_4.\end{aligned}\tag{63}$$

We shall apply the following lemma to estimate H_2 and H_4 :

Lemma III.4. *For $\lambda\in(\frac{1}{2},1)$, there exists $C_\lambda>0$ such that*

$$|H_i|\leqslant C_\lambda|x-s|^{1-\lambda}|||f|||\tag{64}$$

for $i=2,4$.

Proof. To prove (64), we first observe that, for each $\lambda>0$,

$$\sup_{t\geqslant0}t^\lambda e^{-t}=\lambda^\lambda e^{-\lambda}<\infty.\tag{65}$$

By (65), we have, for $\lambda > 0$,

$$\begin{aligned}
&\int_s^x \left|\int_{\xi_{*1}>0} \nu(\xi_*) \left|\frac{1}{\xi_{*1}} e^{-\frac{\nu(\xi_*)}{\xi_{*1}}|x-t|} K(f)(t,\xi_*)\right|^2 d\xi_*\right|^{\frac{1}{2}} dt \\
&= \int_s^x \left|\int_{\xi_{*1}>0} \nu(\xi_*) \left|\left(\frac{\nu(\xi_*)|x-t|}{\xi_{*1}}\right)^{\lambda} e^{-\frac{\nu(\xi_*)}{\xi_{*1}}|x-t|} \frac{|x-t|^{-\lambda}}{\nu(\xi_*)^{\lambda}\xi_{*1}^{1-\lambda}} K(f)(t,\xi_*)\right|^2 d\xi_*\right|^{\frac{1}{2}} dt \quad (66) \\
&\lesssim_\lambda \int_s^x \frac{1}{|x-t|^{\lambda}} \left(\int_{\xi_{*1}>0} \nu(\xi_*) \left|\frac{1}{\nu(\xi_*)^{\lambda}\xi_{*1}^{1-\lambda}} K(f)(t,\xi_*)\right|^2 d\xi_*\right)^{\frac{1}{2}} dt.
\end{aligned}$$

To estimate the ξ_*-integral in the right hand side of (66), we recall the following lemma from Ref. Lemma 4.2 of Ref. 13.

Lemma III.5. *Let $-\frac{3}{2} < \gamma < 0$. For $\lambda \in (\frac{1}{2}, 1)$, there exists $D_\lambda > 0$ depending on λ only such that*

$$\int |\xi_{*1}|^{2\lambda-2} \nu(\xi_*)^{1-2\lambda} (K(f)(x,\xi_*))^2 d\xi_* \leqslant D_\lambda \|f(x,\cdot)\|_{L^*}^2. \quad (67)$$

Combining (66) and (67), we get

$$\begin{aligned}
&\int_s^x \left|\int_{\xi_{*1}>0} \nu(\xi_*) \left|\frac{1}{\xi_{*1}} e^{-\frac{\nu(\xi_*)}{\xi_{*1}}|x-t|} K(f)(t,\xi_*)\right|^2 d\xi_*\right|^{\frac{1}{2}} dt \\
&\lesssim_\lambda \int_s^x \frac{1}{|x-t|^{\lambda}} \|f(t,\cdot)\|_{L^*} dt \quad (68) \\
&\lesssim_\lambda |||f||| |x-s|^{1-\lambda}.
\end{aligned}$$

Finally, we apply (17) and (68) to obtain

$$\begin{aligned}
|H_2| &= \left|\int_s^x \int_{\xi_{*1}>0} k(\xi,\xi_*) \frac{1}{\xi_{*1}} e^{-\frac{\nu(\xi_*)}{\xi_{*1}}|x-t|} K(f)(t,\xi_*) d\xi_* dt\right| \\
&\lesssim_\gamma (1+|\xi|)^{-\frac{3}{2}+\frac{\gamma}{2}} \int_s^x \left|\int_{\xi_{*1}>0} \nu(\xi_*) \left|\frac{1}{\xi_{*1}} e^{-\frac{\nu(\xi_*)}{\xi_{*1}}|x-t|} K(f)(t,\xi_*)\right|^2 d\xi_*\right|^{\frac{1}{2}} dt \quad (69) \\
&\lesssim_\lambda |||f||| |x-s|^{1-\lambda}.
\end{aligned}$$

The proof for the part concerning H_4 can be derived in the same way. Therefore, we complete the proof of Lemma III.4. □

Back to the proof of (57), we let

$$\begin{aligned}
&\int_{\xi_1>0}\int_0^x \phi_\alpha(\xi)\frac{\nu(\xi)}{|\xi_1|^2}e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|}\left(K(f)(x,\xi)-K(s)(x,\xi)\right)dsd\xi \\
&\quad=\int_{\xi_1>0}\int_0^x \phi_\alpha(\xi)\frac{\nu(\xi)}{|\xi_1|^2}e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|}\left(H_1+H_2+H_3+H_4\right)dsd\xi \\
&\quad=: B_1+B_2+B_3+B_4.
\end{aligned} \tag{70}$$

Therefore, we can use Lemma III.4 to estimate B_2 and B_4 as follows: for $i=2$ and 4,

$$\begin{aligned}
|B_i| &\lesssim_{\lambda,\alpha} |||f||| \int_{\xi_1>0}\int_0^x e^{-\frac{|\xi|^2}{4}}\frac{\nu(\xi)}{|\xi_1|^2}e^{-\frac{\nu(\xi)}{|\xi_1|}|x-s|}|x-s|^{1-\lambda}dsd\xi \\
&= |||f||| \int_{\xi_1>0} e^{-\frac{|\xi|^2}{4}}\nu(\xi)^{\lambda-1}|\xi_1|^{-\lambda}\int_0^{\frac{\nu(\xi)x}{|\xi_1|}} e^{-z}z^{1-\lambda}dzd\xi \\
&\lesssim_{\lambda,\alpha} |||f|||.
\end{aligned} \tag{71}$$

The estimates for B_1 and B_3 can be done in a similar fashion as Ref. 10. The main difference is that, to see

$$\|f\|_{L^\infty_{x,\xi}} \lesssim \langle f\rangle', \tag{72}$$

we use the property (17) of the integral operator K instead of (16) to control the integral terms in the right hand side of (22). As a result, we see

$$|B_i| \lesssim \langle f\rangle' \tag{73}$$

for $i=1,3$.

Combing (70), (71), and (73), we obtain (57).

Step 3. In this step, we sketch a proof of (56), which is similar to Step 2. Replacing s by 0 in (63) and using H_i' to denote the term corresponding to H_i, we obtain

$$\begin{aligned}
&\int_{\xi_1>0}\phi_\alpha\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}(K(f)(x,\xi)-K(f)(0,\xi))d\xi \\
&\quad=\int_{\xi_1>0}\phi_\alpha\frac{1}{|\xi_1|}e^{-\frac{\nu(\xi)}{|\xi_1|}x}(H_1'+H_2'+H_3'+H_4')d\xi \\
&\quad=: B_1'+B_2'+B_3'+B_4'.
\end{aligned} \tag{74}$$

From Lemma III.4 with $s=0$, we have, for $\lambda\in(\frac{1}{2},1)$, there exists C_λ such that

$$|H_i'| \leqslant C_\lambda |||f||| |x|^{1-\lambda} \tag{75}$$

for $i = 2, 4$. Then we apply (75) and (65) to (74) to obtain

$$
\begin{aligned}
|B_2' + B_4'| &\lesssim_\lambda |||f||| \int_{\xi_1>0} \phi_\alpha \frac{1}{|\xi_1|} e^{-\frac{\nu(\xi)}{|\xi_1|}x} |x|^{1-\lambda} d\xi \\
&\lesssim_{\alpha,\lambda} |||f||| \int_{\xi_1>0} e^{-\frac{|\xi|^2}{4}} \frac{1}{\nu(\xi)^{1-\lambda}|\xi_1|^\lambda} \left| \frac{\nu(\xi)x}{|\xi_1|} \right|^{1-\lambda} e^{-\frac{\nu(\xi)}{|\xi_1|}x} d\xi \\
&\lesssim_{\alpha,\lambda} |||f||| \int_{\xi_1>0} e^{-\frac{|\xi|^2}{4}} \frac{1}{\nu(|\xi|)^{1-\lambda}|\xi_1|^\lambda} d\xi \\
&\lesssim_{\alpha,\lambda} |||f|||,
\end{aligned}
\tag{76}
$$

where we use the fact $\lambda < 1$ to see the last integral in (76) converges. The estimates for B_1' and B_3' are also done in a similar fashion as Ref. 10. As a result, we have

$$
|B_i'| \lesssim \langle f \rangle', \tag{77}
$$

for $i = 1, 3$. Combining (74), (76), and (77), we obtain (56).

□

Completion of the Proof of Theorem I.6. Combining (56) and (57) in Lemma III.1, (60) in Lemma III.3, and Lemma II.1, we obtain Theorem I.6. □

Remark III.6. *We summary the main difference between the hard and soft potential case:*

(i) To obtain (72)*, we use the estimate* (17) *of the integral operator K instead of* (16)*.*

(ii) The proof for estimates of H_2, H_4, that is, Lemma III.4, is different. More precisely, the formulation (67) *we applied is different from the inequality (4.1) of Ref. 10.*

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