Fourier Analysis, Stein and Shakarchi
Chapter 6 The Fourier Transform on $\mathbb{R}^d$

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Abstract

I complete this solution file when I am a teaching assistant of the course "Analysis II" in NTU 2018 Spring. The following students contribute some answers:

Exercise 13-15: Chin-Bin Hsu.

Problem 1: Chi-An Chen (partially, starting from the view point of solving Bessel equations by power series solution, that is, taking (h) as the definition of J_n instead of the last line of page 197.)

Problem 2(c) is discussed with Ge-Cheng Cheng.

Problem 6 is discussed with Mighty Yeh, Guan-Lin Lin and Chi-An Chen.

Problem 7(a)(b)(d): Andy Huang.

Problem 8: Kuo-Tsan Hsu.

We also refer the reader to Book III's Chapter 7.4 and Book IV's Chapter 8.7 for more discussions on Radon transform.

File complete at 2018.06.19. Slightly improve the explanation for Remark 9 and update the reference [9] to Remark 11 at 2018.12.26.

Exercises

1. Answer: (a) $D := ad - bc = \pm 1$ and $a = dD, b = -cD, c = -bD, d = aD$. (b) $a^2 + b^2 = 1$ since $D = ad - bc = a^2 D + b^2 D, D = \pm 1 \neq 0$. (c) If $D = 1$, then $ze^{-i\varphi} = (x + iy)(a - ib) = (ax + by) + i(-bx + ay) = (ax + by) + i(cx + dy) = R(z)$. Similar for $D = -1$.

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2. **Suppose that $R:\mathbb{R}^3\to\mathbb{R}^3$ is a proper rotation.**

(a) Show that $p(t)=\det(R-tI)$ is a polynomial of degree 3, and prove that there exists $\gamma\in\mathcal{S}^2$ (where $\mathcal{S}^2$ denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma)=\gamma.$$

(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R:\mathcal{P}\to\mathcal{P},$$

and that this linear map is a rotation in $\mathbb{R}^2$.

Proof. (a) As hint, since $p(t)$ is a polynomial, $p(0)=1$ and $p(t)=t^3\det(\frac{1}{t}R-I)\to-\infty$ as $t\to\infty$, we see $p(\lambda)=0$ for some $\lambda>0$ by intermediate value theorem. Then the kernel of $R-\lambda I$ is non-trivial, say $(R-\lambda I)(x)=0$ for some $x\in\mathbb{R}^3\setminus\{0\}$ and hence $R(\frac{x}{\|x\|})=\lambda\frac{x}{\|x\|}$. Use the definition of rotation, we see that $\lambda^2=1$ and hence $0<\lambda=1$.

(b) Given $z\in\mathcal{P}$, we find that $R(z)\in\mathcal{P}$ since $\langle R(z),\gamma\rangle=\langle z,R^t(\gamma)\rangle=\langle z,\gamma\rangle=0$. The linearity and preservation of inner product of $R\downharpoonright_{\mathcal{P}}$ inherit from R. □

3. **Recall the formula**

$$\int_{\mathbb{R}^d}F(x)\,dx=\int_{\mathcal{S}^{d-1}}\int_0^\infty F(r\gamma)r^{d-1}\,dr\,d\sigma(\gamma).$$

Apply this to the special case when $F(x)=g(r)f(\gamma)$, where $x=r\gamma$, to prove that for any rotation R, one has

$$\int_{\mathcal{S}^{d-1}}f(R(\gamma))\,d\sigma(\gamma)=\int_{\mathcal{S}^{d-1}}f(\gamma)\,d\sigma(\gamma).$$

whenever f is a continuous function on the sphere $\mathcal{S}^{d-1}$.

Proof. Consider $F(x)=g(r)f(\gamma)$ where $g\in C_c(\mathbb{R})$ such that $\int_0^\infty g(r)r^{d-1}\,dr>0$. Then we complete the proof since

$$\begin{aligned}\int_0^\infty g(r)r^{d-1}\,dr\int_{\mathcal{S}^{d-1}}f(\gamma)\,d\sigma(\gamma)&=\int_{\mathbb{R}^3}F(x)\,dx=\int_{\mathbb{R}^3}F(R(x))\,dx\\&=\int_0^\infty g(r)r^{d-1}\,dr\int_{\mathcal{S}^{d-1}}f(R(\gamma))\,d\sigma(\gamma).\end{aligned}$$

□

4. Let A_d and V_d denote the area and volume of the unit sphere and unit ball in $\mathbb{R}^d$, respectively. It's standard to prove (a) the formula $A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and (b) $dV_d = A_d$.

   (Here $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ is the Gamma function).

5. **Let A be a $d \times d$ positive definite symmetric matrix with real coefficients. Show that**

$$\int_{\mathbb{R}^d} e^{-\pi(x,A(x))}\,dx = (\det A)^{-1/2}$$

   When $A = I$, it's the standard gauss integral [Hint: Apply the spectral theorem to write $A = RDR^{-1}$ where R is a rotation and, D is diagonal with entries $\lambda_1, \cdots, \lambda_d$, where $\{\lambda_i\}$ are the eigenvalues of A.]

   Proof. Since A is real symmetric, positive definite, $A = PDP^{-1}$ for some orthogonal matrix P and diagonal matrix D with positive diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_d$. The rest is to use change of variables. $\square$

6. **Suppose $\psi \in S(\mathbb{R}^d)$ satisfies $\int |\psi(x)|^2\,dx = 1$. Show the Heisenberg uncertainty principle in $\mathbb{R}^d$:**

$$\Big(\int_{\mathbb{R}^d} |x|^2|\psi(x)|^2\,dx\Big)\Big(\int_{\mathbb{R}^d} |\xi|^2|\widehat{\psi}(\xi)|^2\,d\xi\Big) \geq \frac{d^2}{16\pi^2}.$$

   Extremal functions are $C_\gamma e^{-\gamma|x|^2}$ for any $\gamma > 0$.

   Proof. Since $\psi \in S(\mathbb{R}^d)$, divergence theorem implies

$$0 = \int_{\mathbb{R}^d} \nabla\cdot(x|\psi(x)|^2) = d + \int_{\mathbb{R}^d} (x\cdot\nabla\psi)\overline{\psi} + (x\cdot\nabla\overline{\psi})\psi$$

   Therefore,

$$d \leq 2\int_{\mathbb{R}^d} |x||\nabla\psi||\psi|$$

   The rest is the same as the case $d = 1$. $\square$

   Remark 1. Take a look at [12, Section 11.3]. One can also try to prove the following theorem taken from [7, Exercise 8.19]:

   Theorem 2. *If $f \in L^2(\mathbb{R}^d)$ and the set $S = \{x : f(x) \neq 0\}$ has finite measure, then for any measurable $E \subset \mathbb{R}^d$, $\int_E |\widehat{f}|^2 \leq \|f\|_2^2 m(S)m(E)$.*

   This is the so-called local uncertainty principle. The original Heinsenberg's inequality says that if f is highly localized, then $\widehat{f}$ cannot be concentrated near a single point. This exercise says more: it cannot concentrated in a small neighborhood of two or more widely separated points. See [6] for a comprehensive discussion.

7. **Consider the time-dependent heat equation in $\mathbb{R}^d$:**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \text{ where } t > 0,$$

with boundary values $u(x,0) = f(x) \in S(\mathbb{R}^d)$. If

$$\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi$$

is the d-dimensional heat kernel, show that the convolution

$$u(x,t) = (f * \mathcal{H}_t^{(d)})(x)$$

is indefinitely differentiable when $x \in \mathbb{R}^d$ and $t > 0$. Moreover, u solves the heat equation, and is continuous up to the boundary $t = 0$ in the following two senses: (1) $u(x,t) \to f(x)$ uniform in x as $t \to 0$; (2) $u(x,t) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $t \to 0$.

Moreover, $u(\cdot,t) \in S(\mathbb{R}^d)$ uniformly in t, in the sense that for any $T > 0$, for each $k, l \geq 0$, there is some constant $M_{k,l,T}$ such that

$$\sup_{x\in\mathbb{R}^d, 0<t<T} |x|^k \left|\frac{\partial^l u}{\partial x^l}(x,t)\right| \leq M_{k,l,T}$$

Proof. Modified the proof for $d = 1$ given in Chapter 5 slightly. □

8. See [17, Page 7]. The full proof for the subordination principle needs the theory of residues.

9. **A spherical wave is a solution $u(x,t)$ of the Cauchy problem for the wave equation in $\mathbb{R}^d$, which as a function of x is radial. Prove that u is a spherical wave if and only if the initial data $f, g \in \mathcal{S}$ are both radial.**

Proof. If u is a spherical wave, then the desired result is a consequence of preservation of radial symmetry under pointwise convergence, that is, $h(x) = \lim_{n\to\infty} h_n(x)$ (pointwisely) is radial provided all h_n are radial.

Conversely, one can use the fact that the Fourier and inverse Fourier transform of radial functions are radial, and hence $u(x,t)$ is radial for each $t > 0$ since

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$$

is radial for each $t > 0$ (due to $\widehat{f}, \widehat{g}$ are radial). □

10. Let $u(x,t)$ be a smooth solution of the wave equation (may be examined from the smoothness of initial data), and let $E(t)$ denote the energy of this wave

$$E(t) = \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 + \sum_{j=1}^{d} \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial x_j}(x,t)\right|^2 dx.$$

We have seen that $E(t)$ is constant using Plancherel's formula. One can give an alternate proof of this fact by differentiating the integral with respect to t and showing that

$$\frac{dE}{dt} = 0.$$

11. **Show that the solution of the wave equation**

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial 2u}{\partial x_3^2}$$

subject to $u(x,0) = f(x)$ **and** $\frac{\partial u}{\partial t}(x,0) = g(x)$**, where** $f, g \in \mathcal{S}(\mathbb{R}^3)$**, is given by**

$$u(x,t) = \frac{1}{|S(x,t)|} \int_{S(x,t)} [tg(y) + f(y) + rf(y) \cdot (y-x)] d\sigma(y),$$

where $S(x,t)$ **denotes the sphere of center** x **and radius** t**, and** $|S(x,t)|$ **its area. This is an alternate expression for the solution of the wave equation given in Theorem 3.6. It is sometimes called Kirchhoff's formula.**

Proof. See [3, Section 2.4.1(b)]. The basic strategy is the usage of average operator that can transform the three-dimensional wave equation into one-dimensional wave-like equation which can be solved explicitly. The higher dimensional cases are stated in Problem 4 and 5. □

12. **Establish the identity (14) about the dual transform given in the text. In other words, prove that**

$$\iint_{\mathbb{R}\times S^2} \mathcal{R}(f)(t,\gamma)\overline{F(t,\gamma)}\, d(t,\sigma(\gamma)) = \iiint_{\mathbb{R}^3} f(x)\overline{\mathcal{R}^*(F)(x)}\, dx$$

where $f \in \mathcal{S}(\mathbb{R}^3), F \in \mathcal{S}(\mathbb{R}\times S^2)$**, and**

$$\mathcal{R}(f) = \int_{P_{t,\gamma}} f \quad \text{and} \quad \mathcal{R}^*(F)(x) = \int_{S^2} F(x\cdot\gamma,\gamma)\, d\sigma(\gamma).$$

Proof. We note that $\mathcal{R}(f)$ is bounded, so we can use Fubini's theorem freely. For fixed $\gamma \in S^2$, we will use the change of coordinates $(t, u_1, u_2) \mapsto x = t\gamma + u_1 e_1^\gamma + u_2 e_2^\gamma$ from $\mathbb{R}^3$ to $\mathbb{R}^3$, where

$\{\gamma, e_1^\gamma, e_2^\gamma\}$ forms an orthonormal basis for $\mathbb{R}^3$. So we have $t = x \cdot \gamma$ and

$$\iint_{\mathbb{R}\times S^2} \mathcal{R}(f)(t,\gamma)\overline{F(t,\gamma)}\, d(t,\sigma(\gamma)) = \int_{S^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(t\gamma + u_1 e_1^\gamma + u_2 e_2^\gamma)\overline{F(t,\gamma)}\, d(u_1,u_2)\, dt\, d\sigma(\gamma)$$
$$= \int_{S^2}\iiint_{\mathbb{R}^3} f(t\gamma + u_1 e_1^\gamma + u_2 e_2^\gamma)\overline{F(t,\gamma)}\, d(u_1,u_2,t)\, d\sigma(\gamma)$$
$$= \int_{S^2}\iiint_{\mathbb{R}^3} f(x)\overline{F(x\cdot\gamma,\gamma)}\, dx\, d\sigma(\gamma) = \iint_{\mathbb{R}^2} f(x,y)\int_{S^2}\overline{F(x\cdot\gamma,\gamma)}\, d\sigma(\gamma)\, dx$$
$$= \iiint_{\mathbb{R}^3} f(x)\overline{\mathcal{R}^*(F)(x,y)}\, dx.$$

□

13. **For each (t,θ) with $t\in\mathbb{R}$ and $|\theta|\le\pi$, let $L = L_{t,\theta}$ denote the line in the (x,y)-plane given by**

$$x\cos\theta + y\sin\theta = t.$$

This is the line perpendicular to the direction $(\cos\theta,\sin\theta)$ at "distance" t from the origin (we allow negative t). For $f\in\mathcal{S}(\mathbb{R}^2)$ the X-ray transform or two- dimensional Radon transform of f is defined by

$$X(f)(t,\theta) = \int_{L_{t,\theta}} f = \int_{-\infty}^{\infty} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\, du.$$

Calculate the X-ray transform of the function $f(x,y) = e^{-\pi(x^2+y^2)}$.

Proof. $X(f)(t,\theta) = \int_{\mathbb{R}} e^{-\pi(t^2+u^2)}\, du = e^{-\pi t^2}$ for all $(t,\theta)\in\mathbb{R}\times S^1$. □

14. **Let X be the X-ray transform. Show that if $f\in L^1(\mathbb{R}^2)$ and $X(f)=0$, then $f=0$ a.e., by taking the Fourier transform in one variable.**

Remark 3. There is a non-trivial entire function f such that $X(f)=0$, see [11, Section 4.3] for a quick review of uniqueness and non-uniqueness for 2D Radon transform.

Proof. We will see that for all $(\cos\theta,\sin\theta)\in S^1$ and $\xi\in\mathbb{R}$

$$(\mathcal{F}_t X(f))(\xi,\theta) = (\mathcal{F}_{x,y}(f))(\xi(\cos\theta,\sin\theta)),$$

where $\mathcal{F}_t$ is Fourier transform in $t\in\mathbb{R}^1$ and $\mathcal{F}_{x,y}$ is Fourier transform in $(x,y)\in\mathbb{R}^2$. Then the desired result follows from inverse Fourier transform.

However the above formula is just a consequence of Fubini's theorem and change of coordinates $(t,u)\mapsto(x,y) = (t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)$. Note that the hypothesis $f\in L^1(\mathbb{R}^2)$ is used for the validity of Fubini's theorem.

□

15. **For $F \in \mathcal{S}(\mathbb{R} \times S^1)$, define the dual X-ray transform $X^*(F)$ by integrating F over all lines that pass through the point (x, y) (that is, those lines $L_{t,\theta}$ with $x\cos\theta + y\sin\theta = t$):**

$$X^*(F)(x,y) = \int_0^{2\pi} F(x\cos\theta + y\sin\theta, \theta)\, d\theta.$$

Check that in this case, if $f \in \mathcal{S}(\mathbb{R}^2)$ and $F \in \mathcal{S}(\mathbb{R} \times S^1)$, then

$$\iint_{\mathbb{R}\times S^1} X(f)(t,\theta)\overline{F(t,\theta)}\, d(t,\theta) = \iint_{\mathbb{R}^2} f(x,y)\overline{X^*(F)(x,y)}\, d(x,y).$$

Proof. We note that $X(f)$ is bounded, so we can use Fubini's theorem freely. Using the same change of coordinates as Exercise 14, we have

$$\iint_{\mathbb{R}\times S^1} X(f)(t,\theta)\overline{F(t,\theta)}\, d(t,\theta) = \int_{S^1}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\overline{F(t,\theta)}\, du\, dt\, d\theta$$
$$= \int_{S^1}\iint_{\mathbb{R}^2} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\overline{F(t,\theta)}\, d(u,t)\, d\theta$$
$$= \int_{S^1}\iint_{\mathbb{R}^2} f(x,y)\overline{F(x\cos\theta + y\sin\theta, \theta)}\, d(x,y)\, d\theta = \iint_{\mathbb{R}^2} f(x,y)\int_{S^1}\overline{F(x\cos\theta + y\sin\theta, \theta)}\, d\theta\, d(x,y)$$
$$= \iint_{\mathbb{R}^2} f(x,y)\overline{X^*(F)(x,y)}\, d(x,y).$$

□

Problems

Problem 3-6 are referred to [5, Chapter 5].

1. **Let J_n denote the n-th order Bessel function, for $n \in \mathbb{Z}$. Prove that**

   (a) $J_n(\rho)$ **is real for all real** ρ. **(b)** $J_{-n}(\rho) = (-1)^n J_n(\rho)$. **(c)** $2J_n'(\rho) = J_{n-1}(\rho) - J_{n+1}(\rho)$.

   (d) $\left(\frac{2n}{\rho}\right) J_n(\rho) = J_{n-1}(\rho) + J_{n+1}(\rho)$. **(e)** $(\rho^{-n}J_n(\rho))' = -\rho^{-n}J_{n+1}(\rho)$.

   (f) $(\rho^n J_n(\rho))' = \rho^n J_{n-1}(\rho)$.

   (g) $J_n(\rho)$ **satisfies the second order differential equation**

   $$J_n''(\rho) + \frac{1}{\rho}J_n'(\rho) + (1 - \frac{n^2}{\rho^2})J_n(\rho) = 0.$$

   (h) Show that

   $$J_n(\rho) = \left(\frac{\rho}{2}\right)^n \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{2m}}{2^{2m} m!(n+m)!},$$

   where the series sums from $m = -n$ if $n < 0$.

(i) Show that for all integers n and all real numbers a and b we have

$$J_n(a+b) = \sum_{\ell\in\mathbb{Z}} J_\ell(a)J_{n-\ell}(b).$$

This is called addition formula or addition theorem.

***Remark* 4.** A source for Bessel equations described in (g) is the Ginzburg-Landau equation $\Delta_{(x,y)}u + (1-|u|^2)u = 0$, where $u = u(x,y) \in \mathbb{C}$. When one search solutions of the form $u(r,\theta) = \phi(r)e^{in\theta}$, ϕ will satisfies the n-th Bessel equation. Another source is Laplace equation in cylindrical coordinates. One can also see [4, Chapter 5] and [20] for more disscusions on Bessel equations.

Proof. (a) is trivial (by observing $\int_0^{2\pi} = \int_{-\pi}^{\pi}$ and oddness of the integrand. (b) Using change of variables $\theta = \pi - t$. (c) The computation is standard if one can interchange the order of differentiation and integration. This problem can be solved by starting from difference quotient minus the target function and then use Mean Value Theorem and continuity of the derivative of the integrand to see the limit tends to zero.

(d) is proved by integration by parts. (e)(f) are proved by (c) and (d).

(g) Computing $[\rho^{-2n+1}(\rho^n J_n(\rho))']'$ in two ways: directly and using (e)(f).

(h) Because the series e^z converges on every compact subset of $\mathbb{C}$ uniformly, we have

$$\begin{aligned}
J_n(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{\infty}\frac{(i\rho\sin\theta)^m}{m!}e^{-in\theta}\,d\theta = \sum_{m=0}^{\infty}\frac{\rho^m}{2^m m!}\frac{1}{2\pi}\int_0^{2\pi}(e^{i\theta}-e^{-i\theta})^m e^{-in\theta}\,d\theta \\
&= \sum_{m=0}^{\infty}\frac{\rho^m}{2^m m!}\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{m}(-1)^k\frac{m!}{k!(m-k)!}e^{-ik\theta}e^{i(m-k)\theta}e^{-in\theta}\,d\theta \\
&= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\frac{\rho^m(-1)^k}{2^m k!(m-k)!}\frac{1}{2\pi}\int_0^{2\pi}e^{m-2k-n}\,d\theta \\
&= \sum_{k=0}^{\infty}\frac{\rho^{2k+n}(-1)^k}{2^{2k+n}k!(k+n)!}, \quad \text{if } n\geq 0.
\end{aligned}$$

For $n<0$, we use (a).

(i) (Method I: Comparing Fourier series + Cauchy product) Since $e^{i(a+b)t}, e^{iat}, e^{ibt}$ are smooth in t, their Fourier series converges absolutely and equals to themself, using the Cauchy product, we have

$$\begin{aligned}
\sum_n J_n(a+b)e^{int} = e^{i(a+b)\sin t} = e^{ia\sin t}e^{ib\sin t} &= \Big(\sum_{k\in\mathbb{Z}} J_n(a)e^{ikt}\Big)\Big(\sum_{m\in\mathbb{Z}} J_m(b)e^{imt}\Big) \\
&= \sum_{n\in\mathbb{Z}}\Big(\sum_{k+m=n} J_k(a)J_m(b)\Big)e^{int}
\end{aligned}$$

[Remark: What I mean for "using the Cauchy product" is that one can apply [15, Theorem 3.50 or 3.51] to each term of

$$\sum_{n\in\mathbb{Z}} a_n \sum_{m\in\mathbb{Z}} b_m = \Big(\sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}\Big)\Big(\sum_{m=0}^{\infty} b_m + \sum_{m=1}^{\infty} b_{-m}\Big).]$$

By uniqueness of Fourier series, we have the desired result.

(Method II: Generating function + (h)) See [20, Page 30]. □

2. **Another formula for $J_n(\rho)$ that allows one to define Bessel functions for non-integral values of $\mu, (\mu > -1/2)$ is**

$$J_\mu(\rho) = \frac{(\rho/2)^\mu}{\Gamma(\mu+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{i\rho t}(1-t^2)^{\mu-(1/2)}\, dt, \quad \rho \in \mathbb{C}.$$

(a) Check that the above formula agrees with the definition of $J_n(\rho)$ for integral $n \geq 0$.

(b) Note that $J_{1/2}(\rho) = \sqrt{\frac{2}{\pi\rho}} \sin\rho$.

(c) Prove that

$$\lim_{\mu\to(-1/2)^+} J_\mu(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\rho, \quad \rho > 0.$$

(d) Observe that the formulas we have proved in the text giving F_0 in terms of f_0 (when describing the Fourier transform of a radial function) take the form

$$F_0(\rho) = 2\pi\rho^{-(d/2)+1} \int_0^\infty J_{(d/2)-1}(2\pi\rho r) f_0(r) r^{d/2}\, dr,$$

for $d = 1, 2,$ and 3, if one uses the formulas above with the understanding that $J_{-1/2}(\rho) = \lim_{\mu\to-1/2} J_\mu(\rho)$. It turns out that the relation between F_0 and f_0 given by the above formula is valid in all dimensions d.

Remark 5. This definition is called the Poisson representation formula or the Mehler-Sonine integral, see [20, Page 24 and Chapter VI]. This formula seems to be useful when work with Fourier transform.

Proof. (a) As hint, one verify the case $n = 0$ easily by change of variables $t = \sin x$ and then use integration by parts to check the recursion formula (e) in Problem 1 holds under this definition, that is, for $n \in \mathbb{N}$

$$\rho^{-n} J_n(\rho) = \frac{\frac{1}{2^n}}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} \cos(\rho t)(1-t^2)^{n-\frac{1}{2}}\, dt.$$

(Note that the integral in the definition is well-defined for $\mu > -\frac{1}{2}$ and the interchange of the order of differentiation and integration can be rigorous verified as we mentioned in Problem 1(c).)

(b) is trivial.

(c) As (a), we are motivated by the recursion formula in Problem 1(f) to show for $\mu > \frac{1}{2}$

$$(\rho^{\mu} J_{\mu}(\rho))' = \rho^{\mu} J_{\mu-1}(\rho)$$

rigorously by using integration by parts. Therefore, we have "formally"

$$\lim_{\mu \to (-\frac{1}{2})^+} J_{\mu}(\rho) = \lim_{\mu \to (\frac{1}{2})^+} J_{\mu-1}(\rho) = \lim_{\mu \to (\frac{1}{2})^+} \rho^{-\mu}(\rho^{\mu} J_{\mu}(\rho))'\text{''} = \text{''}\rho^{-\frac{1}{2}}(\rho^{\frac{1}{2}} J_{\frac{1}{2}}(\rho))' = \sqrt{\frac{2}{\pi\rho}} \cos\rho.$$

To show the above red equality, one has to show $\rho^{\mu} J_{\mu}(\rho) \to \rho^{\frac{1}{2}} J_{\frac{1}{2}}(\rho)$ pointwisely and the derivative $(\rho_{\mu} J_{\mu}(\rho))' \to (\rho^{\frac{1}{2}} J_{\frac{1}{2}}(\rho))'$ uniformly on any compact interval $[a, b]$, see [15, Theorem 7.17]. Since the computation is routine, we leave it to the readers.

(d) Since f_0 is radial, its Fourier transform is radial $\widehat{f}_0(\xi) = \widehat{f}_0((|\xi|, 0, \cdots, 0))$ and hence

$$\begin{aligned}
F_0(\xi) = \widehat{f}_0((|\xi|, 0, \cdots, 0)) &= \int_{\mathbb{R}^d} f_0(x) e^{-2\pi i x \cdot (|\xi|, \cdots, 0)}\, dx \\
&= \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} e^{i(2\pi|\xi|r)(-\cos\phi_1)} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \cdots \sin(\phi_{d-2}) \\
&\quad d\phi_1\, d\phi_2 \cdots d\phi_{d-1} f_0(r) r^{d-1}\, dr
\end{aligned}$$

Note that we have, by using the change of coordinates $t = -\cos\phi_1$,

$$\int_0^{\pi} e^{i(2\pi|\xi|r)(-\cos\phi_1)} \sin^{d-2}(\phi_1)\, d\phi_1 = \int_{-1}^{1} e^{i(2\pi|\xi|r)t} (\sqrt{1-t^2})^{d-3}\, dt = J_{\frac{d}{2}-1}(2\pi|\xi|r) \frac{\Gamma(\frac{d-1}{2})\sqrt{\pi}}{(\frac{\rho}{2})^{\frac{d}{2}-1}}.$$

The rest integrals can be computed by Beta functions, that is,

$$\int_0^{\pi} (\sin\theta)^k\, d\theta = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^k\, d\theta = B(\frac{k+1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+2}{2})}.$$

□

Remark 6. Also see Book II's Appendix A and Book IV's Section 8.1 & Exercise 8.1.

3. This problem explains the finite propagation speed of solutions of wave equations, see [3, Section 2.4.3(b) and 7.2.4] and [14, Chapter 1-2].

Exercise 4 and 5 are Hadamard's method of descent and Huygens' principle, their proofs can be found in many PDE textbooks, e.g. [5, Section 5.2] or [3, Section 2.4]. We state the results here and post another related problem $5\frac{1}{2}$ concerning the decay rate in time t.

4. There exist formulas for the solution of the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial_{x_1^2}} + \cdots + \frac{\partial^2 u}{\partial x_d^2} \quad \text{with } u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x)$$

in $\mathbb{R}^d \times \mathbb{R}$ in terms of spherical means which generalize the formula given in the text for $d = 3$. In fact, the solution for even dimensions is deduced from that for odd dimensions, so we discuss this case first.

Suppose that $d > 1$ is odd and let $h \in \mathcal{S}(R^d)$. The spherical mean of h on the ball centered at x of radius t is defined by

$$M_r h(x) = Mh(x,r) = \frac{1}{A_d} \int_{S^{d-1}} h(x - r\gamma)\, d\sigma(\gamma),$$

where A_d denotes the area of the unit sphere S^{d-1} in $\mathbb{R}^d$.

(a) Show that

$$\Delta_x Mh(x,r) = \left[\partial_r^2 + \frac{d-1}{r}\right] Mh(x,r),$$

where Δ_x denotes the Laplacian in the space variables x, and $\partial_r = \partial/\partial_r$.

(b) Show that a twice differentiable function $u(x,t)$ satisfies the wave equation if and only if

$$\left[\partial_r^2 + \frac{d-1}{r}\right] Mu(x,r,t) = \partial_t^2 Mu(x,r,t),$$

where $Mu(x,r,t)$ denote the spherical means of the function $u(x,t)$.

(c) If $d = 2k+1$, define $T\varphi(r) = (r^{-1}\partial_r)^{k-1}[r^{2k-1}\varphi(r)]$, and let $\widetilde{u} = TMu$. Then this function solves the one-dimensional wave equation for each fixed x:

$$\partial_t^2 \widetilde{u}(x,r,t) = \partial_r^2 \widetilde{u}(x,r,t).$$

One can then use d'Alembert's formula to find the solution $\widetilde{u}(x,r,t)$ of this problem expressed in terms of the initial data.

(d) Now show that

$$u(x,t) = Mu(x,0,t) = \lim_{r \to 0} \frac{\widetilde{u}(x,r,t)}{\alpha r}$$

where $\alpha = 1 \cdot 3 \cdots (d-2)$.

(e) Conclude that the solution of the Cauchy problem for the d-dimensional wave equation, when $d > 1$ is odd, is

$$u(x,t) = \frac{1}{1 \cdot 3 \cdots (d-2)} \left[\partial_t (t^{-1}\partial_t)^{(d-3)/2} \Big(t^{d-2} M_t f(x) \Big) + (t^{-1}\partial_t)^{(d-3)/2} \Big(t^{d-2} M_t g(x) \Big) \right],$$

5. The method of descent can be used to prove that the solution of the Cauchy problem for the wave equation in the case when d is even is given by the formula

$$u(x,t) = \frac{1}{1 \cdot 3 \cdots (d-2)} \Big[\partial_t (t^{-1}\partial_t)^{(d-3)/2} \Big(t^{d-2} \widetilde{M}_t f(x) \Big) + (t^{-1}\partial_t)^{(d-3)/2} \Big(t^{d-2} \widetilde{M}_t g(x) \Big) \Big],$$

where $\widetilde{M}_t$ denotes the modified spherical means defined by

$$\widetilde{M}_t h(x) = \frac{2}{A_{d+1}} \int_{B^d} \frac{h(x+ty)}{\sqrt{1-|y|^2}} \, dy.$$

$5\frac{1}{2}$. **This exercise shows that the decay rates for solutions of wave equation in 3-dimension and 2-dimension are different.**

(a) Let u be the solution of

$$\begin{cases} u_{tt} - \Delta u = 0 \ \text{ in } \ \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = f(x), u_t(x,0) = g(x) \ \text{ on } \ \mathbb{R}^3 \end{cases}$$

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ have compact support. Show there exists a constant $C > 0$ such that for each $x \in \mathbb{R}^3, t > 0$,

$$|u(x,t)| \le C/t$$

(b) Let u be the unique solution of

$$\begin{cases} u_{tt} - \Delta u = 0 \ \text{ in } \ \mathbb{R}^2 \times (0,\infty) \\ u(x,0) = f(x), u_t(x,0) = g(x) \ \text{ on } \ \mathbb{R}^2 \end{cases}$$

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ have compact support. Prove that for each $x \in \mathbb{R}^2$, there exists a constant C_x, such that for each $t > 0$

$$|u(x,t)| \le C_x/t$$

(c) Let u be the same function as (b). Prove that there exists a constant $C > 0$ such that for each $x \in \mathbb{R}^2, t > 0$,

$$|u(x,t)| \le C/t^{1/2}$$

Remark 7. In general, the time decay rate is $t^{\frac{1-d}{2}}$ (where d is the spatial dimension). This can be shown by modifying the proof given here for $d = 2, 3$.

Remark 8. Strichartz estimates are another important estimates for wave equations. See [10] and [16, Chapter IV] for the proofs and applications to semilinear wave equations.

Proof. We suppose both the supports of f and g are included in $B_R(0)$.

(a) From uniqueness theorem (proved by the conservation law of energy) and the solution formula given in the textbook and Problem 4, that is,

$$\begin{aligned} u(x,t) &= \partial_t\Big(\frac{t}{4\pi}\int_{S^2} f(x-t\gamma)\,d\sigma(\gamma)\Big) + \frac{t}{4\pi}\int_{S^2} g(x-t\gamma)\,d\sigma(\gamma) \\ &= \frac{1}{4\pi}\int_{S^2} f(x-t\gamma)\,d\sigma(\gamma) + \frac{t}{4\pi}\Big(\int_{S^2} g(x-t\gamma)\,d\sigma(\gamma) - \int_{S^2} (\nabla f)(x-t\gamma)\cdot\gamma\,d\sigma(\gamma)\Big). \end{aligned}$$

We use the divergence theorem as follows:

$$\begin{aligned} \int_{S^2} f(x-t\gamma)\gamma\cdot\gamma\,d\sigma(\gamma) &= \frac{1}{t^3}\int_{\partial B_t(x)} f(y)(y-x)\cdot\frac{y-x}{|y-x|}\,d\sigma(y) = \frac{1}{t^3}\int_{B_t(x)} \mathrm{div}_y(f(y)(y-x))\,dy \\ &= \frac{1}{t^3}\int_{B_t(x)} \nabla f(y)\cdot(y-x) + 3f(y)\,dy \le \frac{1}{t^2}\|\nabla f\|_{L^1} + \frac{3}{t^3}\|f\|_{L^1}. \end{aligned}$$

Similarly, $t\int_{S^2} g(x-t\gamma)\,d\sigma(\gamma) \le \frac{1}{t}\|\nabla g\|_{L^1} + \frac{3}{t^2}\|g\|_{L^1}$. To estimate the rest term, we take advantage that f have compact support:

$$t\int_{S^2}(\nabla f)(x-t\gamma)\cdot\gamma\,d\sigma(\gamma) = \frac{1}{t}\int_{\partial B_t(x)} \nabla f(y)\cdot\frac{y-x}{|y-x|}\,d\sigma(y) = \frac{1}{t}\|\Delta f\|_{L^1(B_R)},$$

(Or bounded by $\frac{4\pi R^2}{t}\|\nabla f\|_\infty$ since the intersection of $\partial B_t(x)$ and $B_R(0)$ has area at most $4\pi R^2$).

(We remark that for small time, we can find a better estimate that u is obviously dominated by $(\|\nabla f\|_\infty + \|g\|_\infty)t + \|f\|_\infty)$.

(b) Again, the uniqueness theorem implies

$$\begin{aligned} u(x,t) &= \partial_t\Big(\frac{t}{2\pi}\int_{B_1(0)} \frac{f(x-ty)}{\sqrt{1-|y|^2}}\,dy\Big) + \frac{t}{2\pi}\int_{B_1(0)} \frac{g(x-ty)}{\sqrt{1-|y|^2}}\,dy \\ &= \frac{1}{2\pi}\int_{B_1(0)} \frac{f(x-ty)}{\sqrt{1-|y|^2}}\,dy + \frac{t}{2\pi}\int_{B_1(0)} \frac{(\nabla f)(x-ty)\cdot(-y) + g(x-ty)}{\sqrt{1-|y|^2}}\,dy \\ &= \frac{1}{2\pi t}\int_{B_t(x)} \frac{f(z)}{\sqrt{t^2-|x-z|^2}}\,dz + \frac{1}{2\pi}\int_{B_t(x)} \frac{(\nabla f)(z)\cdot\frac{x-z}{t} + g(z)}{\sqrt{t^2-|x-z|^2}}\,dz \end{aligned}$$

One notes that the integral $\int_0^t \frac{r}{\sqrt{t^2-r^2}}\,dr = t$, so we need to consider the large time and small time cases separately. Because it is a little delicate issue how to determine a suitable threshold for t, we don't describe how to find it here.

If $|x| + 2R < t$, then $B_R(0) \subset B_t(x)$

$$\int_{B_t(x)} \frac{|f(z)|}{\sqrt{t^2-|x-z|^2}}\,dz = \frac{1}{t}\int_{B_R(0)} \frac{|f(z)|}{\sqrt{1-\frac{|x-z|^2}{t^2}}}\,dz \le \frac{\|f\|_\infty}{t}\frac{\pi R^2}{\sqrt{1-\frac{(R+|x|)^2}{(2R+|x|)^2}}}$$

If $0 < t < |x| + 2R$, then

$$\int_{B_t(x)} \frac{|f(z)|}{\sqrt{t^2 - |x-z|^2}}\, dz \le 2\pi \|f\|_\infty \int_0^t \frac{r}{\sqrt{t^2 - r^2}}\, dr = 2\pi\|f\|_\infty t \le 2\pi \|f\|_\infty \frac{(|x|+2R)^2}{t} \tag{1}$$

Similar for the terms involving g and $|\nabla f|$.

(c) To obtain the L^∞_x bound, we use the divergence theorem as (a) and separate the ball $B_t(x)$ into the inner ball $B_{t-1}(x)$ of radius $t-1$ and the outer annulus $A = B_t(x) \setminus \overline{B_{t-1}(x)}$.

Note that for $t \ge 2$ ($\Leftrightarrow t - 1 \ge \frac{t}{2}$) ,

$$\int_{B_{t-1}(x)} \frac{|f(z)|}{\sqrt{t^2 - |x-z|^2}}\, dz \le \int_{B_{t-1}(x)} \frac{|f(z)|}{\sqrt{t^2 - (t-1)^2}}\, dz \le \frac{\|f\|_{L^1}}{\sqrt{2t-1}} \le \frac{\|f\|_{L^1}}{\sqrt{t}}.$$

For the integral over A, one observes that $\nabla_y \sqrt{t^2 - y^2} = -\frac{y}{\sqrt{t^2 - y^2}}$ and $\nabla_z \frac{x-z}{|x-z|^2} \equiv 0$ on $\mathbb{R}^2$, so

$$\begin{aligned}
&\int_A \frac{f(z)}{\sqrt{t^2 - |x-z|^2}}\, dz = -\int_A f(z) \frac{x-z}{|x-z|^2} \cdot \nabla_z \sqrt{t^2 - |x-z|^2}\, dz \\
&= \int_A \mathrm{div}(f(z) \frac{x-z}{|x-z|^2}) \sqrt{t^2 - |x-z|^2} - \int_{\partial A} f(z) \sqrt{t^2 - |x-z|^2} \frac{x-z}{|x-z|^2} \cdot n\, d\sigma(z) \\
&= \int_A (\nabla f)(z) \cdot \frac{x-z}{|x-z|^2} \sqrt{t^2 - |x-z|^2} - \int_{\partial B_{t-1}(x)} f(z) \sqrt{2t-1} \frac{1}{t-1}\, d\sigma(z).
\end{aligned}$$

Note that the first term is dominated by $\frac{\sqrt{2t-1}}{t-1} \|\nabla f\|_{L^1(A)} \le \frac{\sqrt{2t}}{t/2} \|\nabla f\|_{L^1(\mathbb{R}^2)}$. For the second term, we find

$$\begin{aligned}
&\Big| \int_{\partial B_{t-1}(x)} f(z) \frac{(x-z)}{t-1} \cdot \frac{(x-z)}{t-1}\, d\sigma(z) \Big| = \Big| \frac{1}{t-1} \int_{B_{t-1}(x)} \mathrm{div}_z [f(z)(x-z)]\, dz \Big| \\
&= \frac{1}{t-1} \Big| \int_{B_{t-1}(x)} \nabla f(z) \cdot (x-z) - 2f(z)\, dz \Big| \le \int_{\mathbb{R}^2} |\nabla f(z)|\, dz + \frac{2}{t-1} \int_{\mathbb{R}^2} |f(z)|\, dz.
\end{aligned}$$

So for $t \ge 2$,

$$\Big| \int_{B_t(x)} \frac{f(z)}{\sqrt{t^2 - |x-z|^2}}\, dz \Big| \le \frac{1}{\sqrt{t}} \Big(4\sqrt{2} \|\nabla f\|_{L^1(\mathbb{R}^2)} + (1 + \frac{2}{t-1}) \|f\|_{L^1} \Big).$$

For $t \in (0, 2)$, we see $\Big| \int_{B_t(x)} \frac{f(z)}{\sqrt{t^2 - |x-z|^2}}\, dz \Big| \le 2\pi \|f\|_\infty t$ from (1).

Similar for the terms involving g and $|\nabla f|$. □

Remark 9. One should also try to prove this time decay estimate from the Fourier representation formula (3) in page 186, where one can learn how to apply stationary phase formula to estimate certain oscillatory integrals. For more results in this direction, see [19, Section 2.2].

6. **Given initial data f and g of the form**

$$f(x) = F(x \cdot \eta) \ \ \textbf{and} \ \ g(x) = G(x \cdot \eta),$$

check that the plane wave given by

$$u(x,t) = \frac{F(x\cdot\eta + t) + F(x\cdot\eta - t)}{2} + \frac{1}{2}\int_{x\cdot\eta - t}^{x\cdot\eta + t} G(s)\,ds$$

is a solution of the Cauchy problem for the d-dimensional wave equation.

In general, the solution is given as a superposition of plane waves. For the case $d = 3$, this can be expressed in terms of the Radon transform as follows.

Let

$$\widetilde{R}(f)(t,\gamma) = -\frac{1}{8\pi^2}\Big(\frac{d}{dt}\Big)^2 R(f)(t,\gamma).$$

Then

$$u(x,t) = \frac{1}{2}\int_{S^2}\Big[\widetilde{R}(f)(x\cdot\gamma - t,\gamma) + \widetilde{R}(f)(x\cdot\gamma + t,\gamma) + \int_{x\cdot\gamma - t}^{x\cdot\gamma + t}\widetilde{R}(g)(s,\gamma)\,ds\Big]\,d\sigma(\gamma)$$

where $f, g \in \mathcal{S}(\mathbb{R}^3)$.

Remark 10. Can one derive the time decay results in Problem $5\frac{1}{2}$ and the Huygens principle from this solution representation?

Proof. First we prove that $f(x) = \int_{S^2}\widetilde{R}(f)(x\cdot\gamma,\gamma)\,d\sigma(\gamma)$ from the reconstruction formula as follows:

$$\begin{aligned}-8\pi^2 f(x) = \Delta_x\int_{S^2} R(f)(x\cdot\gamma,\gamma)\,d\sigma(\gamma) &= \int_{S^2}\Big(\frac{d}{dt}\Big)^2 R(f)(x\cdot\gamma,\gamma)\gamma\cdot\gamma\,d\sigma(\gamma)\\ &= -8\pi^2\int_{S^2}\widetilde{R}(f)(x\cdot\gamma,\gamma)\,d\sigma(\gamma).\end{aligned}$$

Similarly, we have $g(x) = \int_{S^2}\widetilde{R}(g)(x\cdot\gamma,\gamma)\,d\sigma(\gamma)$. So the given $u(x,t)$ satisfies $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$.

Second, we point out the first part of this problem can be proven by reading the Laplacian in different coordinates systems, that is, the standard coordinates system $(e_1, e_2, \cdots, e_d)$ and $(\eta, f_1, f_2, \cdots, f_{d-1})$ (which forms a orthogonal basis for $\mathbb{R}^d$). However, the form of Laplacian is unchanged under this change of coordinate (since it's just a rotation), but the d-dimensional wave equation is now reduced to the 1-dimensional wave equation (which can be solved by d'Alembert's formula). Finally, one can exam the given solution representations solve the wave equation easily. ∎

7. **For every real number $a > 0$, define the operator $(-\Delta)^a$ by the formula**

$$(-\Delta)^a f(x) = \int_{\mathbb{R}^d}(2\pi|\xi|)^{2a}\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,d\xi$$

whenever $f \in \mathcal{S}(\mathbb{R}^d)$.

(a) Check that $(-\Delta)^a$ agrees with the usual definition of the a-th power of $-\Delta$ (that is, a compositions of minus the Laplacian) when a is a positive integer.

(b) Verify that $(-\Delta)^a(f)$ is indefinitely differentiable.

(c) Prove that if a is not an integer, then in general $(-\Delta)^a(f)$ is not rapidly decreasing.

(d) Let $u(x, y)$ be the solution of the steady-state heat equation

$$\frac{\partial^2 u}{\partial y^2} + \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} = 0, \quad \textbf{with} \quad u(x, 0) = f(x)$$

given by convolving f with the Poisson kernel (see Exercise 8). Check that

$$(-\Delta)^{1/2} f(x) = -\lim_{y \to 0^+} \frac{\partial u}{\partial y}(x, y),$$

and more generally that

$$(-\Delta)^{k/2} f(x) = (-1)^k \lim_{y \to 0^+} \frac{\partial^k u}{\partial y^k}(x, y)$$

for any positive integer k.

Remark 11. This is a basic nonlocal operator. Note that (d) is a basic fact for nonlocal problems. In 2007, Caffarelli and Silvestre [2] found an important extension theorem for fractional Laplacian, that is, for *every* case $a \in (0, 1)$. This makes many techniques from degenerate elliptic PDE applicable to this field. Later on, Stinga and Torrea [18] extend Caffarelli-Silvestre's to fractional harmonic oscillator by semigroups method, where heat kernel is an essential tool in their analysis. Surveys for nonlocal operators can be found in [13], [1], [9] and [8].

Proof. (a) We note that if $a = 1$, then for each $x \in \mathbb{R}^d$

$$\begin{aligned}
(-\Delta)(f)(x) &= \int_{\mathbb{R}^d} (2\pi|\xi|)^2 \widehat{f}(\xi) e^{-2\pi i \xi \cdot (-x)} \, d\xi = 4\pi^2 \widehat{|\cdot|^2 \widehat{f}(\cdot)}(-x) \\
&= -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) \widehat{\widehat{f}}(-x) \\
&= -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) f(x).
\end{aligned}$$

Then we apply the induction argument to conclude this is true for all $a \in \mathbb{N}$.

(b) Like (a), we realize $(-\Delta)^a(f)$ as some function's Fourier transform. Then the rest of proof is the same as the one for Proposition 1.2(v) of Chapter 5 (with an induction argument).

(c) If $(-\Delta)^a f$ is in Schwartz space, then its Fourier transform $(2\pi|\xi|)^{2a}\widehat{f}(-\xi)$ is in Schwartz space. However, the function $\xi \mapsto |\xi|^{2a}$ is not smooth at zero if a is not an integer.

(d) Since $\widehat{u}(\xi, y) = \widehat{f}(\xi)e^{-2\pi|\xi|y} \in L^1(\mathbb{R}^{d+1})$ (note that the Dirichlet problem for Laplace equation on upper-half plane has no uniqueness theorem, e.g. $f \equiv 0$ but $u(x, y) = y \not\equiv 0$), we have

$$u(x, y) = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{-2\pi|\xi|y}e^{2\pi i x\cdot\xi}\, d\xi.$$

Since $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$, we have, for any $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$,

$$\frac{\partial^k u}{\partial y^k}(x, y) = -\int_{\mathbb{R}^d} (2\pi|\xi|)^k \widehat{f}(\xi)e^{-2\pi|\xi|y}e^{2\pi i x\cdot\xi}\, d\xi$$

Since $|\xi|^k\widehat{f} \in L^1(\mathbb{R}^d)$, we have (from LDCT)

$$-\lim_{y\to 0^+} \frac{\partial^k u}{\partial y^k}(x, y) = \int_{\mathbb{R}^d} (2\pi|\xi|)^k \widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi = (-\Delta)^{\frac{k}{2}}(f)(x).$$

(Another classical way to prove this is using the advantage of fast decay of $\widehat{f}(\xi)$ to decompose the integration over $\mathbb{R}^d$ into "large" and "small" ξ, like the proof for Proposition 1.2(v) of Chapter 5.) □

8. **The reconstruction formula for the Radon transform in $\mathbb{R}^d$ is as follows:**

(a) When $d = 2$,

$$\frac{(-\Delta)^{1/2}}{4\pi}\mathcal{R}^*(\mathcal{R}(f)) = f.$$

where $(-\Delta)^{1/2}$ is defined in Problem 7.

(b) If the Radon transform and its dual are defined by analogy to the cases $d = 2$ and $d = 3$, then for general d,

$$\frac{(2\pi)^{1-d}}{2}(-\Delta)^{(d-1)/2}\mathcal{R}^*(\mathcal{R}(f)) = f.$$

Proof. We only prove (b) since it involves (a). One needs to establish the relationship between $\widehat{R(f)}(s, \gamma)$ and $\widehat{f}(s\gamma)$. However, the same proof as Lemma 5.2 implies that $\widehat{R(g)}(s, \gamma) = \widehat{g}(s\gamma)$ for $\gamma \in S^{d-1}$ and $s > 0$. Moreover, one can find that the proof only uses the Fubini's theorem, so it works for $g \in L^1(\mathbb{R}^d)$ (note that $P_y, \widehat{P_y} \in L^1(\mathbb{R}^d)\backslash S(\mathbb{R}^d)$). So for each $x \in \mathbb{R}^d$, $t \in \mathbb{R}, \gamma \in S^{d-1}$,

$$R(P_y(\cdot - x))(t, \gamma) = \int_{\mathbb{R}} \widehat{P_y(\cdot - x)}(s\gamma)e^{2\pi i t s}\, ds = \int_{\mathbb{R}} e^{-2\pi y|s|}e^{-2\pi i x\cdot s\gamma}e^{2\pi i t s}\, ds.$$

Hence (using the symmetry of P_y)

$$\begin{aligned}
(-\Delta)^{\frac{d-1}{2}}(R^*R(f))(x) &= \lim_{y\to 0^+}(-\frac{\partial}{\partial y})^{d-1}\int_{\mathbb{R}^d}(R^*R(f))(z)P_y(z-x)\,dz \\
&= \lim_{y\to 0^+}(-\frac{\partial}{\partial y})^{d-1}\int_{S^{d-1}}\int_{\mathbb{R}} R(f))(t,\gamma)\overline{R(P_y(\cdot-x))(t,\gamma)}\,dt\,d\sigma(\gamma) \\
&= \lim_{y\to 0^+}(-\frac{\partial}{\partial y})^{d-1}\int_{S^d}\int_{\mathbb{R}} R(f)(t,\gamma)\int_{\mathbb{R}} e^{-2\pi y|s|}e^{2\pi i x\cdot s\gamma}e^{-2\pi i t s}\,ds\,dt\,d\sigma(\gamma) \\
&= \lim_{y\to 0^+}\int_{S^d}\int_{\mathbb{R}} R(f)(t,\gamma)\int_{\mathbb{R}} e^{-2\pi y|s|}e^{2\pi i x\cdot s\gamma}e^{-2\pi i t s}(2\pi|s|)^{d-1}\,ds\,dt\,d\sigma(\gamma) \\
&= (2\pi)^{d-1}\lim_{y\to 0^+}\int_{S^{d-1}}\int_{\mathbb{R}} e^{-2\pi y|s|}e^{2\pi i x\cdot s\gamma}\int_{\mathbb{R}} R(f)(t,\gamma)e^{-2\pi i t s}\,dt|s|^{d-1}\,ds\,d\sigma(\gamma) \\
&= (2\pi)^{d-1}\lim_{y\to 0^+}\int_{S^{d-1}}\int_{\mathbb{R}} e^{-2\pi y|s|}e^{2\pi i x\cdot s\gamma}\widehat{f}(s\gamma)|s|^{d-1}\,ds\,d\sigma(\gamma) \\
&= 2(2\pi)^{d-1}\lim_{y\to 0^+}\int_{S^{d-1}}\int_0^{\infty} e^{-2\pi y s}e^{2\pi i x\cdot s\gamma}\widehat{f}(s\gamma)s^{d-1}\,ds\,d\sigma(\gamma) \\
&= 2(2\pi)^{d-1}\lim_{y\to 0^+}\int_{\mathbb{R}^d} e^{-2\pi y|\xi|}e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\,d\xi \\
&= 2(2\pi)^{d-1}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\,d\xi \\
&= 2(2\pi)^{d-1}f(x).
\end{aligned}$$

□

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