Real Analysis, 2nd Edition, G.B.Folland

Chapter 3 Signed Measures and Differentiation*

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3.1 Signed Measures

1. *Proof.* The first part is proved by using addivitiy and consider $F_j = E_j - E_{j-1}, E_0 = \emptyset$. For the second part, say $E_j \searrow E$, note $\mu(E_1) = \mu(E) + \mu(E_1 - E_2) + \mu(E_2 - E_3) + \cdots = \mu(E) + \mu(E_1) - \mu(E_2) + \mu(E_2) - \mu(E_3) + \cdots = \mu(E) + \mu(E_1) - \lim_{k\to\infty}\mu(E_k)$. The second equality is by finiteness. □

2. *Proof.* Let $E \in \mathscr{M}$ with $|\nu|(E) = 0 = \nu^+(E)+\nu^-(E)$. Since $\nu^+(E)$ and $\nu^-(E)$ are nonnegative, $\nu^+(E) = \nu^-(E) = 0$. So $\nu(E) = \nu^+(E) - \nu^-(E) = 0$. Conversely, if E is ν-null, then $\nu^+(E) = \nu(E \cap P) = 0 = \nu(E \cap N) = \nu^-(E)$ since $E \cap P$ and $E \cap N$ are contained in E. So $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.
   The second assertion is to proved the followings are equivalent,
   (1)$\nu \perp \mu$, (2)$\nu^+ \perp \mu$ and $\nu^- \perp \mu$, (3) $|\nu| \perp \mu$:
   (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are by the definition and the first part of this exercise. To prove (2) $\Rightarrow$ (3), we note that there are measurable sets A_1, B_1, A_2, B_2 such that $A_1 \cup B_1 = X = A_2 \cup B_2$, $A_1 \cap B_1 = \emptyset = A_2 \cap B_2$, and $\mu(A_1) = 0 = \nu^+(B_1) = \mu(A_2) = \nu^-(B_2)$. Let $E = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2)$ and $F = B_1 \cap B_2$, then $E \cup F = X, E \cap F = \emptyset$, and by definition, $\mu(E) = 0 = \nu^+(F) = \nu^-(F)$. Hence $|\nu|(F) = 0$, and therefore $|\nu| \perp \mu$. □

3. *Proof.* (a)(b) and (c) $|\nu|(E) \geq \sup\{|\int_E f\, d\nu| : |f| \leq 1\}$ are trivial. To prove that they are equal, let $X = P \cup N$ be the Hahn decomposition for ν. We note that $|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = \int_E \chi_P - \chi_N\, d\nu$. Since P and N are disjoint, $|\chi_P - \chi_N| \leq 1$. Hence the proof is completed. □

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4. (The minimality of Jordan decomposition of ν.)

   Proof. Let $X = P \cup N$ be the Hahn decomposition for ν. For any measurable set E, we have $\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P)$. If $\nu^+(E) = \infty$, then $\lambda(E \cap P) = \infty$ and hence $\lambda(E) = \infty$. So $\nu^+(E) \le \lambda(E)$. If $\nu^+(E) < \infty$, then $\lambda(E \cap P), \mu(E \cap P) < \infty$. This implies $\nu^+(E) \le \nu^+(E) + \mu(E \cap P) = \lambda(E \cap P) \le \lambda(E)$. The second assertion $\nu^- \le \mu$ is proved in a similar way. □

5. *Proof.* Since $\nu_1 = \nu_1^+ - \nu_1^-$ and $\nu_2 = \nu_2^+ - \nu_2^-(E), \nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$. By ex 4, $(\nu_1^+ + \nu_2^+) \ge (\nu_1 + \nu_2)^+$ and $(\nu_1^- + \nu_2^-) \ge (\nu_1 + \nu_2)^-$. The proof is completed by adding these two inequalities. □

6. *Proof.* The Hahn decomposition is $P \cup N := \{f \ge 0\} \cup \{f < 0\}$. $\nu^\pm(E) = \int_E f^\pm \, d\mu$. □

7. *Proof.* Let $X = P \cup N$ be the Hahn decomposition.

   (a)Given $F \subset E, \nu(F) = \nu(F \cap P) + \nu(F \cap N) \le \nu(F \cap P) \le \nu(E \cap P)$ which yields the inequality. The equality is true by taking $F = E \cap P$ directly. The second case is similar.

   (b)Given $E_1, \cdots E_n$ partition E, note that $|\nu(E_1)| + \cdots |\nu(E_n)| \le |\nu|(E_1) + \cdots + |\nu|(E_n) = |\nu|(E) = |\nu(F \cap P)| + |\nu(F \cap N)|$ (the proof of this inequality is similar to exercise 3). □

3.2 The Lebesgue-Radon-Nikodym Theorem

8. *Proof.* $|\nu| \ll \mu \Leftrightarrow \nu \ll \mu$ is proved by exercise 2. Using this equivalence, we know $\nu \ll \mu \Rightarrow \nu^+ \ll \mu$ and $\nu^- \ll \mu$. The converse is by definition. □

9. *Proof.* For first assertion, we know from the assumption there is a sequence of $\{(E_j, F_j\}$ such that for each $j, X = E_j \cup F_j, \nu_j(F_j) = 0$ and $\mu(E_j) = 0$. Take $E = \cup_j E_j$ and $F = \cap_j F_j$, then $\mu(E) = 0$ and $\sum_j \nu_j(F) = 0$. The second assertion is much easier. □

10. Nothing to comment.

11. This problem is easy, we omit the proof here and remark that the converse of (b) is also true (try to apply Egoroff's Theorem). This is known as **Vitali Convergence Theorem** stated in Exercise 6.15, page 187. Necessity of the hypothesis and further discussions are given in Rudin [3, Exercise 6.10-11]. Also see the additional exercises to Chapter 6.

12. *Proof.* □

13. *Proof.* (σ-finiteness condition can NOT be omitted in Radon-Nikodym Theorem.)

(a) Suppose there is an extended μ-integrable function f such that $dm = fd\mu$. Since for each $x \in [0,1], 0 = m(\{x\}) = \int_{\{x\}} f\, d\mu = f(x)$. This leads to a contradiction since

$$1 = m([0,1]) = \int_{[0,1]} f\, d\mu = 0.$$

(b) Suppose that μ has a Lebesgue decomposition $\lambda + \rho$ with respect to m, with $\lambda \perp m$ and $\rho \ll m$. Then for all $x \in [0,1], \rho(\{x\}) = 0$ and hence $\lambda(\{x\}) = 1$. Since there exists disjoint measurable sets A, B partition $[0,1]$ with A is λ-null, and $m(B) = 0$, we see for any $x \in A, 0 = \lambda(\{x\}) = 1$, so $A = \emptyset$. But then $0 = m(B) = m([0,1]) = 1$, a contradiciton! □

14. *Proof.* □

15. Note that some textbooks referred to call this measure **almost decomposable** since (iii) is required to be true for $\mu(E) < \infty$ only, and they call a measure decomposable if (iii) is true for any measurable set E. A subtle difference is marked in the remark after Exercise 6.25.

Proof. (a) Since $X = \cup_{n=1} X_n$ for some $X_n \in \mathscr{M}$ and $\mu(X_n) < \infty$. Take $\mathscr{F}$ be the collection of $F_k = X_k \setminus \cup_{j=1}^{k-1} X_j$ which possess the desired property (i)-(iv).

(b) □

16. According to Prof. Folland's errata sheet, we assume μ, ν are σ-finite.

Proof. □

17. (Existence of conditional expectation of f on $\mathscr{N}$.)

***Remark* 1.** I think we need to assume $\nu = \mu \lfloor_{\mathscr{N}}$ is σ-finite on $(X, \mathscr{N})$ to make every hypothesis in Radon-Nikodym Theorem satisfied. A counterexample is $\mu =$ Lebesgue measure on real line and $\mathscr{N} =$ the σ-algebra of countable or co-countable sets.

Proof. Let $(X, \mathscr{M}, \mu)$ be a σ- finite measure space, $\mathscr{N}$ be a sub-algebra of $\mathscr{M}$, $\nu = \mu \lfloor_{\mathscr{N}}$ and $f \in L^1(\mu)$. We define $\lambda(E) := \int_E f\, d\mu$ to be a signed measure on $(X, \mathscr{N})$ (by considering $\int_E f^+\, d\mu, \int_E f^-\, d\mu$) Note that given $A \in \mathscr{N}$ with $\nu(A) = 0$, then $\mu(A) = 0$, and hence $\lambda \ll \nu$.

By Radon-Nikodym Theorem, there exists extended ν-integrable g such that $d\lambda = g\, d\nu$ and any two such functions are equal ν- a.e.. Since one of $\int g^+\, d\nu, \int g^-\, d\nu$ is finite and

$$|\int g\, d\nu| = |\int f\, d\mu| < \infty,$$

we know $\int |g|\,d\nu$ is finite. Hence $g \in L^1(\nu)$.

The uniqueness assertion is easy to proved (cf: Proposition 2.16.) $\square$

Remark 2. The following alternative approach to this problem is taken from Williams [5, chapter 9]. A little difference here is that we work on the σ-finite measure space $(X, \mathscr{M}, \mu)$ instead of finite (probability) measure space.

Remark 3. Williams [5, p.85] mentions the following theorem to explain that conditional expectation (and the martingale theory) is crucial in filtering and control- of space-ships, of industrial process, or whatever.

Theorem 4. *(Conditional expectation as least-squares-best predictor)*
Let $f \in L^1(\mu) \cap L^2(\mu), \mathscr{N}$ is a sub σ-algebra of $\mathscr{M}$ and $\nu = \mu\lfloor_{\mathscr{N}}$. Then the conditional expectation $E(f|\mathscr{N})$ is the minimizer of mean square error to f, $E(f-g)^2 := \int_X (f-g)^2\,d\mu$ in the space of $L^1(\nu) \cap L^2(\nu)$.

We need the following lemma :

Lemma 5. *If f, g is $\mathscr{N}$- measurable and $\int |g|, \int |fg| < \infty$, then*

$$E(fg|\mathscr{N}) = fE(g|\mathscr{N}).$$

Proof. It's easy to see the right-hand side is $\mathscr{N}$- measurable. Given $E \in \mathscr{N}$, if $f = \chi_B$ with $B \in \mathscr{N}$, then

$$\int_E \chi_B E(g|\mathscr{N})\,d\nu = \int_{E\cap B} E(g|\mathscr{N})\,d\nu = \int_{E\cap B} g\,d\nu = \int_E \chi_B g\,d\nu = \int_E \chi_B g\,d\mu.$$

By linearity and monotone convergence theorem, this can be extended to any $f, g \geq 0$, and then the desired result is true by splitting them into the positive and negative parts. $\square$

Proof of Theorem 4. (Taken from Durrett [2, p.229])

For any $h \in L^1(\nu) \cap L^2(\nu), E(hf|\mathscr{N}) = hE(f|\mathscr{N})$, (by Cauchy-Schwarz, $\int fh < \infty$.) Integrate both sides with respect to μ, we see

$$\int_X hE(f|\mathscr{N})\,d\mu = \int_X E(hf|\mathscr{N})\,d\mu = \int_X hf\,d\mu$$

And hence we see

$$\int_X h\Big(E(f|\mathscr{N}) - f\Big)\,d\mu = 0$$

Then given $g \in L^1(\nu) \cap L^2(\nu)$, take $h = E(f|\mathscr{N}) - g \in L^1(\nu)$. Note $h \in L^2(\nu)$ is equivalent to $E(f|\mathscr{N}) \in L^2(\nu)$, and the latter is proved by the Jensen's inequality.

Therefore,

$$\int_X (f-g)^2\,d\mu = \int_X \Big(f - E(f|\mathscr{N}) + E(f|\mathscr{N}) - g\Big)^2\,d\mu = \int_X \Big(f - E(f|\mathscr{N}) + h\Big)^2\,d\mu$$
$$= \int_X \Big(f - E(f|\mathscr{N})\Big)^2\,d\mu + 0 + \int_X h^2\,d\mu$$

which shows the desired result. ☐

Proof of Jensen's inequality. The proof is taken from Chung, [1, p318-319]. For any x and y:

$$\varphi(x) - \varphi(y) \geq \varphi'(y)(x-y)$$

where φ' is the right-hand derivative of φ. Hence

$$\varphi(f) - \varphi(E(f|\mathscr{N}) \geq \varphi'(E(f|\mathscr{N}))(f - E(f|\mathscr{N}))$$

The right member may not be integrable; but let $\Lambda = \{w : |E(f|\mathscr{N})(w)| \leq A\}$ for $A > 0$. Replace f by $f\chi_\Lambda$ in the above, Observe that ???????????????????????????????? ☐

3.3 Complex Measures

18. *Proof.* From Exercise 3, we know that $L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i) = L^1(|\nu_r|) \cap L^1(|\nu_i|)$. By Radon-Nikodym Theorem, for $\mu := |\nu_r| + |\nu_i|$, there are real-valued μ-integrable functions f, g such that $d\nu_r = f d\mu$ and $d\nu_i = g d\mu$. Then we have

$$\nu(E) = \int_E f + ig\,d\mu =: \int_E h\,d\mu.$$

From this equation, we have $d|\nu| = |h| d\mu$. Since $|f|, |g| \leq |h|$, we have

$$|\nu_r|(E) \leq |\nu|(E),\ |\nu_i|(E) \leq |\nu|(E)$$

By Radon-Nikodym Theorem again, there exist real-valued μ-integrable functions ϕ, φ such that $d\nu_r = \phi d|\nu|$ and $d\nu_i = \varphi d|\nu|$. Therefore

$$\int_E h\,d\mu = \nu(E) = \int_E \phi + i\varphi\,d|\nu| = \int_E [\phi + i\varphi]|h|\,d\mu.$$

By the uniqueness part of Radon-Nikodym Theorem, $[\phi + i\varphi]|h| = h$, μ-a.e., and hence $|\nu|$-a.e. Let Z be the set where $h = 0$, then it has $|\nu|$ measure zero since

$$|\nu|(Z) = \int_Z |h|\,d\mu = 0$$

This shows that $|\phi + i\varphi| = 1, |\nu|$-a.e.

Now suppose that $f \in L^1(|\nu|)$. Since $d|\nu_r| = |\phi| d|\nu|$ and $d|\nu_i| = |\varphi| d|\nu|$, we have

$$\int |f|\, d|\nu_r| = \int |f||\phi| d|\nu| \leq \int |f| d|\nu| < \infty$$

and

$$\int |f|\, d|\nu_i| = \int |f||\varphi| d|\nu| \leq \int |f| d|\nu| < \infty.$$

So $f \in L^1(|\nu_r|) \cap L^1(|\nu_i|)$.

Conversely, suppose $f \in L^1(|\nu_r|) \cap L^1(|\nu_i|)$, then we have

$$\int |f| d|\nu| = \int |f||\phi + i\varphi| d|\nu| \leq \int |f|[|\phi| + |\varphi|]\, d|\nu| = \int |f|\, d|\nu_r| + \int |f|\, d|\nu_i| < \infty.$$

So $f \in L^1(|\nu|)$.

Finally, supppose that $f \in L^1(\nu) = L^1(|\nu|)$, we then have

$$\begin{aligned} |\int f\, d\nu| &= |\int f\, d\nu_r + i \int f\, d\nu_i| = |\int f[\phi + i\varphi]\, d|\nu|| \\ &\leq \int |f||\phi + i\varphi|\, d|\nu| = \int |f|\, d|\nu|. \end{aligned}$$

□

19. If ν, μ are complex measures and λ is a positive measure, then $\nu \ll \lambda \Leftrightarrow |\nu| \ll \lambda$ and $\nu \perp \mu \Leftrightarrow |\nu| \perp |\mu|$.

Proof. The first $(\Rightarrow)$ is proved by Radon-Nikodym Theorem. Both $(\Leftarrow)$ are proved by definition and the fact $|\nu|(E) \geq |\nu(E)|$. To prove the second $(\Rightarrow)$, we note there exist positive measure $\rho = |\mu_r| + |\mu_i|$ and $\sigma = |\nu_r| + |\nu_i|$, and some functions $f \in L^1(\rho)$ and $g \in L^1(\sigma)$ such that $d\mu = f d\rho, d|\mu| = |f| d\rho, d\nu = g d\sigma$, and $d|\nu| = |g| d\sigma$. Since $\nu \perp \mu$, for each pair $a, b \in \{r, i\}$ there exists disjoint measurable sets A_{ab}, B_{ab} such that $X = A_{ab} \cup B_{ab}$, A_{ab} is ν_a-null and B_{ab} is μ_a-null. It follows that $A := (A_{rr} \cup A_{ri}) \cap (A_{ir} \cup A_{ii})$ is both ν_r-null and ν_i-null. In particular, for each $n \in \mathbb{N}$ the subsets $\{x \in A : \mathrm{Re}(f(x)) > n^{-1}\}$ and $\{x \in A : \mathrm{Re}(f(x)) < n^{-1}\}$ of A has ν_r and ν_i- measure zero, which implies $|f| = 0$ on A and hence A is $|\nu|$-null. Moreover

$$B := A^c = (B_{rr} \cap B_{ri}) \cup (B_{ir} \cap B_{ii})$$

is both μ_r-null and μ_i-null, and a similar argument implies B is $|\mu|$-null. □

20. *Proof.* Given $E \in \mathscr{M}$, since $|\nu|(E) + |\nu|(X - E) = |\nu|(X) = \nu(X) = \nu(X - E) + \nu(E)$, we have

$$|\nu|(E) - \nu(E) = \nu(X - E) - |\nu|(X - E).$$

Since the left-hand side have nonnegative real part and the right-hand have nonpositive one, $\mathrm{Re}(\nu(E)) = |\nu|(E) \geq |\nu(E)| = \sqrt{\mathrm{Re}(\nu(E))^2 + \mathrm{Im}(\nu(E))^2}$. So $\mathrm{Im}(\nu(E)) = 0$ and therefore $\nu(E) = \mathrm{Re}(\nu(E)) = |\nu|(E)$. Since E is arbitrary, $\nu = |\nu|$. $\square$

21. *Proof.* We are going to show $\mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_1$ and then $\mu_3 = |\nu|$. The first inequality is trivial. For the second one, since there exists $w_j \in \mathbb{C}, |w_j| = 1$ such that $|\nu(E_j)| = w_j \nu(E_j)$. Consider the function $f = \sum_j w_j \chi_{E_j}$. Since $\{E_j\}$ is mutually disjoint, $|f| \leq 1$. Take $f_n = \sum_1^n w_j \chi_{E_j}$. We then have, by dominate convergence theorem,

$$|\int f_n - \int f \, d\nu| \leq \int |f_n - f| \, d|\nu| \to 0,$$

We then have

$$\int_E f \, d\nu = \lim_{n\to\infty} \int_E f_n \, d\nu = \lim_{n\to\infty} \sum_1^n w_j \nu(E_j) = \sum_1^\infty |\nu(E_j)|.$$

Then $\sum_1^\infty |\nu(E_j)| = |\int_E f \, d\nu| \leq \mu_3(E)$. Since $\{E_j\}$ is arbitrary, $\mu_2 \leq \mu_3$. For the third inequality, given $\epsilon > 0$, then we can find some f with $|f| \leq 1$ such that

$$\mu_3(E) < |\int_E f \, d\nu| + \epsilon$$

We approximate f by a simple function as follows. Let $D \subset \mathbb{C}$ be the closed unit disc. The compactness of D implies that there are finite many $z_j \in D$ such that $B_\epsilon(z_j)$ covers D. Define $B_j = f^{-1}(B_\epsilon(z_j)) \subset X$, which is measurable since f is. The union of B_j is X. Let

$$A_1 = B_1, \; A_j = B_j \setminus \cup_{i=1}^{j-1} B_i$$

be the disjoint sets with $A_j \subset B_j$ and $\cup A_j = X$. Define the simple function $\phi = \sum_1^m z_j \chi_{A_j}$, then $|\phi| \leq 1$ and $|f(x) - \phi(x)| < \epsilon$ for all x by the construction. Then

$$|\int_E f \, d\nu| - |\int_E \phi \, d\nu| \leq |\int_E f - \phi \, d\nu| < |\nu|(E).$$

$$\mu_3(E) < |\int_E f \, d\nu| + \epsilon < |\int_E \phi \, d\nu| + \epsilon + \epsilon|\nu|(E)$$

Now we define $F_j = A_j \cap E$, then F_j is a finite partition of E and

$$|\int_E \phi \, d\nu| = |\int \sum_j z_j \chi_{A_j \cap E} \, d\nu| = |\sum_j z_j \nu(F_j)| \leq \sum_j |\nu(F_j)| \leq \mu_1(E).$$

Since ϵ is arbitrary, we have $\mu_3(E) \leq \mu_1(E)$.

We have already show $\mu_3 \leq |\nu|$ in the above argument. To get the reverse inequality, let $g = d\nu/d|\nu|$. We know $|g| = 1$ $|\nu|$-a.e., and hence

$$\mu_3(E) \geq |\int_E \bar{g} \, d\nu| = |\int_E \bar{g} g \, d|\nu|| = |\nu(E)|.$$

$\square$

3.4 Differentiation on Euclidean Space

22. *Proof.* Since $M := \int |f| > 0$, there exists $R > 0$ such that $\int_{B_R(0)} |f| > M/2$. For $|x| > R$, the ball $B_{2|x|}(x) \supset B_R(0)$ and hence

$$Hf(x) \geq \frac{1}{2^n|x|^n} \int_{B_{2|x|}(x)} |f| \geq \frac{1}{2^n|x|^n} \int_{B_R(0)} |f| > \frac{M}{2^{n+1}|x|^n}.$$

For every small $\alpha > 0$, there is an inclusion

$$\emptyset \neq \{x : R \leq |x| < (c/\alpha)^{1/n}\} = \{x : |x| \geq R, \text{ and } \frac{C}{|x|^n} > \alpha\} \subset \{x : Hf(x) > \alpha\}$$

Thus, $m(\{x : Hf(x) > \alpha\}) \geq m(\{x : R \leq |x| < (C/\alpha)^{1/n}\}) = w_n(C - R^n\alpha)/\alpha > w_nC/2\alpha$, provided $C - R^n\alpha > C/2$, that is, $\alpha < C/2R^n$. $\square$

Remark 6. maximal inequality

23. *Proof.* $Hf \leq H^*f$ is trivial. $H^*f \leq 2^nHf$ is proved by the fact $x \in B_r(y) \subset B_{2r}(x)$. $\square$

24. Obvious.

25. *Proof.* (a) Apply Lebesgue Differentiation Theorem to χ_E and χ_{E^c}

(b)The first example can be found by considering E as sector of angle $2\pi\alpha$ and x is the origin, the second example is $E = \cup_1^\infty [2^{-n}, 2^{-n} + 2^{-n-1}]$, $x = 0$.

Fixed $N \in \mathbb{N}$, we compute $\frac{m(B_{2^{-N}}(0))\cap E)}{m(B_{2^{-N}}(0))} = 1/4$ and $\frac{m(B_{2^{-N}+2^{-N-1}}(0)\cap E)}{m(B_{2^{-N}+2^{-N-1}}(0))} = 1/3$. So the limit does NOT exist. $\square$

26. *Proof.* Given compact set $K, \nu(K), \lambda(K) < (\nu + \lambda)(K) < \infty$. Let $X = A \amalg B$ be the singular decomposition of λ and ν. Given Borel set $E \subset \mathbb{R}^n$, there exists open sets $U_n \supset E$ such that $(\lambda + \nu)(U_n) - (\lambda + \nu)(E) \to 0$. Note that $(\lambda + \nu)(U_n) = \lambda(U_n \cap A) + \nu(U_n \cap B)$ and $(\lambda+\nu)(E) = \lambda(E\cap A)+\nu(E\cap B)$. Since $\nu(U_n\cap B)-\nu(E\cap B) \geq 0$ and $\lambda(U_n\cap A)-\lambda(E\cap A) \geq 0$, $\lambda(U_n\cap A)+\nu(U_n\cap B)-\lambda(E\cap A)-\nu(E\cap B) \geq \lambda(U_n\cap A)-\lambda(E\cap A) = \lambda(U_n)-\lambda(E) \geq 0$. Since the left-hand side tends to 0 as $n \to \infty$, so does $\lambda(U_n) - \lambda(E)$. Similarly, $\nu(U_n) \to \nu(E)$. $\square$

3.5 Functions of Bounded Variation

27. *Proof.* $\square$

28. *Proof.* $\square$

29. *Proof.* $\square$

30. *Proof.* Let $\{r_n\}$ be the set of all rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto \sum_{\{j:r_j<x\}} \frac{1}{2^j}$. It's easy to see it's increasing and discontinuous at any r_n since $f(z) - f(r_n) \geq \frac{1}{2^n}$ for all $z > r_n$. Given $x \in \mathbb{Q}^c$ and $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\sum_N^\infty \frac{1}{2^j} < \epsilon$.

    Since x is irrational, let $\delta = \frac{1}{2}\min\{|x - r_j| : j = 1, \cdots N-1\} > 0$. Then for any $y \in B_\delta(x)$,

$$|f(x) - f(y)| \leq \sum_N^\infty \frac{1}{2^j} < \epsilon.$$

    Therefore, f is continuous at any irrational number. $\square$

31. We omit (a) since it is standard. (b) is included in the following problem taken from Stein-Shakarchi [4, Exercise 2.11].

    If $a, b > 0$, let

    Proof. $\square$

32. *Proof.* $\square$

33. *Proof.* By Theorem 3.23, we know $0 \leq F'$ exists a.e. We may instead F by the function, still call it F, which equal to $F(x)$ if $x < b$ and equal to F(b) if $x \geq b$.

    Consider $f_k(x) = \{F(x+h) - F(x)\}/h$ where $h = 1/k$, then $f_k \to f$ a.e. and Fatou's lemma implies

$$\begin{aligned}\int_a^b F'(x)\,dx &\leq \liminf_{k\to\infty} \int_a^b f_k(x)\,dx = \liminf_{h\to 0^+} \int_a^b \frac{F(x+h) - F(x)}{h}\,dx \\ &= \liminf_{h\to 0^+} \left(\frac{1}{h}\int_b^{b+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx\right) \leq F(b) - F(a).\end{aligned}$$

    In fact, we have proved that $\int_a^b F'(x)\,dx \leq F(b-) - F(a+)$. $\square$

34. *Proof.* $\square$

35. *Proof.* $\square$

36. *Proof.* $\square$

37. *Proof.* $\square$

38. *Proof.* $\square$

39. *Proof.* $\square$

40. *Proof.* □

41. *Proof.* □

42. *Proof.* □

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