Real and Complex Analysis, 3rd Edition, W.Rudin
Chapter 4 Elementary Hilbert Space Theory*

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1. *Proof.* It's easy to see $M \subset (M^\perp)^\perp$ and the latter is a closed subspace of H. Given F be a closed subspace containing M, then $F^\perp \subset M^\perp$ and hence $(M^\perp)^\perp \subset (F^\perp)^\perp$. Since F is closed, $H = F \bigoplus F^\perp$. Given $z \in (F^\perp)^\perp, z = x + y$ with $x \in F$ and $y \in F^\perp$. Since $0 = (z, y) = (y, y), y = 0$. Hence $(M^\perp)^\perp \subset (F^\perp)^\perp \subset F$. So $(M^\perp)^\perp$ is the smallest closed subspace containing M, that is, $\overline{M}$! (even for non-closed subspace) □

2. Gram-Schmidt Process.

3. *Proof.* The case $1 \leq p < \infty$ is separable since $C(T)$ is dense in $L^p(T)$ and by Weierstrass' theorem the set of trigonometric polynomials with rational coefficients is dense in $(C(T), \|\cdot\|_\infty)$ and hence in $(C(T), \|\cdot\|_p)$.

   Next, consider $\mathscr{F} = \{f_h(x) = \chi_{(0,\frac{1}{2})}(x + h), h \in (0, \frac{1}{4})\}$, then given $f \neq g \in \mathscr{F}, \|f - g\|_\infty = 1$. This shows the dense subset of $L^\infty([0, 1), m) \cong L^\infty(T)$ must be uncountable. □

4. *Proof.* ($\Rightarrow$) There exists a countable dense set $\{w_n\}$ in H. We may choose a subset $\{w_{n_j}\}$ that are linearly independent by kick out the linear dependent element inductively and assume they are orthonormal by Gram-Schmidt process. Note that span$\{w_{n_j}\}$ will be dense in H since its span contains $\{w_n\}$. By Theorem 4.18, we find a countable maximal orthonormal system span$_{\mathbb{Q}}\{w_{n_j}\}$. Note this proof does NOT rely on Hausdorff maximality principle.

   ($\Leftarrow$) Let $\{u_n\}$ be tha maximal orthonormal system. By theorem 4.18, span$\{u_n\}$ is dense in H. So does span$_{\mathbb{Q}}\{u_n\}$ which is countable. □

5. *Proof.* We assume $M \lneq H$, that is, $L \neq 0$. Then there is a unique $0 \neq y \in H$ such that $Lx = (x, y)$ for all $x \in H$. Then $M = \{x : (x, y) = 0\} = Y^\perp$, where Y is the span of $\{y\}$ which is a closed one-dimensional subspace. So $M^\perp = (Y^\perp)^\perp = Y$ by Exercise 1. □

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6. *Proof.* (a) $\{u_n\}$ is norm-bounded by 1 and closed (since every pair of distinct elements has distance $\sqrt{2}$), but the open cover $\{B(u_n, 1)\}$ has no finite subcover.

(b) ($\Leftarrow, \sum \delta_n^2 < \infty$) By Bozano-Weierstrass theorem, it's enough to show S is sequentially compact.

Since for each sequence c_n with $|c_n| \leq \delta_n$, the partial sums of $\sum c_n u_n$ form a Cauchy sequence in H, by completeness of H, $\sum c_n u_n$ exists. Moreover its norm is $\sum_n |c_n|^2$.

Given $\{x_j = \sum c_n^j u_n\} \subseteq S$. Since for each n, $|c_n^j| \leq \delta_n$ for all j, by diagonal process, there is a subsequence, still denoted by x_j, such that $c_n^j \to c_n$ for each n. Define $x = \sum_n c_n u_n$, clearly it belongs to S. Since $\sum \delta_n^2 < \infty$ and

$$\|x_j - x\|^2 = \sum_n |c_n^j - c_n|^2 \leq \sum_{n=1}^{M} |c_n^j - c_n|^2 + \sum_{n=M+1}^{\infty} 4\delta_n^2.$$

We may choose M large enough so that the second term is small uniformly in j and then pick large J such that for all $j > J$ the first term is always small.

Therefore, there is a convergent subsequence of any sequence in S.

(c) ($\Rightarrow$) Let $x_j = \sum_{n=1}^{j} \delta_n u_n \in S$, then $\|x_j\|^2 = \sum_{n=1}^{j} \delta_n^2$. Since S is compact, there is $M > 0$ such that $M \geq \|x_j\|^2 = \sum_{n=1}^{j} \delta_n^2$ for all j, that is, $M \geq \sum_{n=1}^{\infty} \delta_n^2$.

(d) Suppose H is locally compact. Since we know the set of all balls forms the local base of every Banach space, there is $\overline{B_r(0)} \subset K$ which is a compact neighborhood of 0. So $\overline{B_r(0)}$ is compact, which is a contradiction by applying the argument of (a) to rescaled $\{ru_n\}$. $\square$

Remark 0.1. For (d), there is a general result states that every locally compact topological vector space is finite dimensional. See Rudin [5, Theorem 1.22].

7. This is a special case of Exercise 6.4. See my remarks made there.

Proof. If $(a_n) \notin l^2$, then there are $1 < n_1 < n_2 < \cdots$ such that $\sum_1^{n_k} a_j^2 > k$. Then for each k

$$c_k^2 := \sum_{n_k+1}^{n_{k+1}} a_j^2 > 1$$

Define $b_n = \frac{a_n}{kc_k}$ if $n_k < n \leq n_{k+1}$ and 0 if $1 \leq n \leq n_1$. Then

$$\sum b_n^2 = \sum_k \sum_{n_k+1}^{n_{k+1}} \frac{a_n^2}{k^2 c_k^2} = \sum_k \frac{1}{k^2} < \infty.$$

$$\sum a_n b_n = \sum_k \sum_{n_k+1}^{n_{k+1}} \frac{a_n^2}{kc_k} \geq \sum_k \frac{1}{k} = \infty.$$

$\square$

8. Also see Folland [3, Exercise 5.65].

   Proof. ☐

9. *Proof.* Apply Bessel's ineqality with the orthonormal set $\{\sin nx, \cos nx\}_{n=0}^{\infty}$ to $\chi_A \in L^2[0, 2\pi]$. ☐

10. *Proof.* By Hint, $E = E^+ \cup E^-$ where $\sin n_k x \to \pm\frac{1}{\sqrt{2}}$ on $E^\pm$ respectively. Then LDCT and Exercise 9 implies that $0 = \lim_{k\to\infty} \int_{E^\pm} \sin n_k x = \int_{E^\pm} \pm\frac{1}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}} m(E^\pm)$. ☐

11. This exercise shows that the convexity assumption can NOT be dropped from Theorem 4.10 without other restrictions. The desired $E = \{\frac{n+1}{n} e_n\}$ where $e_n(i) = \delta_n^i$, the Kronecker delta. On the other hand, the existence and uniqueness assertions in Theorem 4.10 are not true for every Banach spaces, see Exercise 5.4-5.

12. *1st proof.* Using Stirling Formula, we see

$$\begin{aligned} 1 &= \frac{c_k}{\pi}\int_0^\pi \left(\frac{1+\cos t}{2}\right)^k dt = \frac{2c_k}{\pi}\int_0^{\pi/2} (\cos^2 w)^k\, dw = \frac{2c_k}{\pi}\int_0^1 (1-z^2)^{k-\frac{1}{2}}\, dz \\ &= \frac{c_k}{\pi}\int_0^1 (1-t)^{k-\frac{1}{2}} t^{-\frac{1}{2}}\, dt = \frac{c_k}{\pi}\frac{\Gamma(k+\frac{1}{2})\sqrt{\pi}}{\Gamma(k+1)} \sim \sqrt{\frac{e}{\pi}}\frac{c_k}{\sqrt{k}}\left(1+\frac{-\frac{1}{2}}{k}\right)^k. \end{aligned}$$

☐

2nd proof. $\limsup_{k\to\infty} \frac{c_k}{\sqrt{k}} \le \frac{3\pi}{4}$ since

$$\begin{aligned} 1 &= \frac{c_k}{\pi}\int_0^\pi \left(\frac{1+\cos t}{2}\right)^k dt = \frac{2c_k}{\pi}\int_0^{\pi/2} (\cos^2 w)^k\, dw = \frac{2c_k}{\pi}\int_0^1 (1-z^2)^{k-\frac{1}{2}}\, dz \\ &\ge \frac{2c_k}{\pi}\int_0^{1/\sqrt{k}} 1-(k-\frac{1}{2})z^2\, dz = \frac{2c_k}{\pi}\left(\frac{2}{3\sqrt{k}}+\frac{6}{\sqrt{k}^3}\right) > \frac{c_k}{\sqrt{k}}\frac{4}{3\pi}. \end{aligned}$$

I DO NOT have any good idea to get the lower bound with a simpler method than Stirling. ☐

13. This exercise also appears in Baby Rudin [4, Exercise 8.19]. Note this is related to the notion of equidistributed sequence and ergodic theorem. See Stein-Shakarchi [6, Section 4.2].

    Proof. By direct computation, this is valid for all $f = e^{2\pi n x}$ $(n = 0, 1, \cdots)$ and hence for all trigonometric polynomials. We pass the validity to all continuous functions by Weierstrass approximation theorem. ☐

Remark 0.2. This is valid for all step function $\chi_(a, b)$ by approximate it from above and below by continuous function. And hence being valid for simple functions of Riemann upper and lower sum of Riemann integrable function. Finally, it's valid for all Riemann integrable function. However, this theorem is not true for Lebesgue integrable function $f = \chi_{\{(n\alpha)\}}$, where $(n\alpha)$ is the fractional part of $n\alpha$, since the right-hand integral is zero but the left-hand sum is always 1.

14. *Proof.* Due to the special structure of minimal problem, it's enough to consider the real a, b, c and real-valued function g. Consider Hilbert space $L^2([-1, 1], dx)$ with inner product $(f, h) = \frac{1}{2}\int_{-1}^{1} f(x)h(x)\, dx$. Let A be the span of $\{1, x, x^2\}$ which is the same as span of the orthonormal set $\{1, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\}$. The corresponding Fourier series of x^3 is $\frac{3}{5}x$.

    Thus, $(x^3 - \frac{3}{5}x) \perp A$,

    $$\inf_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\, dx = \int_{-1}^{1} |x^3 - \frac{3}{5}x|^2\, dx = \frac{8}{175},$$

    that is, minima exists and equal to $\frac{8}{175}$. Moreover,

    $$\sup_{g \perp A, \|g\|=1} \int_{-1}^{1} x^3 g(x)\, dx = \sup_{g \perp A, \|g\|=1} (g, x^3)_{L^2} = \sup_{g \perp A, \|g\|=1} (g, x^3 - \frac{3}{5}x)_{L^2} \leq \sqrt{\frac{8}{175}},$$

    by Cauchy-Schwarz. Note that the supreme is attained by $g(x) = \frac{1}{\sqrt{\frac{8}{175}}}(x^3 - \frac{3}{5}x)$. □

15. *Proof.* Due to the special structure of minimal problem, it's enough to consider the real a, b, c. Consider Hilbert space $L^2([0, \infty], e^{-x/2}dx)$ with inner product $(f, h) = \int_0^\infty f(x)h(x)e^{-x}\, dx$. Let A be the span of $\{1, x, x^2\}$ which equals to span of the orthonormal set $\{1, x-1, \frac{1}{2}(x^2-4x+2)\}$. The corresponding Fourier series of x^3 is $9x^2 - 18x + 6$.

    Thus, $x^3 - (9x^2 - 18x + 6) \perp A$,

    $$\inf_{a,b,c} \int_0^\infty |x^3 - a - bx - cx^2|^2 e^{-x}\, dx = \int_0^\infty |x^3 - (9x^2 - 18x + 6)|^2 e^{-x}\, dx = 36,$$

    that is, minima exists and equal to 36.

    Next, we try to maximize

    $$\int_0^\infty x^3 g(x) e^{-x}\, dx$$

    where g is subject to the restrictions

    $$\int_0^\infty |g(x)|^2 e^{-x} = 1, \int_0^\infty g(x)e^{-x} = \int_0^\infty x g(x) e^{-x} = \int_0^\infty x^2 g(x) e^{-x} = 0.$$

    That is, $g \perp A$, $\|g\| = 1$. Note that it's enough to consider real-valued g and

    $$\sup_{g \perp A, \|g\|=1} \int_0^\infty x^3 g(x) e^{-x}\, dx = \sup_{g \perp A, \|g\|=1} (g, x^3)_{L^2} = \sup_{g \perp A, \|g\|=1} (g, x^3 - (9x^2 - 18x + 6))_{L^2} \leq 6,$$

    by Cauchy-Schwarz. Moreover, the supreme is attained by $g(x) = \frac{1}{6}[x^3 - (9x^2 - 18x + 6)]$. □

16. This is a general result for the computations of Exercise 14-15.

Proof. □

17. *Proof.* □

18. Uniform limit on $\mathbb{R}$ of members of X are called "almost periodic". See Stein [7, Problem 4.2], Besicovitch[1] and Bohr [2].

Proof. □

19. *Proof.* □

References

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