Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 8 The Calculus of Variations*

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Notation: U denotes a bounded smooth, open subset of $\mathbb{R}^n$. All given functions are assumed smooth, unless otherwise stated.

1. This is called Rademacher's functions. Consult Brezis [1, Exercise 4.18].

   (a) This is Riemann-Lebesgue lemma, which holds not only for L^2 but also for $L^p, (p \in [1, \infty))$, and weak-* convergence in L^∞; I put my comment on various proofs below:

   (1) I think the easy way to understand is through Bessel's inequality. But this method does not work on L^p, except L^2.

   (2) The second way is through integration by parts, and one needs the density theorem (simple functions or C_c^∞ functions). This method is adapted to our claim, except for weak convergence in L^1. But for L^1 case, it's a simple consequence of squeeze theorem in freshman's Calculus. It's also easy to see $f_n \not\to 0$ a.e. or in measure.

   Proof. (b)You can apply (2)'s method, start with step functions χ_B. □

2. $$L(p, z, x) = e^{-\phi(x)}\Big(\frac{1}{2}|p|^2 - f(x)z\Big).$$

   A good intuition is explained in http://math.stackexchange.com/questions/270110/.

3. The equation can be rewrite as $0 = \frac{-1}{\epsilon}u_t + \frac{1}{\epsilon}\Delta_x u + u_{tt}$ which is equivalent to

   $$0 = (e^{-t/\epsilon}u_t)_t + \frac{1}{\epsilon}\Delta_x(e^{-t/\epsilon}u).$$

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*Completed: Ex 1,2,3,7,9,10,15.

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So this motivates us to define

$$L = \frac{1}{2} e^{-t/\epsilon} \Big(u_t^2 + \frac{1}{\epsilon} |D_x u|^2 \Big),$$

and then we calculate the first variation to conclude that, for each $v \in C_c^\infty(U \times (0, T))$

$$0 = \int_0^T \int_U v \cdot e^{-t/\epsilon} (u_t - \Delta_x u - \epsilon u_{tt}).$$

4. **Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is C^1.**

   (a) Show $L(p, z, x) = \eta(z)$ det $P(P \in M^{n \times n}, z \in \mathbb{R}^n)$ is a null Lagrangian.

   (b) Deduce that if $u : \mathbb{R}^n \to \mathbb{R}^n$ is C^2, then

$$\int_U \eta(\mathbf{u}) \mathbf{det} D\mathbf{u} \, dx$$

   depends only on $\mathbf{u}|_{\partial U}$.

Proof. (a) We use the results and notations in p.462-463 without mentions: for $k = 1, \cdots n$,

$$\begin{aligned}
& -\sum_{j=1}^n (L_{P_j^l}(P, Z, x))_{x_j} + L_{Z^l}(P, Z, x) \\
&= -\sum_{j=1}^n (\eta (\mathrm{cof} P)_j^k)_{x_j} + \frac{\partial \eta}{\partial Z^k} \det P \\
&= -\sum_{j=1}^n \Big[\sum_{l=1}^n \frac{\partial \eta}{\partial Z^l} \frac{\partial Z^l}{\partial x_j} (\mathrm{cof} P)_j^k + \eta (\mathrm{cof} P)_{j, x_j}^k \Big] + \frac{\partial \eta}{\partial Z^k} \det P.
\end{aligned}$$

Given $u \in C^\infty(U \subseteq \mathbb{R}^n; \mathbb{R}^n)$, set $Z = u, \ P = Du$, then the Euler-Lagrange equation becomes

$$\begin{aligned}
& -\sum_{j=1}^n \sum_{l=1}^n \frac{\partial \eta}{\partial u^l} \frac{\partial u^l}{\partial x_j} (\mathrm{cof} Du)_j^k - \eta \sum_{j=1}^n (\mathrm{cof} Du)_{j, x_j}^k + \frac{\partial \eta}{\partial u^k} \det Du \\
&= -\sum_{l=1}^n \frac{\partial \eta}{\partial u^l} \sum_{j=1}^n \frac{\partial u^l}{\partial x_j} (\mathrm{cof} Du)_j^k + 0 + \frac{\partial \eta}{\partial u^k} \det P \\
&= -\sum_{l=1}^n \frac{\partial \eta}{\partial u^l} \delta_k^l \det P + \frac{\partial \eta}{\partial u^k} \det P = 0,
\end{aligned}$$

where δ_k^l is the Kronecker delta.

(b) follows from the alternative characterization of null Lagrangians, that is, Theorem 1 in p.461, and the fact that the above computations are suitable to C^2 functions. □

5. **(Continuation) Fix $x_0 \notin \mathbf{u}(\partial U)$, and choose a function η as above so that $\int_{\mathbb{R}^n} \eta dz = 1$, spt$\eta \subset B(x_0, r), r$ taken so small that $B(x_0, r) \cap u(\partial U) = \emptyset$. Define**

$$\mathbf{deg}(\mathbf{u}, x_0) = \int_U \eta(\mathbf{u}) \mathbf{det} D\mathbf{u} \, dx,$$

   the degree of u relative to x_0. Prove the degree is an integer.

Proof. This is not an easy exercise (at least for me). I would do the following: (1) the integral does not depend on η as long as η satisfies the constraints of the problem. (Try to differentiate the integral with $t\eta_1 + (1-t)\eta_2$ with respect to t, and see that the derivative is 0). (2) by Sard's lemma, we can make sure that the support of η does not contain any critical values of u and fits within a neighborhood which u covers nicely (as a covering map) (3) change the variables to get a finite sum of integrals of the kind $\pm \int \eta(x) dx$. - user53153 Dec 25 '12 at 6:31 □

6. *Proof.* □

7. *Proof.* Expand L in coordinate form directly, we have

$$L(P) = \sum_{i=1}^{n}\sum_{k=1}^{n} P_k^i P_i^k - P_i^i P_k^k$$

Then we see for $l = 1, \cdots n$,

$$\sum_{j=1}^{n} (L_{P_j^l}(P))_{x_j} = 2\sum_{j=1}^{n} (P_l^j)_{x_j} - 2\sum_{i=1}^{n} (P_i^i)_{x_l}.$$

For any $u \in C^\infty(U \subseteq \mathbb{R}^n; \mathbb{R}^n)$, we plug $P_l^j = (u^j)_{x_l}$ into the above identity, and see

$$\sum_{j=1}^{n} (L_{P_j^l}(Du))_{x_j} = 2\sum_{j=1}^{n} ((u^j)_{x_l})_{x_j} - 2\sum_{i=1}^{n} ((u^i)_{x_i})_{x_l} = 0.$$

□

Remark 1. Does this problem have any useful application?

Remark 2. See Giaquinta-Hildebrandt [3, Chapter 1] for more discussions on Null Lagrangians.

8. **Explain why the methods in Section 8.2 will not work to prove the existence of a minimizer of the functional**

$$I[w] := \int_U (1 + |Dw|^2)^{\frac{1}{2}}\, dx$$

over $\mathcal{A} := \{W^{1,q}(U) | w = g \text{ on } \partial U\}$**, for any** $1 \le q < \infty$**.**

Remark 3. This is the minimal surface problem, many great textbooks treat it, e.g. Murray, Giusti, or Colding-Minicozzi.

Proof. The best coercive estimate one can get is $|I(w)| \ge \|Dw\|_{L^1}$ since $\lim_{s\to\infty} \frac{(1+|s|^2)^{\frac{1}{2}}}{s} = 1$. However, L^1 (or $W^{1,1}$) are not reflexive spaces. So the boundedness of minimizing sequence in L^1 does not imply existence of a weakly convergent sequence, e.g. approximation of the identity. If $\phi_k(x) = k\chi_{\{(0,\frac{1}{k})\}}(x)$ is assumed to be converge weakly in L^1 up to a subsequence ϕ_{k_j} and $\frac{k_{j+1}}{k_j} \to \infty$ as $j \to \infty$. Then for the bounded test function $g(x) = (-1)^j$ if $x \in (\frac{1}{k_{j+1}}, \frac{1}{k_j})$, $\int \phi_{k_j} g \to -1$ as odd $j \to \infty$ and $\int \phi_{k_j} g \to 1$ as even $j \to \infty$. □

9. *Proof.* (a) Let $\mathbf{u} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be the minimizer of $I[\mathbf{w}] := \int_U L(D\mathbf{w}, \mathbf{w}, x)\, dx$, then for each $\mathbf{v} \in C_c^\infty(U; \mathbb{R}^m)$

$$0 \leq \frac{d^2}{dt^2} I[\mathbf{u} + t\mathbf{v}] = \int_U \left\{ L_{p_i^k p_j^l} D_i v^k D_j v^l + 2L_{p_i^k u^l} v^l D_i v^k + L_{z^k z^l} v^k v^l \right\}$$

The above identity is true for all Lipschitz continuous $\mathbb{R}^m$-valued function v with compact support. In particular, we take

$$v(x) = \epsilon \rho(\frac{x \cdot \xi}{\epsilon}) \eta \zeta(x),$$

where $\xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m, \zeta \in C_c^\infty(U; \mathbb{R})$ and $\rho : \mathbb{R} \to \mathbb{R}$ is the same tent map in Section 8.1.3. Thus, $|\rho'| = 1$ a.e. and

$$D_i v^k = \rho'(\frac{x \cdot \xi}{\epsilon}) \xi_i \eta^k \zeta + O(\epsilon), \text{ as } \epsilon \to 0.$$

After substituting this into the first expression and sending $\epsilon \to 0$, we obtain

$$0 \leq \int_U L_{p_i^k p_j^l} \xi_i \eta^k \xi_j \eta^l \zeta^2 \, dx.$$

Since this holds for all $\zeta \in C_c^\infty(U)$, we deduce that $0 \leq L_{p_i^k p_j^l} \xi_i \eta^k \xi_j \eta^l$.

(b) Consider $L(P) = \det P = p_1^1 p_2^2 - p_2^1 p_1^2$ on $M^{2\times 2}$. This function is not convex since for $t \in (0, 1)$,

$$L\Big(t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (1-t) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\Big) = \det \begin{pmatrix} 1-t & 0 \\ 1-t & t \end{pmatrix} > 0. \tag{1}$$

But

$$tL \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (1-t) L \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0. \tag{2}$$

Direct computation shows that L satisfies the Legendre-Hadamard condition, that is, for all $P \in M^{2\times 2}, \xi \in \mathbb{R}^2, \eta \in \mathbb{R}^2$,

$$\sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \frac{\partial^2 L(P)}{\partial p_i^k \partial p_j^l} \eta_k \eta_l \xi_i \xi_j = 2\eta_1\eta_2\xi_1\xi_2 - 2\eta_1\eta_2\xi_2\xi_1 = 0.$$

□

Remark 4. Dacorogna [2, Theorem 5.3] says that for C^2 function $L : \mathbb{R}^{mn} \to \mathbb{R} \cup \{\infty\}$, the Legendre-Hadamard condition $\Longleftrightarrow$ rank one convexity, that is,

$$L(\lambda\xi + (1-\lambda)\eta) \leq \lambda L(\xi) + (1-\lambda) L(\eta),$$

for every $\lambda \in [0, 1]$ and $\text{rank}(\xi - \eta) \leq 1$.

Proof. ($\Rightarrow$) Mean-Value theorem and note that $m \times n$ matrix $\mathbf{A}$ is of rank 1 iff $\mathbf{A} = \mathbf{v}\mathbf{w}^T$ for some $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$.

($\Leftarrow$) Given$\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$, consider $\phi(t) = f(\mathbf{P} + t \cdot \xi\eta^T)$ which is C^2 and convex, and hence $\phi''(0) \geq 0$, which is exactly the Legendre-Hadamard condition. □

10. **Use the methods of §8.4.1 to show the existence of a nontrivial weak solution $u \in H_0^1(U), u \not\equiv 0$, of $-\Delta u = |u|^{q-1}u$ in U and $u = 0$ on $\partial\Omega$ for $1 < q < \frac{n+2}{n-2}, n \geq 3$.**

Our method is not a direct application of theorems in Section 8.4.1. Since the corresponding g does not satisfy the growth condition.

Proof. First, we are going to show the existence of minimizer for the energy functional E on the admissible class $\mathscr{A}$ where

$$E(u) = \frac{1}{2}\int_U |\nabla u|^2 \, dx - \frac{1}{q+1}\int_U |u|^{q+1}$$

and

$$\mathscr{A} = \{u \in H_0^1(U) : \|u\|_{L^{q+1}} = 1\}.$$

Then E is coercive on $\mathscr{A}$ since $E \lfloor_{\mathscr{A}} = \frac{1}{2}\int_U |\nabla u|^2 \, dx - \frac{1}{q+1}$. Let u_m be the minimizing sequence, which is bounded in $H_0^1(\Omega)$ by coercivity. Hence weak compactness theorem and Rellich's compactness theorem imply that, up to a subsequence, $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_m \to u$ in $L^{q+1}(\Omega)$. Hence $\|u\|_{L^{q+1}} = 1$, that is, $u \in \mathscr{A}$. Therefore u is a minimizer since

$$\liminf_{m\to\infty} E(u_m) \geq E(u) \geq \inf_{\mathscr{A}} E = \lim_{m\to\infty} E(u_m)$$

Finally, a almost identically argument to Lagrange mulitiplier theorem to show this minimizer solves $-\Delta v = \lambda|v|^{q-1}v$ for some $\lambda \in \mathbb{R}$ with zero Dirichlet boundary condition weakly. If $\lambda = 0$, by uniqueness theorem for laplace equation, we see $u \equiv 0$. But this contradicts to $\|u\|_{L^{q+1}} = 1$. Hence we see a nontrivial function $\tilde{u} = \lambda^{\frac{1}{q-1}}u$ solves the given boundary value problem weakly. We remark two details in the beginning of the proof of Lagrange multiplier theorem:
(1.)Since $\|u\|_{L^{q+1}} = 1 \neq 0, |u|^{q-1}u$ is not identical zero a.e.
(2.)For any $v \in H_0^1(U)$, the integral $\int_U |u|^{q-1}uv \, dx$ is well-defined by Sobolev embedding theorem, the assumption that U is a bounded domain, and Hölder's inequality to exponent pair $(\frac{q+1}{q}, q+1)$. □

11. If $u \in C^\infty(\overline{U})$ and Multiply the equation with $v \in C^\infty(\overline{U})$, then we see

$$(f, v)_{L^2} = \int_U \nabla u \nabla v \, dx - \int_{\partial U} v\frac{\partial u}{\partial \nu} \, ds = \int_U \nabla u \nabla v \, dx + \int_{\partial U} v\beta(u) \, ds$$

For $u \in H^1(U)$, since for any $z, w \in \mathbb{R}$, $|\beta(z) - \beta(w)| \leq b|z - w|$, trace theorem and standard approximation argument implies that we can define $u \in H^1(U)$ is a weak solution to our nonlinear boundary-value problem provided for each $v \in H^1(U)$,

$$(f, v)_{L^2} = \int_U \nabla u \nabla v \, dx + \int_{\partial U} Tv \cdot \beta(Tu) \, ds$$

where $T : H^1(U) \to L^2(\partial U)$ is the trace operator.

Proof. □

12. *Proof.* □

13. *Proof.* □

14. *Proof.* □

15. **(Pointwise gradient constraint)**

    (a) Show tere exists a unique minimizer $u \in \mathcal{A}$ of

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - fw \, dx,$$

where $f \in L^2(U)$ and

$$\mathcal{A} := \{w \in H_0^1(U) : |Dw| \leq 1 \textbf{ a.e.}\}.$$

(b) Prove

$$\int_U Du \cdot D(w - u) \, dx \geq \int_U f(w - u)$$

for all $w \in \mathcal{A}$

Proof. The proof is almost identical to the proof of Theorem 3 and 4 in Section 8.4.2, except we have to check $\mathcal{A}$ is weakly closed by Mazur's theorem, a corollary of the Hahn-Banach Theorem:

Theorem 5. *[1, Section 3.3] Let C be a convex set in a Banach space E, then C is weakly closed if and only if it is closed.*

One can see the convexity of $\mathcal{A}$ easily. The closedness of $\mathcal{A}$ is proved by the same argument as Step 1 of Theorem 3 I referred in the beginning. So we complete the proof. □

16. *Proof.* □

17. *Proof.* □

18. *Proof.* $\square$

19. *Proof.* $\square$

20. *Proof.* $\square$

References

[1] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science & Business Media, 2010.

[2] Bernard Dacorogna. *Direct methods in the calculus of variations*, volume 78. Springer Science & Business Media, 2007.

[3] Mariano Giaquinta and Stefan Hildebrandt. *Calculus of Variations I*, volume 310. Springer Science & Business Media, 2004.