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Chapter 4 Point Set Topology*

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4.1 Topological Spaces

1. **If card$(X) \geq 2$, there is a topology on X that is T_0 but not T_1.**

Proof. □

2. **If X is an infinite set, the cofinite topology on X is T_1 but not T_2, and is first countable iff X is countable.**

Proof. □

3. **Every metric space is normal.**

Proof. □

4. **Let $X = \mathbb{R}$, and let $\mathscr{T}$ be the family of all subsets of $\mathbb{R}$ of the form $U \cup (V \cap \mathbb{Q})$ where U, V are open in the usual sense. Then $\mathscr{T}$ is a topology that is Hausdroff but not regular. (In view of Exercise 3, this shows that a topology stronger than a normal topology need not be normal or even regular.)**

Proof. □

5. **Every separable metric space is second countable.**

Proof. □

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6. **Let $\mathscr{E} = \{(a, b] : -\infty < a < b < \infty\}$.**

   (a) $\mathscr{E}$ is a base for a topology $\mathscr{T}$ on $\mathbb{R}$ in which the members of $\mathscr{E}$ are both open and closed.

   (b) $\mathscr{T}$ is first countable but not second countable. (If $x \in \mathbb{R}$, every neighborhood base at x contains a set whose supremum is x.)

   (c) $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathscr{T}$. (Thus the converse of Proposition 4.5 is false.)

   Proof. □

7. **If X is a topological space, a point $x \in X$ is called a cluster point of the sequence $\{x_j\}$ if for every neighborhood U of $x, x_j \in U$ for infinitely many j. If X is first countable,, x is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to x.**

   Proof. □

8. **If X is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in X, then $x_j \to x$ for every $x \in X$.**

   Proof. □

9. **If X is a linearly order set, the topology $\mathscr{T}$ generated by the sets $\{x : x < a\}$ and $\{x : x > a\}(a \in X)$ is called the order topology.**

   (a) If $a, b \in X$ and $a < b$, there exist $U, V \in \mathscr{T}$ with $a \in U, b \in V$, and $x < y$ for all $x \in U$ and $y \in V$. the order topology is the weakest topology with this property.

   (b) If $Y \subset X$, the order topology on Y is never stronger than, but may be weaker than, the relative topology on Y induced by the order topology on X.

   (c) The order topology on $\mathbb{R}$ is the usual topology.

   Proof. □

10. **A topological space X is called disconnected if there exist nonempty open sets U, V such that $U \cap V = \emptyset$ and $U \cup V = X$; otherwise, X is connected/ When we speak of connected or disconnected subsets of X, we refer to the relative topology on them.**

(a) X is connected iff $\emptyset$ and X are the only subsets of X that are both open and closed.

(b) If $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected subsets of X such that $\cap_{\alpha\in A}E_\alpha \neq \emptyset$, then $\cup_{\alpha\in A}E_\alpha$ is connected.

(c) If $A \subset X$ is connected, then $\overline{A}$ is connected.

(d) Every point $x \in X$ is contained in a unique maximal connected subset of X, and this subset is closed. (It is called the connected component of x)

Proof. (a) is trivial. (b) Let $E := \cup_{\alpha\in A}E_\alpha$. Given O_1, O_2 open in X such that $O_1 \cap E, O_2 \cap E$ are non-empty, $O_1 \cup O_2 \supset E$. Choose $x \in \cap_{\alpha\in A}E_\alpha$, and WLOG assume $x \in O_1$. Choose $y \in O_2 \cap E$, then $y \in E_\beta$ for some $\beta \in A$. Note that $(O_1 \cap E_\beta) \cup (O_2 \cap E_\beta) = (O_1 \cup O_2) \cap E_\beta = E_\beta$ and $x \in O_1 \cap E_\beta, y \in O_2 \cap E_\beta$. By the connectedness of E_β, $O_1 \cap O_2 \neq \emptyset$. Since O_1, O_2 are arbitrary chosen, E is connected.

(c) Given O_1, O_2 open in X such that $O_1 \cap \overline{A}, O_2 \cap \overline{A}$ are non-empty, $O_1 \cup O_2 \supset \overline{A} \supset A$. Given $x \in O_1 \cap \overline{A}$, since O_1 is an open neighborhood of x, $O_1 \cap A \neq \emptyset$. Similar for $O_2 \cap A \neq \emptyset$. By the connectedness of A, $O_1 \cap O_2 \neq \emptyset$. Since O_1, O_2 are arbitrary chosen, $\overline{A}$ is connected.

(d) Let $\mathscr{C} := \{A \subset X : A \text{ is connected and contains } x\}$. Let $C = \cup_{A\in\mathscr{C}}A$. By (b), C is connected. Then it's obvious maximal, unique. The closedness follows from (c). □

11. **If $E_1, \cdots E_n$ are subsets of a topological space, then $\overline{\cup_1^n E_j} = \cup_1^n \overline{E_j}$.**

(It's not true for intersection, e.g. $E_1 = \mathbb{Q}$ and $E_2 = \mathbb{Q}^c$; it's also not true for infinitely many E_j, e.g. $E_j = \{\frac{1}{j}\}$.)

Proof. This is easy, we omit it. □

12. **Let X be a set. A Kuratowski closure operator on X is a map $A \mapsto A^*$ from $\mathscr{P}(X)$ to itself satisfying (i) $\emptyset^* = \emptyset$, (ii) $A \subset A^*$ for all A, (iii) $(A^*)^* = A^*$ for all A, and (iv) $(A \cup B)^* = A^* \cup B^*$ for all A, B.**

(a) If X is a topological space, the map $A \mapsto \overline{A}$ is a Kuratowski closure operator.

(b) Conversely, given a Kuratowski closure operator, let $\mathscr{F} = \{A \subset X : A = A^*\}$ and $\mathscr{T} = \{U \subset X : U^c \in \mathscr{F}\}$. Then $\mathscr{T}$ is a topology, and for any set $A \subset X, A^*$ is its closure with respect to $\mathscr{T}$.

Proof. □

13. **If X is a topological space, U is open in X and A is dense in X, then $\overline{U} = \overline{U \cap A}$. (It's not true if U is not open, e.g. $X = \mathbb{R}, U = \mathbb{R} \setminus \mathbb{Q}$ and $A = \mathbb{Q}$.)**

    Proof. Of course, $\overline{U} \supset \overline{U \cap A}$.

    Given $x \in U$, if $x \in A$, then $x \in U \cap A$. If $x \notin A$, then $x \in$acc(A). Given V open in X, since U is open, $V \cap U$ is open in X and hence there is a point y lying in $A \cap (V \cap U) = V \cap (A \cap U)$ and hence $y \neq x$. Since V is given, $x \in$acc$(A \cap U)$. Consequently, $x \in \overline{U \cap A}$.

    Since x is given, $U \subset \overline{U \cap A}$. Since $\overline{U \cap A}$ is closed, $\overline{U} \subset \overline{U \cap A}$. □

4.2 Continuous Maps

14. *Proof.* □

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27. *Proof.* □

28. **Let X be a topological space equipped with an equivalence relation, $\widetilde{X}$ the set of equivalence classes, $\pi : X \to \widetilde{X}$ the map taking each $x \in X$ to its equivalence class, and $\mathscr{T} = \{U \subset \widetilde{X} : \pi^{-1}(U)$ is open in $X\}$.**

    (a) $\mathscr{T}$ is a topology on $\widetilde{X}$. (It is called the quotient topology.)

    (b) If Y is a topological space, $f : \widetilde{X} \to Y$ is continuous iff $f \circ \pi$ is continuous

    (c) $\widetilde{X}$ is T_1 iff every equivalence class is closed in X.

    Proof. (a) Since $\pi^{-1}(\emptyset_{\widetilde{X}}) = \emptyset_X$ and $\pi^{-1}(\widetilde{X}) = X$, $\emptyset_{\widetilde{X}}$ and $\tilde{X}$ are in $\mathscr{T}$. Given $\{U_\alpha\} \subset \mathscr{T}$, then

    $$\pi^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha \pi^{-1} U_\alpha$$

    hence $\cup_\alpha U_\alpha \in \mathscr{T}$. Similarly $\cap_1^n U_i \in \mathscr{T}$ for every finite set $\{U_1, \cdots U_n\} \subset \mathscr{T}$.

    (b) By definition, π is continuous. So $f \circ \pi$ is continuous provided f is. Conversely, if $f \circ \pi$ is continuous, then for every open set $V \subset Y$, $f^{-1}(V)$ is open in $\widetilde{X}$ since $\pi^{-1} f^{-1}(V) = (f \circ \pi)^{-1}(V)$ is open in X. Therefore, f is continuous.

    (c) Suppose $\widetilde{X}$ is T_1. Given any equivalence class $\pi^{-1}(\{[x]\})$ for some $[x] \in \widetilde{X}$, since the singleton $\{[x]\}$ is closed, by continuity, $\pi^{-1}(\{[x]\})$ is closed .

    Conversely, suppose every equivalence class is closed. Given $[x] \in \tilde{X}$, then $\pi^{-1}(\{[x]\})$ is closed. Since $\pi^{-1}(\{[x]\}^c) = \left(\pi^{-1}(\{[x]\})\right)^c$ is open, $\{[x]\}^c \in \mathscr{T}$. So $\{[x]\}$ is closed. Hence $\widetilde{X}$ is T_1. ☐

29. *Proof.* ☐

4.3 Nets

30. *Proof.* ☐

31. *Proof.* ☐

32. *Proof.* ☐

33. *Proof.* ☐

34. *Proof.* ☐

35. *Proof.* ☐

36. *Proof.* ☐

4.4 Compact Spaces

37. *Proof.* □

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4.5 Locally Compact Hausdorff Spaces

46. *Proof.* □

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4.6 Two Compactness Theorems

58. *Proof.* □

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62. *Proof.* □

63. *Proof.* □

64. By Arezla-Ascoli Theorem.

65. *Proof.* □

4.7 The Stone-Weierstrass Theorem

66. *Proof.* □

67. *Proof.* □

68. *Proof.* □

69. *Proof.* □

70. *Proof.* □

71. *Proof.* □

References