# Real and Complex Analysis, 3rd Edition, W.Rudin

# Chapter 5 Examples of Banach Space Techniques

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1. **Let $X$ consists of two points $a$ and $b$, put $\mu(\{a\}) = \mu(\{b\}) = \frac{1}{2}$, and let $L^p(\mu)$ be the resulting *real* $L^p$ space. Identify each real function $f$ on $X$ with the point $(f(a), f(b))$ in the plane, and sketch the unit balls of $L^p(\mu)$, for $0 < p \leq \infty$. Note that they are convex if and only if $1 \leq p \leq \infty$. For which $p$ is this unit ball a square? A circle? If $\mu(\{a\}) \neq \mu(\{b\})$, how does this situation differ from the preceding one?**

   *Proof.* Let $U = \{f : \|f\|_p \leq 1\}$. In the equal mass case, the figuare is a square if and only if $p = 1$ or $\infty$ and is a circle if and only if $p = 2$. The convexity part is also standard. For non-equal mass case, it is not a square nor a circle for any $p$ (by comparing the side length or examining the candidate for "center of circle"). However, for $p = \infty$, $U$ is a square. □

2. Triangle inequality.

3. **If $1 < p < \infty$, prove that the unit ball in $L^p(\mu)$ is strictly convex. Show that this fails in every $L^1(\mu)$, in every $L^\infty(\mu)$, and in every $C(X)$. (Ignore trivialities, such as spaces consisting of only one point.)**

   ***Remark*** 1. Actually, one can prove $L^p$ is uniformly convex by Clarkson's inequality. See [4, Problem 1.6] and [1, Section 4.3]. Moreover, Milman-Pettis Theorem states every uniformly convex space is reflexive. See [1, Section 3.7].

   *Proof.* For $1 < p < \infty$. Given $f, g \in L^p(\mu)$ with $\|f\|_p = \|g\|_p = 1$, $f \neq g$. Minkowski inequality implies $h = \frac{1}{2}(f + g)$ has $L^p$ norm less than or equal to 1. Moreover, the equality holds if and

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only if there is $\lambda \geq 0$ such that $f = \lambda g$, which is equivalent to $f = g$ since both functions has the same $L^p$ norm. So $\|h\|_p < 1$.

For $L^1(\mu)$ and $L^\infty(\mu)$. Suppose $0 < \mu(A) < \infty$ and $0 < \mu(B) < \infty$, and $A \cap B = \emptyset$. Let $f(x) = 1/\mu(A)$ when $x \in A$ and $f(x) = 0$ when $x \notin A$. Let $g(x) = 1/\mu(B)$ when $x \in B$ and $g(x) = 0$ when $x \notin B$. Then $f, g$ are linearly independent, and $\|\frac{f+g}{2}\|_1 = \|f\|_1 = \|g\|_1 = 1$. Similarly, let $F(x) = 1$ when $x \in A$ and $f(x) = 0$ when $x \notin A$. Let $g(x) = 1$ when $x \in A \cup B$ and $g(x) = 0$ when $x \notin A \cup B$. Then $f, g$ are linearly independent and $\|\frac{f+g}{2}\|_\infty = \|f\|_\infty = \|g\|_\infty = 1$.

For $C(X)$, suppose $X$ is a compact metric space with more than one point. (Note that if $X$ is not compact, it's not clear that $\|\cdot\|_\infty$ is a norm on $C(X)$.) Consider $f(x) = 1/(1 + d(x, a))$, $g(x) = 1/(1 + d(x, a))^2$. Then $\|\frac{f+g}{2}\|_\infty = \|f\|_\infty = \|g\|_\infty = 1$. □

4. **Let $C$ be the space of all continuous functions on $[0, 1]$ with the supreme norm. Let $M$ consists of all $f \in C$ for which**

$$\int_0^{\frac{1}{2}} f(t)\,dt - \int_{\frac{1}{2}}^1 f(t)\,dt = 1.$$

**Prove that $M$ is a closed convex subset of $C$ which contains no elements of minimal norm.**

***Remark* 2.** This is related to Theorem 4.10 and Exercise 4.11 and 5.5.

*Proof.* Convexity of $M$ is trivial. Closedness of $M$ is through theorem on uniform convergence and integration. Next, we try to calculate the infimum. By triangle inequality, we see

$$1 \leq \inf\{\|f\|_\infty : f \in M\}.$$

On the other hand, for $n \geq 2$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1+n}{1-n} & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1], \\ \text{linear} & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]. \end{cases}$$

Then $f_n \in M$ and $\|f_n\|_\infty = \frac{n+1}{n-1} \to 1$ as $n \to \infty$. So

$$1 = \inf\{\|f\|_\infty : f \in M\}.$$

If there is $f \in C$ such that $\|f\|_\infty = 1 = \int_0^{\frac{1}{2}} f(t)\,dt - \int_{\frac{1}{2}}^1 f(s)\,ds$, then $f(t) = 1$ on $[0, 1/2)$ and $f(t) = -1$ on $(1/2, 1]$. But this contradictions to the continuity of $f$. □

5. **Let $M$ be the set of all $f \in L^1([0,1])$, relative to Lebesgue measure, such that**

$$\int_0^1 f(t)\,dt = 1.$$

**Show that $M$ is a closed convex subset of $L^1([0,1])$ which contains infinitely many elements of minimal norm. (Compare this and Exercise 4 with Theorem 4.10.)**

*Proof.* Convexity of $M$ is trivial. Closedness of $M$ is through triangle inequality. Note that $\inf\{\|f\|_1 : f \in M\} \geq 1$ and for each $n \in \mathbb{N}$, $f_n = n\chi_{(0,\frac{1}{n})} \in M$, distinct from other $f_m$ and has norm 1. □

6. *Proof.* Using the continuity of $f$, we can extend it to $\overline{M}$ and preserve its norm.

Since $H = \overline{M} \bigoplus M^{\perp}$, we extend $f$ to $H$ by define $F(x) = F(x^M + x^{M^{\perp}}) := f(x^M)$. Then it's easy to check all the assertions on $F$ are satisfied. The uniqueness is easy to prove. □

7. **Construct a bounded linear functional on some subspace of some $L^1(\mu)$ which has two (hence infinitely many) distinct norm-preserving linear extensions to $L^1(\mu)$.**

***Remark*** 3. In contrast to Exercise 6, this exercise shows that such unique extension result is not true for every Banach space. Hence, no uniqueness assertion in Hahn-Banach Theorem.

*Proof.* Consider $L^1 = L^1([-1,1], m)$ and $M := \{f \in L^1 : f \equiv 0 \text{ on } [-1,0]\} \leqslant L^1$. Consider the functional $T(f) = \int_{-1}^1 f\,dx$ on $M$ which is linear, bounded with norm 1. Such $T$ has two distinct norm-preserving extensions, one is $T_1(f) = \int_{-1}^1 f\,dx$, another one is $T_2(f) = \int_{-1}^1 f\chi_{(-\frac{1}{2},1)}\,dx$. □

8. *Proof.* (a) Standard result. It's true even for the range $\mathbb{R}$ or $\mathbb{C}$ is replaced by a Banach space.

(b) It's easy to see the norm $\leq \|x\|$. To see it's actually an equality, we use Hahn-Banach Theorem (more precisely, Theorem 5.20).

(c) By (a)(b) and Banach-Steinhaus Theorem. □

9. *Proof.* (a)It's easy to see $\|\Lambda_y\| \leq \|y\|_1$. To get the reverse inequality, we let $x_n = (\xi_i)_{i=1}^n$, where $\xi_i = \frac{\overline{\eta_i}}{|\eta_i|}$ if $\eta_i \neq 0$ and $\xi_i = 0$ if $\eta_i = 0$. Then $x_n \in c_0$, $\|x_n\|_\infty \leq 1$ and $\|\Lambda_y\| \geq |\Lambda_y x_n| = \sum_{i=1}^n |\eta_i|$ for every $n$. Therefore, $\|y\|_1 \leq \|\Lambda_y\|$.

Now we prove the linear map $y \mapsto \Lambda_y$ is surjective. Given $\Lambda \in (c_0)^*$, we let $\eta_i = \Lambda e_i$ for each $i$. Let $x_n = (\xi_i)_{i=1}^n$, where $\xi_i = \frac{\overline{\eta_i}}{|\eta_i|}$ if $\eta_i \neq 0$ and $\xi_i = 0$ if $\eta_i = 0$. Then $x_n \in c_0$, $\|x_n\|_\infty \leq 1$ and

$\|\Lambda\| \geq |\Lambda x_n| = \sum_{i=1}^n |\eta_i|$ for every $n$. Therefore, $\sum_{i=1}^\infty |\eta_i| \leq \|\Lambda\|$, that is, $y = (\eta_i) \in l^1$. Given $x = (\xi_i)_{i=1}^\infty \in c_0$, we see $x_n = (\xi_i)_{i=1}^n \to x$ in $l^\infty$. By the continuity of $\Lambda$,

$$\Lambda x = \lim_{n\to\infty} \Lambda x_n = \lim_{n\to\infty} \sum_{i=1}^n \xi_i \Lambda e_i = \lim_{n\to\infty} \sum_{i=1}^n \xi_i \eta_i = \Lambda_y x.$$

(b) It's easy to see $\|\Lambda_y\| \leq \|y\|_\infty$. To get the reverse inequality, we may assume $\|y\|_\infty > 0$, then for each small $\epsilon > 0$, there is some $|\xi_i| > \|y\|_\infty - \epsilon$. Take $x = e_i$, then $\|x\|_1 = 1$ and hence $\|\Lambda\| \geq |\Lambda x| = |\xi_i| > \|y\|_\infty - \epsilon$. Letting $\epsilon \to 0$, we see $\|\Lambda\| \geq \|y\|_\infty$.

Now we prove the linear map $y \mapsto \Lambda_y$ is surjective. Given $\Lambda \in (l^1)^*$, we let $\eta_i = \Lambda e_i$ and hence $|\eta_i| \leq \|\Lambda\|$ for each $i$, that is, $y = (\eta_i) \in l^\infty$. Given $x = (\xi_i)_{i=1}^\infty \in l^1$, we see $x_n = (\xi_i)_{i=1}^n \to x$ in $l^1$. By the continuity of $\Lambda$,

$$\Lambda x = \lim_{n\to\infty} \Lambda x_n = \lim_{n\to\infty} \sum_{i=1}^n \xi_i \Lambda e_i = \lim_{n\to\infty} \sum_{i=1}^n \xi_i \eta_i = \Lambda_y x.$$

(c) By Hahn-Banach Theorem (Theorem 5.20) and (a).

(d) Consider the collection $S$ of all elements $(x_i)_{i=1}^\infty, x_i = 1$ or $0$ for each $i$ which is an uncountable subset of $l^\infty$ and for each $x, y \in S, x \neq y, \|x - y\|_\infty = 1$. So every dense subset of $l^\infty$ is uncountable.

Let $S$ be the collection of all elements $(x_i)_{i=1}^\infty, x_i \in \mathbb{Q}$ for each $i$ and $x_i = 0$ for all $i \geq N$ for some $N$. By understanding the collection $S$ is countable and dense in $l^1$ and $c_0$, we see $l^1$ and $c_0$ are separable. $\square$

10. **If $\sum \alpha_i \xi_i$ converges for every sequence $\{\xi_i\}$ such that $\xi_i \to 0$ as $i \to \infty$, prove that $\sum |\alpha_i| < \infty$.**

***Remark*** 4. The assumption is weaker than Exercise 6.4 and Folland [2, Theorem 6.14].

*Proof.* For each $n \in \mathbb{N}$, define $\Lambda_n : c_0 \subset l^\infty \to \mathbb{C}$ by

$$\Lambda_n(\xi) = \sum_{i=1}^n \alpha_i \xi_i.$$

It's easy to check $c_0$ with sup-norm is a Banach space, each $\Lambda_n$ is linear, bounded with norm $\|\Lambda_n\| = \sum_{i=1}^n |\alpha_i|$ and $\{\Lambda_n\}$ is pointwisely bounded. By Banach-Steinhaus Theorem, we see there is $M > 0$ such that $M \geq \|\Lambda_n\| = \sum_{i=1}^n |\alpha_i|$ for all $n$. Therefore, $\sum_{i=1}^\infty |\alpha_i|$ exists (due to monotonicity and boundedness) and $M \geq \sum_{i=1}^\infty |\alpha_i|$. $\square$

11. **Let $\beta \in (0,1)$. Prove that $C^{0,\beta}([a,b];\mathbb{C})$ are Banach spaces with norms $\|f\|_1 = |f(a)| + [f]_{0,\beta}$ and $\|f\|_2 = \|f\|_\infty + [f]_{0,\beta}$, where $[f]_{0,\beta} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}$.**

*Proof.* Let $U = [a, b]$. For simplificity, we only consider the completeness with respect to $\|\cdot\|_2$. Given a Cauchy sequence $\{f_n\} \subset C^{0,\beta}(\overline{U})$, then there is a $f \in C(\overline{U})$ such that $f_n \to f$ uniformly and a constant $M > 0$ such that for each $n \in \mathbb{N}$ and $s \neq t$,

$$\frac{|f_n(s) - f_n(t)|}{|s - t|^\beta} \le M.$$

Then for each $s \neq t$, there is some $N = N(s, t) \in \mathbb{N}$ such that

$$\begin{aligned} |f(s) - f(t)| &\le |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)| \\ &\le 2|s - t|^\beta + M|s - t|^\beta. \end{aligned}$$

Therefore, $f \in C^{0,\beta}(\overline{U})$. It remains to show $f_n \to f$ in $C^{0,\beta}(\overline{U})$. For each $s, t \in U$ and $\epsilon > 0$, by Cauchy's criteria, we can find a $N = N(\epsilon)$ such that for $k, n \ge N$

$$|f_k(s) - f_n(s) - f_k(t) + f_n(t)| \le \epsilon |s - t|^\beta$$

For each $\eta > 0$, we can find a $K = K(\eta, s, t) > N$, such that $|f(x) - f_K(x)| \le \eta$ for $x = s$, or $x = t$. Therefore, for every $n \ge N$

$$|f(s)-f_n(s)-f(t)+f_n(t)| \le |f(s)-f_K(s)|+|f_K(s)-f_n(s)-f_K(t)+f_n(t)|+|f(t)-f_K(t)| \le 2\eta+\epsilon|s-t|^\beta.$$

Letting $\eta \to 0$, we obtain that $|f(s) - f_n(s) - f(t) + f_n(t)| \le \epsilon|s - t|^\beta$ for all $n \ge N(\epsilon)$. □

***Remark* 5.** It is true for the general case $C^{k,\beta}(U)$, $U$ is connected in $\mathbb{R}^d$, with an almost identical proof as the above case $k = 0$, except we need the standard convergence theorem between $\{f_n\}$ and $\{Df_n\}$.

12.

***Remark* 6.** See Notes and Comments for Section 5.22.

*Proof.* □

13. *Proof.* (a) By the pointwise convergence assumption,

$$X = \cup_M E_M := \cup_M \cap_n \{x : |f_n(x)| \le M\}.$$

Note that $E_M \subset E_{M+1}$ is nonempty for large M, and each $E_M$ is closed since it's intersection of closed sets. Therefore, by Baire Category Theorem, there is some $M$ such that $E_M$ contains an open set $V \neq \emptyset$. (b)'s proof is similar to (a) (see Hint.) □

14. *Proof.* □

15. *Proof.* □

16.

***Remark*** *7.* See Rudin [3, Theorem 2.11, 2.14-15 and Theorem 1.24].

**Theorem 8** (Closed Graph Theorem)**.** *Let $L$ be a linear map between two $F$-spaces (complete translation-invariant metric vector spaces), then the graph of $L$ is closed iff $L$ is continuous.*

The corresponding open mapping theorem has a setting in $F$-spaces.

17. *Proof.* (a) It's easy to see $\|f\|_\infty$ dominates the norm of multiplication operator $M_f(\cdot) = f\cdot$

(b) □

18. *Proof.* Given $x \in X$, for each $\epsilon > 0$, there exists $y = y(x, \epsilon) \in E$ such that $\|y - x\|_X < \frac{\epsilon}{M}$, then

$$\|\Lambda_n x - \Lambda_m x\|_Y \leq \|\Lambda_n x - \Lambda_n y\|_Y + \|\Lambda_n y - \Lambda_m y\|_Y + \|\Lambda_m y - \Lambda_m y\|_Y \leq 2M\frac{\epsilon}{M} + \|\Lambda_n y - \Lambda_m y\|_Y$$

The last term is less than $\epsilon$ if $n, m > N$ for some $N = N(y) = N(x, \epsilon)$. So $\{\Lambda_n x\}$ is Cauchy in the Banach space $Y$. □

19. *Proof.* (a) We try to apply Exercise 18 with $X = Y = C(T), \Lambda_n f = \frac{s_n(f)}{\log n}$. Then we see $\Lambda_n$ converges to 0 pointwisely on the dense set $E = P(T)$, the set of all trigonometric polynomials. It remains to show that $\Lambda_n$ is uniformly bounded which is due to

$$(\log n)\|\Lambda_n\| = \int_0^\pi |\frac{\sin(n+\frac{1}{2})x}{\sin x/2}|\, dx \leq \frac{2}{1-\frac{\pi^2}{12}}\int_0^\pi \frac{|\sin(n+\frac{1}{2})x|}{x}\, dx \leq C\sum_{k=1}^n \frac{1}{k\pi}.$$

(b) Use the Banach-Steinhaus Theorem in exactly the same way as Section 5.11 with the change of the linear functionals to $\Lambda_n f = \frac{1}{\lambda_n} s_n(f; 0)$. Note that $\|\Lambda_n\| = \|D_n\|_1/\lambda_n$. Note that

$$\frac{\|D_n\|_1}{\lambda_n} \geq \frac{4}{\pi^2|\lambda_n|}\sum_{k=1}^n \frac{1}{k} \geq \frac{4}{\pi^2|\lambda_n|}(\ln(n) + \gamma) = \frac{4}{\pi^2}(\frac{\ln(n)}{|\lambda_n|} + \frac{\gamma}{|\lambda_n|}) \to \infty,$$

since you are given $\frac{\lambda_n}{\ln(n)} \to 0$ . Here $\gamma$ is the Euler Mascheroni constant.

So actually, it's unbounded for all $f$ in some dense $G_\delta$ set in $C(T)$. □

20. *Proof.* (a) No, since $\mathbb{Q}$ is not a $G_\delta$ set, but the set of points $A$ at which a sequence of positive continuous functions is unbounded is $\cap_m \cup_n \{x : f_n(x) > m\}$ which is a $G_\delta$ set. (If $\mathbb{Q} = \cap_n V_n$, $V_n$ open, then $\mathbb{R} = \{r_m\} \cup (\cup_n V_n^c)$ which is of first category. A contradiction!)

(b)Let $\mathbb{Q} = \{q_k\}$, we consider

$$f_n(x) = \min_{1\leq k\leq n}\{k + n|x - q_k|\} \geq 1.$$

Then for each $q_m \in \mathbb{Q}$, we see for $n \geq m$, $f_n(q_m) \leq m + n|q_m - q_m| = m$, so $\{f_n(q_m)\}$ is bounded. On the other hand, if $x \in \mathbb{Q}^c$, then given $M > 0$, there is some $N = N(M) > M$ such that $n|x - q_i| > M$ for all $1 \leq i \leq M$ provided $n > N$. Therefore,

$$f_n(x) = \min_{1 \leq k \leq n} \{k + n|x - q_k|\} > M,$$

that is, $f_n(x) \to \infty$.

(c) Irrational part is answered in (b). The answer is affirmative for rational part:

Let $A_n = \cup_{i=1}^{n}(q_i - \epsilon_n, q_i + \epsilon_n)$, where $0 < \epsilon_n = \frac{1}{4}\min\{|q_i - q_j| : 1 \leq i < j \leq n+1\} \searrow 0$.

Take $f_n$ to be the zig-zag continuous function that eqauls to $n$ at $q_1, \cdots q_n$ and 0 outside $A_n$. Then for every $x \in \mathbb{Q}, f_n(x) = n \to \infty$. On the other hand, given $x \in \mathbb{Q}^c$, suppose for some $n_0 \in \mathbb{N}$, $x \in A_n$ for all $n \geq n_0$. Then for some fixed $1 \leq i \leq n_0$, $x \in (q_i - \epsilon_{n_0}, q_i + \epsilon_{n_0})$. Since $x \in A_{n_0+1}$, by construction of $\epsilon_n$, $x \in (q_i - \epsilon_{n_0+1}, q_i + \epsilon_{n_0+1})$. Inductively, we know $x \in (q_i - \epsilon_n . q_i + \epsilon_n)$ for all $n \geq n_0$, and hence $x = q_i$ since $\epsilon_n \to 0$. This contradicts to $x \in \mathbb{Q}^c$.

So for every $M \in \mathbb{N}$, there is some $m > M$ such that $x \notin A_m$, that is, $f_m(x) = 0$, and therefore $f_n(x) \not\to \infty$. □

21. *Proof.* □

22. *Proof.* □

# References


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