Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 4 Poisson's Equation and the Newtonian Potential

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1. *Proof.* (a) WLOG, we assume $\gamma = \min(\alpha, \beta) = \alpha$.

(1) Given $x, y \in \Omega$, then

$$\begin{aligned}|u(x)v(x) - u(y)v(y)| &\le |u(x)||v(x) - v(y)| + |v(y)||u(x) - u(y)| \\ &\le \|u\|_0[v]_\beta|x-y|^\beta + \|v\|_0[u]_\alpha|x-y|^\alpha \\ &\le |x-y|^\gamma \max(1, d^{\alpha+\beta-2\gamma})\Big(\|u\|_0[v]_\beta + \|v\|_0[u]_\alpha\Big).\end{aligned}$$

Hence, $[uv]_\gamma \le \max(1, d^{\alpha+\beta-2\gamma})\Big(\|u\|_0[v]_\beta + \|v\|_0[u]_\alpha\Big)$. Furthermore,

$$\|uv\|_\gamma \le \max(1, d^{\alpha+\beta-2\gamma})\Big(\|u\|_0\|v\|_0 + \|u\|_0[v]_\beta + \|v\|_0[u]_\alpha\Big) \le \max(1, d^{\alpha+\beta-2\gamma})\|u\|_\alpha\|v\|_\beta.$$

(2)Given $x, y \in \Omega$, then

$$\begin{aligned}\frac{|u(x)v(x) - u(y)v(y)|}{|x-y|^\gamma} &\le \frac{|u(x)||v(x) - v(y)| + |v(y)||u(x) - u(y)|}{|x-y|^\gamma} \\ \le \|u\|_0\frac{[v]'_\beta|x-y|^\beta}{d^\beta|x-y|^\gamma} + \|v\|_0\frac{[u]'_\alpha|x-y|^\alpha}{d^\alpha|x-y|^\gamma} &= \|u\|_0\frac{[v]'_\beta|x-y|^{\beta-\gamma}}{d^{\beta-\gamma}d^\gamma} + \|v\|_0\frac{[u]'_\alpha}{d^\gamma} \\ &\le \|u\|_0\frac{[v]'_\beta}{d^\gamma} + \|v\|_0\frac{[u]'_\alpha}{d^\gamma}.\end{aligned}$$

Hence, $[uv]'_\gamma \le \|u\|_0[v]'_\beta + \|v\|_0[u]'_\alpha$ and therefore $\|uv\|'_\gamma \le \|u\|'_\alpha\|v\|'_\beta$.

(b) Given $W \subset\subset \Omega$ and two distinct points $x, y \in W$, denote I be the interval between $g(x)$ and $g(y)$ and $L_{f,I}$ be the Holder constant of f on I, then

$$|f(g(x)) - f(g(y))| \le L_{f,I}|g(x) - g(y)|^\alpha \le L_{f,I}(L_{g,W}|x-y|^\beta)^\alpha.$$

□

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2. Go back to check the convergence of the integrals in the proof of Lemma 4.2.

3. I think we need to assume $p > n$ rather than $p > n/2$. See Exercise 8. Also see Lieb and Loss, [3, Chapter 10].

   Proof. □

4. *Proof.* □

5. *Proof.* Denote fundamental solution for Laplaian with pole y by $\Gamma_y(x) := \Gamma(x-y)$. Put $v(x) = \Gamma_y(x) - \Gamma(R')$, where $0 < R' \le R$, in the Green's identity

$$\int_\Omega u\Delta v - v\Delta u \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, dS_x,$$

with $\Omega = B_{R'}(y) \setminus B_\epsilon(y)$ and letting $\epsilon \to 0$, we have

$$\begin{aligned} u(y) &= \int_{B_{R'}} \Delta u(x)\Big[\Gamma_y(x) - \Gamma(R')\Big]\, dx + \int_{\partial B_{R'}} u\frac{\partial \Gamma_y(x)}{\partial n} - \Big[\Gamma_y(x) - \Gamma(R')\Big]\frac{\partial u}{\partial n}\, dS_x \\ &= (\le,\ge)\frac{1}{nw_n(R')^{n-1}}\int_{\partial B_{R'}} u\, dS_x + \int_{B_{R'}} f(x)\Big[\Gamma_y(x) - \Gamma(R')\Big]\, dx, \end{aligned}$$

where $\Delta u = (\ge,\le)f$. Next, we only consider the equality case since the other cases are similar. Multiply both sides with $(R')^{n-1}$ and integrate w.r.t R' from 0 to R, we have, for $n > 3$,

$$\begin{aligned} u(y) &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{(2-n)w_nR^n}\int_0^R\int_{B_{R'}(y)} f(x)\Big[|x-y|^{2-n} - (R')^{2-n}\Big]\, dx (R')^{n-1}\, dR' \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{(2-n)w_nR^n}\int_0^R\int_0^{R'}\int_{\partial B_1(0)} f(y+rw)\, dw\Big[r^{2-n}(R')^{n-1} - R'\Big]r^{n-1}\, dr\, dR' \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{(2-n)w_nR^n}\int_0^R\int_{\partial B_1(0)}\int_r^R \Big[r^{2-n}(R')^{n-1} - R'\Big]\, dR' f(y+rw)\, dw r^{n-1}\, dr \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{(2-n)w_nR^n}\int_0^R\int_{\partial B_1(0)} \Big[\frac{r^{2-n}}{n}(R^n - r^n) - \frac{1}{2}(R^2 - r^2)\Big] f(y+rw)\, dw r^{n-1}\, dr \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{n(2-n)w_n}\int_0^R\int_{\partial B_1(0)} \Big[r^{2-n} + \frac{n-2}{2}\frac{r^2}{R^n} - \frac{n}{2}R^{2-n}\Big] f(y+rw)\, dw r^{n-1}\, dr \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx + \frac{1}{n(2-n)w_n}\int_{B_R(0)} \Big[|Z|^{2-n} + \frac{n-2}{2}\frac{|Z|^2}{R^n} - \frac{n-2}{2}\frac{R^2}{R^n} - R^{2-n}\Big] f(y+Z)\, dZ \\ &= \frac{1}{w_nR^n}\int_{B_R(y)} u\, dx - \frac{1}{nw_n}\int_{B_R(0)} \Big[\frac{1}{n-2}(|Z|^{2-n} - R^{2-n}) + \frac{1}{2R^n}(|Z|^2 - R^2)\Big] f(y+Z)\, dZ. \end{aligned}$$

The calculation for $n = 2$ is similar. □

6. *Proof.* (incomplete!!!!!) Since Ω is C^2 and bounded, we can find a neighborhood Γ of $\partial\Omega$ in Ω such that $\operatorname{dist}(x, \partial\Omega) =: d(x) \in C^2(\overline{\Gamma})$ and hence $\nabla d, \Delta d$ are bounded in Γ. Moreover, since Ω is compact, there exists $\delta > 0$ such that $B(x,\delta) \cap \Omega \subset \Gamma$ for every $x \in \Omega$.

Let $\eta \in C_c^\infty(B_\delta(0))$ with $0 \le \eta \le \eta(0) = (\beta(1-\beta))^{-1}$. Given $x \in \partial\Omega$ and let $\eta_x(y) := \eta(y-x)$. Since

$$
\begin{aligned}
\Delta(d^\beta \eta_x) &= d^\beta \Delta\eta_x + 2\nabla d^\beta \nabla \eta_x + \Delta(d^\beta)\eta_x \\
&= -d^{\beta-2}[\beta(1-\beta)\eta_x - \beta d(\Delta d)\eta_x - 2\beta d \nabla d \nabla \eta_x - d^2 \Delta \eta_x],
\end{aligned}
$$

we can find a small $r > 0$ indepdent of x such that for $y \in B_r(x)$, $\Delta(d^\beta \eta_x)(y) \le \frac{-1}{2} d^{\beta-2}$. On the other hand, for $y \notin B_r(x)$, $|\Delta(d^\beta \eta_x)(y)| \le C(\beta, \|d\|_{C^2(\Gamma)}, r) =: C$ (Note $\eta_x = 0$ on $\Omega \setminus B_\delta(x)$.)

Since $\partial\Omega$ is compact, there exists finite many $x_1 \cdots x_m$ such that $\{B_r(x_i)\}_i$ covers $\partial\Omega$. Let v be the solution of $\Delta v = -mC$ in Ω and $v = 0$ on $\partial\Omega$

Define $w = \sum \eta_{x_i} d^\beta + v$, then $w = 0$ on $\partial\Omega$ and $\Delta w \le -\frac{1}{2} d^{\beta-2}$. So $\Delta(2Nw \pm u) \le 0$ in Ω and $2Nw \pm u = 0$ on $\partial\Omega$. So $|u(x)| \le 2Nw(x)$ in Ω by the maximum principle. It remains to estimate $v(x)$.

Note that since $|\nabla d(y)| \to 1$ as $y \to \partial\Omega$,

$$
\Delta(d^\beta)(y) = d(y)^{\beta-2}[\beta(\beta-1)|\nabla d(y)|^2 + \beta d(y)\Delta d(y)] \to -\infty \quad \text{as } y \to \partial\Omega.
$$

So there exists a neighborhood $\Gamma' \subset \Gamma$ of $\partial\Omega$ and C' such that

$$
\Delta(C' d^\beta - v) \le 0 \quad \text{in } \Gamma' \text{ and } C' d^\beta - v \ge 0 \text{ on } \partial\Gamma'.
$$

By the maximum principle, $v(x) \le C' d(x)^\beta$ in Γ'. □

7. Standard change of variables. I think this is the same as the derivation of Laplace-Beltrami operator in Riemannian geometry.

8. See Lieb and Loss, [3, Chapter 10].

   Proof. □

9. *Proof.* (a) Since $\Delta(\eta P) = (\Delta\eta)P + 2\nabla\eta\nabla P$, $\operatorname{supp}(\Delta(\eta P)) \subset \{1 \le |x| \le 2\}$. Then for any $x \ne 0$ and $y \in B_{\frac{1}{2}|x|}(x)$, for all but finitely many k, $\Delta(\eta P)(t_k y) \ne 0$. So f is continuous at any $x \ne 0$. At the origin, we know $f(0) = 0$ from the definition. Since $|f(x)| = |c_k \Delta(\eta P)(t_k x)| \le M|c_k|$ if $2^{-k} \le x \le 2^{-k+1}$ and $c_k \to 0$, f is continuous at the origin.

   Next, we define $v(x) = \sum \frac{c_k}{t_k^2}(\eta P)(t_k x)$. For each $x \ne 0$ and $y \in B_{\frac{1}{2}|x|}(x)$, we see only finite terms contribute $v(y)$ and hence $v \in C^2(\mathbb{R}^n \setminus \{0\})$. Since $\sum \frac{|c_k|}{t_k^2}$ converges and ηP is bounded, v is continuous everywhere (and hence bounded near the origin).

Since for each $x \neq 0$, there is only one k_0 such that $1 \leq |2^{k_0}x| \leq 2$, then for some $|\alpha| = 2, D^\alpha P \equiv P_\alpha \neq 0$ and

$$\partial^\alpha v(x) = \sum^{k_0-1} c_k P_\alpha + c_{k_0}\eta(2^{k_0}x)P_\alpha + \sum_{i,\alpha_i=1} c_{k_0}(\partial_i\eta)(2^{k_0}x)(\partial^{\alpha-\alpha_i}P)(2^{k_0}x)$$

Since $k_0(x) \to \infty$ as $|x| \to 0$, $c_{k_0(x)} \to 0$ as $|x| \to 0$. Moreover, since $\sum c_k$ diverges, $\lim_{|x|\to 0} \partial^\alpha v(x)$ does not exists.

Given $\epsilon > 0$. Suppose there exist classical solution to $\Delta u = f$ in B_ϵ, then $u - v$ is bounded harmonic in $B_{\epsilon/2} \setminus \{0\}$. By removable singularity, we know $u - v$ has a harmonic extension to the origin, which implies the contradiction that v has a C^2 extension to the origin.

(b) Similarly, we see $w(x) := \sum \frac{c_k}{t_k^3}(\eta Q)(t_k x)$ is $C^3(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$. We also note that for each $x \neq 0, \Delta w(x) = g(x) = \sum \frac{c_k}{t_k}(\Delta(\eta Q))(t_k x)$ and $D_i g(x) = \sum c_k (D_i \Delta(\eta Q))(t_k x)$, so $g \in C^1(\mathbb{R}^n \setminus \{0\})$. At the origin, we know $g(0) = 0$ from the definition and for each $h \neq 0$, there is only one k_0 such that $1 \leq |2^{k_0}h| \leq 2$. Note that $k_0(h) \to \infty$ as $h \to 0$ and

$$|\frac{g(he_i) - g(0)}{h}| = |c_{k_0}\frac{\Delta(\eta P)(2^{k_0}he_i)}{2^{k_0}h}| \leq |c_{k_0}|M \to 0 \text{ as } h \to 0.$$

So $D_i g(0) = 0$. Since $|D_i g(x)| = |c_k D_i \Delta(\eta P)(t_k x)| \leq M'|c_k|$ if $x \in [2^{-k}, 2^{-k+1})$ and $c_k \to 0, D_i g$ is continuous at the origin for each i. Therefore, $g \in C^1(\mathbb{R}^n)$.

Since for each $x \neq 0$, there is only one k_0 such that $1 \leq |2^{k_0}x| \leq 2$, then for some $|\alpha| = 3, D^\alpha Q \equiv Q_\alpha \neq 0$ and

$$\partial^\alpha w(x) = \sum^{k_0-1} c_k Q_\alpha + c_{k_0}\eta(2^{k_0}x)Q_\alpha + \sum_{\beta\leq\alpha} c_{k_0}\frac{\alpha!}{\beta!(\alpha-\beta)!}(\partial^\beta\eta)(2^{k_0}x)(\partial^{\alpha-\beta}P)(2^{k_0}x)$$

Since $k_0(x) \to \infty$ as $|x| \to 0$, $c_{k_0(x)} \to 0$ as $|x| \to 0$. Moreover, since I assume $|\sum c_k| = \infty$, $\lim_{|x|\to 0} |\partial^\alpha w(x)| = \infty$ and hence w is not $C^{2,1}$ in any neighnorhood of the origin by the mean value theorem (MVT). □

Remark 1. Another example is given in [2, Section 3.4] where $u = (x_1^2 - x_2^2)(-\log|x|)^{1/2}$ on $B_R(0), R < 1$.

Remark 2. This problem is concern the existence of $C^2(\Omega)$ solution to Dirichlet problem in B_1. Another problem one may ask is whether the C^2-global regularity theorem true ? That is, if $u \in C^2(B_1) \cap C^0(\overline{B_1})$ solves $\Delta u = f \in C^0(\overline{B_1})$ and $u = g \in C^2(\overline{B_1})$, can we conclude that $u \in C^2(\overline{B_1})$?

This question is related to the analytic continuation, I find it's answered negatively in [1, Chapter II.3]. The example is the following:

Consider a conformal map $f : D \subset \mathbb{C} \to \Omega$ where $\Omega = \{x + iy : 0 < x < \frac{1}{1+|y|}\}$. Clearly, f is unbounded. On the other hand, Re f has a continuous extension to $\overline{D}$ because it has a finite limit. Write $f(z) = \sum_{n=0} c_n z^n$ and define $F(z) = \sum_{n=1} c_n n^{-2} z^n$. Re$F$ will be the counterexample. The reason is:

If all the second partial derivatives of ReF are bounded on $\overline{D}$, then F'' is bounded by Cauchy-Riemann equations. But this is impossible since $f(z) - f(0) = z(zF')' = zF' + z^2F''$ where the left hand side is unbounded and the right hand side is bounded by MVT.

10. I think the denominator in (a) should be $2(n-2)$, not $2n$. (Of course, this is for $n \geq 3$, and for $n = 2$, we use the same technique as the proof for Theorem 4.6 to show the denominator can be $2(3-2) = 2$). For example, take radial function $f = f(r) \in C_c^\infty(B_R(0))$ such that $-1 \leq f \leq 0, f \equiv -1$ on $r \leq R - 2\epsilon$ and $f \equiv 0$ on $r > R - \epsilon$. Then $|u(0)| = \int_B \frac{|x-y|^{2-n} f(y)}{nw_n(2-n)}\, dy = \int_0^R r f(r)\, dr/(2-n) \in (\frac{(R-2\epsilon)^2}{2(n-2)}, \frac{(R-\epsilon)^2}{2(n-2)})$.

Proof. Since $u \in C_0^2(B), f \in C_0(B)$ and hence for $n \geq 3$ and for each $x \in B$,

$$|u(x)| = |\int_B \frac{|x-y|^{2-n} f(y)}{nw_n(2-n)}\, dy| \leq |f|_0 \frac{R^2}{2(n-2)}$$

On the other hand, for each $x \in B$ and for $n \geq 2$,

$$|D_i u(x)| = |\int_B \frac{|x-y|^{-n}(x_i - y_i) f(y)}{nw_n}\, dy| \leq \int_{B_R(x)} \frac{|x-y|^{1-n}}{nw_n}\, dy |f|_0 = R|f|_0.$$

□

References

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[3] Elliott H Lieb and Michael Loss. *Analysis*, volume 14. American Mathematical Society, Providence, RI,, 2nd edition, 2001.