Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 6 Classical Solutions; the Schauder Approach*

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1. *Proof.* (a) follows from (b), we only present a proof for (b) by mathematical induction here.

We assume this is true for k and try to prove it's true for $k+1$, that is, we assume

$$|a^{ij}|^{(0)}_{k+1,\alpha;\Omega}, |b^i|^{(1)}_{k+1,\alpha;\Omega}, |c|^{(2)}_{k+1,\alpha;\Omega} \leq \Lambda.$$

and there is some $C_k(n,\alpha,\lambda,\Lambda)$ such that for any open $V \subseteq \Omega$, if $Lv = f$ in V with $|a^{ij}|^{(0)}_{k,\alpha;V}, |b^i|^{(1)}_{k,\alpha;V}, |c|^{(2)}_{k,\alpha;V} \leq \Lambda$, then

$$|v|^{(0)}_{k+2,\alpha;V} \leq C_k(n,\alpha,\lambda,\Lambda)(|u|_0 + |f|^{(2)}_{k,\alpha;V}). \tag{1}$$

Now we try to show

$$|u|^{(0)}_{k+3,\alpha;\Omega} \leq C_{k+1}(n,\alpha,\lambda,\Lambda)(|u|_0 + |f|^{(2)}_{k+1,\alpha;\Omega}).$$

for some constant $C_{k+1}(n,\alpha,\lambda,\Lambda)$.

Given $x \in \Omega$ and $B = B_{\frac{d_x}{2}}(x)$, we note that, for each $z \in B$,

(A) $\frac{d_x}{2} \leq d_z$, since $d_x \leq d(x,w) \leq d(x,z) + d(z,w) \leq \frac{d_x}{2} + d(z,w)$ for any $w \in \partial\Omega$.

(B) $d_{z,B} \leq \frac{d_x}{2} \leq d_z$, by (A).

To prove the desired inequality, we need to apply the interior Schauder estimate in the ball.

First, we note that by (B), $[p]^{(s)}_{m;B} \leq [p]^{(s)}_{m;\Omega}$ and by MVT and (A), $[p]^{(s)}_{m,\alpha;B} \leq |p|^{(s)}_{m+1;\Omega}$ for each $0 \leq m \leq k, 0 \leq s$. Hence, we apply this result for $p = a^{ij}, b^i, c$ and $s = 0,1,2$ in the following without mentions.

Second, after differentiating the equation, we have for each $l = 1,2,\cdots n$,

$$L(D_l u)(x) = -D_l a^{ij}(x) D_{ij} u(x) - D_l b^i(x) D_i u(x) - D_l c(x) u(x) + D_l f(x); \tag{2}$$

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by the inductive assumption, the k-th interior Schauder's estimates (1) to (2) in the ball B implies that,

$$\begin{aligned}&d_{x,B}|Du(x)|+d_{x,B}^2|D^2u(x)|+\cdots+d_{x,B}^{k+3}|D^{k+3}u(x)|\\&\le C_k(d_{x,B}\sup_{z\in B}|Du(z)|+d_{x,B}|Da^{ij}(z)D_{ij}u(z)+Db^i(z)D_iu(z)+Dc(z)u(z)-Df(z)|^{(2)}_{k,\alpha;B})\end{aligned}$$

By (A), we know $d_{x,B}\sup_{z\in B}|Du(z)|\le\sup_{z\in B}|Du(z)|d_z$ and by (A)(B)(1), for each $0\le m\le k$,

$$\begin{aligned}&d_{x,B}[Da^{ij}(z)D_{ij}u(z)+Db^i(z)D_iu(z)+Dc(z)u(z)-Df(z)]^{(2)}_{m,B}\\&\le\sup_{|\beta|=m}\sup_{z\in B}\sum_{\gamma\le\beta}|D^\gamma Da^{ij}(z)|d_z^{|\gamma|+1}|D^{\beta-\gamma}D_{ij}u(z)|d_z^{|\beta-\gamma|+2}+|D^\gamma Db^i(z)|d_z^{|\gamma|+2}|D^{\beta-\gamma}D_iu(z)|d_z^{|\beta-\gamma|+1}\\&+|D^\gamma Dc(z)|d_z^{|\gamma|+3}|D^{\beta-\gamma}u(z)|d_z^{|\beta-\gamma|}+|D^\beta Df(z)|d_z^{m+3}\\&\le\sup_{|\beta|=m}\sum_{\gamma\le\beta}[a^{ij}]^{(0)}_{|\gamma|+1}[u]^{(0)}_{|\beta-\gamma|+2}+[b^i]^{(1)}_{|\gamma|+1}[u]^{(0)}_{|\beta-\gamma|+1}+[c]^{(2)}_{|\gamma|+1}[u]^{(0)}_{|\beta-\gamma|}+[f]^{(2)}_{m+1}\\&\le\sup_{|\beta|=m}\sum_{\gamma\le\beta}([a^{ij}]^{(0)}_{|\gamma|+1}+[b^i]^{(1)}_{|\gamma|+1}+[c]^{(2)}_{|\gamma|+1})([u]^{(0)}_{|\beta-\gamma|+2}+[u]^{(0)}_{|\beta-\gamma|+1}+[u]^{(0)}_{|\beta-\gamma|})+[f]^{(2)}_{m+1}\\&\le[f]^{(2)}_{m+1}+\left(|a^{ij}|^{(0)}_{m+1}+|b^i|^{(1)}_{m+1}+|c|^{(2)}_{m+1}\right)\cdot 3|u|^{(0)}_{m+2}\le[f]^{(2)}_{m+1}+9\Lambda|u|^{(0)}_{m+2},\end{aligned}$$

and

$$
\begin{aligned}
&d_{x,B}[Da^{ij}D_{ij}u(y)+Db^iD_iu(y)+Dcu(y)-Df(y)]^{(2)}_{k,\alpha;B}\\
&\leq \sup_{|\beta|=k}\sup_{y,z\in B;y\neq z}\Bigg(\sum_{\gamma\leq\beta}\frac{|D^{\beta-\gamma}Da^{ij}(y)D^\gamma D_{ij}u(y)-D^{\beta-\gamma}Da^{ij}(z)D^\gamma D_{ij}u(z)|}{|y-z|^\alpha}\\
&+\frac{|D^{\beta-\gamma}Db^i(y)D^\gamma D_iu(y)-D^{\beta-\gamma}Db^i(z)D^\gamma D_iu(z)|}{|y-z|^\alpha}+\frac{|D^{\beta-\gamma}Dc(y)D^\gamma u(y)-D^{\beta-\gamma}Dc(z)D^\gamma u(z)|}{|y-z|^\alpha}\\
&+\frac{|D^\beta Df(y)-D^\beta Df(z)|}{|y-z|^\alpha}\Bigg)d^{k+2+\alpha}_{y,z;B}d_{y,z}\\
&\leq \sup_{|\beta|=k}\sup_{y,z\in\Omega;y\neq z}\Bigg(\sum_{\gamma\leq\beta}\frac{|D^{\beta-\gamma}Da^{ij}(y)-D^{\beta-\gamma}Da^{ij}(z)|}{|y-z|^\alpha}d^{|\beta-\gamma|+1+\alpha}_{y,z;B}|D^\gamma D_{ij}u(y)|d^{|\gamma|+2}_{y,z}\\
&+|D^{\beta-\gamma}Da^{ij}(z)|d^{|\beta-\gamma|+1}_{y,z}\frac{|D^\gamma D_{ij}u(y)-D^\gamma D_{ij}u(z)|}{|y-z|^\alpha}d^{|\gamma|+2+\alpha}_{y,z;B}\\
&+\frac{|D^{\beta-\gamma}Db^i(y)-D^{\beta-\gamma}Db^i(z)|}{|y-z|^\alpha}d^{|\beta-\gamma|+2+\alpha}_{y,z;B}|D^\gamma D_iu(y)|d^{|\gamma|+1}_{y,z}\\
&+d^{|\beta-\gamma|+2}_{y,z}|D^{\beta-\gamma}Db^i(z)|\frac{|D^\gamma D_iu(y)-D^\gamma D_iu(z)|}{|y-z|^\alpha}d^{|\gamma|+1+\alpha}_{y,z;B}\\
&+\frac{|D^{\beta-\gamma}Dc(y)-D^{\beta-\gamma}Dc(z)|}{|y-z|^\alpha}d^{|\beta-\gamma|+3+\alpha}_{y,z;B}|D^\gamma u(y)|d^{|\gamma|}_{y,z}\\
&+|D^{\beta-\gamma}Dc(z)|d^{|\beta-\gamma|+3}_{y,z}\frac{|D^\gamma u(y)-D^\gamma u(z)|}{|y-z|^\alpha}d^{|\gamma|+\alpha}_{y,z;B}+\frac{|D^\beta Df(y)-D^\beta Df(z)|}{|y-z|^\alpha}d^{k+3+\alpha}_{y,z}\Bigg)\\
&\leq \sup_{|\beta|=k}\Bigg(\sum_{\gamma<\beta}[a^{ij}]^{(0)}_{|\beta-\gamma|+1,\alpha;B}[u]^{(0)}_{|\gamma|+2}+[a^{ij}]^{(0)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|+2,\alpha;B}+[b^i]^{(1)}_{|\beta-\gamma|+1,\alpha;B}[u]^{(0)}_{|\gamma|+1}+[b^i]^{(1)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|+1,\alpha;B}\\
&+[c]^{(2)}_{|\beta-\gamma|+1,\alpha;B}[u]^{(0)}_{|\gamma|}+[c]^{(2)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|,\alpha;B}\Bigg)+\Bigg([a^{ij}]^{(0)}_{1,\alpha;B}[u]^{(0)}_{|\beta|+2}+[a^{ij}]^{(0)}_1[u]^{(0)}_{|\beta|+2,\alpha;B}+[b^i]^{(1)}_{1,\alpha;B}[u]^{(0)}_{|\beta|+1}\\
&+[b^i]^{(1)}_1[u]^{(0)}_{|\beta|+1,\alpha;B}+[c]^{(2)}_{1,\alpha;B}[u]^{(0)}_{|\beta|}+[c]^{(2)}_1[u]^{(0)}_{|\beta|,\alpha;B}\Bigg)+[f]^{(2)}_{k+1,\alpha}\\
&\leq\Bigg([a^{ij}]^{(0)}_{k+1,\alpha}[u]^{(0)}_2+[a^{ij}]^{(0)}_{k+1}[u]^{(0)}_{3;\Omega}+[b^i]^{(1)}_{k+1,\alpha}[u]^{(0)}_1+[b^i]^{(1)}_{k+1}[u]^{(0)}_{2;\Omega}+[c]^{(2)}_{k+1,\alpha}[u]^{(0)}_0+[c]^{(2)}_{k+1}[u]^{(0)}_{1;\Omega}\Bigg)\\
&+\sup_{|\beta|=k}\Bigg(\sum_{0<\gamma<\beta}[a^{ij}]^{(0)}_{|\beta-\gamma|+2;\Omega}[u]^{(0)}_{|\gamma|+2}+[a^{ij}]^{(0)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|+3;\Omega}+[b^i]^{(1)}_{|\beta-\gamma|+2;\Omega}[u]^{(0)}_{|\gamma|+1}+[b^i]^{(1)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|+2;\Omega}\\
&+[c]^{(2)}_{|\beta-\gamma|+2;\Omega}[u]^{(0)}_{|\gamma|}+[c]^{(2)}_{|\beta-\gamma|+1}[u]^{(0)}_{|\gamma|+1;\Omega}\Bigg)\\
&+\Bigg([a^{ij}]^{(0)}_{2;\Omega}[u]^{(0)}_{k+2}+[a^{ij}]^{(0)}_1[u]^{(0)}_{k+2,\alpha;\Omega}+[b^i]^{(1)}_{2;\Omega}[u]^{(0)}_{k+1}+[b^i]^{(1)}_1[u]^{(0)}_{k+2;\Omega}+[c]^{(2)}_{2;\Omega}[u]^{(0)}_k+[c]^{(2)}_1[u]^{(0)}_{k+1;\Omega}\Bigg)+[f]^{(2)}_{k+1,\alpha}\\
&\leq 2\Big(|a^{ij}|^{(0)}_{k+1,\alpha}+|b^i|^{(1)}_{k+1,\alpha}+|c|^{(2)}_{k+1,\alpha}\Big)|u|^{(0)}_{k+2,\alpha}+[f]^{(2)}_{k+1,\alpha}\leq 6\Lambda C_k\Big(|u|_0+|f|^{(2)}_{k+1,\alpha}\Big)+[f]^{(2)}_{k+1,\alpha}.
\end{aligned}
$$

Then

$$
|u|^{(0)}_{k+3,\alpha}\leq 2^{k+3}\Big\{|u|_0+C_k(C_k+(9+6)\Lambda C_k+1)(|u|_0+|f|^{(2)}_{k+1,\alpha})\Big\}\leq C_{k+1}(|u|_0+|f|^{(2)}_{k+1,\alpha}),
$$

where $C_{k+1}:=2^{k+3}\Big(1+C_k+C_k^2+15\Lambda C_k^2\Big)$. □

2. *Proof.* □

3. One of the counterparts of exterior cone condition for parabolic equations is the exterior tusk condition. See Lieberman [1, Exercise 3.11] and Lorenz [2, Section 3.11.4].

   Proof. □

4. If one go through the details of the construction of Perron solution, we will find out that the condition that $\frac{b^i}{\lambda}$ is bounded in Ω is only used to show $v^{\pm} = \pm \sup_{\partial\Omega} |\varphi| \pm (e^{\gamma d} - e^{\gamma x_1}) \sup_{\Omega} \frac{|f|}{\lambda}$ are super-(sub-)function of the Dirichlet problem $Lu = f$ in $\Omega, u = \varphi$ on $\partial\Omega$. However, in this problem we know $w^{\pm} \equiv \pm \sup_{\partial\Omega} |\varphi|$ will be a super-(sub-) function even if $\frac{b^i}{\lambda}$ is unbounded.

   Proof. As mentioned above, the existence of Perron solution $u(x)$ is examined in Section 6.3 and 6.6. To see $u(x) \to \varphi(x_0)$ as $x \to x_0$, we follow the Remarks after Lemma 6.12 to establish $w_\epsilon^{\pm} = \varphi(x_0) \pm \epsilon \pm k_\epsilon \nu(x_0) \cdot (x - x_0)$ as a local barrier relative to L, φ and $\sup_\Omega |\varphi|$ at x_0 for some suitable positive constants k_ϵ. First we let $B(x_0) =: B$ be the ball such that $b \cdot \nu(x_0) \geq 0$ in $B \cap \Omega$. Then $w_\epsilon^{\pm}(x_0) \to \varphi(x_0)$ as $\epsilon \to 0$ and $w_\epsilon^{\pm}$ is a sub-(sup-)solution in $\Omega \cap B$ since $Lw_\epsilon^{\pm} = \pm k_\epsilon b(x) \cdot \nu(x_0)$.

   Next, we check $w_\epsilon^+ \geq \sup_\Omega |\varphi|$ on $\partial B \cap \Omega$ and $w_\epsilon^+ \geq \varphi$ on $B \cap \partial\Omega$. (A sign changed argument for w_ϵ^- part is omitted.) By uniform continuity of φ, we know $|\varphi(x) - \varphi(x_0)| < \epsilon$ if $|x - x_0|$ is less than some $\delta = \delta(\epsilon)$. By the strictly convexity of Ω at x_0, we know $\nu(x_0) \cdot (x - x_0) > 0$ for all $x \in \partial(\Omega \cap B) \setminus \{x_0\}$, and hence by the continuity, $\nu(x_0) \cdot (x - x_0) \geq \tau > 0$ on $\partial(\Omega \cap B) \setminus B_\delta(x_0)$ for some $\tau = \tau(\epsilon)$.

   Now, we see that if we pick $k_\epsilon = 2\frac{\sup_\Omega |\varphi|}{\tau}$, then $w_\epsilon^+ = \varphi(x_0) + \epsilon + k_\epsilon \nu(x_0) \cdot (x - x_0) \geq \varphi(x)$ if $x \in B_\delta(x_0) \cap \partial\Omega$ and $\varphi(x_0) + \epsilon + 2\frac{\sup_\Omega |\varphi|}{\tau}\nu(x_0) \cdot (x - x_0) \geq -\sup |\varphi| + \epsilon + 2 \sup |\varphi| > \sup |\varphi|$ if $x \in \partial(\Omega \cap B) \setminus B_\delta(x_0)$. □

   Remark 1. Note that we only use $b \cdot \nu(x_0) \geq 0$ in showing $w_\epsilon^{\pm}$ is sub-(super-)solution. Is it necessary to assume $b \cdot \nu(x_0) > 0$ in a neighborhood of x_0 ?

5. We follows Michael [3].

   Proof. □

6. We follows Michael [3, Section 5].

   Proof. □

7. *Proof.* □

8. *Proof.* ☐

9. *Proof.* ☐

10. This is based on Olejnik and Radkevic[4].

    Proof. ☐

11. *Proof.* ☐

References

[1] Gary M Lieberman. *Second Order Parabolic Differential Equations.* World scientific, revised edition, 2005.

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[3] JH Michael. A general theory for linear elliptic partial differential equations. *Journal of Differential Equations*, 23(1):1–29, 1977.

[4] AO Olejnik and EV Radkevic. *Second order equations with nonnegative characteristic form.* 1973.