Additional Problems to L^p Spaces*

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Abstract

As an appendix to Folland [5, Chapter 6], those exercises are related to the L^p Spaces, collected from Jones [6, Chpater 10], Rudin [11, Chapter 3], Stein-Shakarchi [12, Chapter 1], Wheeden-Zygumnd [14, Chapter 8-10]. The order of sections follows from Folland's book.

6.1 Basic Theory of L^p Spaces

1. (Separability of L^p Spaces, Wheeden-Zygumnd [14, Exercise 10.9])

Let $1 \leq p < \infty$, and $(X, \mathscr{M}, \mu)$ be a positive measure space. Define a function $\rho(A, B) = \mu(A\Delta B)$ on $\mathscr{M} \times \mathscr{M}$. Show that (i) $(\mathscr{M}, \rho)$ is a metric space; (ii) $L^p(X, \mu)$ is separable if $(\mathscr{M}, \rho)$ is; (iii) if μ is finite, then $(\mathscr{M}, \rho)$ is separable provided $L^p(X, \mu)$ is.

Proof. (i)We only check the triangle inequality: Given $E_1, E_2, E_3 \in \mathscr{M}$. Since

$$\begin{aligned} E_1 \setminus E_2 = [(E_1 \setminus E_2) \cap E_3] \cup [(E_1 \setminus E_2) \setminus E_3] &= [(E_1 \cap E_3) \setminus E_2] \cup [E_1 \setminus (E_2 \cup E_3)] \\ &\subseteq (E_3 \setminus E_2) \cup (E_1 \setminus E_3) \end{aligned}$$

$$\begin{aligned} E_2 \setminus E_1 = [(E_2 \setminus E_1) \cap E_3] \cup [(E_2 \setminus E_1) \setminus E_3] &= [(E_2 \cap E_3) \setminus E_1] \cup [E_2 \setminus (E_1 \cup E_3)] \\ &\subseteq (E_3 \setminus E_1) \cup (E_2 \setminus E_3), \end{aligned}$$

we see

$$E_1 \Delta E_2 \subseteq (E_1 \Delta E_3) \cup (E_3 \Delta E_2)$$

which implies the desired triangle inequality.

(ii)Let $\mathscr{A} := \{A_n\}$ be a countable dense subset of $\mathscr{M}$. Since the algebra generated by $\mathscr{A}$ is also countable, we may assume $\mathscr{A}$ is an algebra. Let S be the family of simple function whose

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coefficients are rational and sets are in $\mathscr{A}$, then S is countably many. Since the family of all simple functions with rational coefficients is dense in L^p, we will show S is dense in this set.

Let $g = \sum_k c_k\chi_{E_k}$ for some $c_k \in \mathbb{Q}$ and $E_k \in \mathscr{M}$ with $\cup E_k = X$. Let $c = \max_k|c_k|$, we may assume $c > 0$. Given $\epsilon > 0$, there exist $A_1, \cdots A_n \in \mathscr{A}$ such that $\mu(E_k\Delta A_k) < \frac{1}{2n}\frac{\epsilon^p}{(2c)^p}$ for all k. Since $\mathscr{A}$ is an algebra, $A_k \supset B_k := A_k \setminus \cup_{j=1}^{k-1}A_j \in \mathscr{A}$ for each k, $\cup_j B_j = \cup_j A_j$, and those B_j are mutually disjoint.

Let $f(x) = \sum_k c_k\chi_{B_k} \in S$, then

$$\mu(\{x : f \neq g\}) = \mu(\cup_k\{x \in B_k \subset A_k : f \neq g\} \cup \{x \notin \cup_k B_k = \cup_k A_k : f \neq g\}) \leq n\frac{1}{2n}\frac{\epsilon^p}{(2c)^p}2.$$

So, $\int_X |f - g|^p\, d\mu \leq (2c)^p\frac{\epsilon^p}{(2c)^p} = \epsilon^p$. Therefore S is dense in L^p, and hence L^p is separable.

(iii)Since μ is finite, the family C of all characteristic functions of measurable sets is a subset of $L^p(X, \mathscr{M}, \mu)$. Let d be the metric induced by L^p norm on L^p, since (L^p, d) is separable, (S, d) is separable. (You need some trick to prove this, and realize it's not true in general topological spaces.)

Since the map $T : (\mathscr{M}, \rho) \to (S, d)$ defined by $A \mapsto \chi_A$ is an isometry, $(\mathscr{M}, \rho)$ is separable. □

2. In [10] L.A.Rubel gives a complex-variable proof for Hölder's inequality, which is very similar to the proof of Riesz-Thorin theorem. We present his proof here.

   Proof. □

3. Related to Exercise 6.3 and 6.4, we are going to prove a set equality due to Alvarez [1]:

   Theorem 1. *Let* $L^p_q := L^p + L^q$. *Then (a)* $L^p_q + L^r_s = L^{\min(p,r)}_{\max(q,s)}$ *(b)* $L^p_q \cdot L^r_s = L^u_v$, *where* $u^{-1} = p^{-1} + r^{-1}; v^{-1} = q^{-1} + s^{-1}$ *(c)* $L^p_q(L^{p_0}_{q_0} + L^{p_1}_{q_1}) = L^p_q \cdot L^{p_0}_{q_0} + L^p_q \cdot L^{p_1}_{q_1}$.

   Proof. □

   Remark 2. L^p_q is a special case of Orlicz spaces. Their duality results are given in Stein-Shakarchi [12, Exercise 1.24, 1.26 and Problem 1.5].

4. Related to Exercise 6.5, A. Villani [13] simplified Romero's previous work [9] and show that,

   Theorem 3. *Let* (Ω, Σ, μ) *be the measure space and* $0 < p < q \leq \infty$. *The followings are equivalent: (i)* $L^p(\mu) \subseteq L^q(\mu)$ *for some* $p < q$*; (ii)* $L^p(\mu) \subseteq L^q(\mu)$ *for all* $p < q$*; (iii)* $\inf\{\mu(E) : \mu(E) > 0\} > 0$.

   In the case $q < p$, *condition (iii) is replaced by (iii)'* $\sup\{\mu(E) : \mu(E) < \infty\} < \infty$.

This is then generalized by Miamee [7] to the more geneal relation $L^p(\mu) \subseteq L^q(\nu)$.

Theorem 4. *Let μ and ν be positive measures defined on a measurable space (Ω, Σ) and $0 < p < q < \infty$. The inclusion $L^p(\mu) \subseteq L^q(\nu)$ holds iff ν is absolutely continuous with respect to μ and there exists a constant $C(p,q)$ such that $\|f\|_{L^q(\nu)} \leq C(p,q)\|f\|_{L^p(\mu)}$ for all $f \in L^p(\mu)$.*

proof of Theorem 3. □

proof of Theorem 4. □

5. This result is related to Exercise 6.10 and 6.20.

Theorem 5. *(Brezis-Lieb lemma[4])*

Suppose $f_n \to f$ a.e. and $\|f_n\|_p \leq C < \infty$ for all n and for some $0 < p < \infty$. Then

$$\lim_{n\to\infty} \|f_n\|_p^p - \|f_n - f\|_p^p = \|f\|_p^p.$$

Proof. □

Remark 6.

6. This is another related result to Exercise 6.10. Note that it's weaker than the Schur's property in additional exercise 6.2.2.

Theorem 7. *(Radon-Riesz Property)*

Let $p \in (1, \infty)$. Suppose $f_k \rightharpoonup f$ in $L^p(X, \mu)$ and $\|f_n\|_p \to \|f\|_p$, then $\|f_n - f\|_p \to 0$.

Although there is Schur's property for l^1, this theorem is not true, for example $L^1(\mathbb{R}, m)$. This can be seen by considering $f_n(x) = 1 + \sin(nx)$.

Another classical example to Radon-Riesz property is Hilbert space, whose proof is relatively easy! Actually we have:

Theorem 8. *(Uniform Convexity $\Rightarrow$ Radon-Riesz Property, Brezis [3, Theorem 3.32])*

Let X be a uniformly convex Banach space, Suppose $f_k \rightharpoonup f$ in X and $\|f_n\|_X \to \|f\|_X$, then $\|f_n - f\|_X \to 0$.

proof of Theorem 8. If $f \equiv 0$, then it's done. So assume $f \not\equiv 0$, we may assume $f_n \not\equiv 0$ for all n by deleting the first N terms. Set $g_n = \frac{f_n}{\|f_n\|}$ and $g = \frac{f}{\|f\|}$, then $g_n \rightharpoonup g$.

By the Hahn-Banach Theorem, there exists a functional $x^* \in X^*$ with $\|x^*\| = 1$ and such that $x^*(g) = \|g\| = 1$. Then as $n \to \infty$,

$$1 \geq \|\frac{g_n + g}{2}\| \geq |x^*(\frac{g_n + g}{2})| \to 1.$$

If $\|g_n - g\| \not\to 0$ as $n \to \infty$, then there exists $\epsilon > 0$ and sequences $n_K \in \mathbb{N}$ such that for each $K \in \mathbb{N}, n_K > K$ and $\|g_{n_K} - g\| > \epsilon$. By uniform convexity, there exists $\delta = \delta(\epsilon) > 0$ such that $\|\frac{g_{n_K}+g}{2}\| \leq 1 - \delta$, which contradicts to $\|\frac{g_n+g}{2}\| \to 1$ as $n \to \infty$.

Therefore $\|\frac{f_n}{\|f_n\|} - \frac{f}{\|f\|}\| = \|g_n - g\| \to 0$ which implies $\|f_n - f\| \to 0$ by triangle inequality and the fact $\|f_n\| \to \|f\|$. □

Due to the above proof uses the Hahn-Banach Theorem and Uniform Convexity, which seems unnecessary in the case of L^p spaces $(1 < p < \infty)$, we prove the L^p case without using them.

proof of Theorem 7. (Taken from Riesz-Nagy [8, p.78-80])

□

7. Exercise 6.11 on essential range is extended in Rudin [11, Exercise 3.19].

   Let R_f be the essential range of $f \in L^\infty(X, \Sigma, \mu)$. Let A_f be the set of all averages

$$\frac{1}{\mu(E)} \int_E f \, d\mu$$

   where $E \in \Sigma$ and $\mu(E) > 0$. What relations exist between A_f and R_f? Is A_f always closed?? Are there measures μ such that A_f is convex for every $f \in L^\infty(\mu)$? Are there measures μ such that A_f fails to be convex for some $f \in L^\infty$?

   How about these results affected if $L^\infty(\mu)$ is replaced by $L^1(\mu)$, for instance?

   Proof. □

8. (Exercise 6.15, Vitali Convergence Theorem) These remarks come from Rudin [11, Ex 6.10-11].

   (a) Show that we can not omit the tightness condition (iii): for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all n, even if $\{\|f_n\|_1\}$ is bounded.

   Proof. Consider the Lebesgue measure on $(-\infty, \infty)$ with $f_n = \chi_{(n,n+1)}$. □

   (b)To apply Vitali's theorem in finite measure space, sometimes we see $|f(x)| < \infty$ a.e. is automatically true, but sometimes it's not. Give examples.

Proof. (i) Consider the Lebesgue measure on $[0,1]$, by the uniform integrability we know there exist nonoverlapping closed intervals $I_1, \cdots I_k$ whose union is $[0,1]$, and for all $j = 1, \cdots k$ and $n \in \mathbb{N}$, $\int_{I_j} |f_n| < 1$. (It's easy to show it's equivalent to use $|\int f|$ and $\int |f|$ in the definition of uniform integrability.) By Fatou's Lemma,

$$\int_{[0,1]} |f| \leq \liminf_n \int_{[0,1]} |f_n| < k.$$

(ii) We need to find out a finite measure space $(X, \mathscr{M}, \mu)$ and an uniformly integrable sequence of L^1 functions f_n with $f_n \to f$ a.e., $f(x)$ is not finite a.e. and $f_n \nrightarrow f$ in L^1.

On $\mathbb{R}$, let $\mathscr{M}$ is the σ-algebra of countable or co-countable sets. $\mu(E) := 0$ if E is countable, $\mu(E) := 1$ if E is co-countable, which is easy to check μ is a measure on $\mathscr{M}$. Consider $f_n \equiv n$, which is the desired example. $\square$

(c) It's easy to see Vitali's Theorem implies Lebesgue Dominated Convergence Theorem in finite measure space. The sequence $f_n(x) = \frac{1}{x}\chi_{(\frac{1}{n+1},\frac{1}{n})}(x)$ is an example in which Vitali's theorem applies although the hypothesis of Lebesgue's theorem do not hold.

(d)The sequence $f_n = n\chi_{(0,1/n)} - n\chi_{(1-\frac{1}{n},1)}$ on [0,1] shows the assumption that $f_n \geq 0$ is sometimes important in some applications. Note that $f_n(x) \to 0$ for every $x \in [0,1]$, $\int f_n(x)\,dx = 0$, but f_n is not uniformly integrable.

(e) However, the following converse of Vitali's theorem is true:

Theorem 9. *If $\mu(X) < \infty$, $f_n \in L^1(\mu)$, and $\lim_{n\to\infty} \int_E f_n\,d\mu$ exists for every $E \in \mathscr{M}$, then $\{f_n\}$ is uniformly integrable.*

Proof. As hint by Rudin, we define $\rho(A,B) = \int |\chi_A - \chi_B\,d\mu$. Then $(\mathscr{M}, \rho)$ is a complete metric space (modulo sets of measure zero), and $E \mapsto \int_E f_n\,d\mu$ is continuous for each n, (denote this map by F_n.) If $\epsilon > 0$, consider $A_N = \{E : |F_n(E) - F_m(E)| < \epsilon,\ \text{if } n, m \geq N\}$. Since $X = \cup A_N$ by hypothesis, Baire Category theorem implies that some A_N has nonempty interior, that is, there exist $E_0 \in \mathscr{M}, \delta > 0, N \in \mathbb{N}$ so that

$$|\int_E (f_n - f_N)\,d\mu| < \epsilon \ \text{ if } \rho(E, E_0) < \delta,\ n > N. \tag{1}$$

If $\mu(A) < \delta$, (1) holds with $B = E_0 \setminus A$ and $C = E_0 \cup A$ in place of E. Thus,

$$|\int_A (f_n - f_N)\,d\mu| = |\int_C - \int_B (f_n - f_N)\,d\mu| < 2\epsilon.$$

By considering $\{f_1, \cdots f_N\}$, there exists $\delta' > 0$, such that

$$|\int_A f_n\,d\mu| < 3\epsilon \ \text{ if } \mu(A) < \delta',\ n = 1, 2, 3, \cdots$$

$\square$

(f) The Dunford-Pettis theorem: [3, p.467-472]

9. *Proof.* $\square$

6.2 The Dual of L^p

1. *Proof.* Rudin Ex6.4? $\square$

2. In the content of Proposition 6.13, show that for every measure space (X, Σ, μ) which is not semifinite, there is a $g \in L^\infty$ such that $\|\phi_g\| < \|g\|_\infty$.

   Proof. Let (X, Σ, μ) be our nonsemifinite measure space. Then there is a measurable subset A of X with $\mu(A) = \infty$ that does not have a measurable subset with nonzero finite measure. Consider the function $g = \chi_A \in L^\infty$, given $f \in L^1$, then on A, there is a sequence of simple functions $f_n = \sum_i c_{n,i}\chi_{E_{n,i}}$ such that $c_{n,i} \in \mathbb{R}$, $E_{n,i} \subseteq A$ and $0 \leq f_n \nearrow |f|$ a.e..

   By MCT, $0 \leq |\int fg| \leq \int_A |f| = \lim_n \int_A f_n = \lim_n \sum_i c_{n,i}\mu(E_{n,i})$. Since $|f| \in L^1$, $\mu(E_{n,i}) < \infty$ for each n, i and therefore $\mu(E_{n,i}) = 0$. So $|\int fg| = 0$.

   In particular, $\phi_g = \sup\{|\int fg| : \|f\|_1 = 1\} = 0 < 1 = \|g\|_\infty$. $\square$

3. Schur's property

   Proof. $\square$

4. Uniform convexity and Reflexivity (Brezis [3, Theorem 3.31] Milman-Pettis)

   Proof. $\square$

5. Eberlein-$\breve{S}$mulian theorem, Brezis [3, Theorem 3.19]

   Proof. $\square$

6. Almost decomposable $\not\Leftarrow$ semifinite.

   Proof. $\square$

7. From Bogachev [2, 1.12.134], we know μ is semifinite $\not\Rightarrow$ μ is decomposable.

Proof. □

8. Stein [12, Exercise 1.18]. Mixed norm and its dual space.

   Proof. □

9. *Proof.* □

10. *Proof.* □

6.3 Some Useful Inequalities

1. Hanner's inequality uniform convexity

   Proof. □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

11. *Proof.* □

12. *Proof.* □

6.4 Distribution Functions and Weak L^p

1. Normability of weak L^p spaces

   Proof. □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

6.5 Interpolation of L^p Spaces

1. The $L^p(\mathbb{R}^d;\mathbb{R})$ case in Riesz-Thorin's theorem.

   Proof. □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

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