Real Analysis, E.M.Stein-R.Shakarchi

Chapter 3 Differentiation and Integration*

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Exercises

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

11. **If $a, b > 0$, let**

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \textbf{for } 0 < x \leq 1, \\ 0 & \textbf{if } x = 0. \end{cases} \tag{1}$$

Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$. Then, by taking $a = b$, construct (for each $0 < \alpha < 1$) a function that satisfies the Lipschitz condition of exponent α but which is not of bounded variation.

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Proof. If $a \geq b$, then $\mathrm{Var}(f) \geq \sum_{k=1}^{N} (\frac{2}{2k-1})^{b/a} \to \infty$ as $N \to \infty$. Conversely, one can show f is of bounded variation from $|f'(x)| \leq ax^{a-1} + bx^{a-b-1}$. When $a = b$, one note that if $1 > h > 0$, then by MVT,

$$|(x+h)^a \sin(x+h)^{-a} - x^a \sin x^{-a}| \leq \min\{2a\frac{h}{x}, 2(x+h)^a\}.$$

If $x^{a+1} \geq h$, then $|(x+h)^a \sin(x+h)^{-a} - x^a \sin x^{-a}| \leq 2ah^{1-\frac{1}{a+1}} =: 2ah^{\alpha}$.

If $x^{a+1} < h < 1$, then $|(x+h)^a \sin(x+h)^{-a} - x^a \sin x^{-a}| \leq 2(h^{\frac{1}{a+1}} + h)^a \leq 2^{1+a}h^{\alpha}$. $\square$

12. *Proof.* $\square$

13. *Proof.* $\square$

14. *Proof.* $\square$

15. *Proof.* $\square$

16. *Proof.* $\square$

17. *Proof.* $\square$

18. *Proof.* $\square$

19. *Proof.* $\square$

20. *Proof.* $\square$

    ***Remark* 1.** See also [4, Exercise 7.7] , [1, Example 8.30] and [3, Theorem 1.37] for an advanced result of (a).

21. *Proof.* $\square$

22. *Proof.* $\square$

23. *Proof.* $\square$

24. *Proof.* $\square$

25. *Proof.* $\square$

26. *Proof.* $\square$

27. *Proof.* $\square$

28. *Proof.* $\square$

29. *Proof.* □

30. *Proof.* □

31. *Proof.* □

32. *Proof.* □

Problems

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. **In problem 5.8 of Book I, it's shown that the following Lacunary Fourier series is α-Hölder continuous and nowhere differentiable (and hence not of bounded variation)**

$$f_1(x) := \sum_{n=0}^{\infty} 2^{-n} e^{2\pi i 2^n x}$$

Another simplified proof can be found in Jones [2, Section 16.H].

8. *Proof.* □

References

[1] Bernard R Gelbaum and John MH Olmsted. *Counterexamples in analysis.* Dover Publications, corrected reprint of the second (1965) edition edition, 2003.

[2] Frank Jones. *Lebesgue Integration on Euclidean Space.* Jones & Bartlett Learning, revised edition, 2001.

[3] Giovanni Leoni. *A first course in Sobolev spaces*, volume 181. American Mathematical Society Providence, RI, 2nd edition, 2017.

[4] Walter Rudin. *Real and Complex Analysis.* Tata McGraw-Hill Education, third edition, 1987.