

Complex Analysis, Stein and Shakarchi

Chapter 3 Meromorphic Functions and the Logarithm

Yung-Hsiang Huang*

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Exercises

1. From the identity $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$, it's easy to show its zeros are exactly $n \in \mathbb{Z}$. Each of them is of order 1 by showing $(\frac{d}{dz} \sin \pi z)(n) \neq 0$. The residue of $\frac{1}{\sin \pi z}$ at $z = n$ is $\frac{(-1)^n}{\pi}$ by computing $\lim_{z \rightarrow n} \frac{z-n}{\sin \pi z}$ with the weak form of L'Hôpital's Rule.

2. Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

Proof. The computation is standard, see [1, Section 11.1] □

3. It's standard to compute the Fourier transform of 1-d Poisson Kernel for upper half plane. We omit it.
4. It's also standard to compute the Fourier transform of the conjugated Poisson Kernel (which is related to the Hilbert transform). We omit it.
5. Use contour integration to show that

$$\int_{\mathbb{R}} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}$$

for all $\xi \in \mathbb{R}$.

Proof. □

*Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

6. **Show that**

$$\int_{\mathbb{R}} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.$$

Proof.

□

7. **Prove that whenever $a > 1$,**

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}.$$

Proof.

□

8. **Prove that if $a, b \in \mathbb{R}$ and $a > |b|$, then**

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Proof. The basic idea is to compute the integrand along the unit circle $|z| = 1$. In the integrand, $\cos \theta$ is read as $\frac{z+\bar{z}}{2|z|} = \frac{z+z^{-1}}{2}$ and $d\theta$ is read as $\frac{dz}{iz/|z|} = \frac{dz}{iz}$. So the integrand is a meromorphic function inside S^1 .

□

9. **Show that**

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Proof.

□

10. **Show that if $a > 0$, then**

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Proof.

□

11. **Show that if $|a| < 1$, then**

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0.$$

Then, prove that the above result remains true if we assume only that $|a| \leq 1$.

Proof.

□

12. **Suppose u is not an integer. Prove that**

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$$

by integrating

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

over the circle $|z| = R_N = N + \frac{1}{2}$ ($N \in \mathbb{N}, N \geq |u|$), adding the residues of f inside the circle, and letting $N \rightarrow \infty$.

Remark 1. Check Exercise 3.9 and 5.15 in Book I for other two proofs, based on Fourier series and Poisson's summation formula respectively.

Proof. The computation is standard, see [1, Section 11.2-I,II] for a general principle.

□

Remark 2. See [1, Section 11.2-III and Chapter 12] for more applications.

13. **Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$. Suppose also that**

$$|f(z)| \leq A|z - z_0|^{\epsilon-1}$$

for some $\epsilon > 0$ and all z near z_0 . Show that the singularity of f at z_0 is removable.

Proof. Let $g(z) = (z - z_0)f(z)$. Then g is holomorphic in the punctured disc and $g(z) \rightarrow 0$ as $z \rightarrow z_0$. So g is bounded and hence holomorphic in the unpunctured disc $D_r(z_0)$ by Riemann's theorem on removable singularity with $g(z_0) = 0$. Hence $g(z) = (z - z_0)h(z)$ in some neighborhood of z_0 , where h is a holomorphic function there. So $h = f$ in the punctured neighborhood by uniqueness theorem and hence z_0 is a removable singularity for f .

□

14. **Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}, a \neq 0$. [Hint: Apply the Casorati-Weierstrass Theorem to $f(1/z)$.]**

Proof.

□

15. **Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:**

(a)(Extended Liouville's Theorem) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

(b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg z < \varphi$ as $|z| \rightarrow 1$, then $f \equiv 0$.

(c) Let w_1, \dots, w_n be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points $w_j, 1 \leq j \leq n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points $w_j, 1 \leq j \leq n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

Proof. (a) By the Cauchy inequalities,

$$|f^{(n)}(0)| \leq n! \frac{AR^k + B}{R^n}$$

For $n > k$, taking the limit as $R \rightarrow \infty$ implies $f^{(n)}(0) = 0$. Since f is entire and all derivatives higher than k vanish, f is a polynomial of degree at most k by series expansion.

(b) (Note that we don't need the boundedness assumption.)

(c) Let $f(z) = \prod_{i=1}^n (z - w_i)$. Then f is entire with $|f(0)| = 1$. By maximum modulus theorem, there is some $z_0 \in S^1$ such that $|f(z_0)| \geq 1$. Note that $|f(w_i)| = 0$ and the restriction of $|f|$ on S^1 is a real continuous function on the interval $[0, 1]$, so the intermediate theorem implies $|f(w)| = 1$ for some $w \in S^1$.

(d) It's easy to prove by applying Liouville's theorem to the function $g(z) = e^{f(z)}$. □

Remark 3. The proof for (b) has the same idea as Exercise 2.2 of Gilbarg-Trudinger [3].

16. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z)$$

Show that if ϵ is sufficiently small, then

(a) $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and

(b) if z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

Proof. (a) is an easy consequence of Rouché's Theorem. (Note that $\inf_{|z|=1} f(z) > 0$ by the nonvanishing assumption).

(b) By (a), the map is defined on some $I = [0, \epsilon_0)$. Let $M = \sup_{|z| \leq 1} |g(z)|$. Given $b_0 \in I$ and then for any small $\eta > 0$, $\overline{B_\eta(z_{b_0})} \subset B_1$ and $\inf_{w \in \partial B_\eta(z_{b_0})} |f_{b_0}(w)| =: l_\eta > 0$, we note that for all $b \in B_{\frac{l_\eta}{M}}(b_0) \cap I$, f_b has a zero in $B_\eta(z_{b_0})$ according to Rouché's theorem and for any $z \in \partial B_\eta(z_{b_0})$,

$$|f_b(z) - f_{b_0}(z)| = |b - b_0| |g(z)| \leq |b - b_0| M < l_\eta \leq |f_{b_0}(z)|.$$

So the map $\epsilon \mapsto z_\epsilon$ is continuous. □

Remark 4. Conway [2, Page 125] contains a symmetric form of Rouché's Theorem discovered by Glicksberg (Monthly 1976, pp.186-187).

17. **Let f be non-constant and holomorphic in an open set containing $\overline{\mathbb{D}} = \overline{B_1(0)}$.**

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc. [Hint: One must show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$. To do this, it suffices to show that $f(z) = 0$ has a root (why?). Use the maximum modulus principle to conclude.]

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a point $z_0 \in D$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

Proof. (a) By Rouché's theorem, $f(z)$ and $f(z) - w$ has the same number of zeros inside the unit circle provided $|w| < 1$. So we reduce the problem to show f has a zero inside the unit circle. Note that $|f| < 1$ on \mathbb{D} by the maximum principle. So $|f| > |f(0)|$ on S^1 and hence f and $f - f(0)$ has the same number of zeros on \mathbb{D} . (b) is the same proof except the role $f(0)$ is replaced by $f(z_0)$. □

Remark 5. In (a) we actually show more precisely that $f(\mathbb{D}) = \mathbb{D}$.

Another way to show that you may assume $w_0 = 0$ is to use the function $\varphi(z) = \frac{z-w_0}{1-\overline{z}w_0}$ and the facts about the Möbius transform. (See Section 8.2.1 in Stein-Shakarchi).

18. **Give another proof of the Cauchy integral formula**

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

using homotopy of curves.

[Hint: Deform the circle C to a small circle centered at z , and note that the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded.]

Proof.

□

19. We omit the proof for the strong maximum principle for harmonic functions.
20. This exercise shows how the mean square convergence dominates the uniform convergence of analytic functions. If U is an open subset of \mathbb{C} we use the standard notation $\|f\|_{L^2(U)}$ and $\|f\|_{L^\infty(U)}$ to denote the L^2 norm and sup norm with respect to Lebesgue measure on \mathbb{R}^2 .

(a) If f is holomorphic in a neighborhood of the disc $D_r(z_0)$, show that for any $0 < s < r$ there exists a constant $C > 0$ (which depends on s and r) such that

$$\|f\|_{L^\infty(D_s(z_0))} \leq C\|f\|_{L^2(D_r(z_0))}.$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the L^2 norm, then the sequence $\{f_n\}$ converges uniformly on every compact subset of U to a holomorphic function.

[Hint: Use mean-value property.]

Proof.

□

21. Certain sets have geometric properties that guarantee they are simply connected.
- (a) An open set $\Omega \subset \mathbb{C}$ is convex if for any two points in Ω , the straight line segment between them is contained in Ω . Prove that a convex open set is simply connected.
- (b) More generally, an open set $\Omega \subset \mathbb{C}$ is star-shaped if there exists a point $z_0 \in \Omega$ such that for any $z \in \Omega$, the straight line segment between z and z_0 is contained in Ω . Prove that a star-shaped open set is simply connected.

Conclude that the slit plane $\mathbb{C} \setminus \{(-\infty, 0]\}$ (and more generally any sector, convex or not) is simply connected.

(c) What are other examples of open sets that are simply connected?

Proof.

□

22. Show that there is no holomorphic function f in the unit disc \mathbb{D} that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$.

Proof.

□

Problem

1. *Proof.* □
2. *Proof.* □
3. If $f(z)$ is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0 , then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$. There is a generalization of this expansion that holds even if z_0 is an essential singularity. This is a special case of the *Laurent series expansion*, which is valid in an even more general setting.

Let f be holomorphic in a region containing the annulus $\{z : r_1 \leq |z - z_0| \leq r_2\}$ where $0 < r_1 < r_2$. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where the series converges absolutely in the interior of the annulus. To prove this, it suffices to write

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

when $r_1 < |z - z_0| < r_2$, and argue as in the proof of Theorem 4.4, Chapter 2. Here C_{r_1} and C_{r_2} are the circles bounding the annulus.

Proof. See [1, Section 9.2]. □

4. *Proof.* □
5. *Proof.* □

References

- [1] Joseph Bak and Donald J Newman. *Complex Analysis*. Springer-Verlag, 3rd edition, 2010.
- [2] John B Conway. *Functions of One Complex Variable*, volume 11. Springer, 2nd edition, 1978.
- [3] David Gilbarg and Neil S Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224. Springer Science & Business Media, 2001.