Fourier Analysis, Stein and Shakarchi

Chapter 3 Convergence of Fourier Series

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Abstract

$\mathbb{T}$ denotes $[-\pi,\pi]$ or $[-\frac{1}{2},\frac{1}{2}]$. Some estimates may differ a constant multiple from the real situation because the author is familiar with the Fourier coefficients $\widehat{f}(n) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi i n x}\,dx$ which is different from this textbook.

Note that Exercise 16 (Bernstein's Theorem) contains an extended discussion (mainly on the counterexamples for Hölder exponent $\alpha = \frac{1}{2}$) from page 8 to page 15. We also discuss a Poincaré inequality on $2D$ rectangle in Exercise 11.

Exercises

1. **We omit the proofs for showing that $\mathbb{R}^d$ and $\mathbb{C}^d$ are complete.**

2. **Prove that the vector space $l^2(\mathbb{Z})$ is complete.**

Proof. Suppose $A_k = \{a_{k,n}\}_{n\in Z}$ with $k = 1, 2, \cdots$ is a Cauchy sequence. For each n, $\{a_{k,n}\}_{k=1}$ is a Cauchy sequence of complex numbers since $|a_{k,n} - a_{k',n}| \le \|A_k - A'_k\|_{l^2}$, therefore it converges to a limit, say b_n. Given $\epsilon > 0$, there is $N_\epsilon \in \mathbb{N}$ such that for all $k, k' \ge N_\epsilon$, $\sum_{n=1}^{M} |a_{k,n} - a_{k',n}|^2 \le \|A_k - A_{k'}\|_{l^2}^2 < \frac{\epsilon^2}{2}$ for all M. For each $k > N_\epsilon$ and M, we let $k' \to \infty$ to get $\sum_{n=1}^{M} |a_{k,n} - b_n|^2 < \epsilon^2$. Since M is arbitrary, one has shown that $\|A_k - B\|_{l^2} < \epsilon$ as $k \ge N_\epsilon$, where $B = (\cdots, b_{-1}, b_0, b_1, \cdots)$. Also, $B \in l^2(\mathbb{Z})$ by Minkowski's inequality $\|B\|_{l^2} \le \|B - A_{N_\epsilon}\|_{l^2} + \|A_{N_\epsilon}\|_{l^2} < \infty$. Since ϵ is arbitrary, $A_k \to B$ in $l^2(\mathbb{Z})$ as $k \to \infty$. □

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3. **It's standard to construct a sequence of integrable functions $\{f_k\}$ on $[0, 2\pi]$ such that**

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f_k(\theta)|^2 \, d\theta = 0$$

but $\lim_{k\to\infty} f_k(\theta)$ fails to exist for any θ. For example, take $\{\chi_{I_k}\}_k$, the first interval I_1 be [0,1], the next two be the two halves of [0,1], and the next four be the four quarters, and so on. Note that there is always a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to 0$ almost everywhere.

4. **Recall the vector space $\mathcal{R}$ of Riemann-integrable functions, with its inner product and norm**

$$\|f\| = \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx\Big)^{1/2}.$$

(a) Then there exist non-zero integrable functions f for which $\|f\| = 0$, e.g. $\chi_{\{\pi\}}$.

(b) However, one can use the proof by contradiction to show if $f \in \mathcal{R}$ with $\|f\| = 0$, then $f(x) = 0$ whenever f is continuous at x.

(c) Conversely, by using the Lebesgue criteria, one can show if $f \in \mathcal{R}$ vanishes at all of its points of continuity, then the lower Riemann sum of $|f|^2$ is 0 and hence $\|f\| = 0$.

5. **Let $f(\theta) = \log(1/\theta)$ for $0 < \theta \le 2\pi$ and $f(0) = 0$. Define a sequence of functions in $\mathcal{R}$ by $f_n(\theta) = \chi_{(\frac{1}{n}, 2\pi]} f(\theta)$. Then it's easy to show $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$. However, f does not belong to $\mathcal{R}$ since it's unbounded.**

6. Let $\{a_k\}_{k=-\infty}^{\infty}$ be a sequence defined by $a_k = k^{-1}$ if $k \ge 1$ and $a_k = 0$ if $k \le 0$. Note that $\{a_k\} \in \ell^2(\mathbb{Z})$, but one can show that no Riemann integrable function (or more generally $L^\infty(\mathbb{T})$) has k-th Fourier coefficient equal to a_k for all k by the same proof as page 83-84. **However, it's known to be a Fourier coefficient of some $L^2(\mathbb{T})$ function by the L^2 convergence of the series and completeness of $L^2(\mathbb{T})$. Check Planchel's theorem and [13, Theorem 8.30]**

7. **By using the Abel-Dirichlet test and Planchel's theorem, one can show that the trigonometric series**

$$\sum_{n\ge 2} \frac{1}{\log n} \sin nx$$

converges for every x, yet it is not the Fourier series of a $L^2(\mathbb{T})$ function. However, it's actually not he Fourier series of a $L^1(\mathbb{T})$ function, see [5, Section 14.I], especially

(c)(iii) $\Rightarrow$ (i), which we will provide a proof here. Also see its following remark for cosine series.

The same method can be applied to $\sum \frac{\sin nx}{n^\alpha}$ for $0 < \alpha \le \frac{1}{2}$. For the case $1/2 < \alpha < 1$ is more difficult to show it's not a Fourier series of a Riemann integrable function. See Problem 1.

Proof. The most difficult part is to find a corrected start line. Suppose $\sum_{n=1}^{\infty} b_n \sin nx \sim f(x)$ for some $f \in L^1(\mathbb{T})$. We will prove that $\sum_{n=1}^{\infty} \frac{b_n}{n} < \infty$. Then we see the series we proposed is not a Fourier series of any $L^1(\mathbb{T})$ function. To show the finiteness, one notes that

$$\sum_{n=1}^{N} \frac{b_n}{n} = \int_0^{\pi} \Big(\frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin nx}{n}\Big) f(x)\, dx,$$

where the series in the integrand will converges to the sawtooth function pointwisely and is predicted to be uniformly bounded (also see the Gibbs phenomena near the origin proved in Exercise 20).

Under this prediction, one can use the LDCT (with dominating function $C|f(x)|$) to conclude

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \int_0^{\pi} \Big(1 - \frac{x}{\pi}\Big) f(x)\, dx < \infty.$$

Finally, we show the uniform boundedness of $S_N := \sum_{n=1}^{N} \frac{\sin nx}{n}$ by a method different from comparing with Abel mean, that is, Lemma 2.3 of this chapter. The first strategy comes into mind may be the summation by parts. But one can get an almost optimal upper bound in terms of cosecant function to the partial sum part since the conjugated Dirichlet kernel $\widetilde{D}_N(x) := \sum_{j=1}^{N} \sin jx = \frac{\cos(x/2) - \cos((N+1/2)x)}{2\sin(x/2)}$ evaluated at $x_N = (N + \frac{1}{2})^{-1}\frac{\pi}{2} (\to 0 \text{ as } N \to \infty)$ is of size $\csc x_N (\to \infty \text{ as } N \to \infty)$. This also reflects the significance of Gibbs phenemona near the origin. So it's natural to separate the analysis of $S_N(x)$ into two parts: x near the origin and far away from the origin.

First, one can easily use the summation by parts to show that for $1 \le M \le N$

$$|\sum_{j=M}^{N} \frac{1}{j} \sin jx| \le \frac{1}{2M} |\csc \frac{x}{2}|.$$

Second, we decompose the sum into two parts, we use linear estimate of sine function for both small and large indices, that is $|\sin jx| \le jx$ and $\sin \frac{x}{2} \ge \frac{x}{\pi}$ for $x \in [0, \pi]$. Note that for all $1 \le m \le N$ and $x \in (0, \pi]$

$$\Big|\sum_{j=1}^{N} \frac{1}{j} \sin jx\Big| \le \sum_{j=1}^{m} \frac{1}{j} |\sin jx| + \Big|\sum_{j=m+1}^{N} \frac{1}{j} \sin jx\Big| \le \sum_{j=1}^{m} x + \frac{1}{2(m+1)} |\csc \frac{x}{2}| \le mx + \frac{x}{2\pi(m+1)}.$$

If $N \leq \frac{1}{x}$ (which means x small and hence we don't want to use the cosectant estimate), then we pick $m = N$ and obtain $S_N(x) \leq Nx \leq 1$.

If $N > \frac{1}{x}$, then we choose m to be the unique integer satisfying $\frac{1}{x} - 1 < m \leq \frac{1}{x}$. ($m = 0$ means x is large, so we just use the cosectant estimate.) Then the upper bound becomes

$$|S_N(x)| \leq 1 + \frac{x^2}{2\pi} \leq 1 + \frac{\pi}{2}.$$

Therefore we complete the proof. $\square$

8. Exercise 6 in Chapter 2 dealt with the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Similar sums can be derived using the methods of this chapter.

(a) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Using Parseval's identity one can find the sums of the following two identity:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \pi^4/96 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4/90.$$

(b) Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. As (a), one can find that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Remark 1. . The general expression when k is even for $\sum_{n=1}^{\infty} 1/n^k$ in terms of π^k is given in Problem 4. However, finding a formula for the sum $\sum_{n=1}^{\infty} 1/n^3$, or more generally $\sum_{n=1}^{\infty} 1/n^k$ with k odd, is a famous unresolved question.

9. For α not an integer, the Fourier series of

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

Apply Parseval's formula one see that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

Also one notes that Problem 2.3 (or Jordan's test) implies that $\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan \pi\alpha}$.

Other method to construct these identities are given in Exercise 5.15 and Book II's Exercise 3.12.

10. **Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x,t)$ of the string at time t satisfies the wave equation**

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \tau/\rho$$

The string is subject to the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where we assume that $f \in C^1$ and $g \in C^0$. We define the total energy of the string by

$$E(t) = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The first term corresponds to the "kinetic energy" of the string (in analogy with $(1/2)mv^2$, the kinetic energy of a particle of mass m and velocity v), and the second term corresponds to its "potential energy."

Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,

$$E(t) = E(0) = \frac{1}{2}\rho \int_0^L g(x)^2\, dx + \frac{1}{2}\tau \int_0^L f'(x)^2\, dx.$$

Proof. $dE/dt \equiv 0$ by wave equation. To passing d/dt inside the integral, maybe this problem needs stronger assumptions on smoothness of f, g. □

11. **The inequalities of Wirtinger and Poincaŕe establish a relationship between the norm of a function and that of its derivative.**

(a) If f is T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)\, dt = 0$, show that

$$\int_0^T |f(t)|^2\, dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\, dt,$$

with equality if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$.

(b) If f is as above and g is just C^1 and T-periodic, prove that

$$\left|\int_0^T \overline{f(t)} g(t)\, dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\, dt \int_0^T |g'(t)|^2\, dt,$$

(c) For any compact interval $[a,b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\, dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\, dt,$$

Discuss the case of equality, and prove that the constant $\frac{(b-a)^2}{\pi^2}$ cannot be improved. [Hint: Extend f to be odd with respect to a and periodic of period $T = 2(b-a)$ so that its integral over an interval of length T is 0. Apply part (a) to get the inequality, and conclude that equality holds if and only if $f(t) = A\sin(\pi\frac{t-a}{b-a})$].

Remark 2. The Poincaré inequality, Sobolev inequality and isoperimetric inequality (introduced in Section 4.1 and Book III's Section 3.4) are closely related.

Proof. (a) By hypothesis, $\widehat{f}(0) = 0$ and $\widehat{f'}(n)$ exists for all $n \in \mathbb{N}$ and equals to $\frac{2\pi in}{T}\widehat{f}(n)$ where $\widehat{f}(n) := \frac{1}{T}\int_0^T f(x)e^{-\frac{2\pi in}{T}x}\,dx$. Then Parseval's identity implies that

$$\int_0^T |f(t)|^2\,dt = T\sum_{n\neq 0}|\widehat{f}(n)|^2 = T\sum_{n\neq 0}\frac{T^2}{4\pi^2n^2}|\widehat{f'}(n)|^2 \leq T\sum_{n\neq 0}\frac{T^2}{4\pi^2}|\widehat{f'}(n)|^2 = \frac{T^2}{4\pi^2}\int_0^T |f'(x)|^2\,dx.$$

with equality if and only if $\widehat{f}(n) = 0$ whenever $n \neq \pm 1$, that is, $f(x)$ is a linear combination of $e^{\frac{2\pi ix}{T}}$ and $e^{-\frac{2\pi ix}{T}}$.

(b) Note that Parseval's identity and $\widehat{f}(n) = 0$ imply that

$$\begin{aligned}|\int_0^T \overline{f(t)}g(t)\,dt| = T|\sum_{n\in\mathbb{Z}}\overline{\widehat{f}(n)}\widehat{g}(n)| = T|\sum_{|n|>0}\overline{\widehat{f}(n)}\widehat{g}(n)| &\leq \Big(T\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2\Big)^{\frac{1}{2}}\Big(T\sum_{|n|>0}|\widehat{g}(n)|^2\Big)^{\frac{1}{2}}\\ &\leq \Big(\int_0^T |f(t)|^2\Big)^{\frac{1}{2}}\Big(\frac{T^2}{4\pi^2}\int_0^T |g'(t)|^2\,dt\Big)^{\frac{1}{2}}.\end{aligned}$$

(c)'s proof is nothing more than the hint, so I omit it. $\square$

Remark 3. I also try the following way for (b), but it doesn't work.

Let $F(t) := \int_0^t f(s)\,ds$. Then $F(0) = F(T) = 0$ and hence

$$|\int_0^T \overline{f(t)}g(t)\,dt| = |-\int_0^T \overline{F(t)}g'(t)\,dt| \leq \Big(\int_0^T |F(t)|^2\Big)^{\frac{1}{2}}\Big(\int_0^T |g'(t)|^2\,dt\Big)^{\frac{1}{2}}.$$

But I think $\int_0^T F(t)\,dt = 0$ is not true, so we can't apply (a) directly.

Remark 4. The idea to solve (a) can also be applied to the following problem I saw in the qualifying exam of PDE in NTU. (Test Date: 2012.09.14):

Let $A = [0,a] \times [0,b] \subset \mathbb{R}^2$ and u be a C^1 function on A with $u = 0$ on ∂A.

(a) Prove the Poincare inequality: there exists a constant C idependent of u such that

$$\int_A |u(x)|^2\,dx \leq C\int_A |\nabla u(x)|^2\,dx.$$

(b) What is the best constant C?

Proof. We deal with a general situation first. Let $0 < \mu_1 < \mu_2 < \mu_3 < \cdots \to \infty$ denote all the eigenvalues of $-\Delta$ on the bounded domain Ω (the existence is a consequence of Fredholm's Theorem) and the corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ forms an orthogonal basis on $H_0^1(\Omega)$ (from the theory of symmetric operator.) If $\partial\Omega$ is smooth, then $\phi_n \in C^\infty(\overline{\Omega})$ for all $n \in \mathbb{N}$ by elliptic regularity theory.

The above general facts can be found in [1, Section 6.1-6.3] or other standard PDE testbooks.

To prove the Poincare inequality, it suffices to consider u is a finite linear combination of $\{\phi_n\}_n$ and then use the standard approximation argument. For $u = \sum_{j=1}^k c_j\phi_j$, we have

$$\int_\Omega |\nabla u|^2 = -\int_\Omega u\Delta u = \int_\Omega \sum_{j,k=1}^{m} c_j\phi_j c_k\mu_k\phi_k = \int_\Omega \sum_{i=1}^{m} c_i^2\mu_i\phi_i^2 \ge \mu_1 \int_\Omega \sum_{i=1}^{m} c_i^2\phi_i^2 = \mu_1 \int_\Omega |u|^2.$$

For $\Omega = A = [0,a] \times [0,b]$, it's easy to see that $\mu_{n,m} := \pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$ is an eigenvalue for each $m, n \in \mathbb{N}$. The rest is to show the span of the corresponding eigenfunctions is dense in $H_0^1(A)$.

To arrive this, we note that the span is dense in $L^2(A)$ since the "Cesaro mean" of multiple Fourier series converges in $L^2(A)$. (Note that for multiple Fourier series, the Cesaro mean I used is not standard, it's defined in [4, Definition 3.1.8] through d-dim Fejér kernel which is a multiplication of every $1D$ Fejér kernel.)

To show the span is dense in $H_0^1(A)$, we use the standard theory of charactering orthonormal basis for Hilbert spaces (e.g. Theorem 2.3 (ii)⇒(i) in Chapter 4 of Book III). We have to prove $w \equiv 0$ whenever $(\phi_{n,m}, w)_{H^1} = 0$ for all n, m. This is true since $0 = \int_A \phi_{n,m} w + \int_A \nabla\phi_{n,m} \cdot \nabla w = \int_A \phi_{n,m} w + \int_A (-\Delta\phi_{n,m}) w = (1 + \mu_{n,m})(\phi_{n,m}, w)_{L^2}$, that is, $(\phi_{n,m}, w)_{L^2} \equiv 0 \ \forall\, n, m$. □

12. There are many standard ways to show $\int_0^\infty \frac{\sin x}{x}\, dx$. We omit it.

13. When f is smoother, $\hat{f}$ decays faster and the rate is of little-o (by applying Riemann-Lebesgue lemma).

14. **Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.**

Proof. By Cauchy-Schwarz's inequality and Planchel's theorem,

$$\sum_{m\in\mathbb{Z}} |\widehat{f}(m)| \le |\widehat{f}(0)| + \Big(\sum_{m\in\mathbb{Z}\setminus\{0\}} |im\widehat{f}(m)|^2\Big)^{\frac{1}{2}} \Big(\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{|m|^2}\Big)^{\frac{1}{2}} \le |\widehat{f}(0)| + \frac{1}{\sqrt{2\pi}}\|f'\|_{L^2}(2\frac{\pi^2}{6})^{\frac{1}{2}}.$$

□

***Remark* 5.** It is possible to weaken the assumption $f \in C^1(\mathbb{T})$ to $f \in C^\gamma(\mathbb{T})$ for some $\gamma > \frac{1}{2}$, see Exercise 16.

15. **Let f be 2π-periodic and Riemann integrable on $[-\pi, \pi]$.**

    (a) By change of coordinates, it's easy to show

$$\widehat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \frac{\pi}{n})] e^{-inx}\, dx.$$

   (b) From (a), it's easy to show that if (b) f satisfies a Hölder condition of order α for some $0 < \alpha \le 1$, then $\widehat{f}(n) = O(|n|^{-\alpha})$.

   (c) Prove that the above result cannot be improved by showing that the Lacunary Fourier series

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x} \in C^\alpha([-\pi, \pi]),$$

   where $0 < \alpha < 1$. So for each $\beta > \alpha$, $(2^k)^\beta \widehat{f}(2^k) = 2^{k(\beta-\alpha)} \to \infty$ as $k \to \infty$ and hence f is not of β-Hölder for any $\beta > \alpha$.

Proof. Only (c) is non-trivial. For each $x \in [-\pi, \pi]$ and $h \neq 0$, we pick $N \in \mathbb{N}$ such that $2^{N-1}|h| < 1 \le 2^N |h|$, so

$$\begin{aligned}
|f(x+h) - f(x)| &\le |\sum_{0}^{N} 2^{-k\alpha} e^{i2^k x}(e^{i2^k h} - 1)| + |\sum_{k=N+1}^{\infty} 2^{-k\alpha} e^{i2^k x}(e^{i2^k h} - 1)| \\
&\le \sum_{0}^{N} 2^{-k\alpha} 2^k |h| + 2 \sum_{k=N+1}^{\infty} 2^{-k\alpha} = \frac{2^{(1-\alpha)(N+1)} - 1}{2^{1-\alpha} - 1} |h| + \frac{2^{1-\alpha(N+1)}}{1 - 2^{-\alpha}} \\
&\le \frac{4^{1-\alpha}}{2^{1-\alpha} - 1} |h|^\alpha + \frac{2^{1-\alpha}}{1 - 2^{-\alpha}} |h|^\alpha.
\end{aligned}$$

□

***Remark* 6.** For (c), $f(x)$ is nowhere differentiable, see Theorem 3.1 of Chapter 4 and Problem 5.8. One can also check another characterization in [5, Section 16.H] that seems to be different from this textbook, that is, the methods of delay means.

16. **The outline below actually proves the Bernstein's theorem that the Fourier series of a α-Hölder function f converges absolutely and uniformly if $\alpha > 1/2$, denoted by $f \in A(\mathbb{T})$. $A(\mathbb{T})$ is called the Wiener algebra.**

    (a) For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. One can use Parseval's identity to prove that

$$\frac{1}{2\pi} \int_{0}^{2\pi} |g_h(x)|^2\, dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2 |\widehat{f}(n)|^2,$$

and hence by the Hölder condition with constant $K > 0$

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\widehat{f}(n)|^2 \leq K^2 h^{2\alpha}.$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, then we have

$$\sum_{2^{p-1}<n\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^{2\alpha}}{2^{2(p+1)\alpha-1}}$$

since

$$\frac{1}{2}\sum_{2^{p-1}<n\leq 2^p} |\widehat{f}(n)|^2 \leq \sum_{2^{p-1}<n\leq 2^p}^{\infty} |\sin nh|^2 |\widehat{f}(n)|^2 \leq \frac{K^2\pi^{2\alpha}}{2^{2(p+1)\alpha}}$$

(c) Estimate $\sum_{2^{p-1}<n\leq 2^p} |\widehat{f}(n)|^2$ and conclude that $f \in A(\mathbb{T})$.

***Remark* 7.** In Bernstein's theorem, the pointwise Hölder condition can obviously be replaced by the L^2 Hölder condition at certain special values $h_p = \frac{\pi}{2\cdot 2^p}$, that is,

$$\|f(\cdot + h_p) - f(\cdot)\|_{L^2} \leq K|h|^\alpha, \ ,\forall \ h_p = \frac{\pi}{2\cdot 2^p} \ \forall \ p \geq 1.$$

Next, we show the optimality of $\alpha > 1/2$ by two examples: the first one is Hardy-Littlewood function. It is a little bit complicated than the second one.

(d) Show that there exists a constant $C > 0$ such that for $N = 2, 3, 4, \cdots$

$$\sup_{t\in\mathbb{R}} \left| \sum_{k=2}^{N} e^{ik\log k} e^{ikt} \right| \leq C\sqrt{N}.$$

(e) Show that the Hardy-Littlewood function

$$f(x) := \sum_{k=2}^{\infty} \frac{e^{ik\log k}}{k} e^{ikx}$$

is in $C^{1/2}(\mathbb{T})$ but $f \notin A(\mathbb{T})$.

The second example is Rudin-Shapiro's polynomial. We define the trigonometric polynomials P_m and Q_m inductively as follows: $P_0 = Q_0 = 1$ and

$$P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t)$$
$$Q_{m+1}(t) = P_m(t) - e^{i2^m t} Q_m(t).$$

(f) One can easily see that

$$|P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 = 2\Big(|P_m(t)|^2 + |Q_m(t)|^2\Big).$$

and hence $|P_m(t)|^2 + |Q_m(t)|^2 = 2^{m+1}$ and $\|Q_m\|_{C(\mathbb{T})} \leq 2^{(m+1)/2}$.

(g) For $|n| < 2^m$, $\widehat{P}_{m+1}(n) = \widehat{P}_m(n)$, hence there exists a sequence $\{\epsilon_n\}_{n=0}^{\infty}$ such that ϵ_n is either 1 or -1 and such that $P_m(t) = \sum_0^{2^m-1} \epsilon_n e^{int}$.

(h) Write $f_m = P_m - P_{m-1} = e^{i2^{m-1}t}Q_{m-1}$ and $f = \sum_1^{\infty} 2^{-m} f_m$. Show that $f \in C^{\alpha}(\mathbb{T}) \backslash A(\mathbb{T})$

Finally, the condition $\alpha > \frac{1}{2}$ in Bernstein's theorem can be relaxed to $\alpha > 0$ if f is of bounded variation.

(i) Prove the following Zygmund's theorem: let f be of bounded variation on $\mathbb{T}$ and lies in $C^{\alpha}(\mathbb{T})$ for some $\alpha > 0$. Then $f \in A(\mathbb{T})$.

Note that the second condition imposed on f is not superfluous, as the example

$$f(x) \sim \sum_{n=2}^{\infty} \frac{\sin nx}{n \log n} = \int_0^x \Big(\sum_{n=2}^{\infty} \frac{\cos nt}{\log n} \Big)\, dt,$$

shows. Here f is absolutely continuous, but $f \notin A(\mathbb{T})$.

Proof. (a)(b) are proved in the statement. For (c), we note that

$$\sum_{2^{p-1}<n\le 2^p} |\widehat{f}(n)| \le (2^p - 2^{p-1})^{\frac{1}{2}} \Big(\sum_{2^{p-1}<n\le 2^p} |\widehat{f}(n)|^2 \Big)^{\frac{1}{2}} \le 2^{\frac{p}{2}-(p+1)\alpha} K^2 \pi^{2\alpha}$$

So $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| = 1 + \sum_{p=1}^{\infty} \sum_{2^{p-1}<n\le 2^p} |\widehat{f}(n)| < \infty$ since $\alpha > \frac{1}{2}$.

The proofs for (d)(e) are given in the end.

(f) is obvious. (g) is also obviously proved by mathematical induction.

(h) Note that $f \in C(\mathbb{T})$ since the series converges uniformly by Weierstrass M-test. Moreover, the supports of $\widehat{f_m}$ are pairwisely disjoint for distinct m. Thus, $\|f\|_{A(\mathbb{T})} = \sum_1^{\infty} 2^{-m} 2^{m-1} = \infty$ and hence $f \notin A(\mathbb{T})$. Finally, for $1 > |h| > 0$, there is some $k \in \mathbb{N}$ such that $2^{-k} < |h| \le 2^{1-k}$. Then by (f),

$$|f(x+h) - f(x)| \le \sum_{m=1}^{k} 2^{-m} |f_m(x+h) - f_m(x)| + \sum_{m=k+1}^{\infty} 2^{-m} |f_m(x+h) - f_m(x)|$$

$$\le \sum_{m=1}^{k} 2^{-m} h \|f_m'\|_{\infty} + \sum_{m=k+1}^{\infty} 2^{-m} 2 \|Q_{m-1}\|_{\infty} \le \sum_{m=1}^{k} 2^{-m} h \|f_m'\|_{\infty} + \sum_{m=k+1}^{\infty} 2^{-m} 2 \cdot 2^{m/2}.$$

To estimate the first term, we need to use the following Bernstein's inequality [8, page 99-100], also see [7, Section 6.2].

Theorem 8. *For any trigometric polynomial T of degree N (that is, $T(x) = \sum_{|k|\le N} a_k e^{ikx}$), we always have*

$$\|T'\|_p \le N \|T\|_p,$$

where $1 \le p \le \infty$ and $\|\cdot\|_p$ stands for the $L^p(\mathbb{T})$ norm.

Now we can complete the proof for $f \in C^{\frac{1}{2}}(\mathbb{T})$ as follows:

$$|f(x+h) - f(x)| \leq \sum_{m=1}^{k} 2^{-m} h 2^m 2^{m/2} + \frac{2}{\sqrt{2}-1}|h|^{\frac{1}{2}} \leq \frac{2}{\sqrt{2}-1} 2|h|^{\frac{1}{2}}.$$

Proof for Bernstein's inequality. Consider the function $g(t) = ite^{-it/2}$, for $-\pi \leq t \leq \pi$. One has, on $[-\pi, \pi]$,

$$ite^{-it/2} = \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k e^{ikt}}{(2k+1)^2}.$$

Hence

$$ite^{itx} = \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k e^{itx+ikt+\frac{it}{2}}}{(2k+1)^2} \quad \text{for } |t| \leq \pi,$$

or putting $x = \frac{N\theta}{\pi}$ and $t = \frac{n\pi}{N}$,

$$ine^{in\theta} = \frac{4N}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k e^{in\theta + i(k+\frac{1}{2})n\pi/N}}{(2k+1)^2} \quad \text{for } |n| \leq N,$$

that is,

$$F'(\theta) = \frac{4N}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k F(\theta + (k+\frac{1}{2})\pi/N)}{(2k+1)^2}$$

for $F(\theta) = e^{in\theta}$ with $|n| \leq N$. By linearity, this formula is true for our T. Thus the Minkowski inequality implies that

$$\|T'\|_p \leq \frac{4N}{\pi} \sum_{-\infty}^{\infty} \frac{\|T\|_p}{(2k+1)^2} = N\|T\|_p.$$

□

Remark 9. In [11, Proposition I.1.11], The author says that this Bernstein's inequality can be proved via the de la Vallée Poussin's kernel which is related to the delay means

$$V_n(x) := (1 + e^{inx} + e^{-inx})F_n(x) = 2F_{2n-1} - F_{n-1},$$

where F_n is the Fejér kernel. But I can't prove their claim $\|V_n'\|_1 \leq Cn$ for some $C > 0$.

(i) Let V_f denote the total vatiation of f on $\mathbb{T}$, we can estimate the L^2 modulus of continuity as follows:

$$\|f(\cdot + \frac{\pi}{2N}) - f(\cdot)\|_{L^2}^2 = \frac{1}{2N} \sum_{k=1}^{2N} \int_{\mathbb{T}} \Big| f\Big(x + \frac{k\pi}{2N}\Big) - f\Big(x + \frac{(k-1)\pi}{2N}\Big) \Big|^2 dx$$

$$\leq \frac{1}{2N} \sum_{k=1}^{2N} \int_{\mathbb{T}} \Big| f\Big(x + \frac{k\pi}{2N}\Big) - f\Big(x + \frac{(k-1)\pi}{2N}\Big) \Big| dx \cdot C\Big(\frac{\pi}{2N}\Big)^{\alpha}$$

$$\leq 2CV_f\Big(\frac{\pi}{2N}\Big)^{1+\alpha} = \frac{M}{N^{1+\alpha}}.$$

Therefore f satisfies the L^2 Holder condition with exponent $(1+\alpha)/2 > \frac{1}{2}$ at least along the values $h_p = \frac{\pi}{2^{p+1}}$. We apply the Bernstein's theorem to complete the proof, see Remark 7.

Finally, we give a proof for (d)(e). We assume (d) is true first.

(e) It's clear that $f \notin A(\mathbb{T})$. Using the summation by parts with $a_n = \frac{1}{n}$ and $b_n = e^{in\log n}e^{inx}$ in Exercise 2.7 to deduce the N-th partial sum of f as (where $B_j = \sum_1^j b_n$)

$$f_N(x) = \frac{1}{N}B_N(x) + \sum_{j=1}^{N-1} \frac{1}{j(j+1)}B_j(x).$$

Then f_N converges absolutely by (d) and hence to f uniformly. Also, letting $N \to \infty$ we obtain

$$f(x+h) - f(x) = \sum_{j=1}^{\infty} \Big(B_j(x+h) - B_j(x)\Big)\frac{1}{j(j+1)} = \sum_{j=1}^{N} + \sum_{j=N+1}^{\infty} := P + Q.$$

Let $0 < h < 1$, $N = [\frac{1}{h}]$. The terms of $|Q|$ are $O(\sqrt{j})j^{-2}$ so that

$$|Q| = O(N^{-\frac{1}{2}}) = O(h^{\frac{1}{2}}).$$

On the other hand, we apply the summation by parts again to see

$$B_j'(z) = \sum_{k=1}^{j} ke^{ik\log k}e^{ikz} = jB_j(z) + \sum_{k=1}^{j-1} B_k(z) = O(j^{3/2}).$$

Therefore, applying the mean value theorem to the real and imaginary parts of $B_j(x+h) - B_j(x)$, we get

$$|P| \le \sum_{j=1}^{N} O(hj^{3/2})j^{-2} = O(hN^{1/2}) = O(h^{\frac{1}{2}}).$$

Therefore $|f(x+h) - f(x)| \le O(h^{\frac{1}{2}})$ for each $0 < h < 1$, that is, $f \in C^{\frac{1}{2}}(\mathbb{T})$.

(d) is a consequence of certain lemmas, due to Van der Corput, of considerable interest in themselves. (For applications to oscillatory integrals, see [12, Page 332] and Book IV's Exercise 8.13 with Section 6.2 of Chapter 8).

We introduce some notations and general results first since it will be used in Problem 4.3's special case $\sigma \in (1,2)$, too. Given a real-valued function $f(u)$ and numbers $a < b$, we set $F(u) = e^{2\pi i f(u)}$,

$$I(F;a,b) = \int_a^b F(u)\,du, \;\; S(F;a,b) = \sum_{a<n\le b} F(n), \;\; D(F;a,b) = I(F;a,b) - S(F;a,b).$$

Lemma 10. *(i) If f has a monotone derivative f', and if there is a $\lambda > 0$ such that $f' \ge \lambda$ or $f' \le -\lambda$ in (a,b), then $|I(F;a,b)| < \lambda^{-1}$.*

(ii) If $f'' \ge \rho > 0$ or $f'' \le -\rho < 0$, then $|I(F;a,b)| \le 4\rho^{-\frac{1}{2}}$.

Lemma 11. *If f' is monotone and $|f'| \leq \frac{1}{2}$ in (a, b), then*

$$|D(F; a, b)| \leq A$$

where A is an absolute constant independent of a, b. In fact, we have $A = \sum_{n=1}^{\infty} \frac{2}{\pi n(n-\frac{1}{2})} + 2$.

Lemma 12. *If $f'' \geq \rho > 0$ or $f'' \leq -\rho < 0$, then*

$$|S(F; a, b)| \leq \Big(|f'(b) - f'(a)| + 2\Big)(4\rho^{-\frac{1}{2}} + A).$$

Completion of the proof for (d): The function $f(u) = (2\pi)^{-1}(u \log u + ux)$ has an increasing derivative $f'(u) = (2\pi)^{-1}(\log u + 1 + x)$ since $f''(u) = (2\pi)^{-1}u^{-1} \geq (2\pi)^{-1}2^{-j-1}$ on $[2^j, 2^{j+1}]$ for each $j \in \mathbb{Z}_{\geq 0}$. A simple application of Lemma 12 shows that

$$|S(F; 2^j, 2^{j+1})| \leq \{(2\pi)^{-1} \log 2 + 2\}\{8\sqrt{\pi}2^{j/2} + A\} \leq C2^{j/2}$$

for some $C > 0$ and for each $j \geq 0$. Similarly, $|S(F; 2^n, N)| \leq C2^{n/2}$ if $2^n < N \leq 2^{n+1}$ and hence

$$\begin{aligned} |s_N(x)| := |\sum_{k=2}^{N} e^{ik \log k + ikt}| &\leq 1 + |S(F; 1, 2)| + |S(F; 2, 4)| + \cdots + |S(F; 2^n, N)| \\ &\leq 1 + C(1 + 2^{1/2} + \cdots + 2^{n/2}) \leq C_1 2^{n/2} < C_1 N^{1/2}. \end{aligned}$$

Hence we complete the proof for (d).It remains to give proofs for Lemmas 12,10,11.

Proof of Lemma 12. We may suppose that $f'' \geq \rho$ (otherwise replace f by $-f$ and S by $\overline{S}$). Let α_p be the point where $f'(\alpha_p) = p - \frac{1}{2}$, and for $p = 0, \pm 1, \pm 2, \cdots$, let

$$F_p(u) = e^{2\pi i(f(u) - pu)}$$

Then $|f'(u) - p| \leq \frac{1}{2}$ in (α_p, α_{p+1}). Let $\alpha_r, \alpha_{r+1}, \cdots, \alpha_{r+s}$ be the points, if any such exist, belonging to the interval $[a, b]$. From Lemmas 10(ii) and 11, we have

$$|S(F; \alpha_p, \alpha_{p+1})| = |S(F_p; \alpha_p, \alpha_{p+1})| = |I(F_p; \alpha_p, \alpha_{p+1}) - D(F_p; \alpha_p, \alpha_{p+1})| \leq 4\rho^{-\frac{1}{2}} + A.$$

The same holds for $S(F; a, \alpha_r)$ and $S(F; \alpha_{r+s}, b)$. Since $S(F; a, b)$ is the sum of thses expressions for the intervals $(a, \alpha_r), (\alpha_r, \alpha_{r+1}), \cdots, (\alpha_{r+s}, b)$, whose number is $s+2 = f'(\alpha_{r+s}) - f'(\alpha_r) + 2 \leq f'(b) - f'(a) + 2$, Lemma 12 follows. □

Proof of Lemma 10. (i) Since $I(F; a, b) = (2\pi i)^{-1} \int_a^b \frac{1}{f'(u)} dF(u)$, the second mean-value theorem, applied to the real and imaginary parts of this integral, shows that $|I| \leq 2\frac{2}{2\pi\lambda} < \lambda^{-1}$.

(ii) We may suppose that $f'' \geq \rho > 0$ (otherwise replace f by $-f$ and I by $\overline{I}$). Then f' is increasing. Suppose for the moment that f' is of constant sign in (a,b), say $f' \geq 0$. If $a < \gamma < b$, then $f' \geq (\gamma - a)\rho + 0$ in (γ, b). Therefore from (i) we have

$$|I(F;a,b)| \leq |I(F;a,\gamma)| + |I(F;\gamma,b)| \leq (\gamma - a) + 1/(\gamma - a)\rho.$$

and choosing γ so as to make the last sum minimum, we find that $|I(F;a,b)| \leq 2\rho^{-1/2}$.

In the general case (a,b) is a sum of two intervals in each of which f' is of constant sign, and (ii) follows by adding the inequalities for these two intervals. $\square$

Proof of Lemma 11. Suppose first that a and b are not integers. The sum S is then $\int_a^b F(u)\,d\psi(u)$, where $\psi(u) = [u] + \frac{1}{2}$ for $u \notin \mathbb{Z}$ ($[u]$ being the integral part of u), and $\psi(n) = n$. And hence

$$D(F;a,b) = \int_a^b F(u)\,d\chi(u), \text{ where } \chi(u) = u - [u] - \frac{1}{2}\,(u \notin \mathbb{Z}).$$

The function χ has period 1, and integration by parts gives

$$D(F;a,b) = -I(F'\chi;a,b) + R$$

where $|R| = |F(b)\chi(b) - F(a)\chi(a)| \leq |F(b)||\chi(b)| + |F(a)||\chi(a)| \leq 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$. The partial sums of the Fourier series $S[\chi](x) = -\sum_1^\infty \frac{\sin 2\pi nx}{\pi n}$ are uniformly bounded and converges to χ a.e.. (See my proof for Exercise 7). So we can apply LDCT to conclude

$$D - R = -I(F'\chi;a,b) = \sum_{n=1}^\infty \frac{1}{2\pi i n}\Big\{\int_a^b \frac{f'(x)}{f'(x)+n}\,de^{2\pi i(f(x)+nx)} - \int_a^b \frac{f'(x)}{f'(x)-n}\,de^{2\pi i(f(x)-nx)}\Big\}.$$

The ratios $\frac{f'}{f'\pm n} = 1 \mp \frac{n}{f'\pm n}$ are monotone, the second mean value theorem shows that the absolute value of each terms in $D-R$ does not exceed $2 \cdot 2/(2\pi n(n-\frac{1}{2}))$, and so the series converges absolutely and uniformly. Then we completes the proof for $a, b \notin \mathbb{Z}$ with

$$A = \sum_1^\infty \frac{2}{\pi n(n-\frac{1}{2})} + 1.$$

If a or b is an integer, it is enough to observe that $D(F;a,b)$ differs from $\lim_{\epsilon \to 0} D(F; a+\epsilon, b-\epsilon)$ by $1 = \frac{1}{2}(|F(a)| + |F(b)|)$ at most. So we can pick

$$A = \sum_1^\infty \frac{2}{\pi n(n-\frac{1}{2})} + 2$$

for general case. $\square$

$\square$

Remark 13. The Bernstein's theorem can be rephrased as an embedding into Sobolev space $H^\beta(\mathbb{T})$, see [11, Section 1.4.5]. For optimality in Bernstein's theorem, the first example and its proof are taken from [4, Exercise 3.3.8] and [14, Page 197-199]. The general form of (e) is that $f_\alpha(x) := \sum_{k=1}^\infty \frac{e^{ick\log k}}{k^{\frac{1}{2}+\alpha}} e^{ikx} \in C^\alpha(\mathbb{T}), \forall\, 0<\alpha<1, \forall\, c>0$. The Rudin-Shapiro's polynomial is taken from [6, page 34-35], [9] and [11, Section 1.4.6].

The proof for Zygmund's theorem is taken from [14, section VI.3]. Also see that section for some further remarks and theorems.

Remark 14. One can also apply (d) to illustrate the condition $1 \le p \le 2$ in Hausdorff-Young's inequality (Corollary 2.4 in Chapter 2 of Book IV) is necessary. More precisely, for all $p > 2$

$$g(x) = \sum_{k=2}^\infty \frac{e^{ik\log k}}{\sqrt{k}(\log k)^2} e^{ikx} \in L^p(\mathbb{T}),$$

but $\sum_{m\in\mathbb{Z}} |\widehat{g}(m)|^q = \infty$ for all $q < 2$. I learn this from [4, Exercise 3.2.3].

$16\frac{1}{4}$ **This is an example I saw in [7, Section 7.1], which can serve as an remarkable application of Bernstein's theorem.**

Theorem 15. *Let $c(x)$ be the standard Cantor Lebesgue function and $C(x) = c(x) - x$. Then $C \in A(\mathbb{T})$. In fact, $C \in C^\alpha(\mathbb{T})$ with $\alpha = \log 2/\log 3 > \frac{1}{2}$. However, it seems to be difficult to show the absolutely convergence via the explicit formula $\widehat{C}(0) = 0$, and for $n \neq 0$*

$$\widehat{C}(n) = \frac{(-1)^n}{2\pi i n} \prod_{k=1}^\infty \cos\frac{2\pi n}{3^k}.$$

Note that the above formula has a natural probability explanation, see [7, page 194].

$16\frac{1}{2}$ **Another related theorem is Wiener's $1/f$ theorem, Newman's proof (1975) can be found in [7, Section 7.2]. It can also be an easy corollary to Gelfands theory of commutative Banach algebras, see [11, Section 4.3] or [10, Theorem 11.6].**

Theorem 16. *Let $f \in A(\mathbb{T})$. If f has no zero on $\mathbb{T}$, then $1/f \in A(\mathbb{T})$. Note that the converse is trivial.*

In 1934, Lévy observe that $1/f$ may be replaced by any analytic function on an open set containing the value of f, see [14, page 245-246]. In particular, for $f \in A(\mathbb{T})$

$$\|e^f\|_A \le \exp(\|f\|_A) \text{ and } \|\cos f\|_A \le \cosh \|f\|_A, \tag{1}$$

where $\|f\|_A := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|$.

Proof for (1). The first assertion can be easily proved if we note the following two facts which can also be proved easily,

$$\|\sum g_i\|_{A(\mathbb{T})} \le \sum \|g_i\|_{A(\mathbb{T})}, \quad \|g_1 g_2\|_{A(\mathbb{T})} \le \|g_1\|_{A(\mathbb{T})}\|g_2\|_{A(\mathbb{T})}.$$

(They reflect that $(A(\mathbb{T}), \|\cdot\|_{A(\mathbb{T})})$ is a Banach algebra.)

The second assertion then follows from the first one by noting $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. □

17. **If f is a function of bounded variation (abbr. BV.) on $[-\pi,\pi] =: \mathbb{T}$, then we have $|\widehat{f}(n)| \le \frac{\mathbf{Var}(f,\mathbb{T})}{2\pi|n|}$ by using integration by parts. Moreover, from the Tauberian theorems for Cesaro-Abel sums, we have $S_N(f)(x) \to \frac{1}{2}(f(x+)+f(x-))$ as $N \to \infty$ for each $x \in \mathbb{T}$ if f is of BV($[-\pi,\pi]$), see Problem 2.3. However, the classical Jordan's test says that we still has weaker conclusion under weaker assumption as follows:**

Theorem 17. *(Jordan) Let $t_0 \in \mathbb{T}$. If $f \in L^1(\mathbb{T})$ and of $BV([t_0-\delta, t_0+\delta])$ for some $\delta > 0$, we have $S_N(f)(t_0) \to \frac{1}{2}(f(t_0+)+f(t_0-))$ as $N \to \infty$.*

Remark 18. Also see Problem 4.6.

18. **Here are a few things we have learned about the decay of Fourier coefficients:**

(a) if f is of class C^k, then $\widehat{f}(n) = o(1/|n|^k)$; (b) if f is Lipschitz, then $\widehat{f}(n) = O(1/|n|)$; (c) if f is monotonic, then $\widehat{f}(n) = O(1/|n|)$; (d) if f is satisfies a Hölder condition with exponent α where $0<\alpha<1$, then $\widehat{f}(n) = O(1/|n|^\alpha)$;

(e) if f is merely Riemann integrable, then $\sum|\widehat{f}(n)|^2 < \infty$ and therefore $\widehat{f}(n) = o(1)$.

Nevertheless, show that the Fourier coecients of a continuous function cantend to 0 arbitrarily slowly by proving that for every sequence of nonnegative real numbers $\{\epsilon_n\}$ converging to 0, there exists a continuous function f such that $|\widehat{f}(n)| \ge \epsilon_n$ for infinitely many values of n, e.g. taking $\sum_j \epsilon_{k_j} \le \sum_p \frac{1}{p^2} < \infty$ and letting $f(x) = \sum_j \epsilon_{k_j} e^{2\pi i \epsilon_{k_j}}$ which is continuous on $\mathbb{T}$ since the partial sum converges uniformly by Weierstrass M-test.

Proof. (a) and (c) are applications of integration by parts and Riemann-Lebesgue lemma. (b) and (d) are shown in Exercise 16. (e) is Planchel's Theorem. □

Remark 19. More generally, for arbitrarily slow decay sequence $\{\epsilon\}_{n\in\mathbb{Z}}$ one can construct a $L^1(\mathbb{T})$ function such that $\widehat{f}(n) \ge \epsilon_n$ for all $n \in \mathbb{Z}$, see [4, Section 3.3.1].

19. **Give another proof that the sum $\sum_{0<|n|\le N} e^{inx}/n$ is uniformly bounded in N and $x \in [-\pi, \pi]$ by using the fact that**

$$\frac{1}{2i}\sum_{0<|n|\le N}\frac{e^{inx}}{n} = \sum_{n=1}^{N}\frac{\sin nx}{n} = \frac{1}{2}\int_0^x (D_N(t)-1)\,dt,$$

where D_N is the Dirichlet kernel. Now use the fact that $\int_0^\infty \frac{\sin t}{t}\,dt < \infty$ which was proved in Exercise 12.

Proof. The first identity is easy. The second identity is proved as follows:

$$\frac{1}{2}\sum_{0<|n|\le N}\frac{e^{inx}}{in} = \frac{1}{2}\sum_{0<|n|\le N}\Big(\int_0^x e^{int}\,dt + \frac{1}{in}\Big) = \frac{1}{2}\int_0^x \big(\sum_{|n|\le N} e^{int} - 1\big)\,dt.$$

The boundedness can be estimated as follows: for $x \in [-\pi, \pi]$,

$$\Big|\frac{1}{2}\int_0^x \frac{\sin((N+1/2)t)}{\sin(t/2)}\,dt\Big| = \frac{1}{2}\int_0^{|x|}\frac{\sin((N+1/2)t)}{t/2}\,dt + \frac{1}{2}\int_0^{|x|}\sin((N+1/2)t)\Big(\frac{1}{\sin(t/2)} - \frac{1}{t/2}\Big)\,dt$$
$$= \int_0^{(2N+1)|x|}\frac{\sin t}{t}\,dt + O(|x|^2),$$

where the second integrand is estimated by $|\sin y| \le 1$ and $0 \le \frac{1}{\sin(t/2)} - \frac{1}{t/2} \le \frac{2t}{24-t^2}$ since for $t \ge 0$

$$\frac{t}{2} - \frac{t^3}{48} \le \sin\frac{t}{2} \le \frac{t}{2}.$$

□

20. **Let $f(x)$ denote the sawtooth function defined by $f(x) = (\pi - x)/2$ on the interval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$. The Fourier series of f is**

$$f(x) \sim \frac{1}{2i}\sum_{|n|\ne 0}\frac{e^{inx}}{n} = \sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

and f has a jump discontinuity at the origin with $f(0^+) - f(0^-) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$.

Show that

$$\max_{0<x\le\pi/N} S_N(f)(x) - \frac{\pi}{2} = \int_0^\pi \frac{\sin t}{t}\,dt - \frac{\pi}{2} + O\Big(\frac{1}{N}\Big) \ \textbf{ as } N \to \infty,$$

which is roughly 9% of the jump π. This result is a manifestation of Gibbs's phenomenon which states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump.

Proof. From Exercise 19 we note that for $0 < x \le \frac{\pi}{N}$,

$$2S_N(f)(x) = \int_0^x D_N(t)\,dt - x =: G_N(x) - x$$

Note that $G_N'(x) = D_N(x)$ is postitive on $(0, \frac{\pi}{N+\frac{1}{2}})$ and negative on $(\frac{\pi}{N+\frac{1}{2}}, \frac{\pi}{N})$, so

$$\max_{0<x\le\frac{\pi}{N}} G_N(x) = \int_0^{(N+\frac{1}{2})^{-1}\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt = \int_0^{(N+\frac{1}{2})^{-1}\pi} \frac{\sin(N+\frac{1}{2})t}{\frac{1}{2}t}\,dt + O\Big((N+\frac{1}{2})^{-2}\Big)$$
$$= 2\int_0^{\pi} \frac{\sin t}{t}\,dt + O(N^{-2}).$$

Hence we complete the proof since

$$\int_0^{\pi} \frac{\sin t}{t}\,dt + O(N^{-2}) - \frac{\pi}{N+\frac{1}{2}} = S_N\Big(\frac{\pi}{N+\frac{1}{2}}\Big) \le \max_{0<x\le\frac{\pi}{N}} S_N(x) \le \frac{1}{2}\max_{0<x\le\frac{\pi}{N}} G_N(x) + \frac{\pi}{2N}$$
$$= \int_0^{\pi} \frac{\sin t}{t}\,dt + O(N^{-2}) + \frac{\pi}{2N}.$$

□

Remark 20. This is also true for function of bounded variation, see [4, Theorem 3.5.7]. Also see [3], a review of the Gibbs phenomenon from a different perspective.

Problems

1. **For each** $\frac{1}{2} < \alpha < 1$ **the series**

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$$

converges for every x **and is the Fourier series of some** $f \in L^2(\mathbb{T})$ **by P, but such** f **will be proved to be non-Riemann integrable in this Problem. Note that the case** $0 < \alpha \le \frac{1}{2}$ **has be shown in Exercise 7 by the same method.**

(a) If the conjugate Dirichlet kernel is defined by

$$\widetilde{D}_N(x) = \sum_{|n|\le N} \mathbf{sign}(n) e^{inx}$$

where $\mathbf{sign}(n) = 1$ **if** $n > 0$**;** $\mathbf{sign}(0) = 0$**;** $\mathbf{sign}(n) = -1$ **if** $n < 0$**, then show that**

$$\widetilde{D}_N(x) = i\frac{\cos(x/2) - \cos((N+1/2)x)}{\sin(x/2)},$$

and

$$\int_{-\pi}^{\pi} |\widetilde{D}_N(x)|\,dx = O(\log N).$$

So the conjugated Dirichlet kernel is still not a good kernel. In the following, we only use the upper-bound estimate.

(b) As a result, if f is Riemann integrable, then for some $c > 0$

$$|(f * \widetilde{D}_N)(0)| \leq c\|f\|_\infty \log N.$$

(c) However this leads to a contradiction that

$$\sum_{n=1}^{N} \frac{1}{n^\alpha} = O(\log N).$$

Proof. (a) We omit the first easy assertion. The second assertion is compute as Problem 2.2 as follows:

$$\int_{-\pi}^{\pi} \left|\frac{\cos\frac{x}{2} - \cos((N+\frac{1}{2})x)}{\sin(x/2)}\right| dx = 4\int_0^\pi \left|\frac{\sin^2\frac{x}{4} - \sin^2\frac{2N+1}{4}x}{\sin(x/2)}\right| dx = 4\int_0^\pi \left|\frac{\sin\frac{x}{4}}{2\cos\frac{x}{4}} - \frac{\sin^2\frac{2N+1}{4}x}{\sin\frac{x}{2}}\right| dx.$$

So $|C_1 - 4\int_0^\pi \frac{\sin^2\frac{2N+1}{4}x}{\sin\frac{x}{2}}\, dx| \leq \int_{-\pi}^{\pi} |\widetilde{D}_N| \leq C_1 + 4\int_0^\pi \frac{\sin^2\frac{2N+1}{4}x}{\sin\frac{x}{2}}\, dx$

Now we focus on estimating the integral: from the proof of Problem 2.2, we have

$$\begin{aligned}
\int_0^\pi \frac{\sin^2\frac{2N+1}{4}x}{\sin\frac{x}{2}}\, dx &= \int_0^\pi \frac{\sin^2\frac{2N+1}{4}x}{\frac{x}{2}}\, dx + O(1) = 2\int_0^{\frac{2N+1}{4}\pi} \frac{\sin^2 x}{x}\, dx + O(1)\\
&= 2\sum_{k=1}^{[\frac{2N+1}{4}]} \int_{k=1}^{\infty} \frac{\sin^2 x}{x}\, dx + O(1) + O(\frac{4}{2N+1}) = \log(N+\frac{1}{2}) + O(1) + O(\frac{4}{2N+1}).
\end{aligned}$$

(b) is trivial, we omit it. (c) Direct computation shows that

$$\begin{aligned}
(f * \widetilde{D}_N)(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)\Big(\sum_{N\geq n>0} e^{int} - \sum_{-N\leq n<0} e^{int}\Big)\, dt = \sum_{N\geq n>0} \widehat{f}(n) - \sum_{-N\leq n<0} \widehat{f}(n)\\
&= \frac{1}{i}\sum_{k=1}^{N} \frac{1}{k^\alpha} = O(N^{1-\alpha}).
\end{aligned}$$

□

Remark 21. In Exercise 7, we show if $\sum_1^\infty b_n \sin nx \sim g(x)$ for some $g \in L^1(\mathbb{T})$, then $\sum \frac{b_n}{n}$ converges. On the other hand, the converse is true provided we know b_n decreasing to 0. See [5, Section 14.I]. This implies for the case $0 < \alpha \leq \frac{1}{2}$ here, the series is a Fourier series of some $g_\alpha \in L^1(\mathbb{T}) \setminus L^2(\mathbb{T})$.

Remark 22. Note that for $\alpha = 1$, one know it's the Fourier series of sawtooth function (Exercise 2.8) so $\sum_{n=1}^\infty \frac{\sin nx}{n} > 0$ for $0 < x < \pi$. In [5, page 445], the author says that Landau (1933) discover that the above positivity is also true for all its partial sum, that is, $\sum_{n=1}^N \frac{\sin nx}{n} > 0$ for all $0 < x < \pi$, with an elegant short proof as follow.

Proof. Note that $s_N(0) = s_N(\pi) = 0$ for all N. Let x be a critical point of s_N on $(0,\pi)$. Then $s_N'(x) = \frac{\frac{\sin(N+\frac{1}{x})x}{\sin\frac{1}{2}x} - 1}{2} = 0$.

So at these points $\sin Nx = \sin(N+\frac{1}{2})x\cos\frac{x}{2} - \cos(N+\frac{1}{2})x\sin\frac{x}{2} = 0$ or $\sin x \geq 0$. Hence $s_N(x) \geq s_{N-1}(x)$ at these critical points x of s_N.

If there is some critical points $x \in (0,\pi)$ such that $s_2(x) \leq 0$, then we have a contradiction that $s_1(x) = \sin x \leq 0$. Note that s_2 has a minimum on $[0,\pi]$, if there is an interior minimum x, then $s_2(x) > 0 = s_2(0)$ which is a contradiction, and hence $s_2 > 0$ on $(0,\pi)$. Inductively, we have $s_N > 0$ on $(0,\pi)$ for all $N \in \mathbb{N}$. □

2. **An important fact we have proved is that the family $\{e^{inx}\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2([-1,1])$ and it is also complete, in the sense that the Fourier series of f converges to f in the norm. In this exercise, we consider another family possessing these same properties.**

   On $[-1,1]$ define
$$L_n(x) = \frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0,1,2,\cdots.$$
   Then L_n is a polynomial of degree n which is called the n-th Legendre polynomial. People observed it when they want to solve the Laplacian in $\mathbb{R}^3$ in spherical coordinates, it's basically an equation for angle ϕ between position and z-axis and can be solved by ODE methods for power series solution.

   (a) If f is indefinitely differentiable on $[-1,1]$, then we have, by using integration by parts,
$$\int_{-1}^{1} L_n(x)f(x)\,dx = (-1)^n\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)\,dx.$$
   In particular, show that L_n is orthogonal to x^m whenever $m < n$. Hence $\{L_n\}_{n=0}^{\infty}$ is an orthogonal family.

   (b) Show that
$$\|L_n\|_{L^2}^2 = \int_{-1}^{1}|L_n(x)|^2\,dx = \frac{(n!)^2 2^{2n+1}}{2n+1}.$$
   (c) Prove that any polynomial of degree n that is orthogonal to $1, x, x^2, \cdots, x^{n-1}$ is a constant multiple of L_n.

   (d) Let $\mathcal{L}_n = L_n/\|L_n\|_{L^2}$, which are the normalized Legendre polynomials. Prove that $\{\mathcal{L}_n\}$ is the family obtained by applying the "Gram-Schmidt process" to $\{1, x, \cdots, x^n, \cdots\}$, and conclude that every $f \in L^2[-1,1]$ has a Legendre expansion
$$\sum_{n=0}^{\infty}\langle f, \mathcal{L}_n\rangle_{L^2}\mathcal{L}_n$$

which converges to f in L^2 norm.

Proof. (b) The L^2 norm of L_n is computed through integration by parts as follows:

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^n(x^2-1)^n\cdot\left(\frac{d}{dx}\right)^n(x^2-1)^n\,dx=(2n)!(-1)^n\int_{-1}^{1}(x-1)^n(x+1)^n\,dx$$
$$=(2n)!(-1)^n(-1)\int_{-1}^{1}n(x-1)^{n-1}\frac{1}{n+1}(x+1)^{n+1}\,dx=\cdots$$
$$=(2n)!(-1)^n(-1)^n\int_{-1}^{1}\frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots 2n}(x+1)^{2n}\,dx=\frac{(n!)^2 2^{2n+1}}{2n+1}.$$

(c) Let f be such polynomial. By dimension counting and orthogonality of $\{L_j\}_{j=0}^n$, we see that $\operatorname{span}\{x^j\}_{j=0}^n=\operatorname{span}\{L_j(x)\}_{j=0}^n$ and hence $f(x)=\sum_{i=1}^n c_iL_i(x)$ for some $c_i\in\mathbb{C}$. The hypothesis implies that f is orthogonal to L_j for all $1\le j\le n-1$. So $c_j=0$ for $1\le j\le n-1$.

(d) Let P_n be the polynomial of degree n that generated from the Gram-Schmidt process of $\{x^j\}_{j=0}^n$, then P_n is orthogonal to $\{x_j\}_{j=0}^{n-1}$ and hence by (c) and $\|P_n\|=1$ we have $P_n=\mathcal{L}_n$.

To prove the L^2 convergence, the central idea is stated in a standard theory of charactering orthonormal bases for Hilbert spaces (e.g. Theorem 2.3 in Chapter 4 of Book III). I guess all the methods need the Weierstrass approximation theorem. (e.g. page 78-79 or [2, Theorem 6.3]). Here I just mention how to modify the proof on page 78-79 from Riemann to L^2 and from $\{e^{inx}\}_{n=0}^\infty$ to $\{\mathcal{L}_n\}_{n=0}^\infty$ in the following paragraph.

All the statements for continuous function f are unchanged after one replace $\{e^{inx}\}$ by $\{\mathcal{L}_n\}$ and Corollary 5.4 in Chapter 2 by the Weierstrass approximation theorem stated in Exercise 2.16. It remains to show the space of continuous functions $C[-1,1]$ are dense in $L^2[-1,1]$ which can be proved as follows: given $f\in L^2[-1,1]$, one found that $f_B(x):=f(x)\chi_{\{|f|\le B\}}(x)\to_{B\to\infty}f(x)$ in L^2 by LDCT, where χ_E is the characteristic function on the set E. So the problem is reduced from approximating f to approximating f_B in L^2 sense. However, this reduced problem can now be solved similarly to the Riemann integrable case in page 79. Note that the step functions for approximating f_B in L^1 exists due to the definition of Lebesgue integration and hence one can construct the continuous functions for approximating f_B in L^1 from the step functions by shrinking the intervals and use linear functions to connect different intervals like Lemma 1.5 in Appendix B.

Of course there are other ways to prove that $C[-1,1]$ is dense in $L^2[-1,1]$, e.g. convolution. $\square$

Remark 23. See [2, Chapter 6] for discussions on orthogonal polynomials, e.g. Legendre (this Problem), Hermite (Problem 5.7 and Exercise 5.23) and Laguerre.

3. **Let α be a complex number not equal to an integer.**

(a) Calculate the Fourier series of the 2π-periodic function defined on $[-\pi, \pi]$ by $f(x) = \cos(\alpha x)$.

(b) Prove the following formulas due to Euler:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)}.$$

For all $u \in \mathbb{C} - \pi\mathbb{Z}$,

$$\cot u = \frac{1}{u} + 2\sum_{n=1}^{\infty} \frac{u}{u^2 - n^2\pi^2}.$$

(c) Show that for all $\alpha \in \mathbb{C} - \mathbb{Z}$ we have

$$\frac{\alpha\pi}{\sin(\alpha\pi)} = 1 + 2\alpha^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - \alpha^2}.$$

(d) For all $0 < \alpha < 1$, show that

$$\int_0^{\infty} \frac{t^{\alpha-1}}{t+1}\, dt = \frac{\pi}{\sin(\alpha\pi)}.$$

Proof. (a) Since $\cos \alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$, one can compute directly to show $\widehat{f}(n) = (-1)^n \frac{\alpha \sin \alpha\pi}{\pi(\alpha^2 - n^2)}$ (note that we can't use Exercise 9 since $\cos \alpha x$ is not periodic, so the results on $[0, 2\pi]$ and $[-\pi, \pi]$ are different.)

(b)(c) Convergence of the series and Uniqueness Theorem implies that for all $x \in [-\pi, \pi]$

$$\cos \alpha x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{\alpha \sin \alpha\pi}{\pi(\alpha^2 - n^2)} e^{inx}$$

In particular, $x = \pi$ and $x = 0$ imply that

$$\cos \alpha\pi = \sum_{n=-\infty}^{\infty} \frac{\alpha \sin \alpha\pi}{\pi(\alpha^2 - n^2)} = \frac{\sin \alpha\pi}{\alpha\pi} - 2\sum_{n=1}^{\infty} \frac{\alpha \sin \alpha\pi}{\pi(n^2 - \alpha^2)}.$$

and

$$1 = \sum_{n=-\infty}^{\infty} (-1)^{n-1} \frac{\alpha \sin \alpha\pi}{\pi(n^2 - \alpha^2)} = \frac{\sin \alpha\pi}{\alpha\pi} + \frac{2\alpha \sin \alpha\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - \alpha^2}$$

They are equivalent to (b) and (c) respectively.

(d) Assume we have shown the fact in the hint, we rewrite the integral as

$$\int_0^{\infty} \frac{t^{\alpha-1}}{t+1}\, dt = \int_0^1 \frac{t^{\alpha-1}}{t+1}\, dt + \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)+\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k + (1-\alpha)}$$

$$= \frac{1}{1-\alpha} + \sum_{n=1}^{\infty} (-1)^{n-1} \Big(\frac{1}{n - (1-\alpha)} - \frac{1}{n + (1-\alpha)} \Big)$$

$$= \frac{1}{1-\alpha} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2(1-\alpha)}{n^2 - (1-\alpha)^2} = (1-\alpha)^{-1} \frac{(1-\alpha)\pi}{\sin(1-\alpha)\pi} = \frac{\pi}{\sin \alpha\pi},$$

where the exchange of the order in the series in the third equality is a simple consequence of the fact that $S+T=\lim_{n\to\infty} S_n+\lim_{k\to\infty} T_k=\lim_{n\to\infty} S_n+T_n$, where S_n, T_k will be chosen to be the partial sums of the series.

Finally, we show the fact state in the hint by series expansion, that is

$$\int_0^1 \frac{t^{\gamma-1}}{1+t}\,dt=\int_0^1 t^{\gamma-1}\sum_{k=0}^{\infty}(-t)^k=\sum_{k=0}^{\infty}(-1)^k\int_0^1 t^{k+\gamma-1}\,dt=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+\gamma},$$

where the second equality is due to the uniform convergence of $\sum(-1)^k t^{k+\gamma-1}$ on $[0,1]$. $\square$

4. **In this problem, we find the formula for the sum of the series**

$$\sum_{n=1}^{\infty}\frac{1}{n^k}$$

where k is any even integer. These sums are expressed in terms of the Bernoulli numbers; the related Bernoulli polynomials are discussed in the next problem.

Define the Bernoulli numbers B_n by the formula

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n.$$

(a) Show that $B_0=1, B_1=-1/2, B_2=1/6, B_3=0, B_4=-1/30$, and $B_5=0$.

(b) Show that for $n\geq 1$ we have

$$B_n=-\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k.$$

(c) By writing

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{n=2}^{\infty}\frac{B_n}{n!}z^n,$$

show that $B_n=0$ if n is odd and >1. Also prove that

$$z\cot z=1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n}.$$

(d) The zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \textbf{for all } s>1:$$

Deduce from the result in (c), and the expression for the cotangent function obtained in the previous problem, that for $0<x<\pi$

$$x\cot x=1-2\sum_{m=1}^{\infty}\frac{\zeta(2m)}{\pi^{2m}}x^{2m}.$$

(e) Conclude that

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

Remark 24. Another method (by Poisson summation formula) to calculate $\zeta(2m)$ is given in Exercise 5.19.

Proof. (a) is just computation for the long division of $z/\sum_{n=1}^{\infty}\frac{z^n}{n!}$.

(b) Note that

$$\begin{aligned} 1 = \frac{z}{e^z-1}\frac{e^z-1}{z} &= \Big(\sum_{k=0}^{\infty}\frac{B_k}{k!}z^k\Big)\Big(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}z^k\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{B_k}{k!(n-k+1)!}\Big)z^n. \end{aligned}$$

Note that the last equality is proved by using summation by parts (need to use the absolute convergence of the series $\frac{e^z-1}{z}$). The details can be found in Baby Rudin's theorem 3.50 (also check 3.49-3.51).

Since the above formula is true for all $z \in \mathbb{R}$, one then use the uniqueness of power series (see Baby Rudin's Theorem 8.5) to conclude that $0 = \sum_{k=0}^{n}\frac{B_k}{k!(n-k+1)!}$ for all $n \geq 1$ which is equivalent to (b).

(c) These assertions are easy consequences of the fact that

$$\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}.$$

(d) From Problem 3(b), one has that for $0 < x < \pi$

$$x\cot x = 1 - 2\sum_{n=1}^{\infty}\frac{\frac{x^2}{n^2\pi^2}}{1-\frac{x^2}{n^2\pi^2}} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{2k}}{n^{2k}\pi^{2k}} = 1 - 2\sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\Big)\frac{x^{2k}}{\pi^{2k}},$$

where the Fubini's theorem, that is, the interchange of the order of the sums $\sum_n$ and $\sum_k$ is permitted by the absolute convergence of the double series $\sum_{n,k}\frac{x^{2k}}{n^{2k}\pi^{2k}}$ which can be proved easily.

(e) is an consequence of (c)(d) and the uniqueness theorem of power series (Baby Rudin's Theorem 8.5). $\square$

5. **Define the Bernoulli polynomials $B_n(x)$ by the formula**

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n.$$

(a) The functions $B_n(x)$ are polynomials in x and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Show that $B_0(x) = 1, B_1(x) = x - 1/2, B_2(x) = x^2 - x + 1/6$, and $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$.

(b) If $n \geq 1$, then

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

and if $n \geq 2$, then

$$B_n(0) = B_n(1) = B_n.$$

(c) Define $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m$. Show that

$$(m+1)S_m(n) = B_{m+1}(n) - B_{m+1}.$$

(d) Prove that the Bernoulli polynomials are the only polynomials that satisfy (i) $B_0(x) = 1$, (ii) $B_n'(x) = nB_{n-1}(x)$ for $n \geq 1$, (iii) $\int_0^1 B_n(x)\, dx = 0$ for $n \geq 1$, and show that from (b) one obtains

$$\int_x^{x+1} B_n(t)\, dt = x^n.$$

(e) Calculate the Fourier series of $B_1(x)$ to conclude that for $0 < x < 1$ we have

$$B_1(x) = x - 1/2 = \frac{-1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k}.$$

Integrate and conclude that

$$B_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^2 n},$$

$$B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{2n+1}}.$$

Finally, show that for $0 < x < 1$,

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{e^{2\pi ikx}}{k^n}.$$

We observe that the Bernoulli polynomials are, up to normalization, successive integrals of the sawtooth function.

Proof. (a) From Baby Rudin's Theorem 3.50 again, we have

$$\frac{ze^{xz}}{e^z - 1} = \frac{z}{e^z - 1} e^{xz} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \sum_{n=0}^{\infty} \frac{x^m}{m!} z^m = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{B_k x^{n-k}}{k!(n-k)!} \Big) z^n.$$

Again, one use the uniqueness theorem of power series to complete the proof for (a) (we omit the computations for $B_i(x), i = 0, 1, 2, 3$).

(b) is proved by using uniqueness theorem with the fact $\frac{ze^{(x+1)z}}{e^z-1} - \frac{ze^{xz}}{e^z-1} = ze^{xz} = \sum_{k=0}^{\infty} \frac{x^k}{k!} z^{k+1}$.

(c) follows from (b) since we have

$$\begin{aligned} B_{m+1}(n) &= B_{m+1}(n-1) + (m+1)(n-1)^m = B_{m+1}(n-2) + (m+1)[(n-2)^m + (n-1)^m] \\ &= \cdots = B_{m+1}(1) + (m+1)S_m(n). \end{aligned}$$

(d) From (a), we see that (i)(ii). From (b) and (ii), we have

$$(n+1)\int_0^1 B_n(t)\,dt = \int_0^1 B'_{n+1}(t)\,dt = B_{n+1}(1) - B_{n+1}(0) = 0,$$

and

$$(n+1)\int_x^{x+1} B_n(t)\,dt = \int_x^{x+1} B'_{n+1}(t)\,dt = B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n.$$

(e) is proved by mathematical induction. Note that for $0 < x < 1$ the series converges uniformly on the interval between x and $\frac{1}{2}$, so one can interchange the order of the integration $\int_{\frac{1}{2}}^{x}$ and series summation. $\square$

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