Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 5 Sobolev Spaces

Shih-Hsin Chen*, Yung-Hsiang Huang†

2017.08.13

Abstract

In these exercises U always denote an open set of $\mathbb{R}^n$ with smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

1. **Suppose $k \in \mathbb{Z}_{\geq 0}$, $0 < \gamma \leq 1$. Show that $C^{0,\beta}(\overline{U})$ is a Banach space.**

Proof. We prove the case $k = 0$ for simplicity. The general case is almost the same (except we need the standard convergence theorem between $\{f_n\}$ and $\{Df_n\}$).

Given a Cauchy sequence $\{f_n\} \subset C^{0,\beta}(\overline{U})$, then there is a $f \in C(\overline{U})$ such that $f_n \to f$ uniformly and a constant $M > 0$ such that for each $n \in \mathbb{N}$ and $s \neq t$,

$$\frac{|f_n(s) - f_n(t)|}{|s-t|^\beta} \leq M.$$

Then for each $s \neq t$, there is some $N = N(s,t) \in \mathbb{N}$ such that

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)| \\ &\leq 2|s-t|^\beta + M|s-t|^\beta. \end{aligned}$$

Therefore, $f \in C^{0,\beta}(\overline{U})$. It remains to show $f_n \to f$ in $C^{0,\beta}(\overline{U})$. For each $s, t \in U$ and $\epsilon > 0$, by Cauchy's criteria, we can find a $N = N(\epsilon)$ such that for $k, n \geq N$

$$|f_k(s) - f_n(s) - f_k(t) + f_n(t)| \leq \epsilon |s-t|^\beta$$

*Department of Math., National Taiwan University. Email: d03221002@ntu.edu.tw
†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

For each $\eta > 0$, we can find a $K = K(\eta, s, t) > N$, such that $|f(x) - f_K(x)| \leq \eta$ for $x = s$, or $x = t$. Therefore, for every $n \geq N$

$$|f(s)-f_n(s)-f(t)+f_n(t)| \leq |f(s)-f_K(s)|+|f_K(s)-f_n(s)-f_K(t)+f_n(t)|+|f(t)-f_K(t)| \leq 2\eta+\epsilon|s-t|^\beta.$$

Letting $\eta \to 0$, we obtain that $|f(s) - f_n(s) - f(t) + f_n(t)| \leq \epsilon |s-t|^\beta$ for all $n \geq N(\epsilon)$. □

2. *Proof.* $\|u\|_{C^{0,\gamma}} = \sup |u(x)| + \sup \dfrac{|u(x)-u(y)|}{|x-y|^\gamma}$.
   Set t s.t. $(1-t)\beta + t = \gamma$. Then

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} = \left(\frac{|u(x)-u(y)|}{|x-y|^\beta}\right)^{1-t}\left(\frac{|u(x)-u(y)|}{|x-y|}\right)^t.$$

$$\|u\|_{C^{0,\gamma}} \leq \|u\|_\infty^{1-t}\|u\|_\infty^t + [u]_{C^{0,\beta}}^{1-t}[u]_{C^{0,1}}^t := a_1^{1-t}b_1^t + a_2^{1-t}b_2^t.$$

Since $y = x^t$, $0 < t < 1$, is a concave function, we have

$$\begin{aligned} a_1^{1-t}b_1^t + a_2^{1-t}b_2^t &= (a_1+a_2)\left(\frac{a_1}{a_1+a_2}\left(\frac{b_1}{a_1}\right)^t + \frac{a_2}{a_1+a_2}\left(\frac{b_2}{a_2}\right)^t\right) \\ &\leq (a_1+a_2)\left(\frac{b_1}{a_1+a_2} + \frac{b_2}{a_1+a_2}\right)^t \\ &= (a_1+a_2)^{1-t}(b_1+b_2). \end{aligned}$$

Then

$$\begin{aligned} \|u\|_{C^{0,\gamma}} &\leq \|u\|_\infty^{1-t}\|u\|_\infty^t + [u]_{C^{0,\beta}}^{1-t}[u]_{C^{0,1}}^t \\ &\leq (\|u\|_\infty + [u]_{C^{0,\beta}})^{1-t}(\|u\|_\infty + [u]_{C^{0,1}})^t \\ &= \|u\|_{C^{0,\beta}}^{1-t}\|u\|_{C^{0,1}}^t. \end{aligned}$$

□

Remark 1. There is a general convexity theorem for Higher-order Hölder norm (due to Hörmander), see Helms [4, chapter 8].

3. Omit.

4. **Assume $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.**

   (a) Show that u is equal a.e. to an absolutely continuous function and u' belongs to $L^p(0,1)$.

   (b) Prove that if $1 < p < \infty$, then for a.e. $x, y \in [0,1]$,

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}}\|u'\|_p.$$

Proof. (b) follows easily from (a) and Hölder inequality. For (a), denote the weak derivative of u by u'_w. Define $v(t) = \int_0^t u'_w(s)\,ds$, then v is absolutely continuous. Given $\phi \in C_c^\infty$,

$$\begin{aligned}\int_0^1 [v-u]\phi'\,dx &= \int_0^1 \int_0^x u'_w(t)\,dt\phi'(x)\,dx - \int_0^1 u(x)\phi'(x)\,dx \\ &= \int_0^1 \int_t^1 \phi'(x)\,dx u'_w(t)\,dt + \int_0^1 u'_w(x)\phi(x)\,dx \\ &= -\int_0^1 \phi(t)u'_w(t)\,dt + \int_0^1 u'_w(x)\phi(x)\,dx = 0.\end{aligned}$$

(This argument avoids the use of the Lebesgue Differentiation theorem if we only want to show the 1-d weakly differentiable function $F = H$ a.e., an absolutely continuous function.) We are done if we use Exercise 11. There are many other way to prove the second fundamental lemma of calculus of variations without using mollifications. We also note such statement is true for distributions. For example, see [2]. □

5. *Proof.* For any $V \subset\subset W \subset\subset U$, we can find a smooth function u such that $u \equiv 1$ on V and $\text{supp}(u) \subset W$. Take $\varepsilon = \frac{1}{3}\text{dist}(\overline{V}, W^c) > 0$. Let $\phi_\epsilon(z)$ be the standard mollifier supported in $\{|z| \le \epsilon\}, V' := \{z : \text{dist}(z, \overline{V}) < \epsilon\}$ and

$$u(x) := \chi_{V'} * \phi_\varepsilon(x) = \int_{\mathbb{R}^n} \chi_{V'}(x-y)\phi_\varepsilon(y)\,dy = \int_{x-V'} \phi_\varepsilon(y)\,dy.$$

Then u is smooth (by LDCT), $u = 1$ on V and supported in $\{\text{dist}(x, \overline{V}) \le 2\epsilon\} \subset W$. □

6. Apply Exercise 5. We omit it since it can be found in many Differential Geometry textbooks.

7. *Proof.*

$$\begin{aligned}\int_{\partial U} |u|^p\,dS &\le \int_{\partial U} |u|^p \alpha\cdot\nu\,dS \\ &= \int_U \text{div}\,(|u|^p\alpha)\,dx \\ &= \int_U |u|^p \text{div}\,\alpha + \nabla(|u|^p)\cdot\alpha\,dx\end{aligned}$$

□

8. *Proof.* Consider $u_n(x) = \langle 1 - n^*\text{dist}(x, \partial U)\rangle^+$. Note that $u_n \in C(\bar{U})$ $, 0 \le u_n(x) \le 1$ and $\{u_n(x)\}$ is decreasing to 0 for each $x \in U$, MCT tells us $\|u_n\|_{p,U} \to 0$. On the other hand, $\|u_n\|_{p,\partial U} = \|1\|_{p,\partial U}$ is a positive constant. Hence

$$\frac{\|Tu_n\|_{p,U}}{\|u_n\|_{p,U}} = \frac{\|u_n\|_{p,\partial U}}{\|u_n\|_{p,U}} \to \infty$$

which means T will not be bounded, if T exists. □

9. *Proof.* Given $u \in H^2(U) \cap H_0^1(U)$, there exists $u_k \in C^\infty(\bar{U})$ with $u_k \to u$ in $H^2(U)$, and $v_k \in C_c^\infty(\bar{U})$ with $v_k \to u$ in $H^1(U)$. Hence $\{\nabla u_k\}$ and $\{\Delta u_k\}$ are bounded in L^2. Note

$$\int_{\partial U} u_k \frac{\partial u_k}{\partial n}\, dS = \int_U \operatorname{div}(u_k \nabla u_k)\, dx = \int_U |\nabla u_k|^2\, dx + \int_U u_k \Delta u_k\, dx.$$

The right-hand side tends to $\int_U |\nabla u|^2\, dx + \int_U u\Delta u\, dx$. Note

$$\begin{aligned} \int_{\partial U} u_k \frac{\partial u_k}{\partial n}\, dS &= \int_U \operatorname{div}((u_k - v_k)\nabla u_k)\, dx \\ &= \int_U \nabla(u_k - v_k)\nabla u_k + (u_k - v_k)\Delta u_k\, dx. \end{aligned}$$

which tends to 0 by Hölder and $(u_k - v_k) \to 0$ in $H^1(U)$. Therefore, for all $u \in H^2(U) \cap H_0^1(U)$,

$$\int_U |\nabla u|^2 + u\Delta u\, dx = 0$$

Then we apply Hölder's inequality to get the desired inequality. □

10. *Proof.* (Sketch) (b) is similar to (a). To prove (a), we start with the formula in Hint, do integration by parts once, and then apply the generalized Hölder with exponents pair $(p, p, \frac{p}{p-2})$. □

11. *Proof.* Given $K \subset U$, compact. Then $u \in L^1(K)$. Note that the weak derivative $D(u * \eta_\epsilon) = Du * \eta_\epsilon = 0$. Since U is connected and $u * \eta_\epsilon$ is smooth, $u * \eta_\varepsilon$ is a constant c_ϵ which will converges to u in $L^1(K)$ as $\epsilon \to 0$ and hence $\{c_{\frac{1}{n}}\}_{n=1}^\infty$ forms a Cauchy sequence (note that K has finite measure) that will converges to some constant c. Therefore, $u = c$ a.e. in K and hence in U. □

12. *Proof.* Consider in $\mathbb{R}^1$, let $u(x) = \chi_{(0,1)}(x)$. □

13. *Proof.* Consider $U = B(0,1) - \{(x,0) \big| x > 0\}$ in $\mathbb{R}^2$, let $u(x,y) = r\theta$ in polar coordinate. □

14. Let $n > 1$. Show that $\log\log\left(1 + \frac{1}{|x|}\right) \in W^{1,n}(B_1(0))$.

Proof.

$$\begin{aligned} \int_U |u(x)|^n\, dx &= \omega_n \int_0^1 \left|\log\log\left(1 + \frac{1}{r}\right)\right|^n r^{n-1}\, dr, \text{ let } t = \frac{1}{r} \\ &= \omega_n \int_1^\infty \frac{1}{t^n} \cdot \frac{|\log\log(1+t)|^n}{t}\, dt \end{aligned}$$

Since $\lim\limits_{t\to\infty} \dfrac{|\log\log(1+t)|^n}{t} = 0$, there is $T > 0$ such that $\dfrac{|\log\log(1+t)|^n}{t} < 1$ for $t \geq T$. Then

$$\begin{aligned} \int_U |u(x)|^n\, dx &= \omega_n \int_1^\infty \frac{1}{t^n} \cdot \frac{|\log\log(1+t)|^n}{t}\, dt \\ &\leq \omega_n \int_1^T \frac{1}{t^n} \cdot \frac{|\log\log(1+t)|^n}{t}\, dt + \omega_n \int_1^\infty \frac{1}{t^n}\, dt < \infty. \end{aligned}$$

On the other hand, we omit the proof that the first weak derivative of u exists and

$$u_{x_i} = \frac{1}{\log(1+\frac{1}{|x|})} \cdot \frac{1}{1+\frac{1}{|x|}} \cdot \frac{-x_i}{|x|^3}.$$

So

$$\begin{aligned}
\int_U |u_{x_i}|^n \, dx &\leq \omega_n \int_0^1 \left| \frac{1}{\log(1+\frac{1}{r})} \cdot \frac{1}{1+\frac{1}{r}} \cdot \frac{1}{r^2} \right|^n r^{n-1} \, dr, \text{ let } t = \frac{1}{r} \\
&= \omega_n \int_1^\infty \frac{1}{|\log(1+t)|^n} \cdot \frac{1}{(1+t)^n} \cdot t^{n-1} \, dt \\
&\leq \frac{\omega_n}{n} \int_{\mathtt{log2}}^\infty \frac{1}{s^n} \cdot \frac{1}{e^{sn}} \cdot e^{sn} \, ds \ \ (s = \mathtt{log}(1+t)) \\
&< \infty.
\end{aligned}$$

□

15. *Proof.* Applying Poincaré inequality,

$$\begin{aligned}
\Big(\int_U u^2 \, dx\Big)^{\frac{1}{2}} &\leq \Big(\int_U (u)_U^2 \, dx\Big)^{\frac{1}{2}} + \Big(\int_U |u-(u)_U|^2 \, dx\Big)^{\frac{1}{2}} \\
&\leq |(u)_U||U|^{\frac{1}{2}} + C\Big(\int_U |Du|^2 \, dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Applying Hölider inequality,

$$\begin{aligned}
|(u)_U| &\leq \frac{1}{|U|} \int_{\{u\neq 0\}} |u| \, dx \\
&\leq \frac{1}{|U|} \Big(\int_{\{u\neq 0\}} 1^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{\{u\neq 0\}} u^2 \, dx\Big)^{\frac{1}{2}} \\
&= \frac{1}{|U|} (|U|-\alpha)^{\frac{1}{2}} \Big(\int_U u^2 \, dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\Big(\int_U u^2 \, dx\Big)^{\frac{1}{2}} \leq \sqrt{\frac{|U|-\alpha}{|U|}} \Big(\int_U u^2 \, dx\Big)^{\frac{1}{2}} + C\Big(\int_U |Du|^2 \, dx\Big)^{\frac{1}{2}},$$

Since $0 \leq \sqrt{\frac{|U|-\alpha}{|U|}} < 1$, we proved the desired inequality

$$\Big(1 - \sqrt{\frac{|U|-\alpha}{|U|}}\Big)\Big(\int_U u^2 \, dx\Big)^{\frac{1}{2}} \leq C\Big(\int_U |Du|^2 \, dx\Big)^{\frac{1}{2}}.$$

□

Remark 2. For further discussions on Poincare-Type inequality, see G.Leoni [5, Section 12.2] and L.Tartar [7, Section 10-11].

16. *Proof.* For any $u \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{x}{|x|^2} \cdot 2u\nabla u\, dx = -\int_{\mathbb{R}^n} u^2 \mathrm{div}\Big(\frac{x}{|x|^2}\Big)\, dx, \quad \text{by Divergence theorem}$$
$$= -\int_{\mathbb{R}^n} u^2 \frac{n-2}{|x|^2} dx.$$

The Hölder's inequality implies that

$$\frac{n-2}{2}\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \le \Big(\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} |\nabla u|^2\, dx\Big)^{\frac{1}{2}}.$$

Given $u \in H^1(\mathbb{R}^n)$, there exists a sequence $u_k \in C_c^\infty(\mathbb{R}^n)$ converging to u in H^1. By picking a further subsequence, we may assume u_k converging to u pointwisely. Fatou's lemma implies

$$\int_{\mathbb{R}^n} |\nabla u|^2\, dx = \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\nabla u_k|^2\, dx \ge \liminf_{k\to\infty} \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u_k^2}{|x|^2}\, dx \ge \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx.$$

□

***Remark* 3.** This proof is simpler than the one suggested in the hint. Also note that the same approximation argument should be applied at the end of the proof to Theorem 7 on page 296.

***Remark* 4.** A family of inequalities that interpolate between Hardy and Sobolev inequalities is given by the *Hardy-Sobolev inequality*,

$$\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{n-\frac{n-2}{2}p}}\, dx\right)^{\frac{2}{p}} \le C(p)\int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

for any $u \in C_c^\infty(\mathbb{R}^n)$, where $2 \le p \le 2n/(n-2), n \ge 3$. One huge extensions and improvements of the Hardy-Sobolev inequalities is the *Caffarelli-Kohn-Nirenberg* inequalities established in their well-know paper [1]. See also a further generalization by C.S.Lin [6].

17. *Proof.* For $1 \le p < \infty$, there is a sequence $u_m \in W^{1,p} \cap C^\infty(U)$ such that $u_m \to u$ in $W^{1,p}$. Then

$$\int_U |F(u_m) - F(u)|^p \le \sup|F'|^p \int_U |u_m - u|^p \to 0, \text{as } m \to \infty.$$

$$\int_U |F'(u_m)Du_m - F'(u)Du|^p \le \sup|F'| \int_U |Du_m - Du|^p + \int_U |F'(u_m) - F'(u)|^p |Du|^p.$$

Up to a subsequence, we know u_m converges to u a.e. Since F' is continuous, $F'(u_m)$ converge to $F'(u)$ a.e. Hence the last integral tends to 0 by the dominate convergence theorem. Consequently, $\{F(u_m)\}$ and $\{F'(u_m)Du_m\}$ converge to $F(u)$ and $F'(u)Du$ in L^p. By the uniqueness of weak derivative, $D[F(u)] = F'(u)Du$.

On the other hand, $F(u) \in L^p$ since $|F(u)| \le \|F'\|_\infty |u| + |F(0)|$. □

***Remark* 5.** Note that the hypothesis U is bounded is only used to show $F(u) \in L^p$. Such restriction can be removed if we know $F(0) = 0$. (Still need $F' \in L^\infty$.) A trivial counterexample for unbounded U and $F(0) \neq 0$ is $F \equiv 1$.

18. *Proof.* (a) and (c) follow from (b), (b) is proved by the method given in the hint easily. Note that U can be unbounded. (See the remark for problem 17.) $\square$

19. *Proof.* Let ϕ be a smooth, bounded, nondecreasing function, such that ϕ' is bounded and $\phi(z) = z$, if $-1 \leq z \leq 1$. Let $u^\epsilon(x) := \epsilon\phi(u/\epsilon)$. Since $u \in L^2$, u is finite a.e. Therefore $u^\epsilon \to 0$ as $\epsilon \to 0$ a.e.. Moreover by the mean-value theorem, for a.e. x

$$u^\epsilon(x)^2 = \epsilon^2|\phi(u(x)/\epsilon) - \phi(0/\epsilon)|^2 \leq \sup(\phi')^2 u(x)^2 \in L^1$$

Hence $u^\epsilon \to 0$ in L^2 by dominated convergence theorem. By exercise 17, $\{Du^\epsilon\} = \{\phi'(u/\epsilon)Du\}$ is bounded on L^2. Given $w \in C_c^\infty(U)$,

$$\int_U (Du^\epsilon)w = \int_U u^\epsilon Dw \to 0, \text{ since } \|u^\epsilon\|_{L^2} \to 0$$

Apply the density argument with the help of boundedness of $\{\|Du^\epsilon\|_{L^2}\}$, we know the above is true for any $w \in L^2$. Take $w = (Du)\chi_{\{u=0\}} \in L^2$, we see

$$\int_{\{u=0\}} |Du|^2 = \int_{\{u=0\}} \phi'(u/\epsilon)Du \cdot Du = \int_U Du^\epsilon \cdot (Du)\chi_{\{u=0\}} \to 0, \text{ as } \epsilon \to 0$$

which is the desired result. $\square$

***Remark* 6.** In the next two problems, we define the norm in fractional Sobolev spaces by

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi, \text{ for } s \in \mathbb{R}.$$

which is different from Evans' definition. This definition works for negative power s.

20. **(Sobolev Lemma)**

Proof. Since $u \in H^s \subset L^2$, Fourier inversion formula on L^2 implies that $u(-x) = \hat{\hat{u}}(x)$ in L^2 sense. However

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)| \, d\xi \leq \left(\int_{\mathbb{R}^n} \left[(1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)|\right]^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left[(1 + |\xi|^2)^{-\frac{s}{2}}\right]^2 d\xi \right)^{\frac{1}{2}}$$
$$= \|u\|^2_{H^s(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-s} \, d\xi \right)^{\frac{1}{2}} \leq C\|u\|^2_{H^s(\mathbb{R}^n)}, \text{ since } 2s > n,$$

we know $\hat{u} \in L^1$, and the inversion formula holds pointwisely now, this tells us that for each $x \in \mathbb{R}^n$

$$|u(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |e^{ix\cdot\xi}||\hat{u}(\xi)| \, d\xi \leq C\|u\|^2_{H^s(\mathbb{R}^n)}$$

$\square$

***Remark* 7.** It's true that for $s = k + \alpha + \frac{n}{2}, 0 < \alpha < 1, H^s$ is continuously embedded in $C^{k,\alpha}$.

***Remark* 8.** The converse of Sobolev lemma is true: If H^s is continuously embedded in C^k, then $s > k + \frac{n}{2}$. The proof is sketched in Folland [3, page 195].

21. **(H^s is an algebra if $s > n/2$)**

We need the following convolution identity for Fourier transform on $L^2(\mathbb{R}^n)$:

Lemma 9. $\widehat{uv} = (2\pi)^{n/2}\hat{u} * \hat{v}, \forall\, u, v \in L^2$.

Proof. Since $s \geq 0$,

$$\begin{aligned}(1+|\xi|^2)^{s/2} &\leq \big(1 + |\xi-\eta| + |\eta|)^2\big)^{s/2} \leq \big(1 + 2|\xi-\eta|^2 + 2|\eta|^2\big)^{s/2}\\ &\leq 2^s\big[(1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2}\big].\end{aligned}$$

Hence

$$\begin{aligned}\|uv\|_{H^s}^2 &= \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{uv}(\xi)|^2\,d\xi \leq (2\pi)^n\int\big((1+|\xi|^2)^{s/2}(|u| * |v|)(\xi)\big)^2\,d\xi\\ &\leq C(n,s)\int\Big(\int\big[(1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2}\big]|\hat{u}(\xi-\eta)||\hat{v}(\eta)|\,d\eta\Big)^2\,d\xi\\ &= C\int\Big(\int\big[(1+|\xi-\eta|^2)^{s/2}|\hat{u}(\xi-\eta)||\hat{v}(\eta)|\,d\eta + \int(1+|\eta|^2)^{s/2}\big]|\hat{u}(\eta)||\hat{v}(\xi-\eta)|\,d\eta\Big)^2\,d\xi\\ &\leq 2C\int(|\langle\cdot\rangle^s\hat{u}| * |\hat{v}|(\xi))^2 + (|\hat{u}| * |\langle\cdot\rangle^s\hat{v}(\xi))^2\,d\xi,\ \text{where } \langle x\rangle := (1+|x|^2)^{1/2}.\\ &\leq C\big(\| <\cdot>^s\hat{u}\|_{L^2}^2\|\hat{v}\|_{L^1}^2 + \|\hat{u}\|_{L^1}^2\| <\cdot>^s\hat{v}\|_{L^2}^2\big),\ \text{by Young's inequality .}\\ &\leq C\|u\|_{H^s}^2\|v\|_{H^s}^2,\ \text{since } \|\hat{v}\| \leq C\|v\|_{H^s}\end{aligned}$$

□

Acknowledge

The authors would like to thank Tommaso Seneci for his/her reminder that the original proof for problem 19 is wrong. (2017.08.12)

References

[1] Luis Caffarelli, Robert Kohn, and Louis Nirenberg. First order interpolation inequalities with weights. *Compositio Mathematica*, 53(3):259–275, 1984.

[2] Bernard Dacorogna. *Direct methods in the calculus of variations*, volume 78. Springer Science & Business Media, 2007.

[3] Gerald B Folland. *Introduction to Partial Differential Equations*. Princeton university press, 2nd edition, 1995.

[4] Lester L Helms. *Potential Theory*. Springer Science & Business Media, 2nd edition, 2014.

[5] Giovanni Leoni. *A first course in Sobolev spaces*, volume 105. American Mathematical Society Providence, RI, 2009.

[6] Chang Shou Lin. Interpolation inequalities with weights. *Communications in Partial Differential Equations*, 11(14):1515–1538, 1986.

[7] Luc Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3. Springer Science & Business Media, 2007.