Fourier Analysis, Stein and Shakarchi

Chapter 4 Some Applications of Fourier Series

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Notation: $\mathbb{T} := [-\pi, \pi]$.

Exercises

1. **Let $\gamma : [a, b] \to \mathbb{R}^2$ be a parametrization for the closed curve Γ.**

   (a) Prove that γ is a parametrization by arc-length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ is precisely $s - a$ for each $s \in [a, b]$, that is,

$$\int_a^s |\gamma'(t)|\, dt = s - a.$$

   (b) Prove that any curve Γ admits a parametrization by arc-length. [Hint: If η is any parametrization, let $h(s) = \int_a^s |\eta'(t)|\, dt$ and consider $\gamma = \eta \circ h^{-1}$.]

   Proof. (a) ($\Rightarrow$) is trivial. ($\Leftarrow$) This is the definition of arc-length.

   (b) is proved as hint and (a). □

2. Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is a parametrization for a closed curve Γ, with $\gamma(t) = (x(t), y(t))$.

   (a) Use integration by parts, one can show that

$$\frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\, ds = \int_a^b x(s)y'(s)\, ds = -\int_a^b y(s)x'(s)\, ds.$$

   (b) Define the reverse parametrization of γ by $\gamma^- : [a, b] \to \mathbb{R}^2$ with $\gamma^-(t) = \gamma(b + a - t)$. The image of γ^- is precisely Γ, except that the points $\gamma^-(t)$ and $\gamma(t)$ travel in opposite directions. Thus γ^- "reverses" the orientation of the curve. It's easy to prove that

$$\int_\gamma (xdy - ydx) = -\int_{\gamma^-} (xdy - ydx).$$

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In particular, we may assume (after a possible change in orientation) that

$$A = \frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,ds = \int_a^b x(s)y'(s)\,ds.$$

3. Freshman's Calculus should teach how to calculate the area for

$$\Omega = \{(x, y) : 0 \leq x \leq 1 \text{ and } g(x) \leq y \leq f(x)\}.$$

4. **Observe that with the definition of ℓ and $\mathcal{A}$ given in the text, the isoperimetric inequality continues to hold (with the same proof) even when Γ is not simple.**

   Show that this stronger version of the isoperimetric inequality is equivalent to Wirtinger's inequality, which says that if f is 2π-periodic, of class C^1, and satisfies $\int_0^{2\pi} f(t)\,dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2\,dt \leq \int_0^{2\pi} |f'(t)|^2\,dt$$

   with equality if and only if $f(t) = A\sin t + B\cos t$ (Exercise 11, Chapter 3).

Remark 1. There are some ways to relax the smoothness assumption of Γ, one way is through the geometric measure theory (GMT), see for example Book III's Section 3.4 or others textbook for GMT (e.g. Evans-Graiepy, Simon or Federer).

In 2013, Cabre, Ros-Oton and Serra found that the ABP method for elliptic PDEs (treated in the classical textbooks like Section 9.1 of Gilbarg-Trudinger, Chapter 5 of Lin-Han, or Chapter 3 of Cabre-Caffarelli) can be applied to derive some sharp isoperimetric inequalities, their paper is published in *J. Eur. Math. Soc., 18, 2016, 2971 - 2998.*

Proof. (Isoperimetric $\Rightarrow$ Wirtinger)

If we re-parametrize $t = ks$ with $k = T/(2\pi)$, then the change of variable shows that it is sufficient to prove the claim for the period being 2π.

Given f with mean 0. we found $F(t) := \int_0^t f(s)\,ds$ is a 2π-periodic function. So the isoperimetric and Hölder inequalities imply

$$\begin{aligned}\int_0^{2\pi} f^2(t)\,dt = \int_0^{2\pi} f(t)F'(t)\,dt &\leq \frac{1}{4\pi}\Big(\int_0^{2\pi} \sqrt{(f'(t))^2 + (F'(t))^2}\,dt\Big)^2 \\ &\leq \frac{1}{2}\int_0^{2\pi} (f'(t))^2 + f(t)^2\,dt,\end{aligned}$$

which is equivalent to the Wirtinger's inequality.

(Isoperimetric $\Leftarrow$ Wirtinger)

Assume the closed curve $\Gamma(s) = (x(s), y(s))$ is parametrized by the arc-length from $[0, L] \to \mathbb{R}^2$. Then x, y are L- periodic and

$$\Big(\frac{dx}{ds}\Big)^2 + \Big(\frac{dy}{ds}\Big)^2 \equiv 1.$$

Note that $f(\theta) = x(\frac{L}{2\pi}\theta)$ and $g(\theta) = y(\frac{L}{2\pi}\theta)$ are 2π-periodic and

$$\Big(\frac{df}{d\theta}\Big)^2 + \Big(\frac{dg}{d\theta}\Big)^2 \equiv \frac{L^2}{4\pi^2}.$$

Let $\bar{f}$ denotes the mean of f, then the Wirtinger's inequality and $\int_0^{2\pi} g' = 0 = \int_0^{2\pi}(f - \bar{f})$ imply

$$\begin{aligned}2\mathcal{A} = 2\int_0^{2\pi} fg'\,d\theta = 2\int_0^{2\pi}(f-\bar{f})g'\,d\theta = \int_0^{2\pi}(f-\bar{f})^2 + (g')^2 - (f-\bar{f}-g)^2\,d\theta\\ \le \int_0^{2\pi}(f-\bar{f})^2 + (g')^2\,d\theta \le \int_0^{2\pi}(f')^2 + (g')^2\,d\theta = 2\pi\frac{L^2}{4\pi^2},\end{aligned}$$

which is the isoperimetric inequality stated here. □

Remark 2. [1, 5, 9, 3, 12] are classical textbooks and surveys on isoperimetric inequality and Brunn-Minkowski inequality.

5. **Prove that the sequence $\{\gamma_n\}_{n=1}^\infty$, where γ_n is the fractional part of**

$$W_n := \Big(\frac{1+\sqrt{5}}{2}\Big)^n,$$

is not equidistributed in $[0, 1]$.

Proof. According to the hint, one observes that $U_n = W_n + \overline{W_n}$ satisfies $U_r = U_{r-1} + U_{r-2}$ for $r \ge 2$ and $U_0 = 2, U_1 = 1$. So $U_n \in \mathbb{N}$ for all $n \in \mathbb{Z}_{\ge 0}$. Note that $\overline{W_n} \to 0$ as $n \to \infty$ since $|\frac{1-\sqrt{5}}{2}| < 1$. So we complete the proof since $\gamma_n \notin (\frac{1}{4}, \frac{3}{4})$ for all large n. □

6. **Let $\theta = p/q$ be a rational number where p and q are relatively prime integers (that is, θ is in lowest form). We assume without loss of generality that $q > 0$. Define a sequence of numbers in $[0, 1)$ by $\xi_n = \langle n\theta\rangle$ where $\langle\cdot\rangle$ denotes the fractional part. Show that the sequence $\{\xi_1, \xi_2, \cdots\}$ is equidistributed on the points of the form**

$$0, 1/q, 2/q, \cdots, (q-1)/q.$$

In fact, prove that for any $0 \le a < q$, one has

$$\frac{\#\{n : 1 \le n \le N, \langle n\theta\rangle = a/q\}}{N} = \frac{1}{q} + O(\frac{1}{N}).$$

Proof. By Euclidean algorithm, we know that there is $x, y \in \mathbb{Z}$ such that $xp + yq = 1$, that is $(ax + tq)p + (ay - tp)q = a$ for every $t \in \mathbb{Z}$. For each $k \in \mathbb{Z}_{\geq 0}$, by adjusting t, we can find $n = ax + tq \in [kq, (k+1)q)$ with $\langle n\frac{p}{q}\rangle = \frac{a}{q}$. On the other hand, one can easily show such n is unique. For $N = lq + r$ where $0 \leq l$ and $0 \leq r < q$, one then have the inequalities

$$l \leq \#\{n : 1 \leq n \leq N, \langle n\frac{p}{q}\rangle = \frac{a}{q}\} \leq l + 1.$$

Then the desired conclusion follows easily. □

7. **Prove the second part of Weyl's criterion: if a sequence of numbers $\xi_1, \xi_2, \cdots$ in $[0, 1)$ is equidistributed, then for all $k \in \mathbb{Z} \setminus \{0\}$**

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\xi_n} \to 0 \quad \textbf{as} \quad N \to \infty.$$

Proof. First, the equidistributed property of $\{\xi_n\}$ implies that for each $(a, b) \subset [0, 1)$

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(\xi_n) \to b - a \quad \text{as} \quad N \to \infty.$$

By the linearity, we know for each step function $S(x) = \sum_{k=1}^{M} a_k \chi_{I_k}(x)$.

$$\frac{1}{N}\sum_{n=1}^{N} S(\xi_n) \to \int_0^1 S(x)\, dx \quad \text{as} \quad N \to \infty.$$

Finally, for each continuous function $f(x)$ on $[0, 1]$, its uniform continuity implies that for each $\epsilon > 0$ there is a step function S_ϵ such that $\|f - S_\epsilon\|_\infty < \epsilon$. Then for each $N \in \mathbb{N}$

$$\begin{aligned}
|\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) - \int_0^1 f(x)\, dx| &\leq |\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) - \frac{1}{N}\sum_{n=1}^{N} S_\epsilon(\xi_n)| + |\frac{1}{N}\sum_{n=1}^{N} S_\epsilon(\xi_n) - \int_0^1 S_\epsilon(x)\, dx| \\
&\quad + |\int_0^1 S_\epsilon(x)\, dx - \int_0^1 f(x)\, dx| \\
&\leq 2\epsilon + |\frac{1}{N}\sum_{n=1}^{N} S_\epsilon(\xi_n) - \int_0^1 S_\epsilon(x)\, dx|.
\end{aligned}$$

Hence $|\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) - \int_0^1 f(x)\, dx| \leq 3\epsilon$ for large N. □

8. **Show that for any $a \neq 0$, and σ with $0 < \sigma < 1$, the sequence $\langle an^\sigma\rangle$ is equidistributed in $[0, 1)$. For arbitrary non-integer $\sigma > 0$, see Problem 3.**

Proof. We use mean value theorem in the following calculation. Given $k \in \mathbb{Z}\backslash\{0\}$, one observes

$$\sum_{n=1}^{N}\cos(2\pi k\langle an^\sigma\rangle) = \sum_{n=1}^{N}\cos(2\pi kan^\sigma)$$

$$= \int_1^N \cos(2\pi kax^\sigma)\,dx + \cos(2\pi ka) + \sum_{n=2}^{N}\cos(2\pi kan^\sigma) - \int_1^N \cos(2\pi kax^\sigma)\,dx$$

$$= \frac{\sin 2\pi kax^\sigma}{2\pi ka\sigma x^{\sigma-1}}\Big|_{x=1}^{N} - \int_1^N \frac{\sin 2\pi kax^\sigma}{2\pi ka\sigma}(1-\sigma)x^{-\sigma}\,dx + \cos(2\pi ka) + \sum_{n=2}^{N}\cos(2\pi kan^\sigma) - \cos(2\pi kax_n^\sigma)$$

$$= O(N^{1-\sigma}) + \cos(2\pi ka) - \sum_{n=2}^{N} 2\pi ka\sigma\sin(2\pi ka(y_n)^\sigma)y_n^{\sigma-1}(n-x_n),$$

where $x_n \in [n-1, n]$ and $y_n \in [x_n, n]$ for each n.

So $|\sum_{n=1}^{N}\cos(2\pi k\langle an^\sigma\rangle)| \le O(N^{1-\sigma}) + O(\sum_{n=1}^{N} n^{\sigma-1}) = O(N^{1-\sigma}) + O(N^\sigma)$.

Similar for the sine (imaginary) part, so $\{\langle an^\sigma\rangle\}_n$ is equidistributed by Weyl's criterion. □

***Remark* 3.** Such estimate can not be applied to Problem 3 directly, that is, $\sigma > 1$. For $\sigma \in (1,2)$, One can modify the integral "piecewisely" like van der Corput's lemma, see my solution to Exercise 3.16(d) and the second part of Problem 3.

9. **In contrast with the result in Exercise 8, prove that $\langle a\log n\rangle$ for any a.**

Proof. We use mean value theorem in the following calculation. One observes that

$$\sum_{n=1}^{N}\cos(2\pi\langle a\log n\rangle) = \sum_{n=1}^{N}\cos(2\pi a\log n)$$

$$= \int_1^N \cos(2\pi a\log x)\,dx + \sum_{n=2}^{N}\cos(2\pi a\log n) - \int_1^N \cos(2\pi a\log x)\,dx$$

$$= \int_0^{\log N}\cos(2\pi az)e^z\,dz + \sum_{n=2}^{N}\cos(2\pi a\log n) - \cos(2\pi a\log x_n)$$

$$= \frac{N[\cos(2\pi a\log N) + 2\pi a\sin(2\pi a\log N)] - (1+2\pi a)}{1+(2\pi a)^2} - \sum_{n=2}^{N} 2\pi a\sin(2\pi a\log y_n)y_n^{-1}(n-x_n),$$

where $x_n \in [n-1, n]$ and $y_n \in [x_n, n]$ for each n. So

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi k\langle a\log n\rangle) \ge \frac{1}{2\sqrt{1+(2\pi a)^2}} - \limsup_{N\to\infty} C\frac{1}{N}\sum_{n=1}^{N} n^{-1} = \frac{1}{2\sqrt{1+(2\pi a)^2}}.$$

and $\{\langle a\log n\rangle\}_n$ is not equidistributed by Weyl's criterion. □

10. **Suppose that f is a periodic function on $\mathbb{R}$ of period 1, and $\{\xi_n\}$ is a sequence which is equidistributed in $[0,1)$. Prove that:**

(a) If f is continuous and satisfies $\int_0^1 f(x)\,dx = 0$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n) = 0 \ \ \textbf{uniformly in} \ \ x.$$

(b) If f is merely in $L^2(0,1)$ and satisfies $\int_0^1 f(x)\,dx = 0$, then

$$\lim_{N\to\infty} \int_0^1 \Big|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\Big|^2\,dx = 0.$$

Proof. (a) In general, we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n) = \int_0^1 f(t)\,dt$ uniform in x. First, if $f(x) = e^{2\pi ikx}$, $k \neq 0$, then

$$\lim_{N\to\infty} \Big|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\Big| = \lim_{N\to\infty} |e^{2\pi ikx}|\Big|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ik\xi_n}\Big| = 0 \ \ \text{uniformly in} \ \ x.$$

By linearity, this is true for any trigonometric polynomials without zero order term. We then complete the proof by Weierstrauss approximation theorem.

(b) This is accomplished by (a) and approximate f by continuous functions in L^2 sense. (In the estimations, one may need to use Minkowski's inequality somewhere). The approximation steps are the same as my answer to Problem 3.2(d). Note that this is also true for L^p case where $p \in [1,\infty)$. (In this problem $p = 2$.) $\square$

Remark 4. For more results in ergodic theory, check [2], [6], [10] and [13].

11. **Show that if $u(x,t) = (f * H_t)(x)$ where H_t is the heat kernel, and f is Riemann integrable, then**

$$\int_0^1 |u(x,t) - f(x)|^2\,dx \to 0 \ \ \textbf{as} \ \ t \to 0.$$

Proof. Let $f(x) \sim \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$. Given $\epsilon > 0$, we have, by Parseval's identity

$$\int_0^1 |u(x,t) - f(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |a_n|^2(1-e^{-4\pi^2n^2t})^2 \le \sum_{0<|n|\le N} |a_n|^2(1-e^{-4\pi^2n^2t})^2 + \sum_{|n|>N} |a_n|^2,$$

where $N = N(\epsilon)$ is chosen to be large enough such that the second term is less than ϵ. Then we see that there is a $t_\epsilon > 0$ such that the first term is less than ϵ whenever $0 < t < t_\epsilon$. $\square$

12. A change of variables in (8) leads to the solution

$$u(\theta,\tau)=\sum a_n e^{-n^2\tau}e^{in\theta}=(f*h_\tau)(\theta)$$

of the equation

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial \theta^2}\quad\text{with } 0\le\theta\le 2\pi \text{ and } \tau>0,$$

with boundary condition $u(\theta,0)=f(\theta)\sim\sum a_n e^{in\theta}$. Here $h_\tau(\theta)=\sum_{n=-\infty}^{\infty}e^{-n^2\tau}e^{in\theta}$. This version of the heat kernel on $[0,2\pi]$ is the analogue of the Poisson kernel, which can be written as $P_r(\theta)=\sum_{n=-\infty}^{\infty}e^{-|n|\tau}e^{in\theta}$ with $r=e^{-\tau}$ (and so $0<r<1$ corresponds to $\tau>0$).

13. **The fact that the kernel $H_t(x)$ is a good kernel, hence $u(x,t)\to f(x)$ at each point of continuity of f, is not easy to prove. This will be shown in the next chapter. However, one can prove directly that $H_t(x)$ is "peaked" at $x=0$ as $t\to 0$ in the following sense:**

(a) Show that $\int_{-1/2}^{1/2}|H_t(x)|^2\,dx$ is of the order of magnitude of $t^{-1/2}$ as $t\to 0$. More precisely, prove that $t^{1/2}\int_{-1/2}^{1/2}|H_t(x)|^2\,dx$ converges to a non-zero limit as $t\to 0$.

(b) Prove that $\int_{-1/2}^{1/2}x^2|H_t(x)|^2\,dx=O(t^{1/2})$ as $t\to 0$.

Proof. (a) One note that from Parseval's identity $\int_{-\frac{1}{2}}^{\frac{1}{2}}|H_t(x)|^2\,dx=\sum_{n\in\mathbb{Z}}e^{-4\pi^2n^2t}$. Then we have the following upper estimates

$$\begin{aligned}\sum_{n\in\mathbb{Z}}e^{-4\pi^2n^2t}&=\sum_{n\ge 0}e^{-4\pi^2n^2t}+\sum_{n\le 0}e^{-4\pi^2n^2t}-1\ge\int_0^\infty e^{-4\pi^2x^2t}\,dx+\int_{-\infty}^0 e^{-4\pi^2x^2t}\,dx-1\\&=\sqrt{\pi}(2\pi\sqrt{t})^{-1}-1\end{aligned}$$

$$\begin{aligned}\sum_{n\in\mathbb{Z}}e^{-4\pi^2n^2t}&=\sum_{n\ge 1}e^{-4\pi^2n^2t}+\sum_{n\le -1}e^{-4\pi^2n^2t}+1\le\int_0^\infty e^{-4\pi^2x^2t}\,dx+\int_{-\infty}^0 e^{-4\pi^2x^2t}\,dx+1\\&=\sqrt{\pi}(2\pi\sqrt{t})^{-1}+1\end{aligned}$$

So $t^{1/2}\int_{-1/2}^{1/2}|H_t(x)|^2\,dx\to\frac{1}{2\sqrt{\pi}}$ as $t\to 0^+$.

(b) As hint, $x^2\le\frac{1}{4}\sin^2\pi x$ for $|x|\le\frac{1}{2}$, and hence by mean-value theorem and Parseval's identity (note that $\{e^{2\pi i(n+\frac{1}{2})x}\}_{n\in\mathbb{Z}}$ also form an orthonormal basis for $L^2([-\frac{1}{2},\frac{1}{2}])$, we have

$$
\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2|H_t(x)|^2\,dx \le \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big(\frac{e^{i\pi x}-e^{-i\pi x}}{2i}\Big)^2|H_t(x)|^2 = \frac{1}{16}\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big|\sum_{n\in\mathbb{Z}}(e^{i\pi x}-e^{-i\pi x})e^{-4\pi^2n^2t}e^{2\pi inx}\Big|^2\,dx\\
&=\frac{1}{16}\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big|\sum_{n\in\mathbb{Z}}\Big(e^{-4\pi^2n^2t}-e^{-4\pi^2(n-1)^2t}\Big)e^{2\pi i(n+\frac{1}{2})x}\Big|^2\,dx = \frac{1}{16}\sum_{n\in\mathbb{Z}}|e^{-4\pi^2n^2t}-e^{-4\pi^2(n-1)^2t}|^2\\
&=\frac{1}{16}\sum_{n\in\mathbb{Z}}|e^{-4\pi^2(n-\delta_n^t)^2t}|^2 16\pi^4(2n-1)^2t^2,\ \text{ where } \delta_n^t\in(0,1)\ \forall n\in\mathbb{Z}\\
&\le \pi^4t^2\Big[\sum_{n\le 0}|e^{-4\pi^2n^2t}|^2(2n-1)^2+\sum_{n\ge 1}|e^{-4\pi^2(n-1)^2t}|^2(2n-1)^2\Big]\\
&\le \pi^4t^2\Big[1+\int_{-\infty}^{0}e^{-8\pi^2x^2t}(2x-3)^2\,dx+1+\int_0^\infty e^{-8\pi^2x^2t}(2x+3)^2\,dx\Big]\\
&=2\pi^4t^2\Big[1+\int_0^\infty e^{-8\pi^2x^2t}(4x^2+12x+9)\,dx\Big]=Ct^{\frac{1}{2}}+o(t^{\frac{1}{2}})\ \text{ as } t\to 0^+.
\end{aligned}
$$

Similarly, we use symmetry of the integrand and $\pi^2x^2 \ge \sin^2\pi x$ for $x\in[0,\frac{1}{2}]$ to obtain $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2|H_t(x)|^2\,dx \ge Dt^{\frac{1}{2}}+o(t^{\frac{1}{2}})$ as $t\to 0^+$. □

Problems

Problem 2 and 3 are taken from [4, Section 2,3 of Chapter 1].

1. **This problem explores another relationship between the geometry of a curve and Fourier series. The diameter of a closed curve Γ parametrized by $\gamma(t)=(x(t),y(t))$ on $[-\pi,\pi]$ is defined by**

$$d=\sup_{P,Q\in\Gamma}|P-Q| = \sup_{t_1,t_2\in[-\pi,\pi]}|\gamma(t_1)-\gamma(t_2)|.$$

If a_n is the n-th Fourier coecient of $\gamma(t)=x(t)+iy(t)$ and l denotes the length of Γ, then (a) $2|a_n|\le d$ for all $n\neq 0$. (b) $l\le \pi d$, whenever Γ is convex.

Property (a) follows from the fact $2a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\gamma(t)-\gamma(t+\pi/n)]e^{-int}\,dt$.

The equality $l=\pi d$ is satisfied when Γ is a circle, but surprisingly, this is not the only case. In fact, one finds that $l=\pi d$ is equivalent to $2|a_1|=d$. We re-parametrize γ so that for each t in $[-\pi,\pi]$ the tangent to the curve makes an angle t with the y-axis. Then, if $a_1=1$ we have

$$\gamma'(t)=ie^{it}(1+r(t)),$$

where r is a real-valued function which satisfies $r(t)+r(t+\pi)=0$, $\int_0^\pi e^{ix}r(x)\,dx=0$ and $|r(t)|\le 1$. Figure 7 (a) shows the curve obtained by setting $r(t)=\cos 5t$. Also,

Figure 7 (b) consists of the curve where $r(t) = h(3t)$, with $h(s) = -1$ if $-\pi \le s \le 0$ and $h(s) = 1$ if $0 < s < \pi$. This curve (which is only piecewise of class C^1) is known as the Reuleaux triangle and is the classical example of a convex curve of constant width which is not a circle.

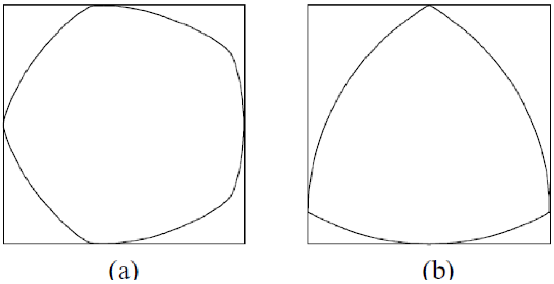

Figure 7. Some curves with maximal length for a given diameter

Remark 5. This is from [7, 8]. Also see Exercise 2.7 of Book II. After checking the details in those papers, I think the condition $\int_0^\pi e^{ix} r(x)\,dx = 0$ omitted in [8] is actually needed.

We summarize [7, 8, Theorem 1] and its proof here. In the end, we give the proof for (b).

Let $R = R_\gamma$ be the radius of the smallest closed disc containing Γ. Then it's easy to see $d \le 2R$ and a weaker inequality $|a_n| \le R$ for all $n \ne 0$. As stated and proved in (a), one have a stronger inequality $|a_n| \le \frac{d}{2}$. Using Fourier series and the Parseval's identity, one can easily see that the equality in weaker inequality holds precisely when $\gamma(t) = a_0 + a_n e^{int}, |a_n| = R$. The papers cited here aim to determine when the equality holds in the above stronger inequality $|a_n| \le \frac{d}{2}$.

Because the case $d = 0$ is trivial, one can normalize the function γ by the condition $a_n = 1$, i.e. to confine ourselves to functions of the type

$$\Gamma(t) = e^{int} + \sum_{k \ne n} a_k e^{ikt},$$

and so to the class

$$\mathcal{F}_n = \{\Gamma : a_n = 1, d_\Gamma = 2\}.$$

From the identity used in (a), $|\Gamma(t) - \Gamma(t + \frac{\pi}{n})| \equiv 2$. Moreover, application of Parseval's identity to $\Gamma(t) - \Gamma(t + \frac{\pi}{n}) \sim \sum_k a_k(1 - e^{\frac{\pi i k}{n}})e^{ikt}$ implies that $a_k(1 - e^{\frac{\pi i k}{n}}) = 0$ for all $k \ne n$ and hence $a_k = 0$ except $k = 2n\nu$ for some $\nu \in \mathbb{Z}$. Therefore

$$\Gamma(t) = e^{int} + \sum_{\nu \in \mathbb{Z}} c_{2n\nu} e^{2n\nu i t}. \tag{1}$$

With $\gamma(t) := \Gamma(\frac{t}{n})$ and $g(t) := \sum_{\nu \in \mathbb{Z}} c_{2n\nu} e^{2\nu i t}$ equation (1) reduces to

$$\gamma(t) = e^{it} + g(t), \;\; g(t + \pi) = g(t) \tag{2}$$

Because the curves γ, Γ have the same diameter, the problem of determing the class $\mathcal{F}_n$ (for any $n \neq 0$) is thus reduced to the particular case of $n = 1$.

One of the main results in those papers are

Theorem 6. *Let the function γ be absolutely continuous on $\mathbb{R}$ and 2π-periodic, and let γ have its first Fourier coefficient normalized by $a_1 = 1$. Then the curve corresponding to this function has a diameter $d_\gamma \geq 2$ and the equality holds if and only if γ' admits the representation*

$$\gamma'(t) = ie^{it}(1 + r(t)),$$

where r is a real-valued function which satisfies $r(t) + r(t + \pi) = 0$, $\int_0^\pi e^{ix} r(x)\, dx = 0$ and $|r(t)| \leq 1$ for all t.

Proof. For the if part, one assume that $\theta \in (0, \pi)$. There is $\alpha = \alpha_{t,\theta}$ such that

$$\gamma(t + \theta) - \gamma(t) = \int_t^{t+\theta} ie^{ix}(1 + r(x))\, dx = -Re^{i\alpha}$$

and $R \geq 0$. This implies

$$\int_t^{t+\theta} (1 + r(x)) \sin(x - \alpha)\, dx = R$$

We are going to show $R = R_{t,\theta} \leq 2$ for all t, θ. Let J denote the subset of $[t, t + \theta]$ on which $\sin(x - \alpha)$ is ≥ 0. Then

$$\begin{aligned} R &\leq \int_J (1 + r(x)) \sin(x - \alpha)\, dx \leq \int_\alpha^{\alpha+\pi} (1 + r(x)) \sin(x - \alpha)\, dx \\ &= 2 + \int_\alpha^{\alpha+\pi} r(x) \sin(x - \alpha)\, dx = 2 \end{aligned}$$

where the last equality is due to the periodicity of r. Then we completes the proof for $d_\gamma = 2$ since $|\gamma(\pi) - \gamma(0)| = |\int_0^\pi ie^{ix} + ie^{ix} r(x)\, dx| = 2\pi$.

For the only if part, we assume first γ is twice differentiable. Let t be arbitrary but fixed, and let $z(x) = \gamma(t) - \gamma(t + x)$. Since γ is of the form (2) by the analysis before (1), we have $z(\pi) = \gamma(t) - \gamma(t + \pi) = 2e^{it}$, and by the assumption, $|z(x)| \leq d_\gamma = 2$ for any x, then function $|z(x)|^2$ attains its maximum at $x = \pi$. Hence

$$\frac{d}{dx}|z(x)|^2 = 2\,\mathrm{Re}\Big(\overline{z(x)}\frac{dz}{dx}(x)\Big) = 0 \text{ at } x = \pi.$$

That is, $\mathrm{Re}\,\{2e^{-it}(ie^{it} - g'(t + \pi))\} = 0$ or $\mathrm{Re}\,\{e^{-it} g'(t))\} = 0$ for all t. Letting

$$e^{-it} g'(t) = ir(t),$$

we conclude that $r(t)$ is real-valued and satisfies $r(t+\pi) = -r(t)$ for all t. Furthermore, $\int_0^\pi e^{ix} r(x)\,dx = -i\int_0^\pi g'(x)\,dx = -i[g(\pi) - g(0)] = 0$.

Finally, as the function $|z(x)|^2$ attains its maximum at $x = \pi$,

$$\frac{d^2}{dx^2}|z(x)|^2 = 2\,\mathrm{Re}\Big(\Big|\frac{dz}{dx}(x)\Big|^2 + \overline{z(x)}\frac{d^2z}{dx^2}(x)\Big) \le 0 \text{ at } x = \pi.$$

That is, $-2(1 - r^2(t)) \le 0$ for all t and is equivalent to $|r(t)| \le 1$. $\square$

Proof for (b) $l \le \pi d$. The proof is taken from here.

Let $\gamma(t) = x(y) + iy(t)$, using complex notation. For every $\theta \in [0,\pi]$ the function $f_\theta(t) = \mathrm{Re}(e^{i\theta}\gamma(t))$ takes values within some interval of length at most d. By convexity, this function has two intervals of monotonicity on the circle (think of $[-\pi,\pi]$ as the circle by gluing the endpoints together)(one can use the notion of support line from the proof of Jensen's inequality to justify this sentence rigorously). Therefore, its total variation is at most $2d$. The total variation is the integral of absolute value of derivative, provided that we chose a sensible (absolutely continuous) parametrization γ: the arclength parametrization would do, for example. Hence,

$$\int_{-\pi}^{\pi} |\mathrm{Re}\,(e^{i\theta}\gamma'(t))|\,dt \le 2d$$

Integrate both sides with respect to θ and exchange the order of integration (by Fubini-Tonelli theorem), we have

$$\int_{-\pi}^{\pi}\int_0^{\pi} |\mathrm{Re}\,(e^{i\theta}\gamma'(t))|\,d\theta dt \le 2\pi d.$$

Note that for any $\zeta \in \mathbb{C}$ we have $\int_0^\pi |\mathrm{Re}(e^{i\theta}\zeta)|\,d\theta = 2|\zeta|$. So the above inequality becomes

$$2l = \int_{-\pi}^{\pi} 2|\gamma'(t)|\,dt \le 2\pi d.$$

$\square$

2. **Here we present an estimate of Weyl which leads to some interesting results.**

   (a) Let $S_N = \sum_{n=1}^N e^{2\pi i f(n)}$. Show that for $H \le N$, one has

   $$|S_N|^2 \le c\frac{N}{H}\sum_{h=0}^{H}\Big|\sum_{n=1}^{N-h} e^{2\pi i(f(n+h)-f(n))}\Big|,$$

   for some constant $c > 0$ independent of N, H, and f.

   (b) Use this estimate to show that the sequence $\langle n^2\gamma\rangle$ is equidistributed in $[0,1)$ whenever γ is irrational.

(c) More generally, show that if $\{\xi_n\}$ is a sequence of real numbers so that for all positive integers h the difference $\langle \xi_{n+h} - \xi_n \rangle$ is equidistributed in $[0,1)$, then $\langle \xi_n \rangle$ is also equidistributed in $[0,1)$.

(d) Suppose that $P(x) = c_k x^k + \cdots + c_0$ is a polynomial with real coefficients, where at least one of $c_1, \cdots, c_k$ is irrational. Then the sequence $\langle P(n) \rangle$ is equidistributed in $[0,1)$.

***Remark* 7.** (a) is a special form of van der Corput's Inequality. (c) is called the Van der Corput's difference theorem.

Proof. (a) As hint, we consider $a_n = e^{2\pi i f(n)}$ if $1 \le n \le N$ and zero for other indices. Then by Cauchy-Schwarz, we have a (general) form of van der Corput's Inequality as follows

$$
\begin{aligned}
H^2 |\sum_{n\in\mathbb{Z}} a_n|^2 &= \Big| \sum_{k=1}^{H} \sum_{n\in\mathbb{Z}} a_{n+k} \Big|^2 = \Big| \sum_{n=1-H}^{N-1} \sum_{k=1}^{H} a_{n+k} \Big|^2 \le (N+H-1) \sum_n \Big| \sum_{k=1}^{H} a_{n+k} \Big|^2 \\
&= (N+H-1) \sum_n \sum_{k=1}^{H} \sum_{j=1}^{H} a_{n+k} \overline{a_{n+j}} = (N+H-1) \sum_{k=1}^{H} \sum_{j=1}^{H} \sum_n a_{n+k} \overline{a_{n+j}} \\
&= (N+H-1) \Big\{ \sum_{1\le j\le k\le H} \sum_n a_{n+k} \overline{a_{n+j}} + \sum_{1\le k<j\le H} \sum_n a_{n+k} \overline{a_{n+j}} \Big\} \\
&= (N+H-1) \Big\{ H \sum_n |a_n|^2 + \sum_{1\le j<k\le H} \sum_n a_{n+k-j} \overline{a_n} + \sum_{1\le k<j\le H} \sum_n a_n \overline{a_{n+j-k}} \Big\} \\
&= (N+H-1) \Big\{ H \sum_n |a_n|^2 + 2\,\mathrm{Re} \sum_{1\le j<k\le H} \sum_n a_{n+k-j} \overline{a_n} \Big\} \\
&= (N+H-1) \Big\{ H \sum_n |a_n|^2 + 2\,\mathrm{Re} \sum_{j=1}^{H-1} \sum_{1\le h\le H-j} \sum_n a_{n+h} \overline{a_n} \Big\} \\
&= (N+H-1) \Big\{ H \sum_n |a_n|^2 + 2\,\mathrm{Re} \sum_{h=1}^{H-1} (H-h) \sum_n a_{n+h} \overline{a_n} \Big\}
\end{aligned}
$$

In particular,

$$
|\sum_{n=1}^{N} e^{2\pi i f(n)}|^2 \le \frac{N+H-1}{H} 2 \sum_{h=0}^{H-1} \Big(1 - \frac{h}{H}\Big) \Big| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \Big| \le \frac{2N}{H} 2 \sum_{h=0}^{H-1} \Big| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \Big|.
$$

(b) Let $k \in \mathbb{Z} \setminus \{0\}$. Note that for $f(n) = kn^2\gamma$ in (a), $f(n+h) - f(n) = k(2nh\gamma + h^2\gamma)$. Since $\langle n\gamma \rangle$ is equidistributed in $[0,1)$, we have, for each $H \in \mathbb{N}$

$$
\begin{aligned}
\limsup_{N\to\infty} |\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n^2 \gamma}|^2 &\le \limsup_{N\to\infty} \frac{c}{H} \sum_{h=0}^{H} \frac{1}{N} \Big| \sum_{n=1}^{N-h} e^{2\pi i 2nkh\gamma} \Big| \\
&= \frac{c}{H} + \limsup_{N\to\infty} \sum_{h=1}^{H} \frac{c(N-h)}{N} \frac{1}{N-h} \Big| \sum_{n=1}^{N-h} e^{2\pi i (2kh) n\gamma} \Big| = \frac{c}{H}.
\end{aligned}
$$

So $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ik\langle n^2\gamma\rangle}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikn^2\gamma}=0$. Since k is arbitrary, $\langle n^2\gamma\rangle$ is equidistributed by Weyl's criterion.

(c) Modify the proof for (b) slightly.

(d) First we consider the case $c_1\in\mathbb{Q}^c$ and $c_2,\cdots c_k\in\mathbb{Q}$. Write $P(x)=Q(x)+c_1x+c_0$. Let D be the least common factor of the denominators of $c_2,\cdots c_k$. Then we have $\langle Q(Dk+d)\rangle=\langle Q(d)\rangle$ for $k\geq 0$ and $d\geq 1$. Therefore, for every $s\in\mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi isP(n)}&=\frac{1}{N}\sum_{n=[\frac{N}{D}]D+1}^{N}e^{2\pi isP(n)}+\frac{1}{N}\sum_{d=1}^{D}\sum_{k=0}^{[\frac{N}{D}]-1}e^{2\pi is(Q(Dk+d)+c_1(Dk+d)+c_0)}\\&=\frac{1}{N}\sum_{n=[\frac{N}{D}]D+1}^{N}e^{2\pi isP(n)}+\Big(\sum_{d=1}^{D}e^{2\pi is(Q(d)+c_1d+c_0)}\Big)\Big(\frac{1}{N}\sum_{k=0}^{[\frac{N}{D}]-1}e^{2\pi isDc_1k}\Big)\end{aligned}$$

Since $c_1\notin\mathbb{Q}$, $\langle c_1k\rangle$ is equidistributed and hence the second term of the above inequality tends to zero as $N\to\infty$. Note that the first term is bounded by $\frac{D}{N}$ which also tends to zero as $N\to\infty$. So $\langle P(n)\rangle$ is equidistributed in $[0,1)$.

For general case, we use induction argument. Let $k\geq q\geq 1$ be the largest index such that $c_q\in\mathbb{Q}^c$. Now we see the claim is true for $k=1$. Now we assume the claim is true for $k=m$ and consider case $k=m+1$, for each $h\in\mathbb{N}$, $P(n+h)-P(n)=Q_h(n)$ for some polynomial Q_h whose indices for irrational coefficients are all $\leq m$. So the Weyl's criterion implies for each $s\in\mathbb{Z}\setminus\{0\}$

$$\frac{1}{N-h}\sum_{n=1}^{N-h}e^{2\pi isQ_h(n)}\to 0$$

and hence, by (a), for each $H\in\mathbb{N}$

$$\limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi isP(n)}\Big|^2\leq\frac{c}{H}+\limsup_{N\to\infty}\sum_{h=1}^{H}\frac{c(N-h)}{N}\frac{1}{N-h}\Big|\sum_{n=1}^{N-h}e^{2\pi isQ_h(n)}\Big|=\frac{c}{H}.$$

Therefore $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi is\langle P(n)\rangle}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi isP(n)}=0$. Since s is arbitrary, $\langle P(n)\rangle$ is equidistributed in $[0,1)$ by Weyl's criterion. □

3. **If $\sigma>0$ is not an integer and $a\neq 0$, then $\langle an^\sigma\rangle$ is equidistributed in $[0,1)$.**

Before the proof taken from [4, Theorem 1.3.5], we present a proof for $\sigma\in(1,2)$ inspired by [4, Exercise 1.2.23] which uses similar idea as Exercise 8. See Remark 3. This reflects the importance of Weyl's estimate (Van der Corput's trick) in Problem 2 for the theory of sequences of uniform distribution.

A proof for the special case $\sigma \in (1,2)$. By Weyl's criterion, it suffices to show for all $k \in \mathbb{Z}\backslash\{0\}$

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a n^\sigma} \to 0 \quad \text{as} \quad N \to \infty.$$

We recall the Van der Corput's lemmas proved in Exercise 3.16 (d).

Given a real-valued function $f(u)$ and numbers $a < b$, we set $F(u) = e^{2\pi i f(u)}$,

$$I(F;a,b) = \int_a^b F(u)\,du, \;\; S(F;a,b) = \sum_{a<n\le b} F(n), \;\; D(F;a,b) = I(F;a,b) - S(F;a,b).$$

Lemma 8. *(i) If f has a monotone derivative f', and if there is a $\lambda > 0$ such that $f' \ge \lambda$ or $f' \le -\lambda$ in (a,b), then $|I(F;a,b)| < \lambda^{-1}$.*

(ii) If $f'' \ge \rho > 0$ or $f'' \le -\rho < 0$, then $|I(F;a,b)| \le 4\rho^{-\frac{1}{2}}$.

Lemma 9. *If f' is monotone and $|f'| \le \frac{1}{2}$ in (a,b), then*

$$|D(F;a,b)| \le A$$

where A is an absolute constant independent of a,b. In fact, we have $A = \sum_{n=1}^{\infty} \frac{2}{\pi n(n-\frac{1}{2})} + 2$.

Lemma 10. *If $f'' \ge \rho > 0$ or $f'' \le -\rho < 0$, then*

$$|S(F;a,b)| \le \Big(|f'(b) - f'(a)| + 2\Big)(4\rho^{-\frac{1}{2}} + A).$$

WLOG, we assume $a > 0$ and $\sigma \in (1,\infty) \setminus \mathbb{Z}$. The function $f(u) = kau^\sigma$ has an increasing derivative $f'(u) = ka\sigma u^{\sigma-1}$. In the following estimates, C stands for different generic constants depending on k, a, σ only.

Now we deal the case $\sigma \in (1,2)$ first, $f''(u) = ka\sigma(\sigma-1)u^{\sigma-2} \ge C2^{j(\sigma-2)}$ on $[2^j, 2^{j+1}]$ for each $j \in \mathbb{Z}_{\ge 0}$. A simple application of Lemma 10 shows that for each $j \ge 0$

$$|S(F;2^j,2^{j+1})| \le \{C2^{j(\sigma-1)} + 2\}\{C2^{j(2-\sigma)/2} + A\} \le C2^{j\sigma/2}.$$

Similarly, $|S(F;2^n,N)| \le C2^{n\sigma/2}$ if $2^n < N \le 2^{n+1}$ and hence

$$\begin{aligned} |\sum_{k=1}^{N} e^{2\pi i k a n^\sigma}| &\le 1 + |S(F;1,2)| + |S(F;2,4)| + \cdots + |S(F;2^n,N)| \\ &\le 1 + C(1 + 2^{\sigma/2} + \cdots + 2^{n\sigma/2}) \le C2^{n\sigma/2} < CN^{\sigma/2}. \end{aligned}$$

Since $\sigma \in (1,2)$, $|\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i k a n^\sigma}| \le CN^{\sigma/2-1} \to 0$ as $N \to \infty$. $\square$

A proof for all $\sigma \in \mathbb{R}^+ \setminus \mathbb{Z}$. This is a special case of the following theorem.

Theorem 11. *Let $k \in \mathbb{N}$, and let $f(x)$ be a function defined for $x \geq 1$, which is k times differentiable for $x \geq x_0$. If $f^{(k)}(x)$ tends monotonically to zero as $x \to \infty$ and if $\lim_{x\to\infty} x|f^{(k)}(x)| = \infty$, then $\langle f(n) \rangle$ is equidistributed in $[0,1)$.*

In our cases, $f(x) = ax^\sigma$ and $k = [\sigma]$.

Proof for Theorem 11. We use induction argument and assuming it's true for $k = 1$ for a while. Now assume it's true for $k = m$ and consider the case $k = m+1$. For each $h \in \mathbb{N}$, we set $g_h(x) = f(x+h) - f(x)$. Then $g_h^{(m)}(x) = f^{(m)}(x+h) - f^{(m)}(x) = \int_x^{x+h} f^{(m+1)}(t)\, dt$ which is monotone since $f^{(m+1)}$ is. Also, $g_h^{(m)}$ decays to 0 since $f^{(m+1)}$ does.

Note that $x|g_h^{(m)}(x)| = x|f^{(m+1)}(x + \theta_x h)|$ for some $\theta_x \in [0,1]$ for each x. So

$$\liminf_{x\to\infty} x|g_h^{(m)}(x)| = \liminf_{x\to\infty} (x+\theta_x h)|f^{(m+1)}(x+\theta_x h)| \frac{x}{x+\theta_x h} = \infty.$$

By the induction hypothesis, $\langle g_h(n) \rangle$ is equidistributed in $[0,1)$ for each $h \in \mathbb{N}$. Hence, $\langle f(n) \rangle$ is equidistributed in $[0,1)$ by Problem 2 (c).

Now we recall the following discrete analogy of $k = 1$ due to Fejér.

Theorem 12 (Fejér's Theorem)**.** *Let $\{f(n)\}_{n=1}^\infty \subset \mathbb{R}$ with $\Delta f(n) := f(n+1) - f(n)$ is monotone in n. If $\lim_{n\to\infty} \Delta f(n) = 0$ and $\lim_{n\to\infty} n|\Delta f(n)| = \infty$, then $\langle f(n) \rangle$ is equidistributed in $[0,1)$.*

Now, for the case $k = 1$, $\Delta f(n)$ satisfies the conditions of Fejér's Theorem by mean-value theorem, at least for sufficiently large n. The finitely many exceptional terms do not influence the uniform distribution of the sequence.

Proof for Fejér's Theorem. First we recall a fundamental inequality (think about Taylor expansion), for each $u, v \in \mathbb{R}$,

$$\begin{aligned} |e^{2\pi i u} - e^{2\pi i v} - 2\pi i(u-v)e^{2\pi i v}| &= |e^{2\pi i(u-v)} - 1 - 2\pi i(u-v)| = 4\pi^2 \left| \int_0^{u-v} (u-v-w) e^{2\pi i w}\, dw \right| \\ &\leq 4\pi^2 \int_0^{u-v} (u-v-w)\, dw = 2\pi^2 (u-v)^2 \end{aligned}$$

Now set $u = sf(n+1)$ and $v = sf(n)$, where $s \in \mathbb{Z} \setminus \{0\}$, in the above inequality, we have

$$\left| \frac{e^{2\pi i s f(n+1)}}{\Delta f(n)} - \frac{e^{2\pi i s f(n)}}{\Delta f(n)} - 2\pi i s e^{2\pi i s f(n)} \right| \leq 2\pi^2 s^2 |\Delta f(n)| \ \ \forall n \geq 1.$$

Hence

$$\left| \frac{e^{2\pi i s f(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi i s f(n)}}{\Delta f(n)} - 2\pi i s e^{2\pi i s f(n)} \right| \leq \left| \frac{1}{\Delta f(n)} - \frac{1}{\Delta f(n+1)} \right| + 2\pi^2 s^2 |\Delta f(n)| \ \ \forall n \geq 1.$$

Then

$$\Big|2\pi is\sum_{n=1}^{N-1}e^{2\pi isf(n)}\Big| = \Big|\sum_{n=1}^{N-1}\Big(2\pi ise^{2\pi isf(n)}-\frac{e^{2\pi isf(n+1)}}{\Delta f(n+1)}+\frac{e^{2\pi isf(n)}}{\Delta f(n)}\Big)+\frac{e^{2\pi isf(N)}}{\Delta f(N)}-\frac{e^{2\pi isf(1)}}{\Delta f(1)}\Big|$$
$$\leq\sum_{n=1}^{N-1}\Big|\frac{1}{\Delta f(n)}-\frac{1}{\Delta f(n+1)}\Big|+\sum_{n=1}^{N-1}2\pi^2s^2|\Delta f(n)|+\frac{1}{|\Delta f(N)|}+\frac{1}{|\Delta f(1)|}$$

By the monotonicity of $\Delta f(n)$, we get

$$\Big|\frac{1}{N}\sum_{n=1}^{N-1}e^{2\pi isf(n)}\Big|\leq\frac{1}{\pi|s|}\Big(\frac{1}{N|\Delta f(N)|}+\frac{1}{N|\Delta f(1)|}\Big)+\frac{\pi|s|}{N}\sum_{n=1}^{N-1}|\Delta f(n)|,$$

which tends to zero as $N\to\infty$. Since s is arbitrary, $\langle f(n)\rangle$ is equidistributed in $[0,1)$. □

□

□

Remark 13. Any direct proof for this problem without using the abstract Theorem 11?

Remark 14. This theorem was first proved by Paul Csillag "Über die gleichmässige Verteilung nichtganzer positiver Potenzen mod. 1." Acta Szeged 5, 13-18 (1930).

4. **An elementary construction of a continuous but nowhere differentiable function is obtained by "piling up singularities," as follows.**

On $[-1,1]$ consider the function

$$\varphi(x)=|x|$$

and extend φ to $\mathbb{R}$ by requiring it to be periodic of period 2. Clearly, φ is continuous on $\mathbb{R}$ and $|\varphi(x)|\leq 1$ for all x so the function f defined by

$$f(x)=\sum_{n=0}^{\infty}\Big(\frac{3}{4}\Big)^n\varphi(4^nx)$$

is continuous on $\mathbb{R}$.

(a) Fix $x_0\in\mathbb{R}$. For every positive integer m, let $\delta_m=\pm\frac{1}{2}4^{-m}$ where the sign is chosen so that no integer lies in between 4^mx_0 and $4^m(x_0+\delta_m)$. Consider the quotient

$$\gamma_n=\frac{\varphi(4^n(x_0+\delta_m))-\varphi(4^nx_0)}{\delta_m}.$$

Prove that if $n>m$, then $\gamma_n=0$, and for $0\leq n\leq m$ one has $|\gamma_n|\leq 4^n$ with $|\gamma_m|=4^m$.

(b) From the above observations prove the estimate

$$\Big|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\Big|\geq\frac{1}{2}(3^m+1),$$

and conclude that f is not differentiable at x_0.

Proof. See [11, Theorem 7.18]. By using Baire Category Theorem, one can show nowhere continuous functions are many in a topological sense, that is, is of second category in the sup-norm topology, see Section 1.2 of Chapter 4 in Book IV. □

5. **Let f be a Riemann integrable function on the interval $[-\pi, \pi]$. We define the generalized delayed means of the Fourier series of f by**

$$\sigma_{N,K} = \frac{S_N + \cdots + S_{N+K-1}}{K}.$$

Note that in particular

$$\sigma_{0,N} = \sigma_N, \sigma_{N,1} = S_N \ \text{ and } \ \sigma_{N,N} = \Delta_N,$$

where Δ_N are the specific delayed means used in Section 3.

(a) Show that

$$\sigma_{N,K} = \frac{1}{K}((N+K)\sigma_{N+K} - N\sigma_N),$$

and

$$\sigma_{N,K} = S_N + \sum_{N+1 \le |\nu| \le N+K-1} \Big(1 - \frac{|\nu| - N}{K}\Big) \widehat{f}(\nu) e^{i\nu\theta}.$$

From this last expression for $\sigma_{N,K}$ conclude that

$$|\sigma_{N,K} - S_M| \le \sum_{N+1 \le |\nu| \le N+K-1} |\widehat{f}(\nu)|$$

for all $N \le M < N + K$.

(b) Use one of the above formulas and Fejér's theorem to show that with $N = kn$ and $K = n$, then

$$\sigma_{kn,n}(f)(\theta) \to f(\theta) \ \text{ as } \ n \to \infty$$

whenever f is continuous at θ, and also

$$\sigma_{kn,n}(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \ \text{ as } \ n \to \infty$$

at a jump discontinuity (refer to the preceding chapters and their exercises for the appropriate definitions and results). In the case when f is continuous on $[-\pi, \pi]$, show that $\sigma_{kn,n}(f) \to f$ uniformly as $n \to \infty$.

(c) Using part (a), show that if $\widehat{f}(\nu) = O(1/|\nu|)$ and $kn \le m < (k+1)n$, we get

$$|\sigma_{kn,n} - S_m| \le \frac{C}{k}$$

for some constant $C > 0$.

(d) Suppose that $\widehat{f}(\nu) = O(1/|\nu|)$. Prove that if f is continuous at θ then

$$S_N(f)(\theta) \to f(\theta) \text{ as } N \to \infty,$$

and if f has a jump discontinuity at θ then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \text{ as } N \to \infty.$$

Also, show that if f is continuous on $[-\pi, \pi]$, then $S_N(f) \to f$ uniformly.

(e) The above arguments show that if $\sum c_n$ is Cesáro summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s. This is a weak version of Littlewood's theorem (Problem 3, Chapter 2).

Remark 15. Another proof for (e) can be found in my Exercise 2.14(d).

Proof. (a) is easy. (For the S_M part, one rewrite the second identity by moving some terms from the second series to S_N, this is why there is a 1 there).

(b) is proved by Fejér's theorem and the first identity in (a).

(c)(d) are consequences of (a)(b).

(e) All the arguments still works when $\widehat{f}(\nu)e^{i\nu\theta}$ is replaced by c_ν. □

6. **Dirichlet's theorem states that the Fourier series of a real continuous periodic function f which has only a finite number of relative maxima and minima converges everywhere to f (and uniformly).**

   Prove this theorem by showing that such a function satisfies $\widehat{f}(n) = O(1/|n|)$.

   [Hint: Argue as in Exercise 17, Chapter 3; then use conclusion (d) in Problem 5 above, or Problem 2.3, the harder Big-O theorem of Littlewood.]

Proof. The key observation is the following:

When $\{x_1, x_2, \cdots x_k\}$ are relative extreme points inside $(-\pi, \pi)$. If x_1 is a relative maximum, then f is increasing on $(-\pi, x_1) \cup (x_2, x_3) \cup \cdots$ and decreasing on $(x_1, x_2) \cup (x_3, x_4) \cup \cdots$. Similar for x_1 is a relative minimum.

By using the integration by parts as Exercise 3.17, we have $\widehat{f}(n) = O(1/|n|)$. □

References

[1] Bandle, Catherine. Isoperimetric inequalities and applications. Vol. 7. Pitman Publishing, 1980.

[2] Cornfeld, Isaac P., Sergej V. Fomin, and Yakov Grigor'evic Sinai. Ergodic theory. Vol. 245. Springer, 1982.

[3] Gardner, Richard. The Brunn-Minkowski inequality. Bulletin of the American Mathematical Society 39.3 (2002): 355-405.

[4] Kuipers, Lauwerens and Niederreiter, Harald. Uniform distribution of sequences. John Wiley & Sons, Inc., 1974.

[5] Osserman, Robert. The isoperimetric inequality. Bulletin of the American Mathematical Society 84.6 (1978): 1182-1238.

[6] Petersen, Karl E. Ergodic theory. Vol. 2. Cambridge university press, 1989.

[7] Pfluger, Albert. On the diameter of planar curves and Fourier coefficients. Zeitschrift fr angewandte Mathematik und Physik ZAMP 30.2 (1979): 305-314.

[8] Pfluger, Albert. On the Diameter of Planar Curves and Fourier Coefficients. Colloquia Mathematica Societatis János Bolyai, Functions, series, operators, 35:957-965, 1983.

[9] Pólya, George, and Gábor Szegö. Isoperimetric inequalities in mathematical physics. No. 27. Princeton University Press, 1951.

[10] Pollicott, Mark, and Michiko Yuri. Dynamical systems and ergodic theory. Vol. 40. Cambridge University Press, 1998.

[11] Rudin, Walter. Principle of Mathematical Analysis, 3rd edition, 1976.

[12] Schneider, Rolf. Convex bodies: the Brunn-Minkowski theory. No. 151. Cambridge university press, 2014.

[13] Walters, Peter. An introduction to ergodic theory. Vol. 79. Springer, 1982.