Fourier Analysis, Stein and Shakarchi
Chapter 8 Dirichlet's Theorem

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Abstract

During the course Analysis II in NTU 2018 Spring, this solution file is latexed by the teaching assistant Yung-Hsiang Huang[1] with the discussions or help from the following contributors:

Exercise 3-5 He-qing Huang; Exercise 7- Mighty Yeh; Exercise 10- ???; Exercise 11- ???; Exercise 12- ???; Exercise 14- ???; Exercise 15- ???; Exercise 16- ???; Problem 1- ???; Problem 2- ???; Problem 3- ???; Problem 4- ???;

1 Exercises

1. **Prove that there are infinitely many primes by observing that there were only finitely many $p_1, \cdots, p_N$,then**

$$\prod_{j=1}^{N} \frac{1}{1 - 1/p_j} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$

Proof. This is a simple consequence of Theorem 1.6. □

2. In the text we showed that there are infinitely many primes of the form $4k+3$ by a modification of Euclid's original argument. One can easily adapt this technique to prove the similar result for primes of the form $3k+2$, and for those of the form $6k+5$.

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3. Using the same map as Problem 1 of Chapter 7 one can prove that if m and n are relatively prime, then $\mathbb{Z}^*(m) \times \mathbb{Z}^*(n)$ is isomorphic to $\mathbb{Z}^*(mn)$. For surjectivity (say, given $(a,b) \in \mathbb{Z}^*(m) \times \mathbb{Z}^*(n)$), one has to verify $k = bmx + any \in \mathbb{Z}^*(mn)$ where $mx + ny = 1$ (comes from Corollary 1.3). This can be verified as follows: suppose not, say there is a prime $p|k$ and $p|m$, then $p|a$ since $p \not| ny$ and hence contradicts to the fact $a \in \mathbb{Z}^*(m)$.

4. **Let $\varphi(n)$ denote the number of positive integers $\leq n$ that are relatively prime to n. Use the order of groups in the previous exercise, one knows that if n and m are relatively prime, then**

$$\varphi(mn) = \varphi(n)\varphi(m).$$

Moreover, one can give a formula for Euler phi-function as follows:

(a) Calculate $\varphi(p)$ when p is a prime by counting the number of elements in $\mathbb{Z}^*(p)$.

(b) Give a formula for $\varphi(p^k)$ when p is a prime and $k \geq 1$ by counting the number of elements in $\mathbb{Z}^*(p^k)$.

(c) Show that

$$\varphi(n) = n \prod_i \left(1 - \frac{1}{p_i}\right)$$

where p_i are the primes that divide n.

Proof. (a) $\varphi(p) = p - 1$ if p is a prime.

(b) Claim: $\varphi(p^k) = p^k - p^{k-1}$ for $k \geq 1$. This can be proved as follows: if $p|s$, then $s \notin \mathbb{Z}^*(p^k)$. On the other hand, if $p \not| s$, since p is a prime, $s \in \mathbb{Z}^*(p^k)$. So $\varphi(p^k) = p^k - p^{k-1}$, the order of $\mathbb{Z}(p^k)$ minus the number of multiples of p that less than p^k.

(c) By the multiplicative property of φ and (b), $\varphi(n) = \varphi(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = \varphi(p_1^{a_1})\varphi(p_2^{a_2}) \cdots \varphi(p_k^{a_k}) = p_1^{a_1}(1 - \frac{1}{p_1}) \cdots p_k^{a_k}(1 - \frac{1}{p_k}) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right)$. □

5. **If n is a positive integer, show that**

$$n = \sum_{d|n} \varphi(d),$$

where φ is the Euler phi-function.

[Hint: There are precisely $\varphi(n/d)$ integers $1 \leq m \leq n$ with gcd$(m,n) = d$.]

Proof. Note that

$$\left\{\frac{i}{n} : 1 \leq i \leq n\right\} = \cup_{d|n} \left\{\frac{j}{d} : 1 \leq j \leq d, \gcd(d,j) = 1\right\} =: \cup_{d|n} A_d$$

and $\{A_d\}_{d|n}$ are pairwisely disjoint. So one completes the proof by computing the cardinality of sets in both sides. $\square$

6. **Write down the characters of the groups $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5), \mathbb{Z}^*(6)$, and $\mathbb{Z}^*(8)$.**

   (a) Which ones are real, or complex?

   (b) Which ones are even ,or odd? (A character is even if $\chi(-1) = 1$, and odd otherwise).

Proof. Since $\mathbb{Z}^*(3), \mathbb{Z}^*(4)$, and $\mathbb{Z}^*(6)$ are all $\cong \mathbb{Z}(2) = \{0, 1\}$, their characters contain the trivial one and the one $\chi(0) = 1, \chi(1) = -1$ only, both are real and even.

For $\mathbb{Z}^*(5) \cong \mathbb{Z}(4) = \{0, 1, 2, 3\}$. The characters are $\chi_j(k) = e^{2\pi i \frac{j}{4} k}(j, k = 0, 1, 2, 3)$. So χ_0, χ_2 are real. χ_1, χ_3 are complex. Only χ_0 is even.

For $\mathbb{Z}^*(8) \cong \mathbb{Z}(2) \times \mathbb{Z}(2) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Because of $(1, 0) + (1, 0) = (0, 0)$, $\chi((1, 0)) = \pm 1$ for each character χ. Same for $(0, 1)$ and $(1, 1)$. So every character is real. Note that $A + B = C$ for $\{A, B, C\} = \{(1, 0), (0, 1), (1, 1)\}$, so -1 appears twice or never appears in the values that each character takes at $\{A, B, C\}$. Hence the even character are the trivial one and the one $\chi((1, 1)) = \chi((0, 0)) = 1$ and $\chi((1, 0)) = \chi((0, 1)) = -1$ $\square$

7. **Recall that for $|z| < 1$,**

$$\log_1 \left(\frac{1}{1 - z} \right) = \sum_{k \geq 1} \frac{z^k}{k}.$$

   We have seen that

$$e^{\log_1 \left(\frac{1}{1-z} \right)} = \frac{1}{1 - z}.$$

   (a) Show that if $w = 1/(1 - z)$, then $|z| < 1$ if and only if $\text{Re}(w) > 1/2$.

   (b) Show that if $\text{Re}(w) > 1/2$ and $w = \rho e^{i\varphi}$ with $\rho > 0, |\varphi| < \pi$, then

$$\log_1 w = \log \rho + i\varphi.$$

   [Hint: If $e^\zeta = w$, then the real part of ζ is uniquely determined and its imaginary part is determined modulo 2π.]

Remark 1. (a) is the Möbius transformation.

Proof. (a) can be proved by brutal computations and Arithmetic-Geometric Means inequality.

(b) As hint, $e^{\log \rho + i\varphi} = \rho e^{i\varphi} = w = \frac{1}{1-z}$ for some $|z| < 1$ from (a). Then

$$e^{\log \rho + i\varphi} = \frac{1}{1 - z} = e^{\log_1 (\frac{1}{1-z})} = e^{\log_1 w}.$$

$\square$

8. **Let ζ denote the zeta function defined for $s > 1$.**

   (a) Compare $\zeta(s)$ with $\int_1^\infty x^{-s}\,dx$ to show that

$$\zeta(s) = \frac{1}{s-1} + O(1) \text{ as } s \to 1^+.$$

   (b) Prove as a consequence that

$$\sum_p \frac{1}{p^s} = \log\left(\frac{1}{s-1}\right) + O(1) \text{ as } s \to 1^+.$$

Proof. (a) Use mean-value theorem, one has

$$|\zeta(s) - \int_1^\infty \frac{1}{x^s}| = \Big|\sum_{n=1}^\infty \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\,dx\Big| = \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}\,dx \leq \sum_{n=1}^\infty \frac{s}{n^{s+1}}.$$

(b) is a consequence of (a) and the fact $\log\zeta(s) = \sum_p \frac{1}{p^s} + O(1)$ proved in Proposition 1.11. $\square$

9. **Let χ_0 denote the trivial Dirichlet character mod q, and $p_1, \cdots, p_k$ the distinct prime divisors of q. Recall that $L(s,\chi_0) = (1-p_1^{-s})\cdots(1-p_k^{-s})\zeta(s)$, and show as a consequence**

$$L(s,\chi_0) = \frac{\varphi(q)}{q}\frac{1}{s-1} + O(1) \text{ as } s \to 1^+$$

Proof. Note that, by Exercise 8 and mean-value theorem to $f(s) = \prod_{j=1}^k (1-p_j^{-s})$,

$$\begin{aligned} L(s,\chi_0) &= \prod_{j=1}^k (1-p_j^{-s})\zeta(s) = \Big[\prod_{j=1}^k (1-p_j^{-s}) - \prod_{j=1}^k (1-p_j)\Big]\zeta(s) + \frac{\varphi(q)}{q}\zeta(s) \\ &= O(s-1)\left(\frac{1}{s-1} + O(1)\right) + \frac{\varphi(q)}{q}\frac{1}{s-1} + O(1). \end{aligned}$$

$\square$

10. **Show that if l is relatively prime to q, then**

$$\sum_{p\equiv l} \frac{1}{p^s} = \frac{1}{\varphi(q)}\log\left(\frac{1}{s-1}\right) + O(1) \text{ as } s \to 1^+.$$

   This is a quantitative version of Dirichlet's Theorem.

Proof. $\square$

11. **Use the characters for $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5)$, and $\mathbb{Z}^*(6)$ to verify directly that $L(1,\chi) \neq 0$ for all non-trivial Dirichlet characters modulo q when $q = 3, 4, 5$, and 6.**

    [Hint: Consider in each case the appropriate alternating series.]

    Proof. □

12. **Suppose χ is real and non-trivial; assuming the theorem that $L(1,\chi) \neq 0$, show directly that $L(1,\chi) > 0$.**

    [Hint: Use the product formula for $L(s,\chi)$.]

    Proof. □

13. **Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_n = a_m$ if $n = m \bmod q$. Show that the series**

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges if and only if $\sum_{n=1}^{q} a_n = 0$.

[Hint: Summation by parts.]

Proof. Let $A_j = \sum_{k=1}^{j} a_k$ with convention that $A_0 = 0$. Recall that

$$\sum_{n=1}^{N} \frac{a_n}{n} = \sum_{n=1}^{N} A_n \frac{1}{n(n+1)} + \frac{A_N}{N+1}$$

The periodicity implies that ($[x]$ is the floor function of x.)

$$A_N = A_q[\frac{N}{q}] + O(1).$$

So the second term is always bounded. Moreover, the first term converges if and only if $A_q = 0$. □

14. **The series**

$$F(\theta) = \sum_{|n|\neq 0} \frac{e^{in\theta}}{n}, \text{ for } |\theta| < \pi,$$

converges for every θ and is the Fourier series of the function defined on $[-\pi, \pi]$ by $F(0) = 0$ and

$$F(\theta) = \begin{cases} i(-\pi - \theta) & \text{if } -\pi \leq \theta < 0 \\ i(\pi - \theta) & \text{if } 0 < \theta \leq \pi, \end{cases}$$

and extended by periodicity (period 2π) to all of $\mathbb{R}$ (see Exercise 8 in Chapter 2).

Show also that if $\theta \neq 0 \mod 2\pi$, then the series

$$E(\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$$

converges, and that

$$E(\theta) = \frac{1}{2}\log\left(\frac{1}{2-2\cos\theta}\right) + \frac{i}{2}F(\theta)$$

Proof. □

15. **To sum the series $\sum_{n=1}^{\infty} a_n/n$ with $a_n = a_m$ if $n = m \mod q$ and $\sum_{n=1}^{q} a_n = 0$, proceed as follows. (a) Define**

$$A(m) = \sum_{n=1}^{q} a_n \zeta^{-mn} \textbf{ where } \zeta = e^{2\pi i/q}$$

Note that $A(q) = 0$. With the notation of the previous exercise, prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{q}\sum_{m=1}^{q-1} A(m)E(2\pi m/q).$$

[Hint: Use Fourier inversion on $\mathbb{Z}(q)$.]

(b) If $\{a_m\}$ is odd, ($a_{-m} = -a_m$) for $m \in \mathbb{Z}$, observe that $a_0 = a_q = 0$ and show that

$$A(m) = \sum_{1 \leq n < q/2} a_n(\zeta^{-mn} - \zeta^{mn}).$$

(c) Still assuming that $\{a_m\}$ is odd, show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{2q}\sum_{m=1}^{q-1} A(m)F(2\pi m/q).$$

[Hint: Define $\tilde{A}(m) = \sum_{n=1}^{q} a_n \zeta^{mn}$ and apply the Fourier inversion formula.]

Proof. □

16. **Use the previous exercises to show that**

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots,$$

which is $L(1, \chi)$ for the non-trivial (odd) Dirichlet character modulo 3.

Proof. □

2 Problems

1. **Here are other series that can be summed by the methods in (a) For the non-trivial Dirichlet character modulo 6, $L(1,\chi)$ equals**

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots,$$

(b) If χ is the odd Dirichlet character modulo 8, then $L(1,\chi)$ equals

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \cdots,$$

(c) For an odd Dirichlet character modulo 7, $L(1,\chi)$ equals

$$\frac{\pi}{\sqrt{7}} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} \cdots,$$

(d) For an even Dirichlet character modulo 8, $L(1,\chi)$ equals

$$\frac{\log(1+\sqrt{2})}{\sqrt{2}} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots,$$

(e) For an even Dirichlet character modulo 5, $L(1,\chi)$ equals

$$\frac{2}{\sqrt{5}} \log\left(\frac{1+\sqrt{5}}{2}\right) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} \cdots,$$

Proof. □

2. **Let $d(k)$ denote the number of positive divisors of k. (a) Show that if $k = p_1^{a_1} \cdots p_n^{a_n}$ is the prime factorization of k, then**

$$d(k) = (a_1 + 1) \cdots (a_n + 1).$$

Although Theorem 3.12 shows that on "average" $d(k)$ is of the order of $\log k$, prove that the following on the basis of (a):

(b) $d(k) = 2$ for infinitely many k.

(c) For any positive integer N, there is a constant $c > 0$ so that $d(k) \geq c(\log k)^N$ for infinitely many k. [Hint: Let $p_1, \cdots p_N$ be N distinct primes, and consider k of the form $(p_1 p_2 \cdots p_N)^m$ for $m = 1, 2, \cdots$.]

Proof. (a)(b) are easy. (c) □

3. **Show that if p is relatively prime to q, then**

$$\prod_{\chi}\Big(1-\frac{\chi(p)}{p^s}\Big)=\Big(\frac{1}{1-p^{fs}}\Big)^g,$$

where $g=\varphi(q)/f$, and f is the order of p in $\mathbb{Z}^*(q)$ (that is, the smallest n for which $p^n\equiv 1$ mod q). Here the product is taken over all Dirichlet characters modulo q.

Proof. □

4. **Prove as a consequence of the previous problem that**

$$\prod_{\chi}L(s,\chi)=\sum_{n\geq 1}\frac{a_n}{n^s},$$

where $a_n\geq 0$, and the product is over all Dirichlet characters modulo q.

Proof. □