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Chapter 5 Elements of Functional Analysis*

Yung-Hsiang Huang†

5.1 Normed Vector Spaces

1. Note for any $x, y \in \mathscr{X}$ and $a, b \in \mathscr{K}$, $\|x+y\| \leq \|x\|+\|y\|$ and $\|by-ax\| \leq |b|\|y-x\|+|b-a|\|x\|$.

2. It's standard.

3. If $\mathscr{Y}$ is complete, so is $L(\mathscr{X}, \mathscr{Y})$.

   Proof. Let $\{T_n\} \subset L(\mathscr{X}, \mathscr{Y})$ be Cauchy. If $x \in \mathscr{X}$, then $\{T_n x\}$ is Cauchy in $\mathscr{Y}$. Define $T : \mathscr{X} \to \mathscr{Y}$ by $Tx = \lim T_n x$ which is clearly linear. Since for any $0 < \epsilon < 1$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\|T_m x - T_N x\| < \epsilon \|x\|$ for all $m > N$ and all $x \in \mathscr{X}$. Let $m \to \infty$, we see $\|Tx - T_N x\| \leq \epsilon \|x\|$ for all $x \in \mathscr{X}$. This implies $T \in L(\mathscr{X}, \mathscr{Y})$ and $\|T - T_n\| \to 0$. □

4. $\|T_n x_n - Tx\| \leq \|T_n(x_n - x)\| + \|(T_n - T)x\|$.

5. **If $\mathscr{X}$ is a normed vector space, the closure of any subspace of $\mathscr{X}$ is a subspace.**

   Proof. Given $M \leq \mathscr{X}$. Given $x, y \in \overline{M}$ and scalars a, b, there exists $x_n, y_n \in M$ such that $x_n \to x, y_n \to y$. By continuity of addition and scalar multiplication, $M \ni ax_n + by_n \to ax + by$ in $\mathscr{X}$. Therefore, $ax + by \in \overline{M}$. □

6. *Proof.* We may assume $\|e_i\|_1 = 1$ for all i. (a)(b) are easy. (c) is due to the subset $\{\sum_i |a_i| = 1\}$ is compact in $\mathbb{R}^n$ and the map in (b) is continuous. For (d), given $\|\cdot\|$ be a norm on $\mathscr{X}$. Then for any $(a_1, \cdots a_n)$,

$$\|\sum a_i e_i\| \leq \max\{\|e_i\|\} \sum |a_i| = \max\{\|e_i\|\} \|\sum a_i e_i\|_1$$

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†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

and hence $\|\cdot\|$ is continuous on $(\mathscr{X}, \|\cdot\|_1)$ which attains a minimum on the compact set $\{\|x\|_1 = 1\} =: B$, say at $u \in B$. Since $u \neq 0, \|u\| > 0$. Therefore, for any $(a_1, \cdots a_n)$,

$$\|\sum a_j e_j\|_1 \|u\| \le \|\sum a_j e_j\|_1 \|\sum \frac{a_i}{\|\sum a_j e_j\|_1} e_i\| = \|\sum a_i e_i\|.$$

□

7. *Proof.* (a) By Exercise 2 and 3, $L(X,X)$ is a Banach space. Since $\|I - T\| < 1$, the series $\sum_n (I-T)^n$ converges absolutely, and hence converges to some $S \in L(X,X)$. Note that

$$\begin{aligned}\|I - ST\| = \|S - I - S(I-T)\| &= \|S - \sum_{j=0}^{N+1}(I-T)^j + \sum_{j=1}^{N+1}(I-T)^j - S(I-T)\| \\ &\le \|S - \sum_{j=0}^{N+1}(I-T)^j\| + \|\sum_{j=0}^{N}(I-T)^j - S\| \cdot \|(I-T)\| \le \epsilon + \epsilon\|I-T\|,\end{aligned}$$

provided N is large enough. Therefore, $\|I - ST\| = 0$, that is, $I = ST$. Similarly, $I = TS$.

(b) Since $\|T^{-1}S - I\| \le \|T^{-1}\| \cdot \|S - T\| < 1$, $T^{-1}S$ has an inverse $R \in L(X,X)$. Since $(RT^{-1})S = R(T^{-1}S) = I$ and $S(RT^{-1}) = (TT^{-1})SRT^{-1} = T(T^{-1}SR)T^{-1} = I$, S has inverse $RT^{-1} \in L(X,X)$. Thus for any invertible T, every element of $B_{\|T^{-1}\|^{-1}}(T)$ is invertible. This is the second assertion. □

8. *Proof.* Clearly, $M(X)$ is a vector space over $\mathbb{C}$. For each $\nu \in M(X)$, by Radon-Nikodym theorem, there is a integrable function f with respect to $\mu = |\nu_r| + |\nu_i|$ such that $d\nu = f d\mu$ and then $d|\nu| = |f| d\mu$. So $|\nu|(X) = \|\nu\| = 0$ iff f is zero μ-a.e. iff $\nu = 0$. For each $a \in \mathbb{C}$, $\|a\nu\| = |a\nu|(X) = \int_X |af|\, d\mu = |a||\nu|(X) = |a|\|\nu\|$. The triangle inequality is proved by the same method with $\mu = \mu_1 + \mu_2 := |\nu_{1,r}| + |\nu_{1,i}| + |\nu_{2,r}| + |\nu_{2,i}|$. Therefore, $\|\cdot\|$ is a norm on $M(X)$.

Given $\sum \nu_n$ be a absolutely convergent series in $M(X)$. Since for each $A \in \Sigma$, $\sum |\nu_n(A)| \le \sum |\nu_n|(A) \le \sum |\nu_n|(X) = \sum \|\nu_n\| < \infty$, we may define $\nu : \Sigma \to \mathbb{C}$ by $\nu(A) = \sum_n \nu_n(A)$. Clearly, $\nu(\emptyset) = 0$. If A_k is a sequence of disjoint measureable sets, since by Tonelli's theorem

$$\sum_{n,k} |\nu_n|(A_k) = \sum_n \sum_k |\nu_n|(A_k) = \sum_n |\nu_n|(\cup_k A_k) \le \sum \|\nu_n\| < \infty$$

and therefore

$$\nu(\cup_k A_k) = \sum_n \nu_n(\cup_k A_k) = \sum_n \sum_k \nu_n(A_k) = \sum_k \sum_n \nu_n(A_k) = \sum_k \nu(A_k).$$

Hence, $\nu \in M(X)$. Given $X_1 \cdots X_m$ covers X,

$$\sum_{j=1}^m |\Big(\nu - \sum_{n=1}^N \nu_n\Big)(X_j)| = \sum_{j=1}^m |\sum_{n=N+1}^\infty \nu_n(X_j)| \le \sum_{j=1}^m \sum_{n=N+1}^\infty |\nu_n|(X_j) = \sum_{n=N+1}^\infty |\nu_n|(X) = \sum_{n=N+1}^\infty \|\nu_n\|.$$

By Exercise 3.21, $\|\nu - \sum_{n=1}^{N} \nu_n\| \le \sum_{n=N+1}^{\infty} \|\nu_n\| \to 0$ as $N \to 0$. So $M(X)$ is complete. $\square$

9. Follow the steps stated in the hint.

10. *Proof.* It's easy to see $\|\cdot\|$ is a norm. The harder part is the completeness.

For $k = 1$. Given $f_n \in L_1^1([0,1])$ with

$$\infty > \sum \|f_n\| = \sum \int_0^1 |f_n(x)| + |f_n'(x)|\, dx = \int_0^1 \sum |f_n(x)| + \sum |f_n'(x)|\, dx,$$

by monotone convergence theorem. Hence $\sum |f_n|$ and $\sum |f_n'|$ are integrable functions and finite a.e. Since for every $x \in [0,1]$, let $a \in [0,1]$ be the point of convergence for $\sum |f_n|$. Then

$$\sum_1^n |f_i(x)| = \sum_1^n |\int_a^x f_i'(t)\, dt + f_i(a)| \le \sum_1^n \int_a^x |f_i'(t)|\, dt + |f_i(a)| \le \int_0^1 \sum |f_n(t)|\, dt + \sum |f_n(a)| < \infty,$$

for each $n \in \mathbb{N}$ and hence $\sum |f_n|$ converges everywhere. By completeness of $\mathbb{R}$, $\sum f_n$ converges everywhere to some function f which is integrable by triangle inequality. Similarly, $\sum f_n'$ converges a.e. to some integrable function F.

Next, we see that f is absolutely continuous on $[0,1]$ with derivative is F since for each $x \in [0,1]$,

$$f(x) = \lim_{n\to\infty} \sum_{i=1}^n f_i(x) = \lim_{n\to\infty} \int_0^x \sum_{i=1}^n f_i'(t)\, dt + \sum_{i=1}^n f_i(0) = \int_0^x F(t)\, dt + f(0).$$

Finally, we see $\sum f_i \to f$ in $L_1^1([0,1])$ since by LDCT

$$\|f - \sum^n f_i\| = \int_0^1 |f(x) - \sum^n f_i(x)| + |F(x) - \sum^n f_i'(x)|\, dx \to 0 \text{ as } n \to \infty.$$

For general $k \in \mathbb{N}$, we use mathematical induction, that is, assume it's true for L_k^1 and replace f in the above argument by $f^{(k)}$ to conclude that it's true for L_{k+1}^1. $\square$

11. *Proof.* (a) By triangle inequality, $\Lambda_\beta([0,1])$ is a vector space and $\|\cdot\|$ is a norm.

Given a Cauchy sequence $\{f_n\} \subset \Lambda_\beta$, then $\{f_n(0)\}$ is Cauchy in $\mathbb{C}$ and hence there is a $f(0)$ such that $f_n(0) \to f(0)$. Moreover, since for each $\epsilon > 0$, there is $N = N(\epsilon) \in \mathbb{N}$ such that for all $n, m > N$ and for all $x \in (0,1]$,

$$|(f_n - f_m)(x) - (f_n - f_m)(0)| < \epsilon |x|^\beta,$$

that is, for all $x \in (0,1]$, $\{f_n(x) - f_n(0)\}$ is Cauchy in $\mathbb{C}$. Therefore, for each $x \in [0,1]$, there exists $f(x) \in \mathbb{C}$ such that $f_n(x) \to f(x)$.

We also note that there is a constant $M > 0$ such that for each $n \in \mathbb{N}$ and $s \ne t$,

$$\frac{|f_n(s) - f_n(t)|}{|s-t|^\beta} \le M.$$

Then for each $s \neq t$, there is some $N = N(s,t) \in \mathbb{N}$ such that

$$\begin{aligned} |f(s) - f(t)| &\le |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)| \\ &\le 2|s-t|^\beta + M|s-t|^\beta. \end{aligned}$$

Therefore, $f \in \Lambda_\beta$. It remains to show $f_n \to f$ in Λ_β. For each $s, t \in [0,1]$ and $\epsilon > 0$, by Cauchy's criteria again, we can find a $N = N(\epsilon)$ such that for $k, n \ge N$

$$|f_k(s) - f_n(s) - f_k(t) + f_n(t)| \le \epsilon |s-t|^\beta$$

For each $\eta > 0$, we can find a $K = K(\eta, s, t) > N$, such that $|f(x) - f_K(x)| \le \eta$ for $x = s$, or $x = t$. Therefore, for every $n \ge N$

$$|f(s) - f_n(s) - f(t) + f_n(t)| \le |f(s) - f_K(s)| + |f_K(s) - f_n(s) - f_K(t) + f_n(t)| + |f(t) - f_K(t)| \le 2\eta + \epsilon |s-t|^\beta.$$

Letting $\eta \to 0$, we obtain that $|f(s) - f_n(s) - f(t) + f_n(t)| \le \epsilon |s-t|^\beta$ for all $n \ge N(\epsilon)$. □

Remark 1. It is true for the general case $C^{k,\beta}(U)$, U is connected in $\mathbb{R}^d$, with an almost identical proof as the above case $k = 0$, except we need the standard convergence theorem between $\{f_n\}$ and $\{Df_n\}$.

Remark 2. If U is a bounded connected set, then we have

$$\|f\| \le \|f\|' := \sup_{x \in U} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} \le \|f\| \Big(\operatorname{diam}(U)^\beta + 1 \Big),$$

that is, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Proof. For $\alpha = 1$, for every $f \in \lambda_\alpha([0,1])$ and $x \in (0,1), f'(x) = 0$ by definition. So f is a constant function.

For $\alpha < 1$, by triangle inequality, λ_α is a vector subspace of Λ_α.

Given $\{f_n\} \subset \lambda_\alpha \to f$ in Λ_α. Given $y \in [0,1]$ and $\epsilon > 0$, by picking some large N we see

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \le \frac{|f(x) - f_N(x) - (f(y) - f_N(y))|}{|x-y|^\alpha} + \frac{|f_N(x) - f_N(y)|}{|x-y|^\alpha} < \frac{\epsilon}{2} + \frac{|f_N(x) - f_N(y)|}{|x-y|^\alpha},$$

For this fixed $f_N \in \lambda_\alpha$, we can choose $\delta = \delta(y, \epsilon)$ such that the second term is less than $\frac{\epsilon}{2}$ if $|x - y| < \delta$. Therefore, $f \in \lambda_\alpha$ and hence λ_α is closed.

Since $\{x^n\}_{n=0}^\infty$ is a linearly independent set in λ_α by Fundamental Theorem of Algebra and Mean-Value Theorem, λ_α is infinite-dimensional. □

12. **Let X be a normed vector space and M a proper closed subspace of X.**

(a) $\|x+M\| = \inf\{\|x+y\| : y \in M\}$ is a norm on X/M.

(b) For any $\epsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\|x+M\| > 1-\epsilon$.

(c) The projection map $\pi(x) = x+M$ from X to X/M has norm 1.

(d) If X is complete, so is X/M. (Use Theorem 5.1.)

(e) The topology defined by the quotient norm is the quotient topology as defined in Exercise 28 in §4.2.

Proof. (a) (i) Given $a \in \mathbb{C}$, since M is a subspace, $\|ax+M\| = |a| \inf\{\|x+a^{-1}y\| : y \in M\} = |a| \inf\{\|x+z\| : z \in M\} = |a|\|x+M\|$. (ii) If $\|x+M\| = 0$, then given $n \in \mathbb{N}$, there is $y_n \in M$ such that $\|x+y_n\| \leq \frac{1}{n}$, and hence $x+y_n \to 0$, that is, $-y_n \to x$. Since M is closed, $x \in M$. (iii) Given $x, y \in X$, then for any $z, w \in M$, $\|x+y+M\| \leq \|x+y+z+w\| \leq \|x+z\|+\|y+w\|$. Take infimum among all $z, w \in M$ respectively, we see $\|x+y+M\| \leq \|x+M\| + \|y+M\|$. (Note that our proof shows that, if M is not closed, then $\|\cdot\|$ is a seminorm. See Exercise 14.)

(b) Given $\epsilon > 0$. Since M is proper, there is a $y \notin M$ and then $\|y+M\| = a > 0$. So $\|\frac{1}{a}y+M\| = 1$, and by definition, there is some $z \in M$ such that $1 \leq \|\frac{1}{a}y+z\| \leq 1+\frac{1}{2}\epsilon$. So there is some $\|w - (\frac{1}{a}y+z)\| < \epsilon$ and $\|w\| = 1$. Now $\|w+M\| \geq 1-\epsilon$ since

$$1 - \|w+M\| = \|\frac{1}{a}y+z+M\| - \|w+M\| \leq \|\frac{1}{a}y+z-w+M\| \leq \|\frac{1}{a}y+z-w\| \leq \epsilon.$$

(c) By (b), $\|\pi\| \geq 1$. On the other hand $\|\pi\| \leq 1$ since $\|x+M\| \leq \|x+0\|$ for all $x \in X$.

(d) Given $\{M_k\}$ Cauchy in X/M, then we can find a subsequence $\{M_{k_i}\}$ such that for all i, $M_{k_i} = x_i + M$ for some $x_i \in X$ and $\|M_{k_{i+1}} - M_{k_i}\| < 2^{-i}$. Let $y_1 = x_1 \in X$ such that $x_1 + M = M_{k_1}$, since $\|M_{k_1} - M_{k_2}\| = \inf\{\|x_1 - x_2 + z\| : z \in M\} < 2^{-1}$, we can take $y_2 = x_2 + w \in X$ for some $w \in M$ such that $\|y_1 - y_2\| < 2^0$ and $y_2 = x_2 + M = M_{k_2}$. Inductively, we can find a sequence $\{y_i\} \subset X$ such that $\|y_i - y_{i+1}\| < 2^{-i+1}$ and $y_i + M = M_{k_i}$. Since X is complete, the series $\sum y_{i+1} - y_i$ converges to $y - y_1 \in X$ for some $y \in X$, that is,

$$y_n - y_1 = \sum_{1}^{n-1} y_{i+1} - y_i \to y - y_1 \text{ as } n \to \infty.$$

Then $y_i + M \to y + M$ by (c), the continuity of projection map π.

(e) ??? ∎

13. *Proof.* By triangle inequality, M is a subspace and the map $g(x+M) = \|x\| \geq 0$ satisfies the triangle inequality. If $g(x+M) = \|x\| = 0$, then $x = 0$ and hence $x + M = M$. Given $\lambda \in \mathbb{C}$, $g(\lambda(x+M)) = g(\lambda x + M) = \|\lambda x\| = |\lambda| g(x+M)$. So g defines a norm on X/M. $\square$

14. *Proof.* $\square$

15. **Suppose that X and Y are normed vector spaces and $T \in L(X,Y)$. Let $N(T) = \{x \in X : Tx = 0\}$. Show that**

    (a) $N(T)$ is a closed subspace of X.

    (b) There is a unique $S \in L(X/N(T), Y)$ such that $T = S \circ \pi$ where $\pi : X \to X/M$ is the projection (see Exercise 12). Moreover, $\|S\| = \|T\|$.

    Proof. $\square$

16. *Proof.* $\square$

5.2 Linear Functionals

17. We may assume $f \not\equiv 0$. The closedness of $f^{-1}(\{0\})$ is due to the continuity of f.

    First proof. Conversely, if the proper subspace $N := f^{-1}(\{0\})$ is closed, then by Exercise 12 (b), there is an $x \in X$ with $\|x\| = 1$ and $\|x + N\| \geq 1 - \frac{1}{2}$, that is, $f(x) \neq 0$. Note that $X = \mathbb{C}x \bigoplus N$ since for each $a \in X$, $a = \frac{f(a)}{f(x)} x + (a - \frac{f(a)}{f(x)} x)$.

    So for each $\lambda x + n \in \mathbb{C}x \bigoplus N$,

    $$\|\lambda x + n\| = |\lambda| \|x + \lambda^{-1} n\| \geq |\lambda| \frac{1}{2}$$

    and therefore

    $$|f(\lambda x + n)| = |\lambda| |f(x)| \leq 2 |f(x)| \|\lambda x + n\|.$$

    That is, f is bounded. $\square$

    Second proof. Suppose on the contrary, for each $n \in \mathbb{N}$, there is $x_n \in X$ with norm 1 such that $|f(x_n)| > n$, by multiplication by -1, we may assume $f(x_n) > n$. Then for each $x \in X$, we see $N \ni x - \frac{f(x)}{f(x_n)} x_n \to x$. The closedness of N implies $x \in N$. This contradicts to $f \not\equiv 0$. $\square$

18. *Proof.* (a) We may assume M is a proper subspace of X. Then Hahn-Banach theorem implies that there is some $f \in X^*$ such that $f(x) \neq 0$ and $f(M) = 0$.

Given $\{y_n + a_n x\} \in M + \mathbb{C}x \to z$ in X. Then $f(z) = \lim_{n\to\infty} f(y_n + a_n x) = \lim_{n\to\infty} f(x)a_n$. So $a_n \to \frac{f(z)}{f(x)}$, $a_n x \to \frac{f(z)}{f(x)}x$, that is $y_n \to z - \frac{f(z)}{f(x)}x \in M$ since M is closed and therefore $z = z - \frac{f(z)}{f(x)}x + \frac{f(z)}{f(x)}x \in M + \mathbb{C}x$. So $M + \mathbb{C}x$ is closed.

(b) From zero-dimensional subspace inductively use (a) to n-dimensional. Another proof is based on the continuity of the map $(M, \|\cdot\|_X) \ni a_1x_1 + a_2x_2 + \cdots a_nx_n \mapsto (a_1, \cdots a_n) \in \mathbb{C}^n$. □

19. *Proof.* □

20. *Proof.* □

21. *Proof.* □

22. *Proof.* □

23. *Proof.* □

24. *Proof.* □

25. *Proof.* Let $\{y_n^*\}$ be a countable dense subset of $\mathscr{X}^*$ and $x_n \in \mathscr{X}$ such that $\|x_n\| = 1$ and $|y_n^*(x_n)| \geq \frac{1}{2}\|y_n^*\|$. Let us denote by L the vector space over $\mathbb{Q}$ generated by $\{x_n\}$, which is easy to see it's countable.

Suppose L is not dense in $\mathscr{X}$, then by Hahn-Banach Theorem, there is $y^* \in \mathscr{X}^*$ such that $y^*(L) = \{0\}$ and $\|y^*\| = 1$. Then for each $n \in \mathbb{N}$,

$$\frac{1}{2}\|y_n^*\| \leq |y_n^*(x_n)| = |y_n^*(x_n) - y^*(x_n)| \leq \|y_n^* - y^*\|.$$

Hence $1 = \|y^*\| \leq \|y^* - y_n^*\| + \|y_n^*\| \leq 3\|y_n^* - y^*\| \to 0$ by picking a suitable subsequence, which is a contradiction. Therefore L is dense in $\mathscr{X}$, that is, $\mathscr{X}$ is separable. □

26. *Proof.* □

5.3 The Baire Category Theorem and Its Consequences

27. *Proof.* Consider Cantor-like sets C_k in $[0,1]$ with Lebesgue measure $1 - \frac{1}{k}$, then $C = \cup C_k$ has measure 1, of the first category and its complement in $[0,1]$ has measure zero. Then we copy this set to each $[m, m+1]$ and the desired set on $\mathbb{R}$ is the union of them. □

28. *Proof.* Part (b) is the same. For (a), given nonempty open set W in X, since X is LCH, there is a nonempty set open set B_1 such that $\overline{B_1} \subset U_1 \cap W$ and $\overline{B_1}$ is compact. Inductively, for $n \geq 2$, we can find a nonempty open set B_n such that $\overline{B_n}$ is compact and $\overline{B_n} \subset U_n \cap B_{n-1}$. Note that $K = \cap \overline{B_n}$ is nonempty due to finite intersection property and $K \subset U_n \cap B_{n-1} \subset U_n \cap W$ for all n. Thus, $\cap_n U_n \cap W \supset K \neq \emptyset$. □

29. Together with Exercise 30, they show the completeness assumption can NOT be dropped from Open Mapping Theorem without other restrictions. Also see Section 5.6, there is an construction to unbounded linear map $X \to Y$ when X is complete (by the axiom of choice).

   Proof. (a) X is proper since $(\frac{1}{n^2}) \in l^1 \setminus X$. By triangle inequality, it's a subspace. The density of X is by cutting the given series.

   (b) T is unbounded since $\|Te_m\|_Y = m = m\|e_m\|_X$.

   Given $f_n \to f$ in X with $Tf_n \to g$ in Y. Given $\epsilon > 0$, then there is $M, N \in \mathbb{N}$ such that $\|Tf_m - g\|_1 < \epsilon$ for all $m > M$ and $\sum_{n=N+1}^{\infty} |g(n)| < \epsilon$ and $\sum_{n=N+1}^{\infty} |nf(n)| < \epsilon$. Moreover, we can choose some $m > M$ such that N such that $\|f_m - f\| < \epsilon/N$, then

$$\begin{aligned}
&\|Tf - g\|_1 \leq \|Tf - Tf_m\|_1 + \|Tf_m - g\|_1 \\
&\leq \sum_{n=1}^{N} |nf(n) - nf_m(n)| + \sum_{n=N+1}^{\infty} |nf(n)| + \sum_{n=N+1}^{\infty} |nf_m(n) - g(n)| + \sum_{n=N+1}^{\infty} |g(n)| + \|Tf_m - g\|_1 \\
&< N\frac{\epsilon}{N} + 4\epsilon = 5\epsilon.
\end{aligned}$$

   So $Tf = g$ in l^1.

   (c) It's easy to see $Sg(n) = \frac{1}{n}g(n)$ for all $g \in Y$. Hence S is surjective and $\|S\| \leq 1$. If S is open, then T is continuous, which contradicts to (b). □

30. *Proof.* (a) Consider $f_n(x) = \sqrt{x + \frac{1}{n}}$, then for all $x \in [0,1]$, $|f_n(x) - \sqrt{x}| \leq \frac{1}{\sqrt{n}} \to 0$. But $\sqrt{x} \notin C^1([0,1])$ and $\{f_n\}$ is Cauchy in $(C^1([0,1]), \|\cdot\|_\infty)$ since $\|f_n - f_m\| \leq |\frac{1}{n} - \frac{1}{m}| \to 0$.

   (b) Consider $f_n = x^n$, then $\|(d/dx)f_n\|_{\mathscr{Y}} = \|nx^{n-1}\|_\infty = n$ and $\|f_n\|_\infty = 1$, which implies that d/dx is not bounded. The graph of d/dx is closed due to the convergence theorem between f_n and f_n'. (For example, see Rudin [1, Theorem 7.17].) □

31. *Proof.* □

32. Open Mapping Theorem to the identity map from $(X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$.

33. *Proof.* □

34. *Proof.* ☐

35. **Let X and Y be Banach spaces, $T \in L(X,Y)$, $N(T) = \{x : Tx = 0\}$, and $M = \text{range}(T)$. Then $X/N(T)$ is isomorphic to M iff M is closed. (See Exercise 15.)**

    Proof. ☐

36. *Proof.* ☐

37. *Proof.* Define $T^* : Y^* \to X^*$ by $T^* f = f \circ T$. For each $x \in X$, we define $\widehat{x} : X^* \to \mathbb{C}$ by $\widehat{x}(x^*) = x^*(x)$.

    Given $x \in X$ with $\|x\| = 1$. By Hahn-Banach theorem, there is some $g \in Y^*$ with $\|g\| = 1$ such that

$$\|Tx\| = g(Tx) \leq \sup_{f \in Y^*, \|f\|=1} |f(Tx)| = \sup_{f \in Y^*, \|f\|=1} |(T^* f)(x)| = \sup_{f \in Y^*, \|f\|=1} |(\widehat{x} T^*)(f)| = \|\widehat{x} \circ T^*\|$$

    Note that for each $f \in Y^*$,

$$\sup_{\|x\|=1} |\widehat{x} \circ T^*(f)| = \sup_{\|x\|=1} |f \circ T(x)| = \|f \circ T\| < \infty.$$

    By uniform boundedness principle, there is some $M > 0$ such that $\sup_{\|x\|=1} \|\widehat{x} \circ T^*\| \leq M$. Therefore, T is bounded since for every $x \in X$ with $\|x\| = 1$, $\|Tx\| \leq \|\widehat{x} \circ T^*\| \leq M$. ☐

38. By uniform boundedness principle.

39. **Let X, Y, Z be Banach spaces and let $B : X \times Y \to Z$ be a separately continuous bilinear map, that is, $B(x, \cdot) \in L(Y, Z)$ for each $x \in X$ and $B(\cdot, y) \in L(X, Z)$ for each $y \in Y$. Then B is jointly continuous, that is, continuous from $X \times Y$ to Z.**

    Remark 3. See Rudin [3, Theorem 2.17 and Exercises 2.9-12] for the discussions around the completeness.

    Proof. Since for each $y \in Y$ with $\|y\| = 1$, the linear map$B(\cdot, y) =: B_y(\cdot) : X \to Z$ is bounded, the uniform boundedness principle implies that there is some $C > 0$ such that for all $x \in X, y \in Y$, $B(x, \frac{y}{\|y\|}) \leq C\|x\|$, that is, $B(x,y) \leq C\|x\|\|y\|$. This implies the jointly continuity since for each $(x_0, y_0) \in X \times Y$ and $\|x - x_0\| \leq 1$,

$$|B(x,y) - B(x_0, y_0)| \leq |B(x, y - y_0)| + |B(x - x_0, y_0)| \leq C(\|x_0\| + 1)\|y - y_0\| + C\|x - x_0\|\|y_0\|.$$

☐

40. *Proof.* □

41. *Proof.* □

42. *Proof.* □

5.4 Topological Vector Spaces

43. *Proof.* □

44. *Proof.* □

45. *Proof.* □

46. *Proof.* □

47. *Proof.* □

48. *Proof.* □

49. *Proof.* □

50. *Proof.* □

51. *Proof.* □

52. *Proof.* □

53. *Proof.* □

5.5 Hilbert Spaces

54. *Proof.* □

55. *Proof.* (a) Direct computation. (b) (i) it's clear that isometry $\Rightarrow$ injectivity. The inner product structure is preserved due to (a). (ii) By definition, unitary $\Rightarrow$ surjectivity. It's proved to be an isometry by taking $x = y$ in the definition. □

56. *Proof.* It's easy to see $E \subset (E^{\perp})^{\perp}$ and the latter is a closed subspace of H. Given F be a closed subspace containing E, then $F^{\perp} \subset E^{\perp}$ and hence $(E^{\perp})^{\perp} \subset (F^{\perp})^{\perp}$. Since F is closed, $H = F \bigoplus F^{\perp}$. Given $z \in (F^{\perp})^{\perp}, z = x + y$ with $x \in F$ and $y \in F^{\perp}$. Since $0 = (z, y) = (y, y), y = 0$. Hence $(E^{\perp})^{\perp} \subset (F^{\perp})^{\perp} \subset F$. □

57. *Proof.* (a) For each $y \in H$, we define the linear functional on H by $x \mapsto \langle Tx, y\rangle$ which is bounded by $\|T\|\|y\|_H$. By Theorem 5.25, there exists a unique $z \in H$ with $\|z\|_H \leq \|T\|\|y\|_H$ such that $\langle Tx, y\rangle = \langle x, z\rangle$ for all $x \in H$. Define T^* by $y \mapsto z$, then T^* is linear and bounded by $\|T\|$.

(b) Note that $\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$ for all $x, y \in H$. By (a) $T = T^{**}$ and $\|T\| \leq \|T^*\|$. Now it's easy to see $\|T^*T\| \leq \|T^*\|\ |T\| = \|T\|^2$. On the other hand, for each $x \in H$, $\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \|T^*T\|\|x\|^2$, so $\|T\| \leq \|T^*T\|^{1/2}$ since $\|Tx\| \leq \|T^*T\|^{1/2}\|x\|$. The rest follow by definition easily.

(c) $z \in R(T)^\perp \iff \langle z, Tx\rangle = 0$ for all $x \in H \iff \langle T^*z, x\rangle = 0$ for all $x \in H \iff T^*z = 0 \iff z \in \mathscr{N}(T^*)$. Then $\mathscr{N}(T)^\perp = \mathscr{N}(T^{**})^\perp = (R(T^*)^\perp)^\perp = \overline{R(T^*)}$ by Exercise 56.

(d) By definition, T is unitary $\Rightarrow$ T is invertible. Moreover $\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle$ for all $x, y \in H$, which implies $T^{-1} = T^*$ by (a). Conversely, if T is invertible and $T^{-1} = T^*$, then $\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, T^{-1}Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$. $\square$

58. *Proof.* By Theorem 5.24, the closedness of M implies P is well-defined.

(a) By definition, we can see P is linear easily. P is bounded since for each $x \in H$, by Pythagorean Theorem,

$$\|Px\|^2 \leq \|Px\|^2 + \|x - Px\|^2 = \|Px + x - Px\|^2 = \|x\|^2.$$

$P = P^*$ since for each $x, y \in H$, $\langle Px, y\rangle = \langle Px, y - Py + Py\rangle = \langle x - x + Px, Py\rangle = \langle x, Py\rangle$.

$P^2 = P$ since for each $x, y \in H$, $\langle P^2x, y\rangle = \langle Px, Py\rangle = \langle Px, Py - y + y\rangle = \langle Px, y\rangle$.

By definition, $R(P) \subset M$. $R(P) \supset M$ since for each $x \in M$, $Px = x$ since $x - x = 0 \in M^\perp$.

Finally, by Exercise 57(c), $\mathscr{N}(P) = \mathscr{N}(P^*) = R(P)^\perp = M^\perp$.

(b) Since $0 = P^2 - P = (P - I)P$, $R(P) = \mathscr{N}(P - I)$ which is closed since $P - I$ is bounded. If $x \in H$, then since $0 = P(I - P)$, $x - Px \in \mathscr{N}(P) = \mathscr{N}(P^*) = R(P)^\perp$. Therefore, P is the orthogonal projection onto $R(P)$.

(c) Let $x \in H$. By Bessel's inequality and closedness of M, $\sum_{\alpha\in\mathscr{A}}\langle x, u_\alpha\rangle u_\alpha$ is well-defined and belongs to M. Given $y \in M$, there exist an index set $\{i\} := I \subset \mathscr{A}$ and $a_i \in \mathbb{C} \setminus \{0\}$ such that $y = \sum a_i u_i$, then

$$\Big\langle x - \sum_{\alpha\in\mathscr{A}}\langle x, u_\alpha\rangle u_\alpha, y\Big\rangle = \sum_{i\in I}\overline{a_i}\Big\langle x - \sum_{\alpha\in\mathscr{A}}\langle x, u_\alpha\rangle u_\alpha, u_i\Big\rangle = \sum_{i\in I}\overline{a_i}(\langle x, u_i\rangle - \langle x, u_i\rangle) = 0.$$

Therefore, $x - \sum_{\alpha\in\mathscr{A}}\langle x, u_\alpha\rangle u_\alpha \in M^\perp$ and hence $Px = \sum_{\alpha\in\mathscr{A}}\langle x, u_\alpha\rangle u_\alpha$. $\square$

59. *Proof.* Existence of minimizer is proved by the same argument in Theorem 5.24 with a notice that $\frac{1}{2}(y_n + y_m) \in K$ due to the convexity of K. To show the uniqueness, let $x, y \in K$ both with minimal norm d, then since $\frac{1}{2}(x+y) \in K$,

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 4d^2 - 4\|\frac{1}{2}(x+y)\|^2 \leq 4d^2 - 4d^2 = 0.$$

$\square$

***Remark* 4.** Without convexity, we see an example for closed, nonempty set without minimal norm element is $E = \{\frac{n+1}{n} e_n\}$ where $e_n(i) = \delta_n^i$, the Kronecker delta. On the other hand, the existence and uniqueness assertions in this theorem are not true for every Banach spaces, see Rudin [2, Exercise 5.4-5].

60. *Proof.* $\square$

61. *Proof.* $\square$

62. *Proof.* $\square$

63. *Proof.* (a) By Riesz Representation Theorem (Theorem 5.25, p.174) and Bessel's inequality.

(b) Given $x \in B$, construct an orthonormal sequence $\{u_n\}$ such that $u_n \perp x$ for all n.(For $x \neq 0$, use Gram-Schmidt with $u_0 = \frac{x}{\|x\|}$.) Set $a = \sqrt{1 - \|x\|^2}$ and $x_n = au_n + x$, then $\|x_n\| = 1$ and $\{x_n - x\}$ is an orthonormal set. By Bessel's inequality,

$$\sum_n |\langle y, x_n - x\rangle|^2 = \sum_n \|x_n - x\|^2 |\langle y, \frac{x_n - x}{\|x_n - x\|}\rangle|^2 \leq \sum_n (\|x_n\| + \|x\|)^2 |\langle y, \frac{x_n - x}{\|x_n - x\|}\rangle|^2 \leq 4\|y\|^2$$

and hence $\lim\langle y, x_n - x\rangle = 0$ for all $y \in \mathscr{H}$. This implies the desired result. $\square$

64. *Proof.* $\square$

65. *Proof.* $\square$

66. *Proof.* $\square$

67. *Proof.* $\square$

References

[1] Walter Rudin. *Principles of Mathematical Analysis*, volume 3. McGraw-Hill New York, 3rd edition, 1976.

[2] Walter Rudin. *Real and Complex Analysis*. Tata McGraw-Hill Education, 3rd edition, 1987.

[3] Walter Rudin. *Functional analysis*. McGraw-Hill, Inc., New York, 2nd edition, 1991.