Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 3 Nonlinear First-Order PDE*

Yung-Hsiang Huang†

1. Check the definition of complete integral directly.

2. *Proof.* □

3. *Proof.* (a) Differentiating the equation $\sum_i a_i x_i^2 + u(x)^3 = 0$ with respect to x_j, we have

$$2a_j x_j + 3u(x)^2 D_j u(x) = 0. \tag{1}$$

Use this equation, we can rewrite (1) as

$$-\frac{3}{2}u^2(x)x \cdot Du(x) + u(x)^3 = 0,$$

which is the desired PDE.

(b) The sphere can be represented by

$$|(x_1, x_2, \cdots, x_n, u(x)) - (a_1, a_2, \cdots a_n, 0)|^2 - 1 = 0 \ (x \in \mathbb{R}^n). \tag{2}$$

Differentiating the equation with respect to x_i, we have

$$(x_i - a_i) + u(x)D_i u(x) = 0.$$

Use this equation, we can rewrite (2) as

$$u^2(|Du|^2 + 1) - 1 = 0,$$

which is the desired PDE. □

4. *Proof.* □

5. *Proof.* □

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*Last Modified: 2017/03/25.

†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

6. *Proof.* (a) First, we note the following Jacobi's identity:

$$\frac{d}{ds}\mathrm{det}A(s) = \mathrm{tr}((\mathrm{cof}A(s))\frac{dA}{ds}(s)).$$

Let $A(s) = (X^i_{x_j}(s,x,t))$ and $B(s) = (b^i_{x_j}(s))$. Then, by the equation,

$$\frac{dA}{ds}(s) = B(X)A(s).$$

Therefore

$$J_s = \mathrm{tr}((\mathrm{cof}A(s))B(X)A(s)) = \mathrm{tr(B(X)A(s)(cof}A(s))) = \mathrm{tr(B(X)det}A(s)) = \mathrm{div}(\mathbf{b})J.$$

(b) The characteristic equations to the PDE are

$$\dot{\mathbf{x}}(t) = \mathbf{b},\ \mathbf{x}(0) = \mathbf{y} \tag{3}$$

$$\dot{z}(t) = -\mathrm{div}\mathbf{b}z,\ z(0) = g(\mathbf{y}).$$

The second ODE is equivalent to

$$\dot{z^{-1}}(t) = \mathrm{div}\mathbf{b}z^{-1},\ z^{-1}(0) = g(\mathbf{y})^{-1}. \tag{4}$$

According to the hypothesis on $\mathbf{b}$, the definition of X and J, and the Jacobi identity in (a), we know (1) these ODEs are uniquenely solvable, (2) $\mathbf{y} = \mathbf{X}(-t,\mathbf{x},0)$, $\mathbf{X}(s,\mathbf{X}(-t,\mathbf{x},0),0)) = \mathbf{X}(s-t,\mathbf{x},0),0))$ for each $s,t \in \mathbb{R}$, (3) $J(-t,\mathbf{x},0) = J(0,\mathbf{x},t)$ and (4) $z(t) = \frac{g(\mathbf{y})}{J(t,\mathbf{y},0)}$.

By the Euler formula again, we have

$$\begin{aligned} J(t,\mathbf{y},0) &= e^{\int_0^t \mathrm{div}\mathbf{b}(\mathbf{X}(s,\mathbf{y},0))\,ds} = e^{\int_0^t \mathrm{div}\mathbf{b}(\mathbf{X}(s,\mathbf{X}(-t,\mathbf{x},0),0))\,ds} = e^{\int_0^t \mathrm{div}\mathbf{b}(\mathbf{X}(s-t,\mathbf{x},0))\,ds} \\ &= e^{-\int_0^{-t} \mathrm{div}\mathbf{b}(\mathbf{X}(\tau,\mathbf{x},0))\,d\tau} = J(-t,\mathbf{x},0)^{-1} = J(0,\mathbf{x},t)^{-1}. \end{aligned}$$

These facts imply $u(x,t) = g(\mathbf{X}(0,\mathbf{x},t))J(0,\mathbf{x},t)$. □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. **Compute the Legendre transformation of convex function $H : \mathbb{R}^n \to \mathbb{R}$:**

(a) $H(p) = \frac{|p|^r}{r}$, $1 < r < \infty$. (b) $H(p) = \sum_{i,j=1}^n a_{ij}p_ip_j + \sum_{i=1}^n b_ip_i$, where (a_{ij}) is a symmetric, positive definite matrix and $b \in \mathbb{R}^n$.

Proof. (a) For each $v \in \mathbb{R}^n$, by the superlinearity of H, we know the only critical point of the map $p \mapsto v \cdot p - H(p)$ is the maximum point. So (a) is proved.

(b) By the ellipticity, we know H is superlinear. For each $v \in \mathbb{R}^n$, we also see the critical point p^* satisfies $v = \frac{1}{2}Ap^* + b$. Since A is invertible, there is only one critical point and hence $p^* = A^{-1}(v-b)$ is the maxima and

$$L(v) = v^T A^{-1}(v-b) - \frac{1}{2}(v-b)^T A^{-T} A A^{-1}(v-b) - b^T A^{-1}(v-b) = \frac{1}{2}(v-b)^T A^{-1}(v-b)$$

□

11. **$H : \mathbb{R}^n \to \mathbb{R}$ is convex. We write $v \in \partial H(p)$ if $H(r) \geq H(p) + v \cdot (r-p)$ for all $r \in \mathbb{R}^n$. Prove that $(1)\ v \in \partial H(p) \Leftrightarrow (2)\ p \in \partial L(v) \Leftrightarrow (3)\ p \cdot v = H(p) + L(v)$, where $L = H^*$.**

Proof. First, we note $L = H^*$ is convex on $\mathbb{R}^n$, since for each $v_1, v_2 \in \mathbb{R}^n$ and $0 \leq t \leq 1$,

$$\begin{aligned} L(tv_1 + (1-t)v_2) &= \sup_{p \in \mathbb{R}^n} \{(tv_1 + (1-t)v_2) \cdot p - H(p)\} \\ &= \sup_{p \in \mathbb{R}^n} \{(t(v_1 \cdot p - H(p)) + (1-t)(v_2 \cdot p - H(p))\} \leq tL(v_1) + (1-t)L(v_2) \end{aligned}$$

Next, we prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. If (1) is true, then for all $r \in \mathbb{R}^n$,

$$H(r) \geq H(p) + v \cdot (r-p).$$

which implies (2) as follows: for each $r' \in \mathbb{R}^n$,

$$\begin{aligned} L(v) + p \cdot (r' - v) &= \sup_{r \in \mathbb{R}^n} \{v \cdot r - H(r) + p \cdot (r' - v)\} \leq \sup_{r \in \mathbb{R}^n} \{v \cdot r - H(p) - v \cdot (r-p) + p \cdot (r' - v)\} \\ &= p \cdot r' - H(p) \leq L(r'). \end{aligned}$$

If (2) is true, then $L(v) + H(p) \geq p \cdot v - H(p) + H(p) = p \cdot v$.

On the other hand, for each $r \in \mathbb{R}^n$, $L(r) \geq L(v) + p \cdot (r - v)$, that is, $p \cdot v \geq L(v) + p \cdot r - L(r)$. Hence,

$$\begin{aligned} p \cdot v &\geq L(v) + \sup_{r \in \mathbb{R}^n} \{p \cdot r - L(r)\} \geq L(v) + \sup_{r \in \mathbb{R}^n} \Big(p \cdot r - \sup_{q \in \mathbb{R}^n} \{r \cdot q - H(q)\}\Big) \\ &= L(v) + \sup_{r \in \mathbb{R}^n} \inf_{q \in \mathbb{R}^n} H(q) + (p-q) \cdot r. \end{aligned}$$

Since H is convex, we can pick $-\infty < D^- H(q) \leq s \leq D^+ H(q) < \infty$ at p such that $H(q) \geq H(p) + (q-p) \cdot s$ for all $q \in \mathbb{R}^n$. Hence $\inf_{q \in \mathbb{R}^n} H(q) + (p-q) \cdot s \geq H(p)$. So $p \cdot v \geq L(v) + H(p)$.

If (3) is true, then (1) is true since for each $r \in \mathbb{R}^n, p \cdot v = L(v) + H(p) \geq v \cdot r - H(r) + H(p)$. □

Remark 1. Related to convex duality, the Fenchel-Moreau Theorem characterize when is a extended real-valued function on a Hausdorff locally convex space equals to its biconjugate (that is, the double Legendre transformation.)

12. **Assume $L_1, L_2 : \mathbb{R}^n \to \mathbb{R}$ are convex,smooth and superlinear. Show that**

$$\min_{v\in\mathbb{R}^n}(L_1(v) + L_2(v)) = \max_{p\in\mathbb{R}^n}(-H_1(p) - H_2(-p)),$$

where $H_1 = L_1^*, H_2 = L_2^*$.

Proof. By the superlinearity of L_1 and L_2, we know both extrema are attainable.

Given $v \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$, then $L_1(v)+L_2(v) \geq pv-H(p)+(-p)v-H(-p)$. So $L_1(v)+L_2(v) \geq \max_{p\in\mathbb{R}^n} -H(p) - H(-p)$ and hence

$$\min_{v\in\mathbb{R}^n}(L_1(v) + L_2(v)) \geq \max_{p\in\mathbb{R}^n}(-H_1(p) - H_2(-p)).$$

Use the same way, we can prove the converse inequality as follows:

$$\max_{p\in\mathbb{R}^n}(-H_1(p)-H_2(-p)) = -\min_{p\in\mathbb{R}^n}(H_1(p)+H_2(-p)) \geq -\max_{v\in\mathbb{R}^n}(-L_1(v)-L_2(v)) = \min_{v\in\mathbb{R}^n} L_1(v)+L_2(v)).$$

□

13. **Let H be the smooth convex Hamiltonian and g be the smooth Lipschitz initial data. Prove that the Hopf-Lax formula reads**

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL(\frac{x-y}{t}) + g(y)\right\} = \min_{y\in B(x,Rt)}\left\{tL(\frac{x-y}{t}) + g(y)\right\},$$

for $R = \sup_{\mathbb{R}^n}|DH(Dg)|, H = L^*$. (This proves finite propagation speed for a Hamilton-Jacobi PDE with convex Hamiltonian and Lipschitz continuous initial data g.)

Proof. Part of the proof is based on the same idea as exercise 11 (suggested by the first edition).

Since

$$\begin{aligned} tL(\frac{x-y}{t}) + g(y) &\geq tL(\frac{x-y}{t}) - [g]_{C^{0,1}}|x-y| - |g(x)| \\ &= |x-y|\left(\frac{L(\frac{x-y}{t})}{\frac{|x-y|}{t}} - [g]_{C^{0,1}} - \frac{|g(x)|}{|x-y|}\right) \to \infty \text{ as } |y| \to \infty, \end{aligned}$$

there is a constant $A > 0$ such that $\phi(y) := tL(\frac{x-y}{t}) + g(y) \geq \phi(0)$ provided $|y| \geq A$. The continuity of ϕ and the fact $\inf_{y\in\mathbb{R}^n}\phi(y) = \inf_{|y|\leq A}$ implies that $\phi(y)$ has a minimizer y^*. Next, we characterize what y^* is. By convex duality,

$$tL(\frac{x-y^*}{t}) = t\sup_{p\in\mathbb{R}^n}\left(p\cdot\frac{x-y^*}{t} - H(p)\right) = \sup_{p\in\mathbb{R}^n}\left(p\cdot(x-y^*) - tH(p)\right) = p^*\cdot(x-y^*) - tH(p^*),$$

where the existence of maximizer p^* is proved similar as y^*. Note that $0 = x - y^* - tDH(p^*)$. Again, we have for each $y \in \mathbb{R}^n$

$$tL(\frac{x-y^*}{t}) = p^* \cdot (x-y^*) - tH(p^*) \leq p^* \cdot (x-y^*) - p^* \cdot (x-y) + tL(\frac{x-y}{t}),$$

that is,

$$\varphi_1(y) := tL(\frac{x-y}{t}) + g(y) \geq tL(\frac{x-y^*}{t}) + p^* \cdot (y-y^*) + g(y) =: \varphi_2(y).$$

Since φ_1 has a global minimizer at $y = y^*, D\varphi_1(y^*) = 0$. If $D_i\varphi_2(y^*) = a > 0$ for some i, then there is $h > 0$ such that $D_i\varphi_1(y) < \frac{a}{2}$ and $D_i\varphi_2(y) > \frac{a}{2}$ for all $|y - y^*| < 2h$ and then $\varphi_2(y^* + he_i) > \varphi_2(y^*) + \frac{a}{2}h = \varphi_1(y^*) + \frac{a}{2}h > \varphi_1(y^* + he_i)$, a contradiction. Considering $\varphi_1(y^* - he_i)$ and $\varphi_2(y^* - he_i)$ for the case $a < 0$, we see the contradiction. So $D\varphi_2(y^*) = 0$, that is, $p^* = Dg(y^*)$. Hence, $|x - y^*| \leq t \sup |DH(Dg)|$. □

14. **Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem**

$$\begin{cases} u_t + |Du|^2 = 0 & \textbf{in } \mathbb{R}^n \times (0,\infty) \\ u = \begin{cases} \infty \textbf{ if } x \notin E \\ 0 \quad \textbf{if } x \in E \end{cases} & \textbf{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

it would give the solution

$$u(x,t) = \frac{1}{4t}\textbf{dist}(x,E)^2.$$

Proof. Let $H(p) = |p|^2$, then H is smooth, convex and superlinear. Note that $L(v) = H^*(v) = \frac{|v|^2}{4}$ by computing the critial point. So

$$u(x,t) = \min_{y\in\mathbb{R}^n} \frac{1}{4}|\frac{x-y}{t}|^2 + g(y).$$

We note that the minima is attained at $y \in E$ since $g = \infty$ on E^c. So

$$u(x,t) = \min_{y\in E} \frac{1}{4}|\frac{x-y}{t}|^2 = \frac{1}{4t}\text{dist}(x,E)^2.$$

□

15. *Proof.* □

16. **Assume u^1, u^2 are two solutions of the initial value problems**

$$\begin{cases} u^i_t + H(Du^i) = 0 \ \textbf{in } \mathbb{R}^n \times (0,\infty) \\ u^i = g^i \textbf{ on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

given by the Hopf-Lax formula. Prove the L^∞-contraction inequality

$$\sup_{\mathbb{R}^n} |u^1(\cdot,t) - u^2(\cdot,t)| \le \sup_{\mathbb{R}^n} |g^1 - g^2| \ (t > 0).$$

Proof. Given $t > 0$ and $x \in \mathbb{R}^n$, then for some $y_1, y_2 \in \mathbb{R}^n$, $u^1(x,t) = tL(\frac{x-y_1}{t}) + g^1(y_1)$ and $u^2(x,t) = tL(\frac{x-y_2}{t}) + g^2(y_2)$. We are almost done since

$$u^1(x,t) - u^2(x,t) \le tL(\frac{x-y_2}{t}) + g^1(y_2) - tL(\frac{x-y_2}{t}) + g^2(y_2) \le \sup_{\mathbb{R}^n} |g^1 - g^2|$$

and

$$u^2(x,t) - u^1(x,t) \le tL(\frac{x-y_1}{t}) + g^2(y_1) - tL(\frac{x-y_1}{t}) + g^1(y_1) \le \sup_{\mathbb{R}^n} |g^1 - g^2|.$$

□

Remark 2. See Exercise 10.4 for an analogy result for viscosity solution.

17. **Show that**

$$u(x,t) := \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) \ \textbf{if } 4x + t^2 > 0 \\ 0 \ \textbf{if } 4x + t^2 < 0 \end{cases} \tag{5}$$

is a (unbounded) entropy solution of $u_t + (\frac{u^2}{2})_x = 0$.

Proof. □

18. Use the definitions of derivative and convolution.

19. **Assume $F(0) = 0$, u is a continuous integral solution of the conservation law**

$$\begin{cases} u_t + F(u)_x = 0 \ \textbf{in } \mathbb{R} \times (0,\infty) \\ u = g \ \textbf{on } \mathbb{R} \times \{t=0\}, \end{cases} \tag{6}$$

and u has compact support in $\mathbb{R} \times [0,T]$ for each time $T > 0$. Prove for all $t > 0$,

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} g(x)\,dx.$$

Proof. For each $t > 0$, we pick the test function $v \in$ □

20. *Proof.* □