Functional Analysis, Stein-Shakarchi
Chapter 1 L^p spaces and Banach Spaces

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Abstract

Many problems are cited to my solution files for Folland [4] and Rudin [6] post here.

1 Exercises

1. The details and generalizations are discussed in my solution files for Folland [4, Exercise 6.6] and Rudin [6, Exercise 3.4-3.5].

2. *Proof.* (a) follows from the hint by noting the concavity of x^p on $(0,1), 0<p<1$. (b) Assume $l \neq 0$, then there exists $f \in L^p$ such that $|l(f)| \geq 1$. Let $g_1 = f\chi_{(-\inf,x)}$ such that $\|g_1\|_p^p = \frac{1}{2}\|f\|_p^p, g_2 = f - g_1$, then $|l(2g_i)| \geq 1$ for some i. Let $f_1 = 2g_i$, then $\|f_1\|_p = 2^{1-1/p}\|f\|_p$ and $|l(f_1)| \geq 1$. Repeat the previous process inductively, we have f_n with $\|f_n\|_p = 2^{1-1/p}\|f_{n-1}\|_p = \cdots = 2^{n(1-1/p)}\|f\|_p \to 0$ and $1 \leq |l(f_n)| \to 0$ by the continuity of l, which is a contradiction. □

3. Track back the condition for the equality in (2), page 4. See also Folland [4, Exercise 6.1].

4. *Proof.* (a) If $g = 0$ a.e., then we are done. If not, then apply Hölder's inequality to $f^p = (fg)^p g^{-p}$ (b) Try to mimic the proof for $p \geq 1$ and use the inequality in Hint of Exercise 2(a). (c) The triangle inequality is proved by using the same inequality we used in (b). □

5. *Proof.* For each $k \in \mathbb{N}$, there exists n_k such that $\|f_{n_k} - f\|_p \leq 2^{-k}$. We may assume n_k increasing and note that for each k, $\|f_{n_k} - f_{n_{k+1}}\|_p \leq \|f_{n_k} - f\|_p + \|f_{n_{k+1}} - f\|_p \leq 2^{1-k}$. Define $h(x) = \sum_{k=1}^\infty f_{n_k} - f_{n_{k-1}}$, then by the proof in completeness of L^p, it's finite a.e. and the partial sum f_{n_K} converges to h a.e. and in L^p. So $h = f$ and f_{n_K} converges to f a.e. □

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6. As Hints, (b) is proved by apply LDCT.

7. *Proof.* (Sketch) (a) Cut-off the domain, approximation by simple function, using outer and inner regularity of Lebesgue measure, and then use Urysohn's lemma. (b) Mollify the C_c functions you construct in (a). See Folland [4, Proposition 7.9 and 8.17] $\square$

8. As Hints. For $p = \infty$ consider $\chi_{[0,1]^d}$.

9. **Suppose X is a measure space and $1 \le p_0 < p_1 \le \infty$. (a) Consider $L^{p_0} \cap L^{p_1}$ equipped with**

$$\|f\|_{L^{p_0} \cap L^{p_1}} = \|f\|_{L^{p_0}} + \|f\|_{L^{p_1}}.$$

Show that $(L^{p_0} \cap L^{p_1}, \|\cdot\|_{L^{p_0} \cap L^{p_1}})$ is a Banach space.

(b) Show that $(L^{p_0} + L^{p_1}, \|\cdot\|_{L^{p_0}+L^{p_1}})$ is a Banach space, where

$$\|f\|_{L^{p_0}+L^{p_1}} := \inf\{\|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}} : f = f_0 + f_1 \in L^{p_0} + L^{p_1}\}.$$

(c) Show that $L^p \subset L^{p_0} + L^{p_1}$ if $p_0 \le p \le p_1$.

Remark 1. Also see Exercise 24 (as an example of Orlicz space) and Folland [4, Exercise 6.3-6.4 and papers cited there].

Proof. (c) $f = f\chi_{\{|x|\ge 1\}} + f\chi_{\{|x|<1\}}$.

(a) it's easy to see this is a norm on the vector space. you can check the completeness through the one of L^p.

(b) Scaling property is trivial. If $\|f\| = 0$, then for every $n \in \mathbb{N}$, there are $f_0^n \in L^{p_0}$ and $f_1^n \in L^{p_1}$ with the L^p norm less than $\frac{1}{n}$ respectively. Then there is a subsequence such that $f_0^{n_k} \to 0$ a.e. and a further subsequence $f_1^{n_{k_j}} \to 0$ a.e. So does f.

Given $f, g \in L^{p_0} + L^{p_1}$, for each $n \in \mathbb{N}$, there exist decompositions $f = f_0^n + f_1^n$, $g = g_0^n + g_1^n$ such that $\|f_0^n\| + \|f_1^n\| < \|f\| + \frac{1}{n}$, $\|g_0^n\| + \|g_1^n\| < \|g\| + \frac{1}{n}$. Therefore,

$$\|f+g\| \le \|f_0^n + g_0^n\| + \|f_1^n + g_1^n\| \le \|f_0^n\| + \|g_0^n\| + \|f_1^n\| + \|g_1^n\| < \|f\| + \|g\| + \frac{2}{n}.$$

Letting $n \to \infty$, we see $\|f+g\| \le \|f\| + \|g\|$.

Given $\{f_k\}$ be a Cauchy sequence in $L^{p_0} + L^{p_1}$, and given $\epsilon > 0$, then there exists $N = N(\epsilon)$ such that $\|f_k - f_j\| < \epsilon$ for all $k, j > N$ and hence there exist $f_k^0, f_j^0, f_k^1, f_j^1$ such that $\|f_k^0 - f_j^0\|_{p_0} + \|f_k^1 - f_j^1\|_{p_1} < 2\epsilon$. By completeness of L^p, there exist $f^0 \in L^{p_0}, f^1 \in L^{p_1}$ such that $f_k^0 \to f^0$ in L^{p_0} and $f_k^1 \to f^1$ in L^{p_1}. Then $0 \le \|f_k - (f^0 + f^1)\| \le \|f_k^0 - f^0\|_{p_0} + \|f_k^1 - f^1\|_{p_1} \to 0$. $\square$

10. See my solution files for Folland [4, Exercise 6.13, Additional Exercise 6.1 and Exercise 5.25].

11. The same as Exercise 10.

12. *Proof.* (a)(b) are standard. (c) Since $L^{p'}$ is separable, there exists a dense subset $\{x_n\}$ in $L^{p'}$. For x_1, since $|(f_k, x_1)| = |\int f_k x_1| \leq M\|x_1\|$, where $M = \sup \|f_n\|_p$, by Bolzano-Weierstrauss, there exists a subsequence $f_{k,1}$ such that $(f_{k,1}, x_1) \to a_1$, some constant. Inductively in x_n, we conclude that for each k, there exists $\{f_{k,n}\} \subseteq \{f_{k-1,n}\}$ such that $(f_{k,n}, x_n) \to a_n$. Choose $\{f_{k,k}\}$ as the desired subsequence and we are going to prove that it converges weakly.

    Given $l \in (L^p)^*$, by Riesz's theorem, it is represented by some $x \in L^{p'}$. Then there are $\{x_n\} \ni x_j \to x$ in $L^{p'}$. Note that

$$|l(f_{k,k}) - l(f_{m,m})| \leq (I) + (II) + (III) :=$$
$$\leq \int |x(t)f_{k,k}(t) - x_j(t)f_{k,k}(t)| + \int |x_j(t)f_{k,k}(t) - x_j(t)f_{m,m}(t)| + \int |x(t)f_{m,m}(t) - x_j(t)f_{m,m}(t)|.$$

    Given $\epsilon > 0$, there exists $J = J(\epsilon)$ such that $\|x - x_J\| < \frac{\epsilon}{M}$. By Hölder, (I) and (III) are less than ϵ. For this J, we can choose a $N = N(\epsilon)$ such that (II) $< \epsilon$ for all $k, m > N$. This implies that, for each $l \in (L^p)^* \cong L^{p'}$, $l(f_{k,k}) \to c(l)$ as $k \to \infty$. Note that the map $l \mapsto c(l)$ is linear and $|c(l)| = \lim_{k\to\infty} |l(f_{k,k})| \leq M\|l\|_{p'}$. By Riesz representation theorem, there is some $f \in L^p$ such that $\lim_{k\to\infty} l(f_{k,k}) = c(l) = l(f)$ for all $l \in L^{p'}$. □

    Remark 2. The weak compactness theorem is true for any reflexive Banach space. The converse also hold, known as Eberlein-Šmulian theorem, whose proof is put in my additional exercise 6.2.5 for Folland [4]. Also see Brezis [2, Theorem 3.19 and its remarks]

13. *Proof.* (a) This is Riemann-Lebesgue lemma, I think it's weak-* convergence in L^∞, not weak convergence; I put my comment on various proofs below:

    (1) I think the easy way to understand is through Bessel's inequality. But this method does not work on L^p, except L^2.

    (2) The second way is through integration by parts, and one needs the density theorem (simple functions or C_c^∞ functions). This method is adapted to our claim, except for weak convergence in L^1. But for L^1 case, it's a simple consequence of squeeze theorem in freshman's Calculus.

    (b) Apply Hölder's inequality and use the absolute continuity of the indefinite integral of L^1 function. (c) is easy. □

14. **Suppose X is a measure space, $1 < p < \infty$, and suppose $\{f_n\}$ is a sequence of functions with $\|f_n\|_{L^p} \leq M < \infty$.**

(a) Prove that if $f_n \to f$ a.e. then $f_n \to f$ weakly.

(b) Show that the above result may fail if $p = 1$

(c) Show that if $f_n \to f_1$ a.e. and $f_n \to f_2$ weakly, then $f_1 = f_2$ a.e.

Proof. (a)'s hint is given in Folland [4, Exercise 20(a)] by using Egoroff's theorem.

(b) $f_n = \chi_{(n,n+1)}$ in $L^1(\mathbb{R}, m)$.

(c) We don't need to bother whether $\|f_n\|_p \leq M$ is a necessary assumption, since by Uniform Boundedness Principle, a weakly convergent sequence is bounded (cf: Exercise 4.13). So by (a), $f_n \rightharpoonup f_1$, that is, $f_n - f_1 \rightharpoonup 0$. Next, we show the weak limit is unique as follows:

Note that $f_n - f_1 \rightharpoonup f_2 - f_1$, too. Now let $E_j = \{|f_2 - f_1| > \frac{1}{j}\}$ which has finite measure by Chebyshev's inequality and hence $\operatorname{sgn}(f_2 - f_1)\chi_{E_j} \in L^{p'}$. The weak convergence to $f_2 - f_1$ and 0 implies E_j has zero measure for each j and hence $f_1 = f_2$ a.e. $\square$

15. *Proof.* $\square$

16. Multiple Hölder inequality is proved by mathematical induction.

17. **(a) Prove Young's convolution inequality, including the measurability, integrability of $f(x - \cdot)g(\cdot)$ for each $x \in \mathbb{R}^d$**

(b) A version of (a) applies when g is replaced by a finite Borel measure μ:

$$(f * \mu)(x) = \int_{\mathbb{R}^d} f(x - y)\, d\mu(y),$$

and show that $\|f * \mu\|_{L^p} \leq \|f\|_{L^p} |\mu|(\mathbb{R}^d)$

(c) Prove that if $f \in L^p$ and $g \in L^q$, where p, q are conjugated exponents, then $\|f * g\|_{L^\infty} \leq \|f\|_{L^p} \|g\|_{L^q}$. Moreover, $f * g$ is uniformly continuous on $\mathbb{R}^d$, and if $1 < p < \infty$, then $(f * g)(x) \to 0$ as $|x| \to \infty$.

***Remark* 3.** The last assertion is not true for endpoint exponents, e.g. $g(x) = \frac{\overline{f(x)}}{|f(x)|}$, then $(f * g)(x) \equiv \|f\|_1$. However, it's true for the case $f \in L^1$ and $g \in L^1 \cap L^\infty$. See Exercise 2.24 in Book III.

Proof. (a) For $p = 1$, this is the Fubini-Tonelli theorem. We consider $1 < p \leq \infty$ now. If $f, g \geq 0$, then Tonelli's theorem implies $f * g(x)$ exists a.e. and measurable. For $1 < p < \infty$, by Hölder's inequality, we have $(f * g)(x) \leq \|g\|_1^{1/p'} (f^p * g)(x)^{1/p}$. Then we integrate both sides and use Minkowski's integral inequality to conclude the desired inequality; for $p = \infty$,

$(f*g)(x) \le \|f\|_\infty \|g\|_1$. For general f, g, then by Minkowski's inequality $|f|*|g| \in L^p(\mathbb{R}^d)$. And then $|f|*|g| < \infty$ a.e., that is, for a.e. x, $f(x-\cdot)g(\cdot) \in L^1$ and hence $f*g$ exists and finite a.e. To see its measurability, Consider $f_N = f\chi_{\{|x|<N\}} \in L^1$, then $f_N * g$ is measurable by Tonelli's theorem. Fixed x, $f_N(x-t)g(t) \to f(x-t)g(t)$ a.e. t and $|f_N(x-\cdot)g(\cdot)| \le |f(x-\cdot)g(\cdot)| \in L^1$. By LDCT, $f_N * g(x) \to f*g(x)$ for a.e. x, and therefore $f*g$ is measurable.

(b)

(c) The inequality is just Hölder's inequality.

WLOG, we assume $f \in L^p$ with $p < \infty$. Given $\epsilon > 0$, by translation is continuous in L^p norm (Exercise 8), there is $\delta > 0$ such that, $\|f(\cdot + h - f(\cdot)\|_{L^p} < \epsilon / \|g\|_{L^q}$ whenever $h < \delta$. Then for all $x \in \mathbb{R}^d$,

$$|(f*g)(x+h) - (f*g)(x)| \le \|f(\cdot + h - f(\cdot)\|_{L^p}\|g\|_{L^q} < \epsilon$$

whenever $h < \delta$.

The vanishing properties is proved as follows: One consider the case of functions with compact support and then use the density argument. □

Remark 4. On 2017.09.23, I saw this problem. One of the answer states a less explicit fact: "Salem and Zygmund proved that convolution map $L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to L^1(\mathbb{T})$ is onto.

This was shown to hold for all locally compact groups by Paul Cohen in 1959. This result was the starting point of an entire industry establishing "factorization theorems".

A nice survey on this topic is Jan Kisynski, On Cohen's proof of the Factorization Theorem, Annales Polonici Mathematici 75, 2 (2000), 177-192."

18. *Proof.* □

19. *Proof.* □

20. *Proof.* □

21. *Proof.* □

22. The proof for Young's inequality for products is immediate if you draw a correct picture.

23. **Let (X, μ) be a measure space and suppose $0 \not\equiv \Phi(t)$ is a continuous convex and increasing function on $[0, \infty)$, with $\Phi(0) = 0$. Define the Orlicz space**

$$L^\Phi := \{f \text{ measurable} : \int_X \Phi(|f(x)|/M)\, d\mu < \infty \text{ for some } M > 0\},$$

and

$$\|f\|_\Phi = \inf_{M>0} \int_X \Phi(|f(x)|/M)\, d\mu \le 1.$$

Prove that : (a) L^Φ is a vector space. (b) $\|\cdot\|_{L^\Phi}$ is a norm. (c) L^Φ is complete in this norm.

Note that in the special case $\Phi(t) = t^p, 1 \le p < \infty$, then $L^\Phi = L^p$.

Remark 5. I think we should assume $\Phi \not\equiv 0$. one also note that the property that there is $A > 0$ so that $\Phi(t) \ge At$ for all $t \ge 0$ may not be true for small t, e.g. $\Phi(t) = t^2$.

Proof. (a)(b) Given $f, g \in L^\Phi$ and $s \in \mathscr{K}$, where $\mathscr{K} = \mathbb{R}$ or $\mathbb{C}$. Then it's obvious that $sf \in L^\Phi$ and $\|sf\|_{L^\Phi} = |s|\|f\|_{L^\Phi}$. Moreover, for each $\epsilon > 0$, there exists $0 < M_f < \|f\|_{L^\Phi} + \epsilon$ and $0 < M_g < \|g\|_{L^\Phi} + \epsilon$ such that $\int_X \Phi(|f(x)|/M_f)\, d\mu \le 1$, and $\int_X \Phi(|g(x)|/M_g)\, d\mu \le 1$. Then by the convexity and monotone increasing property,

$$\int_X \Phi(\frac{|f(x)+g(x)|}{M_f+M_g})\, d\mu \le \int_X \Phi(\frac{|f(x)|+|g(x)|}{M_f+M_g})\, d\mu = \int_X \Phi(\frac{|f(x)|}{M_f}\frac{M_f}{M_f+M_g} + \frac{|g(x)|}{M_g}\frac{M_g}{M_f+M_g})\, d\mu$$
$$\frac{M_f}{M_f+M_g}\int_X \Phi(\frac{|f(x)|}{M_f})\, d\mu + \frac{M_g}{M_f+M_g}\int_X \Phi(\frac{|g(x)|}{M_g})\, d\mu \le 1.$$

That is, $f+g \in L^\Phi$ and $\|f+g\|_{L^\Phi} \le \|f\|_{L^\Phi} + \|g\|_{L^\Phi} + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, $\|f+g\|_{L^\Phi} \le \|f\|_{L^\Phi} + \|g\|_{L^\Phi}$. Finally, if $\|f\|_{L^\Phi} = 0$, then for each $k \in \mathbb{N}$, there exists $0 < \epsilon_k < \frac{1}{k}$ such that $\int_X \Phi(\frac{|f(x)|}{\epsilon_k})\, d\mu \le 1$. Since $\Phi(0) = 0$, for each $\lambda > 0$ and $m \in \mathbb{N}$, pick k large such that $\lambda\epsilon_k < \frac{1}{m}$

$$\int_X \Phi(\lambda|f(x)|)\, d\mu = \int_X \Phi(\lambda\epsilon_k\frac{|f(x)|}{\epsilon_k} + (1-\lambda\epsilon_k)0)\, d\mu \le \lambda\epsilon_k \int_X \Phi(\frac{|f(x)|}{\epsilon_k})\, d\mu < \frac{1}{m}.$$

So $\int_X \Phi(\lambda|f(x)|)\, d\mu = 0$, then $\Phi(\lambda|f(x)|) = 0$ for μ-a.e. x.

Since $\Phi \not\equiv 0$, $A = \sup\{x : \Phi(x) = 0\} \in [0, \infty)$. Therefore $\lambda|f(x)| \le A$ for μ-a.e. x for all $\lambda > 0$. Hence $|f(x)| = 0$ for μ-a.e. x.

(c)(Simplfied the proof on Rao-Ren [5, Theorem 3.3.10].)

Given $\{f_n\}$ be a Cauchy sequence in L^Φ, then there exists $0 < \epsilon_{nm} \le 2\|f_n - f_m\|_{L^\Phi} \to 0$ as $n, m \to \infty$ (we omit the trivial case that $\|f_n - f_m\|_{L^\Phi} = 0$ for all large n, m) such that

$$\int_X \Phi(\frac{|f_n - f_m|}{\epsilon_{nm}})\, d\mu \le 1.$$

For each $\epsilon > 0$, we note that, by monotonicity of Φ, $\{|f_m - f_n| \ge \epsilon\} \subseteq \{\Phi(\frac{|f_m-f_n|}{\epsilon_{nm}}) \ge \Phi(\frac{\epsilon}{\epsilon_{nm}})\}$. Hence

$$\mu(\{|f_m - f_n| \ge \epsilon\}) \le \mu(\{\Phi(\frac{|f_m-f_n|}{\epsilon_{nm}}) \ge \Phi(\frac{\epsilon}{\epsilon_{nm}})\}) \le \frac{1}{\Phi(\frac{\epsilon}{\epsilon_{nm}})}\int_X \Phi(\frac{|f_m-f_n|}{\epsilon_{nm}})\, d\mu \le \frac{1}{\Phi(\frac{\epsilon}{\epsilon_{nm}})}$$

Let A be the same as the above. Note that

$$0 < \Phi(2A) = \Phi(t\frac{2A}{t} + 0 \cdot (1 - \frac{2A}{t})) \leq \frac{2A}{t}\Phi(t)$$

for $t \geq 2A$, so $\Phi(t) \geq t\frac{\Phi(2A)}{2A} \to \infty$ as $t \to \infty$. So $\mu(\{|f_m - f_n| \geq \epsilon\}) \to 0$ as $n, m \to \infty$. Therefore, $\{f_n\}$ is Cauchy in measure and hence there exists a measurable function f and a subsequence $\{f_{n_i}\}$ such that $f_{n_i} \to f$ a.e. on X.

Since $|\|f_n\|_{L^\Phi} - \|f_m\|_{L^\Phi}| \leq \|f_n - f_m\|_{L^\Phi}$, $\{\|f_n\|_{L^\Phi}\}$ forms a Cauchy sequence in $\mathbb{R}$, and then $\|f_n\|_{L^\Phi} \to \rho$ for some $\rho \geq 0$. If $\rho > 0$, then $f \in L^\Phi$ since by Fatou's lemma, the definition of $\|\cdot\|_{L^\Phi}$ and the continuity and convexity of Φ,

$$\int_X \Phi(\frac{|f|}{2\rho})\, d\mu \leq \liminf_{i\to\infty} \int_X \Phi(\frac{|f_{n_i}|}{2\|f_{n_i}\|_{L^\Phi}})\, d\mu \leq 1.$$

If $\rho = 0$, then there exists $0 < \epsilon_{n_i} \to 0$ as $i \to \infty$ such that $\int_X \Phi(\frac{|f_{n_i}(x)|}{\epsilon_{n_i}})\, d\mu \leq 1$. Using Fatou's lemma, we have $f \in L^\Phi$ since

$$\begin{aligned}0 &\leq \int_X \Phi(|f(x)|)\, d\mu \leq \liminf_{i\to\infty} \int_X \Phi(|f_{n_i}(x)|) = \liminf_{i\to\infty} \int_X \Phi(\epsilon_{n_i}\frac{|f_{n_i}(x)|}{\epsilon_{n_i}} + (1-\epsilon_{n_i})0)\, d\mu \\ &\leq \liminf_{i\to\infty} \epsilon_{n_i} \int_X \Phi(\frac{|f_{n_i}(x)|}{\epsilon_{n_i}})\, d\mu \leq 0.\end{aligned}$$

Finally, given $\{f_{n_k}\}_k \subset \{f_n\}$, there is a further subsequence $\{f_{n_{k_m}}\}_m$ that converges to f a.e. (the limit function must be f). For each $k, j \in \mathbb{N}$, $\Phi(k|f_{n_{k_m}} - f_{n_{k_j}}|) \to \Phi(k|f - f_{n_{k_j}}|)$ as $m \to \infty$. Moreover, since $\|f_{n_m} - f_{n_j}\|_{L^\Phi} \to 0$ as $m, j \to \infty$, there exists $N(k) \in \mathbb{N}$ such that if $m, j > N(k)$, there exists $\epsilon_{m,j} < \frac{1}{k}$ such that $\int_X \Phi(\frac{|f_{n_{k_m}} - f_{n_{k_j}}|}{\epsilon_{m,j}}) \leq 1$.

So $\|f - f_{n_{k_j}}\|_{L^\Phi} < \frac{1}{k}$ if $j > N(k)$ since

$$\begin{aligned}\int_X \Phi(\frac{|f - f_{n_{k_j}}|}{1/k})\, d\mu &\leq \liminf_{m\to\infty} \int_X \Phi(\frac{|f_{n_{k_m}} - f_{n_{k_j}}|}{1/k})\, d\mu = \liminf_{m\to\infty} \int_X \Phi(\frac{|f_{n_{k_m}} - f_{n_{k_j}}|}{\epsilon_{m,j}}\frac{\epsilon_{m,j}}{1/k})\, d\mu \\ &\leq \liminf_{m\to\infty} \frac{\epsilon_{m,j}}{1/k} \int_X \Phi(\frac{|f_{n_{k_m}} - f_{n_{k_j}}|}{\epsilon_{m,j}}) \leq 1.\end{aligned}$$

Therefore $\|f - f_{n_{k_j}}\|_{L^\Phi} \to 0$ as $j \to \infty$. Since every subsequence of $\{f_n\}$ contains a further convergent subsubsequence, f_n converges to f in L^Φ. $\square$

24. *Proof.* $\square$

25. **A Banach space is a Hilbert space if and only if it satisfies the parallelogram law. As a consequence, prove that if $L^p(\mathbb{R}^d)$ with Lebesgue measure is a Hilbert space, then $p = 2$. Generalizations of parallelogram law are stated in Problem 6.**

Proof. (Sketch) Use standard polar identity. For L^p part, see my discussion in Folland [4, Exercise 6.12] which works not only for Lebesgue measure but also for general measure space. □

26. *Proof.* □

27. *Proof.* □

28. *Proof.* □

29. *Proof.* □

30. *Proof.* □

31. *Proof.* □

32. **If the dual space $\mathscr{B}^*$ of a Banach space $\mathscr{B}$ is separable, then $\mathscr{B}$ is separable. (The converse is not true, see Exercise 11.)**

Proof. Let $\{y_n^*\}$ be a countable dense subset of $\mathscr{B}^*$ and $x_n \in \mathscr{B}$ such that $\|x_n\| = 1$ and $|y_n^*(x_n)| \geq \frac{1}{2}\|y_n^*\|$. Let us denote by L the vector space over $\mathbb{Q}$ generated by $\{x_n\}$, which is easy to see it's countable.

Suppose L is not dense in $\mathscr{B}$, then by Hahn-Banach Theorem, there is $y^* \in \mathscr{B}^*$ such that $y^*(L) = \{0\}$ and $\|y^*\| = 1$. Then for each $n \in \mathbb{N}$,

$$\frac{1}{2}\|y_n^*\| \leq |y_n^*(x_n)| = |y_n^*(x_n) - y^*(x_n)| \leq \|y_n^* - y^*\|.$$

Hence $1 = \|y^*\| \leq \|y^* - y_n^*\| + \|y_n^*\| \leq 3\|y_n^* - y^*\| \to 0$ by picking a suitable subsequence, which is a contradiction. Therefore L is dense in $\mathscr{B}$, that is, $\mathscr{B}$ is separable. □

33. *Proof.* □

34. *Proof.* □

35. *Proof.* □

36. *Proof.* □

2 Problems

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. **There are generalizations of the parallelogram law for L^2 (see Exercise 25) that hold for L^p. These are the Clarkson inequalities:**

   (a) For $2 \leq p \leq \infty$ the statement is that

$$\|\frac{f+g}{2}\|_{L^p}^p + \|\frac{f-g}{2}\|_{L^p}^p \leq \frac{1}{2}\Big(\|f\|_{L^p}^p + \|g\|_{L^p}^p\Big).$$

   (b) For $1 < p \leq 2$ the statement is that

$$\|\frac{f+g}{2}\|_{L^p}^q + \|\frac{f-g}{2}\|_{L^p}^q \leq \frac{1}{2}\Big(\|f\|_{L^p}^p + \|g\|_{L^p}^p\Big)^{\frac{q}{p}},$$

   where $p^{-1} + q^{-1} = 1$. This is trickier to prove than (a).

   (c) As a result, L^p is uniformly convex when $1 < p < \infty$. This means that there is a delta function $\delta = \delta_p(\epsilon)$, with $0 < \delta < 1$ and $\delta_p(\epsilon) \to 0$ as $\epsilon \to 0$, so that whenever $\|f\|_{L^p} = \|g\|_{L^p} = 1$, then $\|f - g\|_{L^p} \geq \epsilon$ implies that $\|\frac{f+g}{2}\| \leq 1 - \delta$. This is stronger than the conclusion of strict convexity in Exercise 27.

   (d) Using (c) to prove the Radon-Riesz Theorem: suppose $1 < p < \infty$, and the sequence $\{f_n\} \subset L^p$ or arbitrary uniformly convex Banach space X, converges weakly to f. If $\|f_n\|_X \to \|f\|_X$, then $\|f_n - f\|_X \to 0$ as $n \to \infty$.

   ***Remark* 6.** Milman-Pettis Theorem states every uniformly convex space X is reflexive. See [2, Section 3.7].

   ***Remark* 7.** A related result for (d) is the Brezis-Lieb theorem (refined Fatou lemma), see [3].

   ***Remark* 8.** More discussions of Clarkson inequalities can be found in [1].

   Proof. (c) is easy. (d) We may assume $f \not\equiv 0$. Let $F_n = \|f_n\|^{-1} f_n$ and $F = \|f\|^{-1} f$. So $F_n \rightharpoonup F$ weakly. It follows that

$$1 = \|F\| \leq \liminf \|\frac{F_n + F}{2}\| \leq 1.$$

   Then the uniform convexity implies $\|F_n - F\| \to 0$ as $n \to \infty$. So $\|f_n - f\| \to 0$ as $n \to \infty$. □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

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