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Chapter 6 L^p Spaces

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6.1 Basic Theory of L^p Spaces

1. **When does equality hold in Minkowski's inequality?**

Proof. (a) For $p = 1$, since we know the triangle inequality $|(f + g)(x)| \leq |f(x)| + |g(x)|$ is always true, to characteriz the case when the equality holds in Minkowski's inequality is equivalent to seeking the condition for the equality $|(f+g)(x)| = |f(x)|+|g(x)|$ for all x, that is $f(x)\overline{g(x)} \geq 0$ for a.e. x. (We consider the complex-valued functions here.)

(b) For $1 < p < \infty$, in its proof, we know the equality holds in Minkowski inequality if and only if the ones in triangle and Hölder's inequalities hold, that is, if and only if $f(x)\overline{g(x)} \geq 0$ for a.e. x, and there are constants $\alpha, \beta \geq 0$ such that $|f(x)|^p = \alpha|f(x) + g(x)|^p$ and $|g(x)|^p = \beta|f(x) + g(x)|^p$ for a.e. x. Combining them, we know the equality holds in Minkowski if and only if $f(x) = \gamma g(x)$ for a.e. x, for some $\gamma \geq 0$.

(c) For $p = \infty$. Define the set $A_f = \{\{x_n\}\}$ where $\{x_n\}$ is a sequence in X with $|f(x_n)| \to \|f\|_\infty$. We claim that $\|f + g\|_\infty = \|f\|_\infty + \|g\|_\infty$ iff either one of the functions is 0 a.e., or $A_f \cap A_g$ is nonempty and for some $\{x_n\} \in A_f \cap A_g$, where $f(x_n) \to a$ and $g(x_n) \to b$, there is a $\lambda > 0$ such that $b = \lambda a$.

($\Leftarrow$) If one of them is 0 a.e., then it's done. If not, then

$$\|f\|_\infty + \|g\|_\infty = \lim_n |f(x_n)| + \lim_n |g(x_n)| = |a| + |b| = |a + b| = \lim_n |f(x_n) + g(x_n)| \leq \|f + g\|_\infty.$$

Combining with the Minkowski inequality, we have the desired equality.

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(⇒) We assume both functions are not zero, so $\|f\|_\infty$ and $\|g\|_\infty > 0$. Since there exists x_n such that $|(f+g)(x_n)| \to \|f+g\|_\infty = \|f\|_\infty + \|g\|_\infty$. By a contradiction argument, we see $|f(x_n)| \to \|f\|_\infty$ and $|g(x_n)| \to \|g\|_\infty$, that is, $\{x_n\} \in A_f \cap A_g$. If $f(x_n) \to \|f\| > 0$ and $g(x_n) \to -\|g\| < 0$, then $|f(x_n) + g(x_n)| \to |\|f\|_\infty - \|g\|_\infty| < \|f\|_\infty + \|g\|_\infty = \|f+g\|_\infty$, a contradiction. Similarly, it's impossible that $f(x_n) \to -\|f\|$ and $g(x_n) \to \|g\|$. Hence we get the desired result with $\lambda = \frac{\|g\|_\infty}{\|f\|_\infty}$. □

2. **Prove Theorem 6.8.**

Proof. (a),(b) and (c)(⇐) are obvious, we also omit (d) since it's proved in many textbooks.

(c)(⇒): Given $m \in \mathbb{N}$ there exists $M_m \in \mathbb{N}$ such that for all $n \geq M_m$, $\|f_n - f\|_\infty < \frac{1}{m}$. Then for such (m,n), there is $X_n^m \in \mathscr{M}$ such that $\mu(X_n^m) = 0$ and $|f_n - f| < \frac{1}{m}$ on $X \setminus X_n^m$. Take

$$E^c = \cup_{m=1}^\infty \cup_{n=M_m}^\infty X_n^m$$

which has zero measure and $f_n \to f$ uniformly on E.

(e) is proved by partition the given range $[-\|f\|_\infty, \|f\|_\infty)$ equally into 2^k disjoint closed-open subintervals, and for each $k \in \mathbb{N}$, consider the proper simple function f_k corresponding to the sets

$$\Big\{\{x : \|f\|_\infty \frac{j}{2^k} \leq |f(x)| < \|f\|_\infty \frac{j+1}{2^k}\}, \{x : \|f\|_\infty = |f(x)|\}\Big\}_{j=-2^k}^{2^k-1}.$$

□

3. **If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.**

Proof. $\|f\|_q \leq \|f\|_p^\theta \|f\|_r^{1-\theta} \leq \|f\|^\theta \|f\|^{1-\theta}$, where θ is defined by $q^{-1} = \theta p^{-1} + (1-\theta) r^{-1}$. □

4. **If $1 \leq p < r \leq \infty$, $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.**

Remark 1. (2018.04.10) See Sergey Astashkin and Lech Maligranda's paper (arXiv:1804.03469), "*$L^p + L^q$ and $L^p \cap L^q$ are not isomorphic for all* $1 \leq p, q \leq \infty, p \neq q$" and the references therein.

Proof. Since $p < q < r$,

$$\|f\| \leq \|f\chi_{\{|f|>1\}}\|_p + \|f\chi_{\{|f|\leq 1\}}\|_r \leq \|f\chi_{\{|f|>1\}}\|_q + \|f\chi_{\{|f|\leq 1\}}\|_q = \|f\|_q.$$

□

There is a related set equality due to Alvarez [1]:

Theorem 2. *Let $L^p_q := L^p + L^q$. Then (a) $L^p_q + L^r_s = L^{\min(p,r)}_{\max(q,s)}$ (b) $L^p_q \cdot L^r_s = L^u_v$, where $u^{-1} = p^{-1} + r^{-1}; v^{-1} = q^{-1} + s^{-1}$ (c) $L^p_q(L^{p_0}_{q_0} + L^{p_1}_{q_1}) = L^p_q \cdot L^{p_0}_{q_0} + L^p_q \cdot L^{p_1}_{q_1}$.*

We prove this theorem in additional exercise 6.1.3.

Remark 3. L^p_q is a special case of Orlicz spaces. Its duality result to $L^{p'} \cap L^{q'}$ and generalization are given in Stein-Shakarchi [18, Exercise 1.24, 1.26 and Problem 1.5].

5. Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. What about the case $q = \infty$

Proof. Since the first and second assertions are dual to each other, we only prove the first one.

($\Leftarrow$) By assumption, there is a sequence of sets $\{F_n\}$ with $0 < \mu(F_{n+1}) < 3^{-1}\mu(F_n)$ for all $n \in \mathbb{N}$ and $\mu(F_1) < 1/3$. Let $E_n = F_n \setminus \cup_{n+1}^{\infty} F_j$, then we obtain a sequence of disjoint sets E_n with $0 < 2^{-1}\mu(F_n) < \mu(E_n) < 3^{-n}$. Finally we consider the function $f = \sum a_n \chi_{E_n}$ with $a_n = \mu(E_n)^{-1/q}$ which is in L^p but not in L^q.

($\Rightarrow$) Assume there is $c > 0$ such that any set with positive measure has measure $\geq c$. Take $f \in L^p$ and consider the sets $A_n = \{x : n \leq |f(x)| < n+1\}$. By Chebyshev's inequality, for $n \geq 1$, either $\mu(A_n) \geq c$ or $\mu(A_n) = 0$. Since $f \in L^p$, only finitely many n such that $\mu(A_n) \geq c$. So $f \in L^q$ and thus $L^p \subset L^q$, which is a contradiction.

For the case $q = \infty$, $L^\infty \not\subset L^p$ iff $\mu(X) = \infty$. This is easy to prove.

On the other hand, $L^p \not\subset L^\infty$ iff X contains sets of arbitrarily small positive measure. The proof for ($\Leftarrow$) is the same as above except taking $a_n = \mu(E_n)^{-1/(p+1)}$ now. The proof for ($\Rightarrow$) is the same. □

Remark 4. A. Villani [19] simplified Romero's previous work [14] and show that the followings are equivalent: Let $0 < p < q \leq \infty$. (i) $L^p(\mu) \subseteq L^q(\mu)$ for some $p < q$; (ii) $L^p(\mu) \subseteq L^q(\mu)$ for all $p < q$; (iii) $\inf\{\mu(E) : \mu(E) > 0\} > 0$. In the case $q < p$, condition (iii) is replaced by (iii)' $\sup\{\mu(E) : \mu(E) < \infty\} < \infty$.

This is then generalized by Miamee [12] to the more geneal relation $L^p(\mu) \subseteq L^q(\nu)$.

We prove these results in additional exercise 6.1.4.

6. See Exercise 2.64.

Proof. (i) If $p_1 < \infty$, then consider $f(x) = x^{-1/p_1}\chi_{(0,\frac{1}{2})}(x) + x^{-1/p_0}\chi_{(2,\infty)}(x)$. If $p_1 = \infty$, then consider $f(x) = |\log x|\chi_{(0,\frac{1}{2})}(x) + x^{-1/p_0}\chi_{(2,\infty)}(x)$.

(ii) If $p_1 < \infty$, then $f(x) = x^{-1/p_1}|\log x|^{-2/p_1}\chi_{(0,\frac{1}{2})}(x) + x^{-1/p_0}|\log x|^{-2/p_0}\chi_{(2,\infty)}(x)$. If $p_1 = \infty$, then consider $f(x) = x^{-1/p_0}|\log x|^{-2/p_0}\chi_{(2,\infty)}(x)$.

(iii) Set $p_1 = p_0 < \infty$ in (ii). □

7. *Proof.* We may assume $f \not\equiv 0$. For each $q \in (p, \infty)$, we know $\|f\|_q \leq \|f\|_p^{\theta_q}\|f\|_\infty^{1-\theta_q}$ where $\theta_q = q^{-1}p \to 0$ as $q \to \infty$. Therefore $\limsup_{q\to\infty}\|f\|_q \leq \|f\|_\infty$. On the other hand, given $\epsilon \in (0, \|f\|_\infty)$, there exist a measurable set $E = E(\epsilon)$ with $\mu(E) > 0$ such that $\|f\|_\infty - \epsilon < |f|$ on E. Since $f \in L^p, \mu(E) < \infty$. Integrating both sides on E, we see $(\|f\|_\infty - \epsilon)\mu(E) < \int_E |f| \leq \|f\|_q\mu(E)^{1-\frac{1}{q}}$ and hence $\|f\|_\infty - \epsilon < \|f\|_q\mu(E)^{-\frac{1}{q}}$. Therefore, $\|f\|_\infty - \epsilon \leq \liminf_{q\to\infty}\|f\|_q$. Since $\epsilon > 0$ is arbitrary, $\|f\|_\infty \leq \liminf_{q\to\infty}\|f\|_q \leq \limsup_{q\to\infty}\|f\|_q \leq \|f\|_\infty$. □

8. Suppose $\mu(X) = 1$ and $f \in L^p$ for some $p > 0$, so that $f \in L^q$ for $q \in (0, p)$.

Proof. (a) If $-\log|f|^q \in L^1$, then this is Jensen's inequality. See Exercise 3.42.

If $-\log|f|^q \notin L^1$, then $\int|\log|f|^q| = \infty$ and

$$\int|\log|f|^q| = \int\log|f|^q\chi_{\{|f|>1\}} - \int\log|f|^q\chi_{\{|f|\leq1\}} \leq \int|f|^q\chi_{\{|f|>1\}} - \int\log|f|^q\chi_{\{|f|\leq1\}}$$

Since $f \in L^q, \int\log|f|^q\chi_{\{|f|\leq1\}} = -\infty$. Therefore $\int\log|f| = -\infty$ which implies the inequality.

(b) Since $\log(x) \leq x - 1$ for all $x > 0$. Let $x = \|f\|_q^q$, we yields the first inequality. The second assertion is a consequence of monotone convergence theorem.

(c) By (a), $\|f\|_q \geq \exp(\int\log|f|)$ for all $q \in (0, p)$, and then $\liminf_{q\to0}\|f\|_q \geq \exp(\int\log|f|)$. On the other hand, from (b) we know $\limsup_{q\to0}\log\|f\|_q \leq \limsup_{q\to0}(\int|f|^q - 1)/q = \int\log|f|$. Taking the exponential of both sides, we have $\limsup_{q\to0}\|f\|_q \leq \exp(\int\log|f|)$. □

9. *Proof.* The first assertion is Chebyshev's inequality. To prove the second one, we notices the convergence in measure implies convergence a.e., up to a subsequence, and then the limit function $|f| \leq g$. This implies the desired result by LDCT and the fact $|f_n - f| \leq 2g$. □

Remark 5. The first assertion is trivial true for $p = \infty$, but the second one is false.

10. Suppose $f_n, f \in L^p, f_n \to f$ a.e. for some $p \in [1, \infty)$. Then $\|f_n\|_p \to \|f\|_p \iff \|f_n - f\|_p \to 0$.

Proof. "$\Longleftarrow$" part is easy (we don't need to assume pointwise convergence here). "$\Longrightarrow$" part is an application of Fatou's lemma to the functions $2^p|f_n|^p + 2^p|f|^p - |f_n - f|^p \geq 0$. (Another method is through Egoroff's theorem and Fatou's lemma, see Rudin [15, Exercise 3.17(b)].) □

***Remark* 6.** Rudin [15] mentions that the proof seems to be discovered by Novinger [13], where an application to F.Riesz's theorem on mean convergence of H^p functions to their boundary function is mentioned. See Duren [7, Theorem 2.6 and 2.2] and Rudin [15, Theorem 17.11].

If $f_n \to f$ a.e., by Fatou's lemma, we always have $\|f\|_p \le \liminf \|f_n\|_p$. What can be said about $\|f\|_p$. There is a more general and quantitative result:

Theorem 7 (Brezis-Lieb lemma[4])**.** *Suppose $f_n \to f$ a.e. and $\|f_n\|_p \le C < \infty$ for all n and for some $0 < p < \infty$. Then*

$$\lim_{n\to\infty} \|f_n\|_p^p - \|f_n - f\|_p^p = \|f\|_p^p.$$

Another related result is the Radon-Riesz Theorem:

Theorem 8. *Let $p \in (1, \infty)$. Suppose $f_k \rightharpoonup f$ in L^p and $\|f_n\|_p \to \|f\|_p$, then $\|f_n - f\|_p \to 0$.*

11. *Proof.* (a)Given $z_n \to z$ in $\mathbb{C}$ with all $z_n \in R_f$. Then given $\epsilon > 0$, since $|f(x) - z| \le |f(x) - z_n| + |z_n - z|$, by choosing n large such that $|z_n - z| < \epsilon$, we have

$$\{x : |f(x) - z_n| < \epsilon\} \subset \{|f(x) - z| < 2\epsilon\}.$$

Therefore $0 < \mu(\{x : |f(x) - z_n| < \epsilon\}) \le \mu(\{|f(x) - z| < 2\epsilon\})$ and hence $z \in R_f$.

(b) By definition and proof by contradiction, $\|f\|_\infty \ge \sup\{|w| : w \in R_f\}$. The compactness follows from (a) and Bolzano-Weierstrauss. So there is $z \in R_f$, such that $|z| = \sup\{|w| : w \in R_f\}$. If $|z| < \|f\|_\infty$, then $|z| < \frac{1}{2}(|z| + \|f\|_\infty)$.

Hence the set $\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \epsilon_0\}$ has zero measure for some $\epsilon_0 > 0$, but this implies $\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \epsilon\}$ has zero measure for all $0 < \epsilon < \epsilon_0$. In particular, choosing $1 > \delta > 0$ such that $\frac{\delta}{2}(|z| + \|f\|_\infty) < \epsilon_0$. Since

$$\{x : |f(x)| > \frac{1-\delta}{2}(|z| + \|f\|_\infty)\} \subset \{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \frac{\delta}{2}(|z| + \|f\|_\infty)\},$$

$0 = \mu(\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \frac{\delta}{2}(|z| + \|f\|_\infty)\}) \ge \mu(\{x : |f(x)| > \frac{1-\delta}{2}(|z| + \|f\|_\infty)\}) = 0$, which contradicts the definition of $\|f\|_\infty$ since $0 < \frac{1-\delta}{2}(|z| + \|f\|_\infty) < \|f\|_\infty$. Therefore, $\max\{|w| : w \in R_f\} = |z| = \|f\|_\infty$. ∎

***Remark* 9.** See Rudin [15, Exercise 3.19] and additional exercise 6.1.7 for an extended discussion.

12. *Proof.* Let (X, Σ, μ) be the measure space. We assume $1 \le p < \infty$ first. If (i) for every $E \in \Sigma$, $\mu(E) = 0$ or ∞, then L^p has zero dimension, where there is nothing to prove.

Now we assume there exists $A \in \Sigma$ such that $0 < \mu(A) < \infty$. If (ii) for every $E \in \Sigma$, $\mu(E\Delta A) = 0$, then $\dim(L^p) = 1$, that is, every function equals a.e. to a multiple of χ_A. The inner product is defined by $(\alpha\chi_A, \beta\chi_A) = \alpha\beta\mu(A)^2$.

If (iii) there is a set C with $0 < \mu(C) < \infty$ and $\mu(C\Delta A) > 0$. We may assume $B := C \setminus A$ has positive measure. Note $\|\chi_A\|_p^2 = \mu(A)^{2/p}$, $\|\chi_B\|_p^2 = \mu(B)^{2/p}$, $\|\chi_A + \chi_B\|_p^2 = (\mu(A) + \mu(B))^{2/p}$ and $\|\chi_A - \chi_B\|_p^2 = (\mu(A) + \mu(B))^{2/p}$, so $2(\|\chi_A\|_p^2 + \|\chi_B\|_p^2) = 2(\mu(A)^{2/p} + \mu(B)^{2/p})$ which is not equal to $2(\mu(A) + \mu(B))^{2/p} = \|\chi_A + \chi_B\|_p^2 + \|\chi_A - \chi_B\|_p^2$ except $p = 2$.

For $p = \infty$, in case (iii), $\|\chi_A + \chi_B\|_\infty^2 + \|\chi_A - \chi_B\|_\infty^2 = 2 \neq 4 = 2(\|\chi_A\|_\infty^2 + \|\chi_B\|_\infty^2)$. In case (ii), the situation is the same as above, so it possesses inner product structure. In case (i), if $\mu(X) = 0$, then it's done. For $\mu(X) = \infty$, if there are proper subset A of X such that $\mu(A) = \mu(X\setminus A) = \infty$, then we see there is no inner product structure as above with $B = X \setminus A$; if for every set A with $\mu(A) = \infty$, we always have $\mu(X \setminus A) = 0$, then L^p has dimension 1, since for each $f \in L^\infty$, if f takes two distinct values on disjoint sets A, B, then either $\mu(A) = 0$ or $\mu(B) = 0$. That is, $f = \alpha\chi_X$, for some scalar α. □

13. *Proof.* The case $1 \leq p < \infty$, it's easy to show the set of simple functions on dyadic cubes with rational coefficients is countably many, dense in $L^p(\mathbb{R}^d, m)$.

    Next, consider $\mathscr{F} = \{f_h(x) = \chi_{B_1(0)}(x + h), h \in \mathbb{R}^d\}$, then given $f \neq g \in \mathscr{F}$, $\|f - g\|_\infty = 1$. This shows the dense subset of $L^\infty(\mathbb{R}^d, m)$ must be uncountable. □

    Remark 10. See Additional Exercise 6.1.1 to characterize when is $L^p(X, \mathscr{M}, \mu)$ separable.

14. *Proof.* We may assume $g \not\equiv 0$. $\|T\| \leq \|g\|_\infty$ since $\|Tf\|_p = \|fg\|_p \leq \|f\|_p\|g\|_\infty$. For the second part, given $\epsilon \in (0, \|g\|_\infty)$, since μ is semifinite, there is a set E with $0 < \mu(E) < \infty$ such that $\|g\|_\infty - \epsilon < |g|$ on E. Then for $f = \chi_E$, we see $\|T(\chi_E)\|_p = \|g\chi_E\|_p > (\|g\|_\infty - \epsilon)\|\chi_E\|_p$, which means $\|T\| > \|g\|_\infty - \epsilon$. Since $\epsilon > 0$ is arbitrary, $\|g\|_\infty \leq \|T\|$. □

15. This is Vitali Convergence Theorem. Necessity of the hypotheses and further discussions are given in Rudin [15, Exercise 6.10-11]. For example, in finite measure space, Vitali implies LDCT, and there is some case Vitali works but LDCT can't be applied. The details are in additional exercise 6.1.8.

    Proof. (Necessity) Use Chebyshev's inequality to prove (i). Completeness of L^p implies the existence of limit function, which controls the integrability of f_n for the large n, which is the central idea to prove (ii) and (iii).

(Sufficiency) Given $\epsilon > 0$, let E be the set in condition (iii). Take

$$A_{nm} := \{|f_m(x) - f_n(x)| \geq \frac{\epsilon}{\mu(E)}\}.$$

By (i), we know if n, m large enough, then $\mu(A_{nm}) < \delta$, where δ is the modulus of uniform integrability. Hence, for all large n, m

$$\int_{A_{nm}} |f_n - f_m|^p \, d\mu < \epsilon.$$

Therefore, for all large enough n, m

$$\begin{aligned}\int_X |f_n - f_m|^p \, d\mu &= \int_{E^c} |f_n - f_m|^p \, d\mu + \int_{A_{nm}} |f_n - f_m|^p \, d\mu + \int_{E \setminus A_{nm}} |f_n - f_m|^p \, d\mu \\ &\leq 2\epsilon + \epsilon + \frac{\epsilon}{\mu(E)} \mu(E),\end{aligned}$$

which implies the desired result. $\square$

16. *Proof.* The completeness is a revision of the proof of Theorem 6.6, we omit it. The triangle inequality is a consequence of the inequality $(a + b)^p \leq a^p + b^p, a, b \geq 0$, whose proof is straightforward. $\square$

Remark 11. There are several properties for $0 < p < 1$; the first is in finite measure space, the behavior of $\| \cdot \|_q$ as $q \to 0$, answered in exercise 8(c); the second is that the dual space of $L^p([0,1], \mathscr{L}, m)$ is $\{0\}$, that is, it's NOT a locally convex space; the last is the reversed Hölder and Minkowski inequalities, see Jones [10, p.252-253] and Rudin [16, p.37].

Theorem 12. *Let $(X, \mathscr{M}, \mu)$ be a measure space, $0 < p < 1$ and f, g be two nonnegative and measurable functions. Then*

$$\int_X fg \, d\mu \geq \left(f^p \, d\mu\right)^{1/p} \left(g^{p'} \, d\mu\right)^{1/p'},$$

and

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p.$$

6.2 The Dual of L^p

17. *Proof.* First, we note that $M_q(|g|) \le 8M_q(g)$ since for each $f = \sum_{k=1}^n (a_k + ib_k)\chi_{E_k} \in \Sigma$, the set of simple functions that vanish outside a set of finite measure

$$
\begin{aligned}
&|\int f|g||^2 = |\sum_{k=1}^n a_k \int_{E_k} |g| + ib_k \int_{E_k} |g||^2 = \Big(\sum_{k=1}^n a_k \int_{E_k} |g|\Big)^2 + \Big(\sum_{k=1}^n b_k \int_{E_k} |g|\Big)^2 \\
&\le \Big(\sum_{k=1}^n a_k \int_{E_k} |\mathrm{Re}(g)| + |\mathrm{Im}(g)|\Big)^2 + \Big(\sum_{k=1}^n b_k \int_{E_k} |\mathrm{Re}(g)| + |\mathrm{Im}(g)|\Big)^2 \\
&= \Big(\sum_{k=1}^n a_k \big(\int_{E_k^1} \mathrm{Re}(g) - \int_{E_k^2} \mathrm{Re}(g) + \int_{E_k^3} \mathrm{Im}(g) - \int_{E_k^4} \mathrm{Im}(g)\big)\Big)^2 + \Big(\cdots\Big)^2 \\
&\le 2\Big(\sum_{k=1}^n a_k \int_{E_k^1} \mathrm{Re}(g) - a_k \int_{E_k^2} \mathrm{Re}(g)\Big)^2 + 2\Big(\sum_{k=1}^n a_k \int_{E_k^3} \mathrm{Im}(g) - a_k \int_{E_k^4} \mathrm{Im}(g)\Big)^2 + 2\Big(\cdots\Big)^2 + 2\Big(\cdots\Big)^2 \\
&\le 2M_q(g) + 2M_q(g) + 2M_q(g) + 2M_q(g),
\end{aligned}
$$

where $E_k = E_k^1 \cup E_k^2 = E_k^3 \cup E_k^4$ are separated by the sign of $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$ respectively and the last inequality follows from the definition of $M_q(g)$ (by taking $f_r = \sum_{k=1}^n a_k\chi_{E_k^1} - a_k\chi_{E_k^2}$ and $f_i = \sum_{k=1}^n a_k\chi_{E_k^3} - a_k\chi_{E_k^4}$).

Given $\epsilon > 0$. Let $E := \{x : |g(x)| > \epsilon\}$. If $\mu(E) = \infty$, then for each $n \in \mathbb{N}$ there is $E_n \subset E$ such that $n < \mu(E_n) < \infty$ (Exercise 1.14). However this is impossible for large n since

$$\mu(E_n) \le \frac{1}{\epsilon}\int |g(x)|\chi_{E_n}\, d\mu = \frac{(\mu(E_n))^{1/p}}{\epsilon}\int |g(x)| \frac{1}{(\mu(E_n))^{1/p}}\chi_{E_n}\, d\mu \le \frac{(\mu(E_n))^{1/p}}{\epsilon} M_q(|g|),$$

that is, $\mu(E_n)$ is uniformly bounded (since $q < \infty, p > 1$). So $\mu(E) < \infty$. □

Remark 13. I learn from Joonyong Choi that there is another way to show the uniform boundedness as follows:

$$\mu(E_n) \le \frac{1}{\epsilon}\int |g(x)|\chi_{E_n}\, d\mu = \frac{(\mu(E_n))^{1/p}}{\epsilon}\int g(x)\frac{\overline{g(x)}}{|g(x)|(\mu(E_n))^{1/p}}\chi_{E_n}\, d\mu \le \frac{(\mu(E_n))^{1/p}}{\epsilon} M_q(g),$$

where the last inequality is proved in the first paragraph of the proof of Theorem 6.14. (Note that $f(x) = \frac{\overline{g(x)}}{|g(x)|(\mu(E_n))^{1/p}}\chi_{E_n}$ is bounded and vanishes outside the set E_n of finite measure)

18. (Von Neumann's proof for Lebesgue-Radon-Nikodym Theorem)

Proof. (a) It's a consequence of Hölder's inequality since $\lambda(X) < \infty$.

(b) Note that for all $E \in \mathscr{M}$, $\int_E Re(g)\, d\lambda + i\int_E Im(g)\, d\lambda = \int \chi_E g\, d\lambda = \int \chi_E\, d\mu = \mu(E) \ge 0$ So g is real-valued and $\lambda(\{g \le -n^{-1}\}) = 0$ for all $n \in \mathbb{N}$. Similarly, since $\int \chi_E(1-g)\, d\lambda = \lambda(E) - \int g\, d\lambda = \lambda(E) - \int \chi_E\, d\mu = \lambda(E) - \mu(E) \ge 0$, $\lambda(\{1 - g \le -n^{-1}\}) = 0$ for all $n \in \mathbb{N}$. Combining these two results, we have $0 \le g \le 1$ λ-a.e.

(c) Clearly, for each $P \subset B$, $\mu(P) = \int_B g\, d\mu = \int \chi_B(1-1)\, d\nu = 0$ and for each $N \subset A$, $\nu_s(N) = \nu(\emptyset) = 0$. So $\nu_s \perp \mu$.

Since $(1-g)^{-1}\chi_A \geq 0$ λ-a.e., there exists an increasing sequence of simple functions $\phi_n \geq 0$ converging to $\chi_A(1-g)^{-1}$ λ-a.e. Since $\chi_A(1-g)^{-1} < \infty$ λ-a.e., $\phi_n \in L^2(\lambda)$ for all n. For all $E \in \mathscr{M}$, by monotone convergence theorem, we have

$$\nu_a(E) = \nu(A \cap E) = \int_E \frac{\chi_A}{1-g}(1-g)\,d\nu = \lim_{n\to\infty}\int_E \phi_n(1-g)\,d\nu = \lim_{n\to\infty}\int_E \phi_n g\,d\mu = \int_E \frac{\chi_A}{1-g} g\,d\mu,$$

which is the desired result. □

19. *Proof.* Note that $|\phi_n(f)| \leq \|f\|_\infty$, which means $\|\phi_n\|_{(l^\infty)^*} \leq 1$ for all n. Then Banach-Alaoglu's theorem implies the sequence $\{\phi_n\}$ has a weak* cluster point ϕ. If $\phi(f) = \sum_j f(j)g(j)$ for some $g \in l^1$, then by taking $f_i \in l^\infty$ with $f_i(j) = \delta_{ij}$, the Kronecker delta, we see $g \equiv 0$ since $g(i) = \phi(f_i) = \lim_{n\to\infty}\phi_n(f_i) = 0$ for each i. But this leads a contradiction that $0 = \phi(\mathbf{1}) = \lim_{n\to\infty}\phi_n(\mathbf{1}) = 1$, where $\mathbf{1}(j) = 1$ for all $j \in \mathbb{N}$. □

20. **Suppose $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e.**

(a) If $1 < p < \infty$, then $f_n \to f$ weakly in L^p. (Given $g \in L^q$, here q is conjugate to p, and $\epsilon > 0$, there exist (i) $\delta > 0$ such that $\int_E |g|^q < \epsilon$ whenever $\mu(E) < \delta$, (ii) $A \subset X$ such that $\mu(A) < \infty$ and $\int_{X\setminus A} |g|^q < \epsilon$, and (iii) $B \subset A$ such that $\mu(A \setminus B) < \delta$ and $f_n \to f$ uniformly on B.)

(b) The result of (a) is false in general for $p = 1$

Proof. (a) We omit it since every step is given in the hint and should be verified easily. A more precise result about $\|f_n\|$ and $\|f\|$ is Brezis-Lieb lemma, which is proved in additional exercise 6.1.5.

Note that Banach-Alaoglu theorem implies the following similar and important result: (for L^p case, we can prove it by Cantor diagonal process without Banach-Alaoglu, see my solution file for Stein-Shakarchi [18, Exercise 1.12(c)])

Theorem 14. *Let X be a reflexive Banach space. If $\|f_n\|_X \leq M < \infty$ for all n, then there is a subsequence converge weakly.*

Remark 15. The converse is known as Eberlein-Šmulian theorem whose proof is put in additional exercise 6.2.5. Also see Brezis [3, Theorem 3.19 and its remarks]

(b) For $L^1(\mathbb{R}, m)$, consider $f_n = \chi_{(n,n+1)}$ with $1 \equiv g \in L^\infty(\mathbb{R}, m)$.

For l^1, consider $f_n(j) = \frac{1}{n}$ if $j \leq n$, and 0 otherwise. If there is weak convergence for f_n, then by considering $g_i(j) = \delta_{ij} \in l^\infty$ the Kronecker delta, we see $f \equiv 0$. On the other hand, for $1 \equiv g \in l^\infty$, $0 = \int fg \not\leftarrow \int f_n g = 1$, which is a contradiction. □

21. **If $1 < p < \infty$, $f_n \to f$ weakly in $l^p(A)$ iff $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise.**

Proof. ($\Rightarrow$) By weak convergence, for any $g \in l^q$, there exists $M_g < \infty$ such that $\sup_n |\int f_n g| < M_g$. Then uniform boundedness principle implies $\sup_n \|f_n\|_p$ is finite. The pointwise convergence is proved by taking $g = \chi_{\{a\}}$ for each $a \in A$.

($\Leftarrow$) This is exercise 20 (a). $\square$

Remark 16. ($\Rightarrow$) is also true for $p = 1$ and $p = \infty$ due to Proposition 6.13 works in both case (counting measure is semifinite.) An interesting theorem for $l^1(\mathbb{N})$ is the Schur's property which asserts the equivalence between weak convergence and norm convergence. (See additional exercise 6.2.3.)

($\Leftarrow$) is discussed in exercise 20 (b).

22. **Let $X = [0, 1]$ with Lebesgue measure.**

(a) Let $f_n(x) = \cos(2\pi nx)$. Then $f_n \to 0$ weakly in L^2 (See Exercise 63 in Section 5.5) but $f_n \nrightarrow 0$ a.e. or in measure.

(b) Let $f_n(x) = n\chi_{(0,1/n)}$. Then $f_n \to 0$ a.e. and in measure, but $f_n \nrightarrow 0$ weakly in L^p for any p.

(a) This is Riemann-Lebesgue lemma, which holds not only for L^2 but also for L^p, ($p \in [1, \infty)$), and weak-* convergence in L^∞; I put my comment on various proofs below:

(1) I think the easy way to understand is through Bessel's inequality (cf: Exercise 5.63). But this method does not work on L^p, except L^2.

(2) The second way is through integration by parts, and one needs the density theorem (simple functions or C_c^∞ functions). This method is adapted to our claim, except for weak convergence in L^1. But for L^1 case, it's a simple consequence of squeeze theorem in freshman's Calculus. It's easy to see $f_n \not\to 0$ a.e. or in measure.

(b) The first assertion is easy to prove. To see it's not weak convergent, take $g \equiv 1$ in $L^{p'}$.

Remark 17. From this problem, we know for **finite** measure space $(E, \mathscr{M}, \mu)$, we have the following figure from Dibenedetto [6, p.294](1st edition: p.268) with some modifications (the red and green words.)

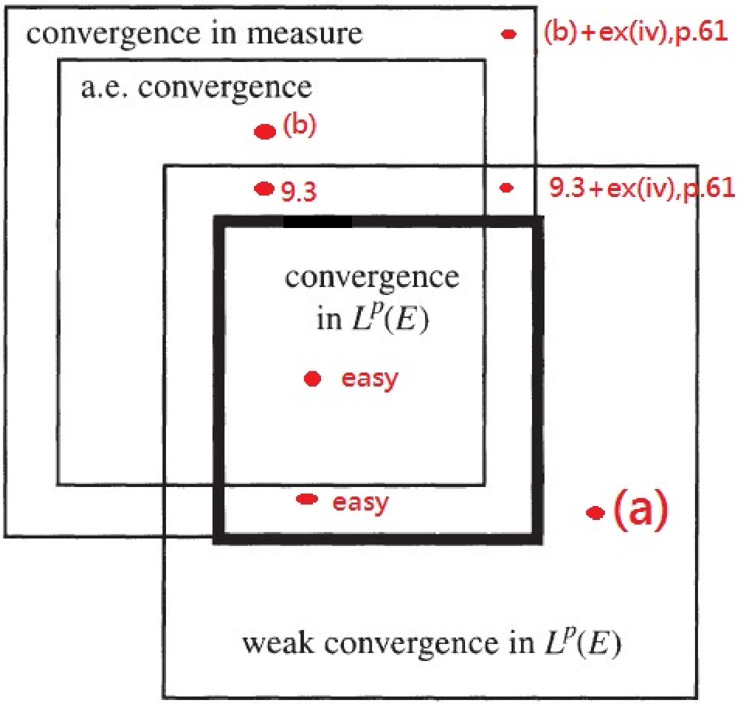

Figure 9.1c.

Important: (a),(b),9.3, and ex(iv)

9.3 Consistent with Figure 9.1c are sequences $\{f_n\}$ of measurable functions in E convergent to zero a.e. in E and weakly in $L^2(E)$ but not convergent in $L^2(E)$. For example, in [0, 1],

$$f_n(x) = \begin{cases} \sqrt{n} & \text{for } x \in \left[0, \frac{1}{n}\right], \\ 0 & \text{for } x \in \left(\frac{1}{n}, 1\right]. \end{cases} \tag{9.6c}$$

23. See Exercise 1.16 for the notion of locally measurable sets.

Proof. (a) If $0 < \mu(E) < \infty$, then $0 = \mu(E \cap E) > 0$, a contradiction. If μ is semifinite, then for every locally null set E with $\mu(E) = \infty$, there exist $F \subset E$ with $0 < \mu(F) < \infty$, but $0 = \mu(E \cap F) = \mu(F) > 0$, a contradiction.

(b) If $\|f\|_* = 0$, then for each $\frac{1}{n}$ there exist $a_n \in (0, \frac{1}{n})$ such that $\{|f| > a_n\}$ is locally null. Since $\{|f| > a_n\} \supseteq \{|f| > \frac{1}{n}\}$, $\{|f| > \frac{1}{n}\}$ is locally null. Given $F \in \mathscr{M}$ with $\mu(F) < \infty$, we have $0 \leq \mu(F \cap \{f \neq 0\}) = \mu(F \cap \cup_n \{|f| > \frac{1}{n}\}) \leq \sum_n \mu(F \cap \{|f| > \frac{1}{n}\}) = 0$, that is, $\{f \neq 0\}$ is locally null.

Given $f, g \in \mathscr{L}^\infty$ and then given $\epsilon > 0$. Note that

$$\{|f+g| > \|f\|_* + \|g\|_* + \epsilon\} \subseteq \{|f|+|g| > \|f\|_* + \|g\|_* + \epsilon\} \subseteq \{|f| > \|f\|_* + \epsilon/2\} \cup \{|g| > \|g\|_* + \epsilon/2\}.$$

Since the latter set is locally null, $\{|f+g| > \|f\|_* + \|g\|_* + \epsilon\}$ is locally null. Then $\|f+g\|_* \leq \|f\|_* + \|g\|_* + \epsilon$ for all $\epsilon > 0$ and therefore $\|f+g\|_* \leq \|f\|_* + \|g\|_*$.

Given $0 \neq \lambda \in \mathbb{C}$ and $f \in \mathscr{L}^\infty$, since $\{|\lambda f| > |\lambda|\|f\|_* + \epsilon\} = \{|f| > \|f\|_* + \frac{\epsilon}{|\lambda|}\}$ is locally null for each $\epsilon > 0$ and $\{|\lambda f| > |\lambda|\|f\|_* - \epsilon\} = \{|f| > \|f\|_* - \frac{\epsilon}{|\lambda|}\}$ is not locally null for each $\epsilon > 0$, we see $\|\lambda f\|_* = |\lambda|\|f\|_*$. Consequently, $\|\cdot\|_*$ is a norm on $\mathscr{L}^\infty$.

Given $\{f_n\} \subset \mathscr{L}^\infty$ with $\sum_n \|f_n\|_* < \infty$. We are going to show $\sum_n f_n \in \mathscr{L}^\infty$. More precisely, $\|\sum_n f_n\|_* \leq \sum_n \|f_n\|_* < \infty$. This is true since for each $\epsilon > 0$

$$\Big\{|\sum_n f_n| > \sum_n \|f_n\|_* + \epsilon\Big\} \subseteq \Big\{\sum_n |f_n| > \sum_n \|f_n\|_* + \epsilon\Big\} \subseteq \cup_n \Big\{|f_n| > \|f_n\|_* + \frac{\epsilon}{2^n}\Big\},$$

where each indexed-set in the last set is locally null. Therefore, $(\mathscr{L}^\infty, \|\cdot\|_*)$ is complete.

The last assertion is due to $\|\cdot\|_* = \|\cdot\|_\infty$ here, which is a consequence of (a). □

24. In contrast to Proposition 6.13 and Theorem 6.14, we do NOT need to assume μ is semifinite or $S_g = \{g \neq 0\}$ is σ-finite here. Also see additional exercise 6.2.2.

Proof. (i) Given $f \in L^1$ with $\|f\|_1 = 1$. Since, for each $n \in \mathbb{N}$, $\{|f| > \frac{1}{n}\}$ has finite measure and the set $\{|g| > \|g\|_*\}$ is locally null, the intersection of these two sets are null. Therefore $|f(x)g(x)| \leq |f(x)|\|g\|_*$ a.e.. Hence $\sup\{|\int fg| : \|f\|_1 = 1\} \leq \|g\|_*$.

Given $\epsilon > 0$, since $\{|g| > \|g\|_* - \epsilon\}$ is not locally null, there is a measurable set A with finite measure such that $B = A \cap \{|g| > \|g\|_* - \epsilon\}$ has nonzero finite measure. Consider $h = \overline{\mathrm{sgn}(g)}\chi_B(\mu(B))^{-1}$, then $\|h\|_1 = 1$ and

$$\sup\{|\int fg| : \|f\|_1 = 1\} \geq |\int hg| = (\mu(B))^{-1}\int_B |g| > \|g\|_* - \epsilon.$$

Therefore $\sup\{|\int fg| : \|f\|_1 = 1\} \geq \|g\|_*$.

(ii) Now we consider the counterpart of Theorem 6.14: Given $\epsilon > 0$. If $\{|g| > M_\infty(g) + \epsilon\}$ is not locally null, then there is a set A with finite measure, such that $B = A \cap \{|g| > M_\infty(g) + \epsilon\}$ has nonzero finite measure. But for $h = \overline{\mathrm{sgn}(g)}\chi_B(\mu(B))^{-1}$, $\|h\|_1 = 1$ and hence

$$M_\infty(g) \geq |\int hg| = (\mu(B))^{-1}\int_B |g| > M_\infty(g) + \epsilon,$$

which is a contradiction. So $\{|g| > M_\infty(g) + \epsilon\}$ is locally null for each $\epsilon > 0$. Therefore $\|g\|_* \leq M_\infty(g) < \infty$. Now by the first paragraph in (i), $M_\infty(g) \leq \|g\|_*$. □

25. *Proof.* If μ is finite, then the proof is essentially the same as the original proof except the role of Theorem 6.14 is replaced by Exercise 24.

Now for decomposable μ, let $\{F_a\}_{a\in\mathscr{A}}$ be the decomposition of μ. Let $g_a \in L^\infty(F_a)$ be so $\phi_{\restriction_{L^1(F_a)}} = \phi_{g_a}$ and define $g = g_a$ on each F_a. Then $g \in \mathscr{L}^\infty$ since for each $\alpha \in \mathbb{R}$, $g^{-1}((\alpha,\infty)) = \cup_a g_a^{-1}(\alpha,\infty))$ is measurable by property (iv) in the definition of decomposable measur and since $\|g\|_* = \sup_a \|g_a\|_* \le \sup_a \|\phi_{\restriction_{L^1(F_a)}}\| \le \|\phi\|$.

Since for each $f \in L^1$, the set $E_n = \{|f| > \frac{1}{n}\}$ has finite measure and $\mu(E_n) = \sum_a \mu(E_n \cap F_a)$. Therefore for each $k \in \mathbb{N}$ only finite many index a such that $\mu(E_n \cap F_a) > \frac{1}{k}$, and only countably many indices a such that $\mu(E_n \cap F_a) > 0$. Furthermore, only countably many indices a such that $\mu(\{|f| > 0\} \cap F_a) > 0$. We call this index set I and thus,

$$\phi(f) = \sum_{i\in I} \phi(f\chi_{F_i}) = \sum_{i\in I} \int_{F_i} f g_i = \int fg,$$

where we apply LDCT in the last equality. □

Remark 18. The definition of decomposable measure we used here should be precisely called **almost decomposable** due to (iii) $\mu(E) = \sum \mu(E\cap F)$ is required to be true for finite measure set E only. (cf: Bogachev [2, 1.12.131-132] and Fremlin [8, Vol. 2].)

A subtle difference is the following facts:

μ is decomposable $\Rightarrow$ μ is semifinite $\not\Leftarrow$ μ is almost decomposable.

($\Rightarrow$) is straightforward. The counterexample for ($\not\Leftarrow$) is recorded in additional exercise 6.2.6.

From Bogachev [2, 1.12.134], we know μ is semifinite $\not\Rightarrow$ μ is decomposable.

Example 19. I learned these examples from Professor Nico Spronk's Notes

(i) The first is an application of Exercise 25 which is not a consequence of Theorem 6.15.

Let I be an uncountable set, and $\mathscr{M}$ be the σ-algebra of sets of countable or co-countable. μ is the counting measure. Then the collection $\tilde{\mathscr{M}}$ of all locally measurable sets is $P(I)$, the power set of I and we can define the saturation $\tilde{\mu}$ of μ on $\tilde{\mathscr{M}}$ (cf: Exercise 1.16).

Note that $L^1(I, \mathscr{M}, \mu) = L^1(I, \tilde{\mathscr{M}}, \tilde{\mu})$ and $\{\{i\}\}_{i\in I}$ is a decomposition of for $\tilde{\mu}$. The theorem we just proved implies that $(L^1)^* \cong \mathscr{L}^\infty$.

(ii) The second example sits outside Theorem 6.15 and Exercise 25!

Let $X = \{f, i\}$ and consider $\mu : P(X) \to [0,\infty]$ be the unique measure satisfying $\mu(\{f\}) = 1$ and $\mu(\{i\}) = \infty$. Note that $\{i\}$ is locally null. Then $\mathscr{L}^\infty$ is the span of $\chi_{\{f\}}$. On the other hand L^1 is also the span of $\chi_{\{f\}}$. Then by definition, $(L^1)^* \cong \mathbb{R} \cong \mathscr{L}^\infty$.

But the existence of infinite atom $\{i\}$ means that we cannot partition X into subsets of finite measure, that is, μ is neither semifinite nor decomposable!

Remark 20. Further results related to Exercise 23-25 may also be confirmed with Cohn [5, Chapter 3].

6.3 Some Useful Inequalities

26. **Complete the proof of Theorem 6.18 for cases $p = 1$ and $p = \infty$.**

Proof. The case $p = 1$ is a consequence of Fubini-Tonelli's Theorem. The case $p = \infty$ is apparent. □

27. **(Hilbert's Inequality) The operator $Tf(x) = \int_0^\infty (x+y)^{-1} f(y)\,dy$ satisfies $\|Tf\|_p \le C_p \|f\|_p$ for $1 < p < \infty$, where $C_p = \int_0^\infty x^{-\frac{1}{p}}(x+1)^{-1}\,dx$. (For those who know about contour integrals: Show that $C_p = \pi \csc(\frac{\pi}{p})$.)** (This is a special case of Schur's lemma, Theorem 6.20, our proof here is essential the same as the one given there.)

Proof. By Minkowski's inequality,

$$\Big(\int_0^\infty |\int_0^\infty \frac{f(y)}{x+y}\,dy|^p\,dx\Big)^{1/p} = \Big(\int_0^\infty |\int_0^\infty \frac{f(xz)}{1+z}\,dz|^p\,dx\Big)^{1/p} \le \int_0^\infty \Big(\int_0^\infty |f(xz)|^p\,dx\Big)^{1/p} \frac{1}{1+z}\,dz$$
$$= \|f\|_p \int_0^\infty z^{-1/p}(1+z)^{-1}\,dz = \|f\|_p \pi \csc(\pi/p).$$

To compute the last integral, one use the residue theorem in one complex variable. □

28. This exercise extends Corollary 6.20 with $T = J_1$. Further generalizations are given in the exercise 29 and 44.

Proof. For $p = \infty$, it is straightforward. For $1 < p < \infty$, using change of variables and Minkowski's inequality, we have

$$\Big(\int_0^\infty \Big|\frac{1}{x^\alpha}\int_0^x f(t)(x-t)^{\alpha-1}\,dt\Big|^p\,dx\Big)^{1/p} \le \Big(\int_0^\infty \Big(\int_0^1 |f(xw)|(1-w)^{\alpha-1}\,dw\Big)^p\,dx\Big)^{1/p}$$
$$\le \int_0^1 \Big(\int_0^\infty |f(xw)|^p(1-w)^{\alpha p-p}\,dx\Big)^{1/p}\,dw = \|f\|_p \int_0^1 (1-w)^{\alpha-1}w^{-1/p}\,dw = \|f\|_p \frac{\Gamma(\alpha)\Gamma(1-p^{-1})}{\Gamma(\alpha+1-p^{-1})}.$$

A counterexample for $p = 1$ is $\chi_{(0,1)}(t)$. Note that

$$\int_0^\infty x^{-\alpha}\int_0^x (x-t)^{\alpha-1}\chi_{(0,1)}(t)\,dt\,dx$$
$$= \int_1^\infty x^{-\alpha}\int_0^1 (x-t)^{\alpha-1}\,dt\,dx + \int_0^1 x^{-\alpha}\int_0^x (x-t)^{\alpha-1}\,dt\,dx$$
$$= \int_1^\infty x^{-\alpha}(x^*)^{\alpha-1}\,dx + \alpha^{-1},$$

where $x^* \in (x-1, x)$ for all $x \in (1, \infty)$ by mean value theorem. We consider $\alpha > 1$ and $0 < \alpha \leq 1$ separately. For $\alpha > 1$, the integrand is larger than $x^{-\alpha}(x-1)^{\alpha-1} = (1-\frac{1}{x})^\alpha (x-1)^{-1}$ which is not integrable on $(2, \infty)$. For $\alpha \leq 1$, the integrand is larger than $x^{-\alpha}x^{\alpha-1} = x^{-1}$ which is not integrable on $(1, \infty)$. So $J_\alpha(\chi_{(0,1)}(t)) \notin L^1(0, \infty)$. □

Remark 21. Is the constant $\frac{\Gamma(1-p^{-1})}{\Gamma(\alpha+1-p^{-1})}$ sharpest ? When does the equality hold? See Rudin [15, Exercise 3.14] for the Hardy's inequality, that is, the case $\alpha = 1$.

Remark 22. A similar operator to fractional integral operator is Riesz and Bessel potential operators, which is strongly related to Sobolev embedding theorem. See Stein [17, Chapter V] and Ziemer [20, Chapter 2]. Also see exercise 44-45.

29. (Hardy's inequality)

Proof. Again, we prove this without using Theorem 6.20. By Minkowski's inequality,

$$\begin{aligned}
&\Big(\int_0^\infty x^{-r-1}\Big[\int_0^x h(y)\,dy\Big]^p dx\Big)^{1/p} = \Big(\int_0^\infty x^{p-r-1}\Big[\int_0^1 h(xz)\,dz\Big]^p dx\Big)^{1/p} \\
&\leq \int_0^1 \Big(\int_0^\infty x^{p-r-1}(h(xz))^p\,dx\Big)^{1/p} dz = \int_0^1 \Big(\int_0^\infty y^{p-r-1}h(y)^p\,dy\Big)^{1/p} z^{\frac{r}{p}-1}\,dz \\
&= \frac{p}{r}\Big(\int_0^\infty y^{p-r-1}h(y)^p\,dy\Big)^{1/p}.
\end{aligned}$$

$$\begin{aligned}
&\Big(\int_0^\infty x^{r-1}\Big[\int_x^\infty h(y)\,dy\Big]^p dx\Big)^{1/p} = \Big(\int_0^\infty x^{p+r-1}\Big[\int_1^\infty h(xz)\,dz\Big]^p dx\Big)^{1/p} \\
&\leq \int_1^\infty \Big(\int_0^\infty x^{p+r-1}(h(xz))^p\,dx\Big)^{1/p} dz = \int_1^\infty \Big(\int_0^\infty y^{p+r-1}h(y)^p\,dy\Big)^{1/p} z^{-\frac{r}{p}-1}\,dz \\
&= \frac{p}{r}\Big(\int_0^\infty y^{p+r-1}h(y)^p\,dy\Big)^{1/p}.
\end{aligned}$$

□

30. *Proof.* (a)Since all terms in the integrand is nonnegative, by Fubini-Tonelli, we can interchange the order of integrations freely.

$$\begin{aligned}
&\int\int K(xy)f(x)g(y)\,dx\,dy = \int\int K(z)f(\frac{z}{y})g(y)y^{-1}\,dz\,dy = \int\int f(\frac{z}{y})g(y)y^{-1}\,dy\ K(z)\,dz \\
&\leq \int(\int[f(\frac{z}{y})]^p y^{-p}\,dy)^{1/p}(\int g(y)^q\,dy)^{1/q}K(z)\,dz \\
&= \int(\int[f(w)]^p(w/z)^p z/w^2\,dw)^{1/p}K(z)\,dz(\int g(y)^q\,dy)^{1/q} \\
&= \int z^{-1+\frac{1}{p}}K(z)\,dz(\int[f(w)]^p w^{p-2}\,dw)^{1/p}(\int g(y)^q\,dy)^{1/q}.
\end{aligned}$$

(b) is proved by (a) with $p = q = 2$ and $g(y) = \int k(zy)f(z)\,dz$. □

31. *Proof.* (Sketch) Mathematical induction and apply the original Hölder's inequality to $\|\cdot\|_r^r$. $\square$

32. (Hilbert-Schmidt integral operator.)

Proof. By Tonelli-Fubini's theorem and Hölder's inequality, we know for μ-a.e. x, the integral

$$\int_Y |K(x,y)||f(y)|\,d\nu(y) \le \Big(\int_Y |K(x,y)|^2\,d\nu(y)\Big)^{1/2}\|f\|_{L^2(\nu)} < \infty.$$

Then $Tf(x)$ is well-defined a.e. and $|Tf(x)| \le \Big(\int_Y |K(x,y)|^2\,d\nu(y)\Big)^{1/2}\|f\|_{L^2(\nu)}$. Integrating Tf with respect to $d\mu(x)$, we see $\|Tf\|_{L^2}^2 \le \|K\|_{L^2}^2\|f\|_{L^2}^2$. $\square$

Theorem 23. *The above $T : L^2(\nu) \to L^2(\mu)$ is compact provided these L^2 space are separable.*

Proof. Let $\{\phi_i(x)\}$ and $\{\varphi_j(y)\}$ be the orthonormal basis of $L^2(\mu)$ and $L^2(\nu)$ respectively. By Fubini's theorem and standard characterizations of complete set in Hilbert space, $\{\phi_i(x)\varphi_j(y) =: \Phi_{ij}(x,y)\}_{i,j}$ is an orthonormal basis for $L^2(\mu\times\nu)$. Hence we can write $K = \sum a_{ij}\Phi_{ij}$. For $N = 1,2,\cdots$, let

$$K_N(x,y) = \sum_{i+j\le N} a_{ij}\Phi_{ij}$$

and T_{K_N} is the corresponding operator with kernel K_N. Then T_{K_N} has finite rank. On the other hand, $\|K - K_N\|_{L^2} \to 0$ as $N\to\infty$. So by the previous exercise $\|T - T_{K_N}\| \to 0$ and hence T is compact. (This standard result is proved by Cantor diagonal process.) $\square$

33. *Proof.* Since $x^{-1/p}$ and $\int_0^x f(t)\,dt$ are continuous, $Tf(x)$ is continuous on $(0,\infty)$. Given $\epsilon > 0$. Since for any $x \in (0,\infty)$

$$|Tf(x) - T(f\chi_{(0,n)})(x)| \le x^{-1/p}\int_0^x |f(t) - f\chi_{(0,n)}(t)|\,dt \le \|f - f\chi_{(0,n)}\|_q,$$

we can choose a large N such that $\|Tf - T(f\chi_{(0,N)})\|_\infty < \epsilon$. Since for $x > N$

$$|T(f\chi_{(0,N)})(x)| \le x^{-1/p}\int_0^N |f(t)|\,dt,$$

the left hand side is less than ϵ if x is sufficient large. Combine these inequalities and since ϵ is arbitrary, $Tf \in C_0(0,\infty)$. The boundedness of T is a consequence of Hölder's inequality. $\square$

34. *Proof.* Since f is absolutely continuous on any $[\epsilon, 1]$, we know for any $[x,y] \subset (0,1)$,

$$|f(x) - f(y)| = |\int_x^y f'(s)ds| \le \Big(\int_x^y s^{\frac{-1}{p}\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\int_x^y s|f'(s)|^p\Big)^{1/p}$$

(i) If $p > 2$, then by the above calculation, for any sequence $x_n \to 0, \{f(x_n)\}$ forms a Cauchy sequence. Therefore $\lim_{x\to 0} f(x)$ exists.

(ii) If $p=2$, given $\epsilon>0$ and pick $y=y(\epsilon)\in(0,\frac{1}{2})$ such that $\int_0^y t|f'(t)|^2\,dt<\epsilon^2$, then for any $x\in(0,y)$,

$$0\leq\frac{|f(x)|}{|\log x|^{1/2}}=\frac{|f(y)-\int_x^y f'(t)dt|}{|\log x|^{1/2}}\leq\frac{|f(y)|}{|\log x|^{1/2}}+\frac{|\log y-\log x|^{1/2}\epsilon}{|\log x|^{1/2}}.$$

Hence there exist $\delta=\delta(\epsilon)\in(0,y)$ such that for any $x\in(0,\delta)$, the last term is less than 2ϵ.

(iii) If $p\in(1,2)$, given $\epsilon>0$ and pick $y=y(\epsilon)\in(0,\frac{1}{2})$ such that $\int_0^y t|f'(t)|^p\,dt<\epsilon^p$, then for any $x\in(0,y)$,

$$0\leq\frac{|f(x)|}{x^{1-(2/p)}}=\frac{|f(y)-\int_x^y f'(t)dt|}{x^{1-(2/p)}}\leq\frac{|f(y)|}{x^{1-(2/p)}}+C_p\frac{|y^{\frac{p-2}{p-1}}-x^{\frac{p-2}{p-1}}|^{\frac{p-1}{p}}\epsilon}{x^{1-(2/p)}}.$$

Hence there exist $\delta=\delta(\epsilon)\in(0,y)$ such that for any $x\in(0,\delta)$, the last term is less than 2ϵ.

(iv)If $p=1$, given $\epsilon>0$ and pick $y=y(\epsilon)\in(0,\frac{1}{2})$ such that $\int_0^y t|f'(t)|\,dt<\epsilon$, then for any $x\in(0,y)$,

$$0\leq\frac{|f(x)|}{x^{-1}}=\frac{|f(y)-\int_x^y f'(t)dt|}{x^{-1}}\leq\frac{|f(y)|}{x^{-1}}+\frac{x^{-1}\epsilon}{x^{-1}}.$$

Hence there exist $\delta=\delta(\epsilon)\in(0,y)$ such that for any $x\in(0,\delta)$, the last term is less than 2ϵ.

From(ii)-(iv), we see the second and third assertions hold. □

6.4 Distribution Functions and Weak L^p

35. **For any measurable f and g we have $[cf]_p=|c|[f]_p$ and $[f+g]_p\leq 2([f]_p^p+[g]_p^p)^{\frac{1}{p}}$; hence weak L^p is a vector space. Moreover, the "balls" $\{g:[g-f]_p<r\}$ ($r>0$, $f\in$ weak L^p) generate a topology on weak L^p that makes weak L^p into a topological vector space.**

Proof. The first assertion follows from definition; the second assertion is through the fact

$$\{|f+g|>\lambda\}\subset\{|f|+|g|>\lambda\}\subset\{|f|>\frac{\lambda}{2}\}\cup\{|g|>\frac{\lambda}{2}\}.$$

The addition (x,y) $\mapsto x+y$ is continuous under the given topology by definition and the second assertion.

The scalar multiplication $(a,x)\mapsto ax$ is continuous by definition and the above two assertions. □

Remark 24. Normability of weak L^p, see Grafakos [9, p.14-15] and additional exercise 6.4.1.

36. **If $f \in$ weak L^p and $\mu(\{x : f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in$(weak $L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.**

Proof. (i) By Proposition 6.24 and assumption, we see $q - p < 0$ and

$$\frac{1}{q}\int |f|^q \, d\mu = \int_0^1 + \int_1^\infty \alpha^{q-1}\mu(\{x : |f| > \alpha\}) \, d\alpha \leq M \int_0^1 \alpha^{q-1} \, d\alpha + C \int_1^\infty \alpha^{q-1-p} \, d\alpha < \infty,$$

for some positive constants M, C.

(ii) Similarly, we see $q - p > 0$ and

$$\frac{1}{q}\int |f|^q \, d\mu = \int_0^{\|f\|_\infty} + \int_{\|f\|_\infty}^\infty \alpha^{q-1}\mu(\{x : |f| > \alpha\}) \, d\alpha \leq C \int_0^{\|f\|_\infty} \alpha^{q-p-1} \, d\alpha < \infty,$$

for some positive constant C. □

37. Draw a picture to prove Proposition 6.25 !!

38. **Suppose $(X, \mathscr{M}, \mu)$ be a measure space, $f \geq 0$, measurable and finite a.e. Let $p \in (0, \infty)$, $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. Prove that**

$$f \in L^p(\mu) \iff \sum_{k=-\infty}^{\infty} 2^{kp}\mu(F_k) < \infty \iff \sum_{k=-\infty}^{\infty} 2^{kp}\mu(E_{2^k}) < \infty.$$

Proof. The first $\iff$ and second $\Longleftarrow$ are trivial. The second $\Longrightarrow$ is by applying Tonelli's theorem (Fubini) to counting measure on $\mathbb{N}$, which is σ-finite: (cf: Exercise 2.46)

$$\sum_{k=-\infty}^{\infty} 2^{kp}\mu(E_{2^k}) = \sum_{k=-\infty}^{\infty} 2^{kp} \sum_{i=k}^{\infty} \mu(F_i) = \sum_{i=-\infty}^{\infty} \mu(F_i) \sum_{k=-\infty}^{i} 2^{kp} = \sum_{i=-\infty}^{\infty} \mu(F_i) 2^{ip} 2^p < \infty.$$

□

Remark 25. We can prove the second $\Longrightarrow$ without using Fubini-Tonelli as follows:

Accoridng to Proposition 6.24, it suffices to show $\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\, d\alpha$ is finite. This is true since

$$\begin{aligned} 2^{k(p-1)+k}\lambda_f(2^{k+1}) = \int_{2^k}^{2^{k+1}} 2^{k(p-1)}\lambda_f(2^{k+1})\, d\alpha &\leq \int_{2^k}^{2^{k+1}} \alpha^{p-1}\lambda_f(\alpha)\, d\alpha \\ &\leq \int_{2^k}^{2^{k+1}} 2^{(k+1)(p-1)}\lambda_f(2^k)\, d\alpha = 2^{(k+1)(p-1)+k}\lambda_f(2^k). \end{aligned}$$

The desired result follows by summing k from $-\infty$ to ∞.

39. **If $f \in L^p$, then $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = \lim_{\alpha \to \infty} \alpha^p \lambda_f(\alpha) = 0$.**

Proof. Since $f \in L^p$, $0 \le \alpha^p \lambda_f(\alpha) \le \int_{\{|f|>\alpha\}} |f|^p \to 0$ as $\alpha \to \infty$. On the other hand, for each $\epsilon > 0$, there is $\delta > 0$ such that $\int_{\{|f|\le\delta\}} |f|^p < \epsilon$. Thus $\alpha^p(\lambda_f(\alpha) - \lambda_f(\delta)) \le \int_{\{\alpha<|f|\le\delta\}} |f|^p < \epsilon$ for all $0 < \alpha < \delta$. We are done by taking $\limsup_{\alpha\to 0}$ on both sides. $\square$

Remark 26. From the above proof, we see that, instead of assuming $f \in L^p$, we can assume either $\int_0^\infty \alpha^p \, d\lambda_f(\alpha)$ or $\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\, d\alpha$ is fintie, although according to Proposition 6.23 and 6.24, they are equivalent.

40. **If f is a measurable function on X, its decreasing rearrangement is the function $f^*(0,\infty) \to [0,\infty]$ defined by**

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \le t\} \;\; (\textbf{where} \;\; \inf = \infty).$$

(a) f^* is decreasing. If $f^*(t) < \infty$ then $\lambda_f(f^*(t)) \le t$, and if $\lambda_f(\alpha) < \infty$ then $f^*(\lambda_f(\alpha)) \le \alpha$.

(b) $\lambda_f = \lambda_{f^*}$ where λ_{f^*} is defined with respect to Lebesgue measure on $(0,\infty)$.

(c) If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \lambda_f(\alpha) = 0$ (so that $f^*(t) < \infty$ for all $t > 0$), and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then $\int_X \phi \circ |f| \, d\mu = \int_0^\infty \phi \circ f^*(t)\, dt$. In particular, $\|f\|_p = \|f^*\|_p$ for $0 < p < \infty$.

(d) If $0 < p < \infty, [f]_p = \sup_{t>0} t^{\frac{1}{p}} f^*(t)$.

(e) The name "rearrangement" for f^* comes from the case where f is a nonnegative function on $(0,\infty)$ assuming four or five different values and draw the graphs of f and f^*.

Proof. (a) It's easy to see f^* is nonincreasing. If $f^*(t) < \infty$, then $\{\alpha : \lambda_f(\alpha) \le t\} \ne \emptyset$ and hence by definition for each n, there is $\epsilon_n \in (0, \frac{1}{n})$ such that $\lambda_f(f^*(t) + \epsilon_n) \le t$. Therefore

$$\lambda_f(f^*(t)) = \lim_{n\to\infty} \lambda_f(f^*(t) + \epsilon_n) \le t.$$

If $\lambda_f(\alpha) < \infty$, then for each $\beta \ge \alpha$, $\lambda_f(\beta) \le \lambda_f(\alpha)$ and hence $f^*(\lambda_f(\alpha)) \le \alpha$.

(b) For $t, y \in (0,\infty)$. Since $f^*(t) > y$ iff $\lambda_f(y) > t$, $\{t : f^*(t) > y\} = (0, \lambda_f(y))$. Therefore, $\lambda_{f^*}(y) = \lambda_f(y)$ for all $y \in (0,\infty)$.

(c) Apply Proposition 6.23.

(d) Given $t > 0$, If $f^*(t) = 0$, then $t^{1/p} f^*(t) \le [f]_p$. If $f^*(t) > 0$ then by definition, for all $0 < \epsilon < f^*(t)$, we have

$$t(f^*(t) - \epsilon)^p < \lambda_f(f^*(t) - \epsilon)(f^*(t) - \epsilon)^p \le [f]_p^p.$$

Taking p-th root and letting $\epsilon \to 0$, we have $t^{1/p} f^*(t) \le [f]_p$. Since $t > 0$ is arbitrary,

$$\sup_{t>0} t^{1/p} f^*(t) \le [f]_p.$$

On the other hand, given $\alpha > 0$ and assume $\lambda_f(\alpha) > 0$ without loss of generality. Since for any $0 < \epsilon < \lambda_f(\alpha)$, we have $f^*(\lambda_f(\alpha) - \epsilon) > \alpha$. Therefore

$$\sup_{t>0} t^{1/p} f^*(t) \ge (\lambda_f(\alpha) - \epsilon)^{1/p} f^*(\lambda_f(\alpha) - \epsilon) > \alpha(\lambda_f(\alpha) - \epsilon)^{1/p}.$$

Letting $\epsilon \to 0$, we have $\sup_{t>0} t^{1/p} f^*(t) \ge [f]_p$. □

Remark 27. See Lieb-Loss [11, Chapter 3-4] and Dibenedetto [6, Chapte 9] for further discussions and applications on higher-dimensional rearrangement and Riesz's rearrangement inequality.

6.5 Interpolation of L^p Spaces

41. **Suppose $1 < p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If T is a bounded linear operator on L^p such that $\int (Tf)g = \int f(Tg)$ for all $f, g \in L^p \cap L^q$, then T extends uniquely to a bounded linear operator on L^r for all $r \in [p, q]$ (if $p < q$) or $[q, p]$ if $q < p$).** We should assume μ is semifinite when $p = \infty$, see the errata sheet provided by Prof. Folland.

Proof. (i) If $q < p < \infty$ or $p < q < \infty$, then we define T on $L^p \cap L^q$ the same as the original bounded linear map on L^p. Given $f \in L^p \cap L^q$ and $g \in L^p \cap L^q$, we have, by Hölder's inequality,

$$|\int (Tf)g| = |\int f(Tg)| \le \|f\|_q \|T\|_{p\to p} \|g\|_p, \tag{1}$$

where $\|T\|_{p\to p}$ is the operator norm of T on L^p. Since $L^p \cap L^q$ is dense in L^p, for any $g \in L^p$, we choose $g_n \in L^p \cap L^q$ such that $g_n \to g$ in L^p, then the sequence $\{\int (Tf)g_n\}$ is Cauchy in $\mathbb{C}$ and we define

$$\int (Tf)g = \lim_{n\to\infty} \int (Tf)g_n.$$

This definition is independent of the choice of g_n since the union of two approximation sequences is still an approximation sequence and the corresponding sequence is still Cauchy in $\mathbb{C}$. Therefore we know (1) holds for any $g \in L^p$ with $\|g\|_p = 1$, and hence

$$\|Tf\|_q \le \|f\|_q \|T\|_{p\to p}$$

for any $f \in L^p \cap L^q$. Since $L^p \cap L^q$ is dense in L^q, this inequality holds for any $f \in L^q$. So T is bounded on L^q. The desired boundedness of T on L^r follows from Riesz-Thorin theorem.

(ii) If $p = \infty$, then $q = 1$. First, we show T maps $L^\infty \cap L^1 \to L^1$ continuously by Theorem 6.14: Let Σ be the set of simple functions that vanish outside a set of finite measure. Given $f \in L^\infty \cap L^1$ and $g \in \Sigma \subset L^\infty \cap L^1$, then $fg \in L^1$ and we have, by Hölder's inequality, for any $\|g\|_\infty = 1$

$$|\int (Tf)g| = |\int f(Tg)| \leq \|f\|_1 \|T\|, \tag{2}$$

where $\|T\|$ is the operator norm of T on L^∞. Therefore

$$M_1(Tf) := \sup\{|\int (Tf)g| : \|g\|_\infty = 1\} \leq \|f\|_1 \|T\|.$$

By Theorem 6.14 and Exercise 17, we have $Tf \in L^1$ and $\|Tf\| \leq \|T\| \|f\|_1$. Since $L^1 \cap L^\infty$ is dense in L^1, we can extend T to L^1 continuously with norm less than $\|T\|$. The desired boundedness of T on L^r follows from Riesz-Thorin again.

The uniqueness part is due to the fact $L^p \cap L^q$ is dense in L^r which is proved by considering cut-off in domain and range simultaneous. $\square$

***Remark* 28.** Here I emphasis that case (i) can also be proved by the method given in (ii), but the density argument in (i) may not be suitable for case (ii).

42. **Prove the Marcinkiewicz theorem in the case $p_0 = p_1$.**

Proof.

$$\begin{aligned}
\int_X |Tf|^q &= q \int_0^\infty s^{q-1} \mu(\{|Tf| > s\})\, ds \\
&= q \int_0^{\|f\|_p} s^{q-1} \mu(\{|Tf| > s\})\, ds + q \int_{\|f\|_p}^\infty s^{q-1} \mu(\{|Tf| > s\})\, ds \\
&\leq q \int_0^{\|f\|_p} s^{q-1} (\frac{C_0 \|f\|_p}{s})^{q_0}\, ds + q \int_{\|f\|_p}^\infty s^{q-1} (\frac{C_1 \|f\|_p}{s})^{q_1}\, ds \\
&= q \Big(\frac{C_0^{q_0}}{q - q_0} + \frac{C_1^{q_1}}{q_1 - q} \Big) \|f\|_p^q
\end{aligned}$$

$\square$

43. **Let H be the Hardy-Littlewood maximal operator on $\mathbb{R}$. Compute $H\chi_{(0,1)}$ explicitly. Show that it is in L^p for all $p > 1$ and in weak L^1 but not in L^1, and that its L^p norm tends to ∞ like $\frac{1}{p-1}$ as $p \to 1^+$, although $\|\chi_{(0,1)}\|_p = 1$ for all p.**

Proof. By direct computations,

$$H\chi_{(0,1)}(x) = \begin{cases} \frac{1}{2(1-x)} & \text{for } x \le 0, \\ 1 & \text{for } x \in (0,1), \\ \frac{1}{2x} & \text{for } x \ge 1. \end{cases} \tag{3}$$

For $t > 0$,

$$m(\{H\chi_{(0,1)}(x) > t\}) = \begin{cases} 0 & \text{for } t > 1, \\ 1 & \text{for } t \in [\frac{1}{2}, 1], \\ \frac{1}{t} - 1 & \text{for } t \in (0, \frac{1}{2}), \end{cases} \tag{4}$$

which is easy to see the weak (1,1) estimate with constant 1 and strong (p,p) estimate with constant $\frac{1}{(p-1)2^{p-1}} + 1$. □

44. **Let I_α be the fractional integration operator of Exercise 61 in §2.6. If $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, and $\frac{1}{r} = \frac{1}{p} - \alpha$, then I_α is weak type $(1, \frac{1}{1-\alpha})$ and strong type (p, r) with respect to Lebesgue measure on $(0, \infty)$.** Also see the last paragraph in section 6.6.

Proof. This exercise is essentially the same as the next one. We change our α in this problem to β and put the following parameters in the next exercise: $n = 1$, $\alpha = 1 - \beta$, $f = g\chi_{(0,\infty)}$ and $K(x) := |x|^{-\alpha}\chi_{(0,\infty)}$. Using the results there, we have the desired weak type estimate. □

45. **If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by**

$$T_\alpha f(x) = \int |x-y|^{-\alpha} f(y)\, dy.$$

Then T_α os weak type $(1, \frac{n}{\alpha})$ and strong type (p, r) with respect to Lebesgue measure on $\mathbb{R}^n$, where $1 < p < \frac{n}{n-\alpha}$ and $\frac{1}{r} = \frac{1}{p} - \frac{n-\alpha}{n}$.

Proof. Given $f \in L^p$, we may assume $\|f\|_p = 1$.

Fix $\mu > 0$ to be determined later, we decompose

$$\begin{aligned} K(x) := |x|^{-\alpha} &= K(x)\chi_{\{|x| \le \mu\}} + K(x)\chi_{\{|x| > \mu\}} \\ &=: K_1(x) + K_\infty(x). \end{aligned}$$

Then $T_\alpha f = K_1 * f + K_\infty * f$. The first term is an L^p function since it's the convolution of an L^1 function K_1 and L^p function f and similarly, the second term is an L^∞ function.

Since for each $\lambda > 0$

$$m(\{|K * f| > 2\lambda\}) \le m(\{|K_1 * f| > \lambda\}) + m(\{|K_\infty * f| > \lambda\})$$

Note that $m(\{|K_1 * f| > \lambda\}) \leq \frac{\|K_1*f\|_p^p}{\lambda^p} \leq (\frac{\|K_1\|_1^p}{\lambda})^p = c_1(\frac{\mu^{n-\alpha}}{\lambda})^p$ and $\|K_\infty\|_{p'} = c_2\mu^{-n/r}$. Choose μ such that $c_2\mu^{-n/r} = \lambda$ and therefore $\|K_\infty * f\|_\infty \leq \lambda$, that is, $m(\{|K_\infty * f| > \lambda\}) = 0$. Then

$$m(\{|K * f| > 2\lambda\}) \leq c_1(\frac{\mu^{n-\alpha}}{\lambda})^p = c_3\lambda^{-r}$$

The special case for $p = 1$ gives the weak $(1, n\alpha^{-1})$ estimate. The strong (p,r) estimate follows from Marcinkiewicz interpolation theorem. $\square$

Remark 29. There are examples for the non-strong type at end point $p = 1$ and non-embedding phenomena at $p = \frac{n}{n-\alpha}$ given in Stein [17, Section V.1]. For simplicity, we construct one-dimensional case only, but they can extend to higher-dimensional case easily.

(i) Take $f = \chi_{(\frac{-1}{2},\frac{1}{2})}$. Then $\|f\|_1 = 1$ and we see $(T_\alpha f)(x) \geq \left(|x| + \frac{1}{2}\right)^{-\alpha} \geq 0$ for all $|x| \geq 1$. Hence

$$\infty = \int_{|x|\geq 1} (2|x|)^{-1} \leq \int_{|x|\geq 1} \left(|x| + \frac{1}{2}\right)^{-\alpha\frac{1}{\alpha}} \leq \int \left|(T_\alpha f)(x)\right|^{\frac{1}{\alpha}}.$$

(ii) Let $f(x) = |x|^{\alpha-1}(\log 1/|x|)^{(\alpha-1)(1+\epsilon)}$ for $|x| \leq \frac{1}{2}$ and 0 for $|x| > \frac{1}{2}$, where $0 < \epsilon \ll 1$. Then $f \in L^{\frac{1}{1-\alpha}}(\mathbb{R})$. But

$$(T_\alpha f)(0) = \int_{|x|\leq\frac{1}{2}} |x|^{-1}(\log\frac{1}{|x|})^{(\alpha-1)(1+\epsilon)}\,dx = \infty,$$

as long as $(\alpha - 1)(1 + \epsilon) \geq -1$. Therefore $T_\alpha f \notin L^\infty$.

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