Partial Differential Equations, 2nd Edition, L.C.Evans

Chapter 6 Second-Order Elliptic Equations

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1. (a) Direct computation. (b) $c = \Delta\sqrt{a}$.

2. *Proof.* Define

$$B[u,v] = \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + cuv\,dx,\ \ u,v \in H_0^1(U).$$

Clearly,

$$|B[u,v]| \le \|a^{ij}\|_\infty \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + \|c\|_\infty \|u\|_{L^2(U)} \|v\|_{L^2(U)} \le \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)},$$

for some positive constant α. For $c(x) \ge -\mu$, where μ is a fixed constant to be assigned later. Using the uniform ellipticity, we have

$$\begin{aligned}\theta \int_U |Du|^2\,dx &\le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} \\ &= B[u,u] - \int_U cu^2\,dx \\ &\le B[u,u] + \mu \int_U u^2\,dx \\ &\le B[u,u] + C\mu \int_U |Du|^2\,dx,\end{aligned}$$

where the Poincaré inequality is applied in the last inequality. Now we set $\mu = \dfrac{\theta}{2C}$ and see that if $c(x) \ge \dfrac{\theta}{2C}$, then

$$\frac{\theta}{2} \int_U |Du|^2\,dx \le B[u,u].$$

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Therefore,

$$\begin{aligned}\|u\|^2_{H^1_0(U)} &= \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \\ &\le (1+C)\|Du\|^2_{L^2(U)} \\ &\le \frac{2}{\theta}(1+C)B[u,u].\end{aligned}$$

□

3. *Proof.* Define

$$B[u,v] = \int_U \sum_{i,j=1}^n u_{x_ix_j} v_{x_ix_j}\, dx,\ \ u,v \in H^2_0(U).$$

Use the integration by part (divergence theorem),

$$\int_U |D^2u|^2\, dx = \int_U \sum_{i,j=1}^n u_{x_ix_j} u_{x_ix_j}\, dx = \int_U \sum_{i,j=1} u_{x_ix_i} u_{x_jx_j}\, dx = \int_U |\Delta u|^2\, dx$$

for $u,v \in H^2_0(U)$. (Fix i, consider $\mathbf{F} = u_{x_i} Du_{x_i}$ and then use $\int_U \operatorname{div} \mathbf{F}\, dx = \int_{\partial U} \mathbf{F}\cdot\nu\, dS$.)

$$\begin{aligned}\|Du\|^2_{L^2(U)} &= -\int_U u\Delta u\, dx \\ &\le \varepsilon\|u\|^2_{L^2(U)} + \frac{1}{4\varepsilon}\|\Delta u\|^2_{L^2(U)} \\ &\le \varepsilon C\|Du\|^2_{L^2(U)} + \frac{1}{4\varepsilon}\|\Delta u\|^2_{L^2(U)}\end{aligned}$$

the Poincar'e inequality is used in the last inequality. We choose $\varepsilon = \dfrac{1}{2C}$ so that

$$\|Du\|^2_{L^2(U)} \le C\|\Delta u\|^2_{L^2(U)}.$$

Therefore,

$$\begin{aligned}\|u\|^2_{H^2_0(U)} &= \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} + \|D^2u\|^2_{L^2(U)} \\ &\le (1+C)\|Du\|^2_{L^2(U)} + \|\Delta u\|^2_{L^2(U)} \\ &\le (1+C+C^2)\|\Delta u\|^2_{L^2(U)} \\ &= (1+C+C^2)B[u,u].\end{aligned}$$

Thus,

$$\frac{1}{1+C+C^2}\|u\|^2_{H^2_0(U)} \le B[u,u],\ \ u \in H^2_0(U).$$

□

Remark 1. The validity of the definition of weak solution to Biharmonic equation with zero Dirichlet and zero Neumann boundary is based on the zero trace theorem for $W^{2,p}_0$ function, which is not proved in Evans' book.

4. *Proof.* " $\Rightarrow$ " take $v = 1$.

   " $\Leftarrow$ " Define

$$B[u,v] = \int_U Du \cdot Dv\, dx, \text{ for } u, v \in H^1(U).$$

Consider a subspace of $H^1(U)$

$$A = \{u \in H^1(U) \big| (u)_U = 0\}$$

Let $l(u) = \int_U u\, dx$, then l is a continuous linear functional on $H^1(U)$ (the boundness of domain U is used here.) Then $A = l^{-1}(\{0\})$ is closed in $H^1(U)$, so A is a Hilbert space with induced inner product. Note that

$$|B[u,v]| \leq \|u\|_A \|v\|_A.$$

and by Poincaré inequality,

$$\|u\|^2_{L^2(U)} = \int_U |u - (u)_U|^2\, dx \leq C \int_U |Du|^2\, dx.$$

Therefore

$$\|u\|_A^2 = \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \leq (1+C)\|Du\|^2_{L^2(U)} = (1+C)B[u,u].$$

By Lax-Milgram, for all $f \in L^2(U)$ (with $\int_U f\, dx = 0$), there exists a unique $u_f \in A$ such that

$$B[u_f, v] = (f, v)_{L^2(U)}, \ \forall v \in A.$$

Let $v \in H^1(U)$, then $v - (v)_U \in A$. Thus, we have

$$\begin{aligned}(f,v)_{L^2(U)} &= (f, (v)_U)_{L^2(U)} + (f, v - (v)_U)_{L^2(U)} \\ &= 0 + B[u_f, v - (v)_U], \ (\text{since} \int_U f\, dx = 0) \\ &= B[u_f, v] = \int_U Du_f \cdot Dv\, dx, \ \forall v \in H^1(U).\end{aligned}$$

Therefore, $u_f \in A \subset H^1(U)$ is a weak solution. In general, $u_f + C$ is always a weak solution for any constant C. □

5. *Motivation.* If $u \in C^\infty(\overline{U})$ solves the given boundary value problem, then for each $v \in C^\infty(\overline{U})$,

$$\int_U fv = \int_U (-\Delta u)v = \int_{\partial U} -v\frac{\partial u}{\partial n} + \int_U \nabla u \nabla v = \int_{\partial U} vu + \int_U \nabla u \nabla v.$$

By density theorem and trace theorem, we define B on $(H^1(U))^2$ by

$$B[u,v] = \int_{\partial U} (Tv)(Tu) + \int_U \nabla u \nabla v,$$

where T is the trace operator with finite operator norm $\|T\|$. We call $u \in H^1(U)$ is a weak solution of the given Robin boundary value problem if $B[u,v] = \int_U fv$ for all $v \in H^1(U)$. □

Proof. We prove uniqueness and existence by Lax-Milgram. Given $u, v \in H^1(U)$, it's easy to see $|B[u,v]| \le \max\{1, \|T\|^2\}\|u\|_{H^1}\|v\|_{H^1}$. Next, we prove B is strictly coercive by a contradiction argument similar to the one for Poincaré inequality.

Assume for each $k \in \mathbb{N}$, there exists $u_k \in H^1$ such that $\|u_k\|_{H^1}^2 > kB[u_k, u_k]$.
Let $v_k = u_k/\|u_k\|_{H^1}$, then $1 = \|v_k\|_{H^1}^2 > kB[v_k, v_k]$, so $\frac{1}{k} > B[v_k, v_k] \ge \|Dv_k\|_{L^2}^2$, $\frac{1}{k} > \|Tv_k\|_{L^2(\partial U)}^2$ and hence $\|v_k\|_{H^1}^2 < 1 + \frac{1}{k} \le 2$. Rellich's compactness theorem asserts there exists a subsequence, still denoted by $\{v_k\}$, converging to a function v in L^2 sense. Note $\|Dv_k\|_{L^2} \to 0$, so for every $\phi \in C_c^\infty(U)$,

$$\int vD\phi = \lim_{k\to\infty} \int v_k D\phi = \lim_{k\to\infty} \int -(Dv_k)\phi = 0$$

Hence $v \in H^1$, $Dv \equiv 0$. Now we know $v_k \to v$ in H^1 sense, so $\|v\|_{H^1} = 1$. Moreover, v is constant in each component of U. But

$$\|Tv\|_{L^2(\partial U)} \le \|T(v - v_k)\|_{L^2(\partial U)} + \|Tv_k\|_{L^2(\partial U)} \le \|T\|\|v - v_k\|_{H^1} + \sqrt{\frac{1}{k}} \to 0, \text{ as } k \to \infty,$$

which means $v \equiv 0$ on ∂U and then $v \equiv 0$ on U. This contradicts to $\|v\|_{H^1} = 1$. Therefore B is strictly coercive. $\square$

6. (This boundary value problem is called Zaremba problem sometimes)

Definition. Similar to Exercise 5, we define $B[u,v] = \int \nabla u \nabla v$ on $H := H^1 \cap \{u : Tu \lfloor_{\Gamma_1} = 0\}$. The linearity and continuity of T ensures that H is a closed subspace of H^1, so H is a Hilbert space with induced inner product. $u \in H$ is a weak solution of the given Robin boundary value problem if $B[u,v] = \int_U fv$ for all $v \in H$. (The zero Newmann boundary condition is involved.) $\square$

Uniqueness and Existence. They are established by Lax-Milgram again, the coercivity of B is established by almost the same argument as Exercise 5. $\square$

7. *Proof.* By definition, we know for each $v \in H^1(\mathbb{R}^n)$, $\int \nabla u \nabla v + c(u)v = \int fv$.
Take $v = -D_i^{-h}D_i^h u, 0 < |h| \ll 1$. Then $v \in H^1$ and has compact support. Hence

$$\int |D_i^h Du|^2 + D_i^h c(u) D_i^h u = -\int f(D_i^{-h}D_i^h u).$$

By mean value theorem,

$$D_i^h c(u)(x) D_i^h u(x) = \frac{c(u(x + he_i)) - c(u(x))}{h} D_i^h u(x) = c'(\cdots)|D_i^h u(x)|^2 \ge 0.$$

Hence

$$\int |D_i^h Du|^2 \le |\int f(D_i^{-h} D_i^h u)| \le \frac{1}{2}\int |f|^2 + \frac{1}{2}\int |D_i^{-h} D_i^h u|^2$$
$$\le \frac{1}{2}\int |f|^2 + \frac{1}{2}\int |DD_i^h u|^2$$

The last inequality follows easily from mean value theorem. Hence $\int |D_i^h Du|^2 \le \int |f|^2$, for all small enough h. This implies $\int |D^2 u|^2 \le \int |f|^2$. (Evans, Page 292) $\square$

8. *Proof.* Differentiate the equation $Lu = 0$, we get $(Da^{ij})u_{x_i x_j} + a^{ij}(Du)_{x_i x_j} = 0$. Multiply this equation by $2Du$, we have $2Du \cdot (Da^{ij})u_{x_i x_j} = -2a^{ij} Du \cdot (Du)_{x_i x_j}$. Adding $-2a^{ij}(Du)_{x_i}(Du)_{x_j}$ to both sides, we find

$$2Du \cdot (Da^{ij})u_{x_i x_j} - 2a^{ij}(Du)_{x_i}(Du)_{x_j} = -a^{ij}(|Du|^2)_{x_i x_j} \tag{1}$$

On the other hand, multiply the origin equation by $2u$ and add $2a^{ij}u_{x_i}u_{x_j}$ to both sides, we have

$$2a^{ij}u_{x_i}u_{x_j} = 2a^{ij}u_{x_i}u_{x_j} + 2a^{ij}uu_{x_i x_j} = a^{ij}(u^2)_{x_i x_j} \tag{2}$$

$(1) - \lambda(2)$ implies

$$-2\lambda a^{ij}u_{x_i}u_{x_j} + 2Du \cdot (Da^{ij})u_{x_i x_j} - 2a^{ij}(Du)_{x_i}(Du)_{x_j} = -a^{ij}(\lambda u^2 + |Du|^2)_{x_i x_j}$$

By uniform ellipticity, boundedness of Da^{ij} and Young's inequality, the left hand side is non-positive for large λ. The second assertion is a consequence of weak maximum principle to $L(|Du|^2 + \lambda u^2) \le 0$

$\square$

9. *Proof.* By weak maximum principle, $\min_{\overline{U}} w = \min_{\partial U} w = w(x^0)$. Let $v_1 = u + \|f\|_\infty w$ and $v_2 = u - \|f\|_\infty w$, then $Lv_1 \ge 0, Lv_2 \le 0$.

By weak maximum principle, $\min_{\overline{U}} v_1 = \min_{\partial U} v_1 = \|f\|_\infty \min_{\partial U} w = \|f\|_\infty w(x^0) = v_1(x^0)$ and $\max_{\overline{U}} v_2 = \max_{\partial U} v_2 = -\|f\|_\infty \min_{\partial U} w = -\|f\|_\infty w(x^0) = v_2(x^0)$. Therefore,

$$0 \ge \frac{\partial v_1}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \|f\|_\infty \frac{\partial w}{\partial \nu}(x^0),$$
$$0 \le \frac{\partial v_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_\infty \frac{\partial w}{\partial \nu}(x^0).$$

Since $u = 0$ on ∂U, $\nabla u /\!/ \nu$. So

$$|\nabla u(x^0)| = |\frac{\partial u}{\partial \nu}(x^0)| \le \|f\|_\infty |\frac{\partial w}{\partial \nu}(x^0)|.$$

$\square$

10. *Proof.* We omit (a) since is standard. For (b), if u attains an interior maximum, then the conclusion follows from strong maximum principle.

If not, then for some $x^0 \in \partial U, u(x^0) > u(x)\ \forall x \in U$. Then Hopf's lemma implies $\frac{\partial u}{\partial \nu}(x^0) > 0$, which is a contradiction. $\square$

Remark 2. *A generalization of this problem to mixed boundary conditions is recorded in Gilbarg-Trudinger, Elliptic PDEs of second order, Problem 3.1.*

11. *Proof.* Define

$$B[u, v] = \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j}\, dx \text{ for } u \in H^1(U), v \in H_0^1(U).$$

By Exercise 5.17, $\phi(u) \in H^1(U)$. Then, for all $v \in C_c^\infty(U)$, $v \geq 0$,

$$\begin{aligned} B[\phi(u), v] &= \int_U \sum_{i,j} a^{ij} (\phi(u))_{x_i} v_{x_j}\, dx \\ &= \int_U \sum_{i,j} a^{ij} \phi'(u) u_{x_i} v_{x_j}\, dx, \ (\phi'(u) \text{ is bounded since u is bounded}) \\ &= \int_U \sum_{i,j} a^{ij} u_{x_i} (\phi'(u) v)_{x_j} - \sum_{i,j} a_{ij} \phi''(u) u_{x_i} u_{x_j} v\, dx \\ &\leq 0 - \int_U \phi''(u) v |Du|^2\, dx \leq 0, \text{ by convexity of } \phi. \end{aligned}$$

(We don't know whether the product of two H^1 functions is weakly differentiable. This is why we do not take $v \in H_0^1$.) Now we complete the proof with the standard density argument. $\square$

12. *Proof.* Given $u \in C^2(U) \cap C(\bar{U})$ with $Lu \leq 0$ in U and $u \leq 0$ on ∂U. Since $\bar{U}$ is compact and $v \in C(\bar{U})$, $v \geq c > 0$. So $w := \frac{u}{v} \in C^2(U) \cap C(\bar{U})$. Brutal computation gives us

$$\begin{aligned} -a^{ij} w_{x_i x_j} &= \frac{-a^{ij} u_{x_i x_j} v + a^{ij} v_{x_i x_j} u}{v^2} + \frac{a^{ij} v_{x_i} u_{x_j} - a^{ij} u_{x_i} v_{x_j}}{v^2} - a^{ij} \frac{2}{v} v_{x_j} \frac{v_{x_i} u - v u_{x_i}}{v^2} \\ &= \frac{(Lu - b^i u_{x_i} - cu) v + (-Lv + b^i v_{x_i} + cv) u}{v^2} + 0 + a^{ij} \frac{2}{v} v_{x_j} w_{x_i}, \text{ since } a^{ij} = a^{ji}. \\ &= \frac{Lu}{v} - \frac{uLv}{v^2} - b^i w_{x_i} + a^{ij} \frac{2}{v} v_{x_j} w_{x_i} \end{aligned}$$

Therefore,

$$Mw := -a^{ij} w_{x_i x_j} + w_{x_i} \left[b^i - a^{ij} \frac{2}{v} v_{x_j} \right] = \frac{Lu}{v} - \frac{uLv}{v^2} \leq 0 \text{ on } \{x \in \bar{U} : u > 0\} \subseteq U$$

If $\{x \in \bar{U} : u > 0\}$ is not empty, Weak maximum principle to the operator M with bounded coefficeints (since $v \in C^1(\overline{U})$) will lead a contradiction that

$$0 < \max_{\{u>0\}} w = \max_{\partial\{u>0\}} w = \frac{0}{v} = 0$$

Hence $u \leq 0$ in U. $\square$

13. (Courant-Fischer's principle)

Proof. First we need to note the same proof of Theorem 2 on Page 356 leads the following theorem:

Theorem 3. *For the symmetric operator $L = -(a^{ij}u_{x_i})_{x_j}$ on a bounded smooth domain U where (a^{ij}) is symmetric and satisfies the uniform ellipticity. Denote k-th zero Dirichlet eigenvalue and eigenfunction by λ_k and ω_k respectively, then*

$$\lambda_k = \min\{B[u,u] \mid u \in H_0^1, \|u\|_{L^2} = 1, (u, \omega_i)_{L^2} = 0, 1 \leq i \leq k-1\}$$

Through this theorem, we know the desired identity is true if we replace equality by $\leq$. Now we prove the converse:

Given $k-1$ dimensional subspace S. By spectral theory for compact self-adjoint operators, there exists $y = \Sigma_1^k c_i\omega_i \in \text{span}\{\omega_1 \cdots \omega_k\}$ with $\|y\|_{L^2} = 1$, such that $y \in S^\perp$, so

$$\min\{B[u,u] \mid u \in S^\perp, \|u\|_{L^2} = 1\} \leq B[y,y] = \Sigma_1^k |c_i|^2\lambda_i \leq \Sigma_1^k |c_i|^2 \lambda_k \leq \lambda_k.$$

Since S is arbitrary chosen, we are done.

□

14. *Proof.* Let $Y := \{u \in C^\infty(\overline{U}) \mid u > 0 \text{ in } U, u = 0 \text{ on } \partial U\}$. Let $u_1 > 0$ be the zero Dirichlet eigenfunction with respect to the eigenvalue λ_1 which exists by Krein-Rutman theorem (Page 361, Theorme 3). By classical Schauder's theory, we know $u_1 \in Y$ and therefore $\sup\limits_{u\in Y} \inf\limits_{x\in U} \frac{Lu}{u} \geq \lambda_1$. Given $v \in Y$, then $w = v - u_1 \in Y$. Note that

$$\frac{Lv}{v} = \frac{L(u_1 + w)}{u_1 + w} = \lambda_1 + \frac{L(w) - \lambda_1 w}{u_1 + w}$$

Consider the adjoint operator

$$L^*w = \partial_{ij}(a^{ij}w) - \partial_i(b^i w) + cw = a^{ij}\partial_{ij}w + [\partial_j(a^{ij} + a^{ji}) - \partial_i b^i]\partial_i w + (c + \partial_{ij}a^{ij} - \partial_i b^i)w,$$

the Krein-Rutman theorem tells us that, for $m > \sup_U |c + \partial_{ij}a^{ij} - \partial_i b^i|$, there exists λ_1^*, such that $\lambda_1^* + m$ is the simple principle zero Dirichlet eigenvalue of $L^* + m$ with positive eigenfunction u_1^*. Since $\lambda_1^*(u_1^*, u_1)_{L^2} = (L^*u_1^*, u_1)_{L^2} = (u_1^*, Lu_1)_{L^2} = \lambda_1(u_1^*, u_1)_{L^2}$, the fact that $u_1^*, u_1 > 0$ implies $\lambda_1 = \lambda_1^*$. Consider

$$(L(w) - \lambda_1 w, u_1^*) = \lambda_1^*(w, u_1^*) - \lambda_1(w, u_1^*) = 0$$

Due to $u_1^* > 0$, there are only two possibilities:

(1) If $L(w) - \lambda_1 w \equiv 0$. Since λ_1 is simple, $w = u_1$. Then $\frac{Lv}{v} = \frac{L(2u_1)}{2u_1} = \lambda_1$.

(2) If $L(w)-\lambda_1 w$ changes sign in U. Then there exist $x\in U$ such that $(L(w)-\lambda_1 w)(x)<0$ and hence $\frac{L(v)(x)}{v(x)}=\lambda_1+\frac{(L(w)-\lambda_1 w)(x)}{(u_1+w)(x)}<\lambda_1$. Since v is arbitrary, $\sup_{u\in Y}\inf_{x\in U}\frac{Lu}{u}\le\lambda_1$.

Hence $\sup_{u\in Y}\inf_{x\in U}\frac{Lu}{u}=\lambda_1$. $\square$

15. *Proof.* (We use Reynolds transport theorem without mentions.)

$$\begin{aligned}\lambda(\tau)=\lambda(\tau)\int_{U(\tau)}|\omega(x,\tau)|^2&=\int_{U(\tau)}-(\Delta_x\omega)(x,\tau)\bar{\omega}(x,\tau)\\&=\int_{U(\tau)}|\nabla_x\omega(x,\tau)|^2.\end{aligned}$$

Differentiate the original equation and $\|w\|_{L^2(U_\tau)}=1$ with respect to τ, we get

$$-\Delta_x\omega_\tau=\lambda\omega_\tau+\dot{\lambda}\omega$$

$$0=\int_{\partial U(\tau)}\omega^2\mathbf{v}\cdot\nu+\int_{U(\tau)}(\omega^2)_\tau=0+\int_{U(\tau)}(\omega^2)_\tau,$$

Hence,

$$\begin{aligned}\dot{\lambda}(\tau)&=\int_{\partial U(\tau)}|\nabla_x\omega|^2\mathbf{v}\cdot\nu+\int_{U(\tau)}(|\nabla_x\omega|^2)_\tau\\&=\int_{\partial U(\tau)}|\frac{\partial\omega}{\partial\nu}|^2\mathbf{v}\cdot\nu+\int_{U(\tau)}2\nabla_x\omega\nabla_x\omega_\tau,\ \text{Since } u=0 \text{ on } \partial U.\\&=\int_{\partial U(\tau)}|\frac{\partial\omega}{\partial\nu}|^2\mathbf{v}\cdot\nu+\int_{U(\tau)}2\omega(-\Delta_x\omega_\tau)\\&=\int_{\partial U(\tau)}|\frac{\partial\omega}{\partial\nu}|^2\mathbf{v}\cdot\nu+\int_{U(\tau)}2\omega(\lambda\omega_\tau+\dot{\lambda}\omega)\\&=\int_{\partial U(\tau)}|\frac{\partial\omega}{\partial\nu}|^2\mathbf{v}\cdot\nu+\lambda\cdot 0+2\dot{\lambda}\end{aligned}$$

$\square$

16. *Proof.* Direct computations tell us $e^{-i\sigma t}v:=e^{i\sigma(\omega\cdot x-t)}$ satisfies the wave equation $u_{tt}-\sigma^2\Delta u=0$ and $r(\frac{\partial v}{\partial r}-i\sigma v)=i\sigma(x\cdot\omega-|x|)v$ which did not tend to 0 as $|x|\to 0$. On the other hand, $r(\frac{\partial\Phi}{\partial r}-i\sigma\Phi)=-\Phi$ which tends to 0 as $|x|\to\infty$. The proof to show Φ is a fundamental solution is almost the same as usual Laplacian case. Try to mimic the proof Evans gives on page 24. $\square$

17. I think we need to assume $w=O(R^{-1})$. and the second expression Evans gives in this problem is wrong, the integrand on the left-hand side should be $|w_r|^2+\sigma^2|w|^2-2\sigma\mathrm{Im}(\bar{w}w_r)$.

Proof. Given $x_0\in\mathbb{R}^3$, let B_R denote the ball centered at x_0 with radius R and $\Phi^{x_0}(x):=\Phi(x-x_0)$, where Φ is defined in Exercise 16 (b). Hence

$$w(x_0)=\int_{B_R}\left[(\Delta+\sigma^2)\Phi^{x_0}\right]w-\left[(\Delta+\sigma^2)w\right]\Phi^{x_0}=\int_{\partial B_R}\frac{\partial\Phi^{x_0}}{\partial r}w-\frac{\partial w}{\partial r}\Phi^{x_0}\tag{3}$$

$$=\int_{\partial B_R}\left[\frac{\partial\Phi^{x_0}}{\partial r}-i\sigma\Phi^{x_0}\right]w-\left[\frac{\partial w}{\partial r}-i\sigma w\right]\Phi^{x_0}\tag{4}$$

the magnitude of the integrand is $o(R^{-1}) * O(R^{-1})$, so its integral tends to 0 as $R \to \infty$. $\square$

Remark 4. Further results for Radiation condition on general dimension $N \geq 2$ can be found in *G. Eskin, Lectures on Linear PDEs, Section 19.* (Need some knowledges on stationary phase formula.)