Real Analysis, Stein and Shakarchi

Chapter 1 Measure Theroy*

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Exercises

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

11. *Proof.* □

12. **Theorem 1.3 states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$ is generally false. Prove the following:**

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(a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

[Hint: What happens to the boundary of any of these rectangles?]

(b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

(c) [1, Theorem 1.11] Every open set in $\mathbb{R}^d$ can be written as a countable union of nonoverlapping closed cubes. It can also be written as a countable union of disjoint partly open cubes.

Proof. (a)(b) If it's possible to do that, it will contradicts the connectedness of the unit disc. $\square$

13. **The following deals with G_δ and F_σ sets.**

(a) Show that a closed set is a G_δ and an open set an F_σ.

(b) Give an example of an F_σ which is not a G_δ.

[Hint: This is more difficult; let F be a denumerable set that is dense.]

(c) Give an example of a Borel set which is not a G_δ nor an F_σ.

Proof. (a) Let F be a closed set. Then $F = \cap_n O_n$ where $O_n = \{x : d(x, F) < \frac{1}{n}\}$ (The reverse inclusion use the closedness of F). The second assertion is proved by taking the complement set operation of the first one.

(b) Consider $\mathbb{Q}$, the following proof for non-G_δ has the same spirit to the proof of Baire Category Theorem: if $\mathbb{Q} = \cup_m \{r_m\} = \cap_n O_n$ for some open sets O_n. Let $V_n = O_n \setminus \{r_n\}$. Then V_n is open and dense and there is a non-empty interval (a_1, b_1) such that $(a_1, b_1) \subset V_1$. Choose now a closed interval $[c_1, d_1] \subset (a_1, b_1)$. Next, as V_2 is open and dense, there is a non-empty interval (a_2, b_2) such that $(a_2, b_2) \subset V_2 \cap (c_1, d_1)$ and choose a closed interval $[c_2, d_2] \subset (a_2, b_2)$. Recursively we can thus obtain two sequences of intervals $(a_n, b_n), [c_n, d_n], n \in \mathbb{N}$, such that

$$[c_{n+1}, d_{n+1}] \subset (a_{n+1}, b_{n+1}) \subset V_{n+1} \cap (c_n, d_n).$$

But $\cap_n V_n = \emptyset$ implies $\cap_n [c_{n+1}, d_{n+1}] = \emptyset$, which is a contradiction.

(c) $E = (\mathbb{Q} \cap (-\infty, 0)) \cup ((0, \infty) \cap \mathbb{Q}^c)$. If E is F_σ, then $E \cap (0, \infty) = \mathbb{Q}^c \cap (0, \infty)$ is F_σ, which contradicts to (b). Similarly, if E is G_δ, then $E \cap (-\infty, 0] = \mathbb{Q} \cap (-\infty, 0)$ is G_δ, which contradicts to (b). $\square$

14. **The purpose of this exercise is to show that covering by a *finite* number of intervals will not suffice in the definition of the outer measure m_*. The outer Jordan content $J_*(E)$ of a set E in $\mathbb{R}$ is defined by**

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

where the inf is taken over every finite covering $E \subset \cup_{j=1}^{N} I_j$, by intervals I_j.

(a) Prove that $J_*(E) = J_*(\overline{E})$ for every set E (here $\overline{E}$ denotes the closure of E).

(b) Exhibit a countable subset $E \subset [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

Proof. □

15. *Proof.* □

16. *Proof.* □

17. *Proof.* □

18. **Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions.**

Proof. □

19. **Here are some observations regarding the set operation $A + B$.**

(a) Show that if either A and B is open, then $A + B$ is open.

(b) Show that if A and B are closed, then $A + B$ is measurable.

(c) Show, however, that $A+B$ might not be closed even though A and B are closed.

Proof. (a) I assume we are in a topological "vector" space V so that $A + B$ is contained in V. WLOG, we assume A is open. Given $x = x_a + x_b \in A + B$, then there exists an open neighborhood N of x_a such that $N \subset A$. So $x_b + N$ is an open (since vector addition is continuous in the TVS V) neighborhood of $x_a + x_b$ containing in $A + B$.

(b) In this problem I think $V = \mathbb{R}^d$. So $B = \cup_k B_k$ with each B_k is compact. For each k, $A + B_k$ is closed since for each $x = x_a + x_b \in \overline{A + B_k}$, there is some sequence $x_a^j + x_b^j \in A + B_k$ converging to x. Up to a subsequence $x_b^j \to z$ for some $z \in B_k$. So $x_a^j \to x - z \in \overline{A} = A$ and $x = x - z + z \in A + B_k$. Since $A + B = \cup_k A + B_k$, $A + B$ is a F_σ set.

(c) Consider $V = \mathbb{R}$ with standard Euclidean metric. $A = \mathbb{N}$ and $B = \{-n + \frac{1}{n}, n \in \mathbb{N}\}$. Then $\frac{1}{n} \in A + B$ and hence $0 \in \overline{A+B} \setminus A + B$. $\square$

Remark 1. Also see exercise 2.13.

20. **Show that there exist closed sets A and B with $m(A) = m(B) = 0$, but $m(A+B) > 0$:**

    (a) In $\mathbb{R}$, let $A = C$ (the Cantor set), $B = C/2$. Note that $A + B \supset [0; 1]$.

    (b) In $\mathbb{R}^2$, observe that if $A = I \times \{0\}$ and $B = \{0\} \times I$ (where $I = [0, 1]$), then $A + B = I \times I$.

    Proof. $\square$

21. *Proof.* $\square$

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37. *Proof.* $\square$

38. **Prove that $(a+b)^\gamma \geq a^\gamma + b^\gamma$ whenever $\gamma \geq 1$ and $a, b \geq 0$. Also, show that the reverse inequality holds when $0 \leq \gamma \leq 1$.**

    Proof. $\square$

39. **Establish the inequality ($x_j \geq 0, j = 1, \cdots, d$)**

$$\frac{x_1 + x_2 + \cdots x_d}{d} \geq \sqrt[d]{x_1 x_2 \cdots x_d}$$

Proof. (Sketch) First from $d = 2$ to $d = 2^k$. For general d, apply the method of induction. $\square$

Problem

1. *Proof.* $\square$

2. *Proof.* $\square$

3. *Proof.* $\square$

4. *Proof.* $\square$

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6. *Proof.* $\square$

7. *Proof.* $\square$

8. *Proof.* $\square$

References

[1] Richard L Wheeden and Antoni Zygmund. *Measure and Integral: An Introduction to Real Analysis*, volume 308. CRC Press, 2nd edition, 2015.