Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 7 Linear Evolution Equations

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Abstract

In the following exercises we assume U is open, bounded set, with smooth boundary, and $T > 0$. Exercise 16 still has some gap to be overcame.

The difficult exercise 9 is solved by mimicking a proof in a paper of Brezis-Evans on 2016/07/31.

1. **Prove there is at most one smooth solution of this initial/boundary-value problem for the heat equation with Neumann boundary conditions**

$$\begin{cases} u_t - \Delta u = f \text{ in } U \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \times [0,T] \\ u = g \text{ on } U \times \{t=0\}. \end{cases} \tag{1}$$

Proof. Consider $E(t) = \int_U (v(x,t))^2\,dx$, where v solves the heat equation with all corresponding boundary conditions are zero. Then we see $E'(t) \le 0$ and $E(0) = 0 \le E(t)$, $\forall t$. So $E(t) \equiv 0$ and hence $v \equiv 0$. □

$1\frac{1}{2}$ **Prove the following weak maximum principle for (1) with $f = 0$:**

If $g \ge 0$ in U, then $u \ge 0$ in U_T.

Proof. The proof strategy is the same as the Dirichlet problem except the contradiction argument for ∂U needs the Hopf boundary lemma.

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Note that we may assume $g \geq c > 0$ by considering $u + c$ and letting $c \to 0^+$ in the final step.

Consider $u_{\epsilon,\delta}(x,t) = u(x,t) + \epsilon t + \delta[|x|^2 + h(x)]$, where $\epsilon, \delta > 0$ and $h(x)$ is the harmonic function which equals to $-|x|^2$ on ∂U (whose existence can be proved by Perron's method or using the Riesz representation theorem with elliptic regularity theory as Chapter 6). Since $|x|^2 + h(x)$ is bounded (due to its continuity on $\overline{U}$), $u_{\epsilon,\delta}(x,0) \geq c - \delta \| |x|^2 + h(x)\|_\infty \geq 0$ for all small $\delta > 0$ (independent of ϵ).

Suppose $\inf_{(x,t)\in\overline{U_T}} u_{\epsilon,\delta}(x,t) < 0$. By continuity of $u_{\epsilon,\delta}$, we know the infimum is attained at (x_0, t_0). It's obvious that $t_0 > 0$. If $(x_0, t_0) \in U \times (0, T]$, then

$$0 \geq (\partial_t - \Delta) u_{\epsilon,\delta}(x_0, t_0) = \epsilon - 2n\delta > 0$$

for all small $\delta > 0$. So $(x_0, t_0) \in \partial U \times (0, T]$. However, the minimality at (x_0, t_0) implies $\frac{\partial u_{\epsilon,\delta}}{\partial \nu}(x_0, t_0) \leq 0$. However, the Hopf boundary lemma implies $\frac{\partial}{\partial \nu}(|x|^2 + h(x))(x_0) > 0$ and hence $\frac{\partial u_{\epsilon,\delta}}{\partial \nu}(x_0, t_0) > 0$.

Therefore $\inf_{(x,t)\in\overline{U_T}} u_{\epsilon,\delta}(x,t) \geq 0$ and hence $\inf_{(x,t)\in\overline{U_T}} u(x,t) \geq 0$ by letting $\epsilon, \delta \to 0$. □

2. *Proof.* Since $\frac{d}{dt}\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2 = -\int_U u(-\Delta u) \leq -\lambda_1 \|u\|_{L^2}^2$, basic Gronwall's inequality shows the desired inequality. □

3. *Proof.*

$$\begin{aligned} 0 &= \int_{U_T} (u_t + Lu) v \, d(x,t) = \int_U \int_0^T u_t v \, dt \, dx + \int_U \int_0^T (Lu) v \, dt \, dx \\ &= \int_U u(x,T)v(x,T) - u(x,0)v(x,0) \, dx - \int_{U_T} u(v_t - L^* u) \, d(x,t) \end{aligned}$$

□

4. *Proof.* Let $\{w_k\}$ be the family of all eigenfunctions of $-\Delta$ with zero Dirichlet boundary conditions. It's an orthogonal basis of $H_0^1(U)$ (See Evans, step 3 in page 357. By Poincaré's inequality,

$$\begin{aligned} \beta \|u_m\|_{H^1}^2 \leq \|Du_m\|_{L^2}^2 = \sum_{k=1}^m d_m^k \int_U Du_m \cdot Dw_k = \int_U = \sum_{k=1}^m d_m^k \int_U f \cdot w_k = \int_U f u_m \\ \leq \|f\|_{L^2} \|u_m\|_{H^1}. \end{aligned}$$

The weak compactness theorem implies there exists $\{u_{m_j}\} \subset \{u_m\}$ converging weakly in H^1 to a function $u \in H_0^1(U)$. For each $k \in \mathbb{N}$,

$$\int_U Du \cdot Dw_k = \lim_{j\to\infty} \int_U Du_{m_j} \cdot Dw_k = \int_U f w_k.$$

The desired result follows from usual density argument. □

Remark 1. My friend told me another method to derive uniform bound of $\|Du_m\|_{L^2}$:
Note that $\|Du_m\|_{L^2}^2 = \sum_{k=1}^m \lambda_k (d_m^k)^2 \le \lambda_1^{-1} \sum_{k=1}^m (\lambda_k d_m^k)^2 = \lambda_1^{-1} \|\Delta u_m\|_{L^2}^2$.

Since $\{w_k\}$ is an orthonormal basis of L^2, given $v \in L^2$ with $\|v\|_{L^2} = 1$, there exists $\{c_j\}$ with $\sum_{j=1}^\infty c_j^2 = 1$ and $v = \sum_{j=1}^\infty c_j w_j$. Then

$$(-\Delta u_m, v)_{L^2} = (-\Delta u_m, \sum_{j=1}^m c_j w_j)_{L^2} = (\nabla u_m, \sum_{j=1}^m c_j \nabla w_j)_{L^2} = \sum_{j=1}^m c_j (f, w_j)_{L^2} \le \|f\|_{L^2}.$$

Hence $\sum_{k=1}^m \lambda_k (d_m^k)^2 = \|Du_m\|_{L^2}^2 \le \lambda_1^{-1} \|\Delta u_m\|_{L^2}^2 \le \lambda_1^{-1} \|f\|_{L^2}^2$, for all $m \in \mathbb{N}$.
Moreover, since $\lambda_k \nearrow \infty$, there exists a constant C independent of m, such that $\|u_m\|_{L^2}^2 = \sum_{k=1}^m (d_m^k)^2 \le C \sum_{k=1}^m \lambda_k (d_m^k)^2 \le C\lambda_1^{-1} \|f\|_{L^2}^2$, for all $m \in \mathbb{N}$. This completes the proof.

Remark 2. The constant in Poincaré inequality is sometimes lack of informations (especially in the case we prove it by contradictions.) The constant in the above method depends on the eigenvalues, which is much concrete than the first one. (But it's still hard to understand eigenvalue problems.)

5. *Proof.* Given $\phi \in C_c^\infty(0,T)$ and $w \in H_0^1(U)$, then $t \mapsto \phi(t)w$ is in $L^2(0,T;H_0^1)$. In the following, $<\cdot,\cdot>$ means the pairing of H^{-1} and H_0^1

$$\begin{aligned}
\Big\langle \int_0^T \phi'(t)u(t)\,dt, w \Big\rangle &= \int_0^T \big\langle \phi'(t)u(t), w \big\rangle\,dt, \text{ by Riemann sum argument,} \\
&= \int_0^T \big\langle u(t), \phi'(t)w \big\rangle\,dt \\
&= \lim_{n\to\infty} \int_0^T \big\langle u_n(t), \phi'(t)w \big\rangle\,dt \\
&= \lim_{n\to\infty} \Big\langle \int_0^T u_n(t)\phi'(t)\,dt, w \Big\rangle \\
&= \lim_{n\to\infty} \Big\langle \int_0^T -u_n'(t)\phi(t)\,dt, w \Big\rangle \\
&= \lim_{n\to\infty} \int_0^T \big\langle -u_n'(t), \phi(t)w \big\rangle\,dt \\
&= \int_0^T \big\langle -v(t), \phi(t)w \big\rangle\,dt \\
&= \Big\langle -\int_0^T v(t)\phi(t)\,dt, w \Big\rangle.
\end{aligned}$$

Hence $u' = v$ in $L^2(0,T;H^{-1})$. □

6. *Proof.* Since $u_k \rightharpoonup u$ in $L^2(E;H)$,

$$\|u\|_{L^2(E;H)}^2 \le \liminf \|u_k\|_{L^2(E;H)}^2 \le C^2|E|,$$

for every measurable subset E of $(0,T)$. Hence $\|u\|_{L^\infty(0,T;H)} \le C$. □

7. *Proof.* Let $v(x,t) := e^{\gamma t}u(x,t)$. Then v solves

$$\begin{cases} v_t - \Delta v + (c-\gamma)v = 0 & \text{in } U \times (0,\infty) \\ v = 0 & \text{on } \partial U \times (0,\infty) \\ v = g & \text{on } U \times \{t=0\} \end{cases}$$

Since $c-\gamma \geq 0$, for each $T>0$ weak maximum and minimum principle implies $|v(x,t)| \leq \|g\|_\infty$ on U_T. Since T is arbitrary chosen, $|u(x,t)| = e^{-\gamma t}|v(x,t)| \leq e^{-\gamma t}\|g\|_\infty$ $\forall (x,t) \in U \times (0,\infty)$. $\square$

8. *Proof.* Similar to exercise 7, the role of γ is replaced by $-\|c\|_\infty$. $\square$

9. On a bounded smooth open set U, we consider a second order differential operator $Lu = -a^{ij}(x)u_{x_ix_j} + b^i(x)u_{x_i} + c(x)u$ with all coefficients are smooth up to the boundary. Show that there exists constants $\beta > 0$ and $\gamma \geq 0$ such that for all $u \in H^2(U) \cap H_0^1(U)$

$$\beta\|u\|_{H^2}^2 \leq (Lu, -\Delta u)_{L^2} + \gamma\|u\|_{L^2}^2$$

(Evans' Hints: About estimating the boundary terms, after changing variables locally and using cutoff functions, you may assume the boundary is flat. This problem is difficult, see *Brezis-Evans, A variational inequality approach to the Bellman-Dirichlet equation for two elliptic operators, Arch. Rational Mech. Analysis, 1979, 1-13*)

Proof.

$$\begin{aligned} \int_U a^{ij}u_{x_ix_j}u_{x_kx_k} &= \int_U -a^{ij}_{x_k}u_{x_ix_j}u_{x_k} - a^{ij}u_{x_ix_jx_k}u_{x_k} + \int_{\partial U} a^{ij}u_{x_ix_j}u_{x_k}\nu_k \\ &= \int_U -a^{ij}_{x_k}u_{x_ix_j}u_{x_k} + a^{ij}_{x_i}u_{x_jx_k}u_{x_k} + \int_U a^{ij}u_{x_jx_k}u_{x_ix_k} + \int_{\partial U} a^{ij}\left(u_{x_ix_j}u_{x_k}\nu_k - u_{x_jx_k}u_{x_k}\nu_i\right) \end{aligned}$$

In the early stage, I can prove this problem if $u \in H_0^2$, since the boundary terms vanish. But it turn to be a difficult problem to me if there are boundary terms! I learn the estimates for boundary terms from page 10-12 in the paper cited above.

We assume $c \equiv 0$ first. Let $\Gamma \subset \partial U$ be some given boundary portion of the boundary, which upo a change of coordinates, if necessary, we may assume to lie in the plane $x_n = 0$ (with $\Omega \subset \{x_n > 0\}$.) Choose a smooth cutoff function $\zeta, 0 \leq \zeta \leq 1$, such that $\zeta(x) = 0$ near $\partial U \setminus \Gamma$. Then

$$-\int_U \zeta Lu\Delta u = \int_U \zeta a^{ij}u_{x_ix_j}u_{x_kx_k}\,dx - \int_U \zeta b^i u_{x_i}u_{x_kx_k}\,dx \tag{2}$$

As above we transform the first term on the right by integration by parts twice:

$$\begin{aligned} \int_U \zeta a^{ij}u_{x_ix_j}u_{x_kx_k}\,dx = &\int_U (\zeta a^{ij})_{x_j}u_{x_ix_k}u_{x_k} - (\zeta a^{ij})_{x_k}u_{x_ix_j}u_{x_k}\,dx + \int_U \zeta a^{ij}u_{x_kx_j}u_{x_ix_k} \\ &+ \int_{\partial U} \zeta a^{ij}(u_{x_ix_j}u_{x_k}\nu_k - u_{x_ix_k}u_{x_k}\nu_j)\,ds. \end{aligned}$$

Call the integrand of the last term I. Then the preceding calculation and (2) together imply

$$\begin{aligned}
&\theta\int_U \zeta\sum_{k,j} u^2_{x_kx_j} \le \int_U \zeta a^{ij}u_{x_kx_j}u_{x_ix_k}\\
&= \int_U \zeta a^{ij}u_{x_ix_j}u_{x_kx_k}\,dx - \int_U (\zeta a^{ij})_{x_j}u_{x_ix_k}u_{x_k} + (\zeta a^{ij})_{x_k}u_{x_ix_j}u_{x_k}\,dx - \int_{\partial U} I\,ds\\
&= -\int_U \zeta Lu\Delta u + \int_U \zeta b^i u_{x_i}u_{x_kx_k}\,dx - \int_U (\zeta a^{ij})_{x_j}u_{x_ix_k}u_{x_k} + (\zeta a^{ij})_{x_k}u_{x_ix_j}u_{x_k}\,dx - \int_{\partial U} I\,ds\\
&\le -\int_U \zeta Lu\Delta u + \epsilon\int_U \sum_{k,j} u^2_{x_kx_j} + \frac{D_1}{\epsilon}\int_U |\nabla u|^2 - \int_{\partial U} I\,ds, \qquad (3)
\end{aligned}$$

where $D_1 > 0$ depends on the dimension and coefficients. Furthermore, since $u \equiv 0$ on ∂U, for $x \in \Gamma$ we have

$$I(x) = \zeta(\sum_{i,j} a^{ij}u_{x_ix_j}u_{x_n} - \sum_i a^{in}u_{x_ix_n}u_{x_n}).$$

Since $1 \le i, j \le n-1, u_{x_ix_j} = 0$ and the terms for $j = n$ is the same as the second term,

$$I(x) = \zeta\sum_{j=1}^{n-1} a^{nj}u_{x_nx_j}u_{x_n} = \zeta\sum_{j=1}^{n-1} a^{nj}\frac{1}{2}\frac{\partial}{\partial x_j}u^2_{x_n}.$$

Thus $(x' = (x_1, \cdots x_{n-1}))$

$$|\int_{\partial U} I(x)\,ds| = |\int_\Gamma I(x)\,dx'| = |\frac{1}{2}\int_\Gamma u^2_{x_n}\sum_{j=1}^{n-1}(a_{nj})_{x_j}\,dx'| \le D_2\int_\Gamma u^2_{x_n}\,dx' \le D_2\int_{\partial U}(\frac{\partial u}{\partial \nu})^2\,ds,$$

where $D_2 > 0$ depends on the dimensions and coefficients. Since $u = 0$ on the boundary, we have the following trace inequality (called by Brezis-Evans)

$$\int_{\partial U}(\frac{\partial u}{\partial\nu})^2\,ds = \int_{\partial U}|\nabla u|\nabla u\cdot\nu\,ds = 2\int_U |\nabla u|\Delta u \le \eta\int_U\sum_{k,j}u^2_{x_kx_j}\,dx + \frac{1}{\eta}\int_U|\nabla u|^2\,dx \qquad (4)$$

Choose ϵ, η small with $\frac{D_1}{\epsilon} - \frac{D_2}{\eta} < 0$, putting them into (3) Then there are $\beta, \alpha > 0$ such that

$$\beta\int_U \zeta\sum_{k,j}u^2_{x_kx_j} + \alpha\int_U|\nabla u|^2 \le -\int_U \zeta Lu\Delta u$$

Next we decompose ∂U into the union of finitely many Γ_i, each of which can be mapped as above by a smooth change of coordinates into the plane $x_n = 0$. Let ζ_i be a smooth partition of unity on U, with $\zeta_i \equiv 0$ near $\partial U \setminus \Gamma_i$. We sum the finite number of inequalities (4), we see there are constants $\beta, \alpha > 0$ depending on the dimension, the coefficients, and the domain U, such that

$$\beta\int_U\sum_{k,j}u^2_{x_kx_j} + \alpha\int_U|\nabla u|^2 \le -\int_U Lu\Delta u$$

For general case $c \not\equiv 0$, denote $\tilde{L} = L - c$ and remember $u = 0$ on the boundary,

$$\begin{aligned}
-\int_U Lu\Delta u\,dx &= -\int_U \tilde{L}u\Delta u\,dx - \int_U cu\Delta u\,dx \\
&\geq \beta\int_U \sum_{k,j} u_{x_kx_j}^2 + \alpha\int_U |\nabla u|^2 + \int_U u\nabla c\cdot\nabla u\,dx + \int_U c|\nabla u|^2\,dx \\
&\geq \beta\int_U \sum_{k,j} u_{x_kx_j}^2 + \alpha\int_U |\nabla u|^2 - D_3\int_U |u||\nabla u|\,dx - \|c\|_\infty\int_U |\nabla u|^2\,dx \\
&\geq \beta\int_U \sum_{k,j} u_{x_kx_j}^2 - \frac{D_3}{2\mu}\int_U |u|^2\,dx + (\alpha - \|c\|_\infty - 2D_3\mu)\int |\nabla u|^2\,dx.
\end{aligned}$$

A careful look back to the constructions of β and α, we know α can be arbitrary large (and β is close to the uniform ellipticity constant at the same time,) so we conclude that there are constants $\tilde{\alpha}, \tilde{\gamma} > 0$, such that

$$-\int_U Lu\Delta u\,dx \geq \beta\int_U \sum_{k,j} u_{x_kx_j}^2 + \tilde{\alpha}\int_U |\nabla u|^2 - \tilde{\gamma}\int_U |u|^2\,dx,$$

which can imply the desired result easily. □

10. *Proof.* Energy method to $E(t) = \frac{1}{2}(\|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)$. □

11. *Proof.* Energy method to $E(t) = \frac{1}{2}(\|u_t\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)$. □

12. Let A be a closed linear operator on a real Banach space X, with domain $D(A)$. If $\lambda, \nu \in \rho(A)$, prove the following resolvent identities

$$R_\lambda - R_\nu = (\nu - \lambda)R_\lambda R_\nu, \text{ and } R_\lambda R_\nu = R_\nu R_\lambda.$$

Proof. We assume $\lambda \neq \nu$ since there is nothing to prove when $\lambda = \nu$. The first identity is true since

$$R_\lambda - R_\nu = R_\lambda(\nu I - A)R_\nu - (\lambda I - A)R_\lambda R_\nu = (\nu - \lambda)R_\lambda R_\nu.$$

Exchange the role of ν and λ, we see $R_\nu - R_\lambda = -(\nu - \lambda)R_\nu R_\lambda$. Hence

$$(\nu - \lambda)R_\lambda R_\nu = R_\lambda - R_\nu = -(R_\nu - R_\lambda) = (\nu - \lambda)R_\nu R_\lambda,$$

which implies the second identity. □

13. *Proof.* Thanks to the exponential decay term and the fact $\|S(t)\| \leq 1$, $\forall t$, we have two things: first, the strong measurability of both integrands come from the uniform continuity of them; second, the existence of both integrals are due to Bochner's theorem (page 734). Note the result

in Appendix E.5 is applicable to time interval $[0,\infty)$, that is, $T=\infty$ (cf: Yosida, Chapter V). We are going to apply the Riemann sum method which restricts us to integrate on finite interval first and note the integrand is bounded.

On each $M\in\mathbb{N}$, we partition $[0,M]$ into $\{[\frac{j}{2^k}M,\frac{j+1}{2^k}M]:j=0,\cdots 2^k-1\}$ and consider the right endpoint Riemann sum functions $f_k(t)$ of $e^{-\lambda t}S(t)u$, which are simple functions. Since $e^{-\lambda t}S(t)u$ is uniform continuous,

$$\sup_{t\in[0,M]}\|f_k(t)-e^{-\lambda t}S(t)u\|_X\searrow 0. \tag{5}$$

Since A is closed,

$$\begin{aligned}A\Big(\int_0^M e^{-\lambda t}S(t)u\,dt\Big)&=A\Big(\lim_{k\to\infty}\int_0^M f_k(t)\,dt\Big)=\lim_{k\to\infty}A\Big(\int_0^M f_k(t)\,dt\Big)=\lim_{k\to\infty}\int_0^M Af_k(t)\,dt\\&=\int_0^M Ae^{-\lambda t}S(t)u\,dt=\int_0^M e^{-\lambda t}AS(t)u\,dt.\end{aligned}$$

Note that the third equality is true since each f_k is simple, the last equality is due to linearity of A. The fourth equality is by monotone convergence theorem, which need a little careful to understand $\|Af_k(t)-Ae^{-\lambda t}S(t)u\|_X\searrow 0$ for every $t\in[0,M]$ through (5).

Due to the exponential decay term, uniform boundedness $\|S(t)\|\le 1$ and the closedness of A, we know the following strong convergences

$$\int_0^M e^{-\lambda t}S(t)u\,dt\underset{M\to\infty}{\longrightarrow}\int_0^\infty e^{-\lambda t}S(t)u\,dt,$$

$$A\Big(\int_0^M e^{-\lambda t}S(t)u\,dt\Big)\underset{M\to\infty}{\longrightarrow}A\Big(\int_0^\infty e^{-\lambda t}S(t)u\,dt\Big),$$

and

$$\int_0^M e^{-\lambda t}AS(t)u\,dt=\int_0^M e^{-\lambda t}S(t)Au\,dt\underset{M\to\infty}{\longrightarrow}\int_0^\infty e^{-\lambda t}S(t)Au\,dt=\int_0^\infty e^{-\lambda t}AS(t)u\,dt.$$

Therefore, the desired result follows. □

14. *Proof.* (a) The continuity of $S(t)$ and contraction property is by Young's convolution inequality; the composition identity can be proved by Fourier Transform or direct computations; for each $g\in L^2(\mathbb{R}^d)$, the continuity of the map $t\mapsto S(t)g$ at $t_0>0$ is through Young's convolution inequality, mean value theorem and dominated convergence theorem; continuity at $t_0=0$ is standard proof for approximation to identity; this also gives us a hint that (b) is not a semigroup, since the map is not continuous at $t_0=0$ (which is needed in some proofs given here, for example, to conclude the second expression in the proof for Theorem 3, page 438). For example, $u(x):=$ characteristic function of first orthant. In one dimension, $[S(t)u](x)=$

$\int_{-\infty}^{x} e^{-\frac{y^2}{4t}}\, dy$ is continuous, and $[S(t)u](0) = \frac{1}{2}$ for all $t > 0$. Hence $\|S(t)u - u\|_\infty \geq \frac{1}{4}$ for all $t > 0$. In higher dimension, $[S(t)u](0) = \frac{1}{2^d}$ and $\|S(t)u - u\|_\infty \geq \frac{1}{2}(1 - \frac{1}{2^d})$. □

15. *Proof.* By definition, for each $1 \leq j \leq k-1$, we have to show $A^j S(t)u \in D(A)$, that is, the existence of the following limit:

$$\lim_{t\to 0^+} \frac{S(t)A^j S(t)u - A^j S(t)u}{t} = \lim_{t\to 0^+} \frac{S(t)S(t)A^j u - S(t)A^j u}{t} = \lim_{t\to 0^+} \frac{S(t)A^j u - A^j u}{t},$$

by $S(0) = I$ and Theorem 1 in Section 7.4 (page 435). This is guaranteed by hypothesis. □

16. *Proof.* The contraction semigroup property of heat kernel $S(t)$ is easy to verified.

Since $g \in C_c^\infty \subset H_0^{2k}$??? $= D(\Delta^k)$ for any $k \in \mathbb{N}$, we know from exercise 15 that for each $t \geq 0, u(\cdot, t) = S(t)g \in H_0^{2k}$ for any $k \in \mathbb{N}$. Therefore, the desired result follows from Sobolev embedding theorem. □