Complex Analysis, Stein and Shakarchi
Chapter 2 Cauchy's Theorem and Its Applications

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September 17, 2019

Exercises

1. **We omit the computations for the fresnel integrals**

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

2. **We omit the computation for the Dirichlet integrals**

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

3. **Evalute the integral $\int_0^\infty e^{-ax}e^{ibx}\,dx$ $(a>0, b\in\mathbb{R})$ by contour integral. (Omit!)**

4. **Prove that for all $\xi \in \mathbb{C}$, we have $e^{-\pi\xi^2} = \int_{\mathbb{R}} e^{-\pi x^2}e^{2\pi i x\xi}\,dx$. (Omit!!)**

5. **Use the Green's theorem to prove Goursat's theorem under the additional assumption that f is continuously complex differentiable on Ω. (Omit)**

6. **Let Ω be a open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then**

$$\int_T f(z)\,dz = 0.$$

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Proof. Divide T into sub-triangles. $\square$

7. **Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies**

$$2|f'(0)| \le d.$$

Moreover, it can be shown that equality holds precisely when $f(z) = f(0) + f'(0)z$. This is a theorem of E. Landau and O. Toeplitz in 1907.

***Remark* 1.** In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chaptr 4, Book I.

***Remark* 2.** I learned the proof for the second assertion from arxiv.0603579.

***Remark* 3.** The proof relies on the strong maximum principle, especially two of its consequences: The open mapping theorem and Schwarz lemma. See page 92 and page 218 for their statements.

***Remark* 4.** Without loss of generality, we may assume $f(0) = 0$ and $d = 2$. Comparing with the Schwarz lemma, we see that it can handle the case such as $f(\mathbb{D})$ is a regular triangle centered at the origin with side length 2 on $\mathbb{C}$ which is not contained in the unit disc. So one may compare them in the following way: if the range of $f(\mathbb{D})$ falls into $\mathbb{D}$, then Schwarz lemma tells more things. However, if the range $f(\mathbb{D})$ is "not so symmetric" as $\mathbb{D}$, but the "diameter" is still the same as $\mathbb{D}$, then we can still obtain a sharper estimate on $|f'(0)|$ than applying Schwarz lemma to $\frac{1}{2}f$, but we get no more information on $|\frac{f(z)}{z}|$ than Schwarz lemma.

Proof. By considering $\frac{2}{d}f$, we may assume $d = 2$.

As hint, the Cauchy's integral formula implies for every $0 < r < 1$,

$$2|f'(0)| = |\frac{1}{2\pi i}\int_{|z|=r} \frac{f(z) - f(-z)}{z^2}\, dz| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|f(z) - f(-z)|}{|z^2|}\, |dz| \le \frac{2}{r} \to 2 \text{ as } r \to 1$$

(Note that the above estimation method can't work for $f'(\xi)$ with $\xi \ne 0$).

If f is linear, then it's easy to see the equality holds.

The converse part is more delicate. We divide it into four steps.

First, we note that applying Schwarz lemma to $\frac{f(z)-f(-z)}{2}$ (where the hypotheses required are obviously satisfied), we have $|f'(0)| \le 1$ again. Moreover, $|f'(0)| = 1$ holds if and only if

$$\frac{f(z) - f(-z)}{2} = f'(0)z \quad \forall\, z \in \mathbb{D}. \tag{1}$$

If one take the derivative, we see

$$f'(z) + f'(-z) = 2f'(0) \quad \forall\, z \in \mathbb{D}$$

This is very similar to what we want $f'(z) + f'(z) = 2f'(0)$. The rest task is towards this assertion, see the final step below.

Second, for $0 \le r < 1$, we let $D_r = \mathrm{diam} f(r\mathbb{D})$ and show that $D_r = \mathrm{diam} f(r\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$. This is done by open mapping theorem as follows: notice that the set

$$\{f(z) - f(w) : |z| < r, |w| < r\} = \bigcup_{|w|<r} [f(r\mathbb{D}) - f(w)]$$

is open since each set in the right-hand side is open by open mapping theorem. Therefore, D_r is NOT attained for any $z, w \in r\mathbb{D}$. However, D_r is attained in $r\overline{\mathbb{D}}$. So we know there is $z_0 \in r\overline{\mathbb{D}}$ and $w_0 \in r\mathbb{T}$ such that $|f(z_0) - f(w_0)| = D_r$. But z_0 must belong to $r\mathbb{T}$ since the set $f(r\mathbb{D}) - f(w_0)$ is open again. So we have proven $D_r = \mathrm{diam} f(r\mathbb{T})$.

The third step is to prove the crucial observation that

$$D_r = 2r, \quad \forall\, 0 \le r < 1. \tag{2}$$

The first task is to observe that $\frac{D_r}{r}$ is non-decreasing in r by applying weak maximum principle as follows:

$$\frac{D_r}{r} = \frac{\mathrm{diam} f(r\mathbb{T})}{r} = \max_{|u|=1} \max_{|z|=r} |\frac{f(z) - f(-uz)}{z}| = \max_{|u|=1} \max_{|z|\le r} |\frac{f(z) - f(-uz)}{z}|,$$

where the last equality holds since $h_u(z) := \begin{cases} \frac{f(z)-f(-uz)}{z}, & z \neq 0 \\ (1+u)f'(0), & z = 0 \end{cases}$ is analytic on $\mathbb{D}$. Moreover, this expression suggests the following lower bound estimate by taking $z = 0$:

$$\frac{D_r}{r} \ge \sup_{|u|=1} |1 + u||f'(0)| = 2.$$

On the other hand, we have the nondecreasing function $\frac{D_r}{r}$ is bounded by $\frac{2}{r}$ which is a decreasing function in r and tends to 2 as r increases to 1. Hence $\frac{D_r}{r} \equiv 2$ for all $0 < r < 1$.

The final step seems intelligent, but I think it's motivated from the words I said after (1) and contains some geometric interpretations (so that we make a lot of efforts in the second and third steps).

Now for every $|w| < 1$, we consider the auxiliary analytic function $g_w(z) = f(z) - f(-w)$ on $\mathbb{D}$. Note that $g_w(w) = 2f'(0)w$ by (1).

We consider the following $\phi(\theta) = |g_w(we^{i\theta})|^2 = g_w(we^{i\theta})\overline{g_w(we^{i\theta})} = g_w(we^{i\theta})g_w^*(\overline{w}e^{-i\theta})$, where $g_w^*(z) := \overline{g_w(\overline{z})}$ is analytic on $\mathbb{D}$. So we see $\phi'(\theta)$ exists and

$$\phi'(\theta) = -2\mathrm{Im}\big[we^{i\theta}g_w'(we^{i\theta})\overline{g_w(we^{i\theta})}\big].$$

In particular,

$$\phi'(0) = -4|w|^2\mathrm{Im}[f'(0)g_w'(w)].$$

On the other hand, ϕ attains it maximum at $\theta = 0$ since

$$\sup_{\theta\in\mathbb{R}} \phi(\theta) = \sup_{z\in|w|\mathbb{D}} |f(z) - f(-w)|^2 = D_{|w|}^2 = (2|w|)^2 = |f(w) - f(-w)|^2 = \phi(0)$$

where (2), (1) and $|f'(0)| = 1$ are used. Hence

$$\mathrm{Im}[f'(0)f'(w)] = \mathrm{Im}[f'(0)g_w'(w)] = 0 \quad \forall\, w \in \mathbb{D}.$$

By open mapping theorem again, we have the analytic function $f'(0)f'(w)$ is a constant and hence $f'(w) \equiv f'(0)$. That is, $f(w) = f(0) + f'(0)w$. □

8. **If f is a holomorphic function on the strip $-1 < y < 1, x \in \mathbb{R}$ with ($\eta \in \mathbb{R}$ is fixed)**

$$|f(z)| \leq A(1 + |z|)^\eta$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta, \text{ for all } x \in \mathbb{R}.$$

Proof. Use Cauchy's integral formula with $C = \partial B_{1/2}(x)$. Note that for all $z \in C$,

$$(1 + |z|)^\eta \leq (1 + |x| + \frac{1}{2})^\eta < 2^\eta(1 + |x|)^\eta$$

So one may pick $A_n = n!2^n A 2^\eta$. □

9. **Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then $\varphi(z) = z$.**

[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof.]

Remark 5. Compare with the Schwarz lemma, e.g. Section 8.2.

Proof. May assume $z_0 = 0$ by considering $f(z) = \varphi(z + z_0) - z_0$.

Assume not, we can suppose $\varphi(z) = z + a_n z^n + g_1(z)$ near the origin with $n > 1$, $a_n \neq 0$ and $g_1(s) = O(s^{n+1})$ as $s \to 0$ by hypothesis. Then as hint, $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$ is true for $k = 1$. Assume it's true for $k = m$ with $|g_k(z)|" = |\varphi_k(z) - (z + ka_n z^n)| \leq M_m |z|^{n+1}$ as $|z| \leq \delta_m$ for some constants $M_m > 0, 1 > \delta_m > 0$. For $k = m + 1$ we have

$$
\begin{aligned}
&\varphi_{m+1}(z) = \varphi_m(z) + a_n \varphi_m(z)^n + g_1(\varphi_m(z)) \\
&= z + ma_n z^n + g_m(z) + a_n(z + ma_n z^n + g_m(z))^n + g_1(z + ma_n z^n + g_m(z)) \\
&= z + (m+1)a_n z^n + \Big(g_m(z) + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} z^{n-j}(ma_n z^n + g_m(z))^j + g_1(z + ma_n z^n + g_m(z)) \Big) \\
&=: z + (m+1)a_n z^n + g_{m+1}(z).
\end{aligned}
$$

Note that for $|z| \leq \delta_m < 1$

$$
\begin{aligned}
&|g_{m+1}(z)| \leq |g_m(z)| + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} |z|^{n-j}(m|a_n||z|^n + |g_m(z)|)^j + |g_1(z + ma_n z^n + g_m(z))| \\
&\leq M_m |z|^{n+1} + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} |z|^{n-j}(m|a_n||z|^n + M_m|z|^{n+1})^j \\
&\leq |z|^{n+1} \Big(M_m + M_1(1 + m|a_n||z|^{n-1} + M_m|z|^n)^{n+1} + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} (m|a_n| + M_m)^j \Big) + M_1(|z| + m|a_n||z|^n \\
&\leq |z|^{n+1} \Big(M_m + M_1(1 + m|a_n| + M_m)^{n+1} + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} (m|a_n| + M_m)^j \Big) =: M_{m+1}|z|^{n+1},
\end{aligned}
$$

with $\delta_{m+1} := \delta_m$. Therefore, the asymptotics holds on a fixed ball $B_{\delta_1} \subset \Omega$. Note that Cauchy's integral formula and the fact that $\varphi_k(\Omega) \subset \Omega$ is bounded by some $A > 0$ for every k imply that as $k \to \infty$

$$
|a_n| = \frac{|(\varphi_k)^{(n)}(0)|}{kn!} \leq \frac{A}{k\delta_1^n} \to 0.
$$

But this contradicts to the non-vanishing assumption for a_n. So $\varphi(z) = z$. □

10. **Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc U be approximated uniformly by polynomials in the variable z**

Proof. If f can be approximate by the polynomials uniformly, then $\int_{\partial(\frac{1}{2}U)} f(z)z\,dz = 0$ since $\int_{\partial(\frac{1}{2}U)} p(z)z\,dz = 0$ for any polynomial p. So the answer is No, since $g(z) = \overline{z}$ does not satisfy the identity. Another proof (with the same tool) is to conclude g is holomorphic by Morera's theorem. □

Remark 6. Also check the complex form of Stone-Weierstrass Theorem.

11. **Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.**

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathbf{Re}\Big(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\Big)\,d\varphi.$$

(b) Show that

$$\mathbf{Re}\Big(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\Big) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

Proof. We omit the proof for Part (b) since it is easy. For part (a), it becomes transparent if one realized that

$$0 = \int_{\partial D_R} \frac{f(w)}{w-(\frac{\bar{z}}{R^2})^{-1}}\,dw$$

since $|\frac{\bar{z}}{R^2}| < R$. More precisely, the desired result follows from the following calculations: for any $|w| = R$

$$\begin{aligned}
\Big(\frac{1}{w-z} - \frac{1}{w-\frac{R^2}{\bar{z}}}\Big)\,dw &= \frac{z-\frac{R^2}{\bar{z}}}{w^2-(z+\frac{R^2}{\bar{z}})w+\frac{R^2z}{\bar{z}}}dw\\
&= \frac{|z|^2-R^2}{\bar{z}w-(|z|^2+R^2)+|w|^2zw^{-1}}\frac{dw}{w}\\
&= \frac{|z|^2-|w|^2}{\bar{z}w-(|z|^2+|w|^2)+|w|^2zw^{-1}}\frac{dw}{w}\\
&= \frac{|w|^2-|z|^2}{|w-z|^2}\frac{dw}{w}.
\end{aligned}$$

Comments: I think, combining with (b), it makes the numerical calculations for Poisson integral formula (and hence the solution to Laplace equation) easier than just use the original Cauchy's formula since the kernel is now real-valued. □

12. **Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is, $\Delta u(x,y) = 0$ for all $(x,y) \in \mathbb{D}$.**

(a) Prove that there exists a holomorphic function f on the unit disc such that $\mathrm{Re}(f) = u$. Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc

and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(e^{i\varphi})\, d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma + r^2}.$$

Proof. We omit (b) since it's an easy consequence of (a) and Exercise 11. For (a), note that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

So $2\frac{\partial u}{\partial z}$ is a holomorphic function. By Theorem 2.1, there is a holomorphic function F such that $F' = 2\frac{\partial u}{\partial z}$. By Cauchy-Rieman equation, we see that

$$\frac{\partial u}{\partial z} = \frac{1}{2}F' = \frac{1}{2}(\frac{\partial \mathrm{Re}F}{\partial x} - i\frac{\partial \mathrm{Re}F}{\partial y}) = \frac{\partial \mathrm{Re}F}{\partial z}.$$

Hence the desired result follow from that ReF and u are real-valued functions. □

Remark 7. Weyl's lemma tells us that u can merely assume to be locally Lebesgue integrable and weakly harmonic.

Remark 8. The domain is not restricted to the open disc, it can be arbitrary simply connected domain Ω (so that Theorem 2.1 holds). However, the condition Ω is simply connected can not be removed, for example $u(z) = \log|z|$ is harmonic in the punctured plane $\mathbb{C}\setminus\{0\}$, but it cannot be written as real part of a holomorphic function. We now give a proof for the last statement **assuming the fact that "the logarithm function Log z defined on $\mathbb{C}\setminus(-\infty, 0]$ can not be analytically extended to the negative real axis"** is proved:

Theorem 9. *The function* $\log|z|$ *has no harmonic conjugation on* $\mathbb{C}\setminus\{0\}$.

Proof. If there is such function $f(z) = \log|z| + iv(x,y)$. Write $v_0(x,y) = \mathrm{Arg}(z) = \arctan\frac{y}{x}$, which is defined on $\mathbb{C}\setminus(-\infty, 0]$. Now the crucial point is that according to the Cauchy-Riemann equation:

$$\frac{\partial v}{\partial x} = -\frac{\partial \log|z|}{\partial y} = -\frac{y}{x^2+y^2} = \frac{\partial v_0}{\partial x}$$
$$\frac{\partial v}{\partial y} = \frac{\partial \log|z|}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v_0}{\partial y}$$

So $v = v_0+$ some constant. This contradicts to the fact that Log$z := \log|z| + i\mathrm{Arg}(z)$ can't be analytically extended to $(-\infty, 0)$. □

13. **Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion**

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

Proof. Assume f is not a polynomial, then $f^{(k)} \not\equiv 0$ for all $k \in Z_{\geq 0}$. Let Z_k denotes the zeros of $f^{(k)}$, which has no accumulation point by unique continuation (Theorem 4.8, Chapter 2). So Z_k is countable. (By constructing a one-to-one map from Z_k to $\mathbb{Q}^2 \subset \mathbb{C}$). However, by hypothesis, $\mathbb{C} = \cup_{k=0}^{\infty} Z_k$ is countable, which is a contradiction. $\square$

14. **Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if**

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Proof. We may assume $z_0 = 1$ by rotating the coordinate. By the hypothesis (i.e. compactness of S^1), there is $\delta > 0$ such that $B_{1+\delta}(0) \subset \Omega$ and the function $g(z) = f(z) - \sum_{j=1}^{N} a_{-j}(z-1)^{-j}$ is holomorphic on Ω, where N is the order of the pole. So for $z \in B_{1+\delta}(0)$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with $\limsup_{k\to\infty} \frac{b_{k+1}}{b_k} < 1$ (So $b_k \to 0$ as $k \to \infty$). For $z \in B_1(0)$,

$$\sum_{n=0} a_n z^n = \sum_{k=0}^{\infty} b_k z^k + \sum_{j=1}^{N} a_{-j}(z-1)^{-j}$$

Note that

$$(z-1)^{-j} = \frac{(-1)^{j-1}}{(j-1)!}\Big(\frac{d}{dz}\Big)^{j-1}(z-1)^{-1} = \frac{(-1)^j}{(j-1)!}\Big(\frac{d}{dz}\Big)^{j-1}\sum_{k=0}^{\infty} z^k = \frac{(-1)^j}{(j-1)!}\sum_{s=0}^{\infty}\frac{(s+j-1)!}{s!}z^s,$$

which converges for every compact subsets of $B_1(0)$, so we can rearrange the series as follows:

$$\begin{aligned}
\sum_{n=0} a_n z^n &= \sum_{k=0}^{\infty} b_k z^k + \sum_{j=1}^{N}\Big(\frac{(-1)^j}{(j-1)!}\sum_{s=0}^{\infty}\frac{(s+j-1)!}{s!}z^s\Big) \\
&= \sum_{k=0}^{\infty} b_k z^k + \sum_{s=0}^{\infty}\Big(\sum_{j=1}^{N}\frac{(-1)^j}{(j-1)!}\frac{(s+j-1)!}{s!}\Big)z^s \\
&= \sum_{k=0}^{\infty}\big(b_k + P(k)\big)z^k,
\end{aligned}$$

where $P(k)$ is a polynomial of order at most $N-1$. By the uniqueness theorem, $a_n = b_n + P(n)$ for each $n \in \mathbb{Z}_{\geq 0}$. Therefore,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{P(n)}{P(n+1)} = 1.$$

$\square$

15. **Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if $|f(z)| = 1$ whenever $|z| = 1$, then f is constant.**

Proof. Define $g(z)$ by $\left(\overline{f(\frac{1}{\bar{z}})}\right)^{-1}$ when $|z| > 1$ and $f(z)$ when $|z| \leq 1$. (Note that $\frac{1}{\bar{z}}$ appears as the reflection of z with respect to the unit circle; conjugate of f makes the function holomorphic on $|z| > 1$; the reciprocal makes g continuous on $|z| = 1$).

If we can show g is an bounded and entire function, then g is a constant by Liouville's theorem. Continuity is easy to show by using $f(S^1) \subset S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Also note that g is bounded by some positive constant M, since f is continuous and non-vanishing, the range of f excludes a neighborhood of zero, so f^{-1} is bounded on $|z| \leq 1$.

Let L denote the left open half plane. Note that the map $\phi : z \mapsto 1/z$ is a bijection from the $L \setminus \overline{B_1(0)}$ onto $L \cap B_1$. Given a triangle $T \subset L \setminus \overline{B_1(0)}$, then it's easy to prove that its interior is contained in $L \setminus \overline{B_1(0)}$ and $\phi(T)$ is a closed contour whose interior is contained in $L \cap B_1$ and

$$\int_T g(z)\, dz = \int_{\phi(T)} \left(w^2 \overline{f(\overline{w})}\right)^{-1} dw = 0.$$

by the holomorphicity of $\left(w^2 \overline{f(\overline{w})}\right)^{-1}$. So g is holomorphic on $L \setminus S^1$ by Morera's theorem. Using the same approximation argument as Schwarz reflection principle, we know $\int_T g(z)\, dz = 0$ is also true even any vertex of the above T lies on the unit disc.

Next, we show the holomorphicity on a $\frac{1}{2}$-tubular neighborhood of $L \cap S^1$. When two vertexes of a triangle T lies in $|z| = 1$ and one lies in $|z| > 1$, we subdivide the triangle into the T_i and T_o denote the part inside and outside $B_1(0)$ respectively. Note that T_i and T_o consists of two straight lines from the origin triangle and the arc from the unit circle, we can approximate T_i by T_i^ϵ which lies in $L \cap B_1(0)$ for each $\epsilon > 0$ and is piecewise smooth (so that the Cauchy's theorem we have in Chapter 2 can be applied) . By the uniform continuity ,finite length of the arc and holomorphicity of g on $L \cap B_1(0)$, $\int_{T_i} g(z) = \lim_{\epsilon\to 0} \int_{T_i^\epsilon} g(z) = 0$. Similarly, $\int_{T_o} g(z) = 0$ and hence $\int_T g(z) = \int_{T_i} g(z) + \int_{T_o} g(z) = 0$. For arbitrary triangle T, one can reduce to the previous case by subdivide it easily.

Similarly, g is holomorphic on the right-, upper-, and lower- open half plane, and of course, holomorphic on the unit disc. Since holomorphicity is a local property, we have shown g is an entire function.

□

Remark 10. This looks like a circle version of Schwarz reflection principle. Also take a look at Problem 3.

Problems

1. **Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be regular for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f, and in this case C is called the natural boundary of f. Here are some typical examples:**

   (a) Let
   $$f(z) = \sum_{n=0}^{\infty} z^{2^n} \text{ for } |z| < 1$$
   Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.]

   (b) fix $0 < \alpha < \infty$. Show that the analytic function f defined by
   $$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \text{ for } |z| < 1$$
   extends continuously to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

Proof. (a)

(b) □

Remark 11. One can see more discussions in Rudin [1, Chapter 16], especially 16.2-16.8 and its exercise 16.1. (And of course, Titchmarsh's *The Theory of functions*)

2. **Let**

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n \text{ for } |z| < 1$$

where $d(n)$ **denotes the number of divisors of** n**. Observe that the radius of convergence of this series is 1. Verify the identity**

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}$$

Using this identity, show that if $z = r$ **with** $0 < r < 1$**, then**

$$|F(r)| \geq c(1 - r) \log(1/(1 - r))$$

as $r \to 1$**. Similarly, if** $\theta = 2\pi p/q$ **where p and q are positive integers and** $z = re^{i\theta}$**, then**

$$|F(re^{i\theta})| \geq c_{p/q} \frac{1}{1 - r} \log(1/(1 - r))$$

as $r \to 1$**. Conclude that** F **cannot be continued analytically past the unit disc.**

Proof. □

3. **Morera's theorem states that if** f **is continuous in** $\mathbb{C}$**, and** $\int_T f(z)\, dz = 0$ **for all triangles** T**, then** f **is holomorphic in** $\mathbb{C}$**. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.**

(a)Suppose that f **is continuous on** $\mathbb{C}$**, and** $\int_C f(z)\, dz = 0$ **for every circle** C**. Prove that** f **is holomorphic.**

(b) More generally, let Γ **be any toy contour, and** f **the collection of all translates and dilates of** Γ**. Show that if** f **is continuous on** $\mathbb{C}$**, and**

$$\int_\gamma f(z)\, dz = 0 \text{ for all } \gamma \in f$$

then f **is holomorphic. In particular, Morera's theorem holds under the weaker assumption that** $\int_T f(z)\, dz = 0$ **for all equilateral triangles.**

[Hint: As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + O(|z - z_0|^2)$ for z near z_0. Integrating this expansion over small circles around z_0 yields $\frac{\partial f}{\partial \bar{z}} = b = 0$ at z_0. Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

In general, let $\varphi(w) = \varphi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leq \varphi(w) \leq 1$, and $\int_{\mathbb{R}^2} \varphi(w)\, dV(w) = 1$, where $dV(w) = dxdy$, and $\int$ denotes the usual integral of a function of two variables in $\mathbb{R}^2$. For each $\epsilon > 0$, let $\varphi_\epsilon(z) = \epsilon^{-2}\varphi(\epsilon^{-1}z)$, as well as

$$f_\epsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\epsilon(w)\, dV(w),$$

where the integral denotes the usual integral of functions of two variables, with $dV(w)$ the area element of $\mathbb{R}^2$. Then f_ϵ is smooth, satisfies condition for circle in (a), and $f_\epsilon \to f$ uniformly on any compact subset of $\mathbb{C}$.]

Proof. □

4. **Prove the converse to Runge's theorem: if K is a compact set whose complement is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomial on K.**

   [Hint: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z0)$. If f can be approximated uniformly by polynomials on K, show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (Chapter 3) to show that this inequality continues to hold for all z in the component of K^c that contains z_0.]

Proof. □

5. **There exists an entire function F with the following "universal" property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers, so that**

$$\lim_{k\to\infty} F(z + N_k) = h(z)$$

   uniformly on every compact subset of $\mathbb{C}$.

   (a) Let $p_1, p_2, \cdots$ denote an enumeration of the collection of polynomials whose coefficients have rational real and imaginary parts. Show that it suffices to find an entire function F and an increasing sequence $\{M_n\}$ of positive integers, such that

$$|F(z) - p_n(z - M_n)| < \frac{1}{n} \text{ whenever } z \in D_n, \tag{3}$$

   where D_n denotes the disc centered at M_n and of radius n. [Hint: Given h entire, there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} p_{n_k}(z) = h(z)$ uniformly on every compact subset of $\mathbb{C}$.]

(b) Construct F satisfying (3) as an infinite series

$$F(z) = \sum_{n=1}^{\infty} u_n(z)$$

where $u_n(z) = p_n(z - M_n)e^{-c_n(z-M_n)^2}$ and the quantities $c_n > 0$ and $M_n > 0$ are chosen appropriately with $c_n \to 0$ and $M_n \to \infty$. [Hint: The function e^{-z^2} vanishes rapidly as $|z| \to \infty$ in the sectors $\{|\arg z| < \pi/4 - \delta\}$ and $\{|\pi - \arg z| < \pi/4 - \delta\}$.]

In the same spirit, there exists an alternate "universal" entire function G with the following property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers, so that

$$\lim_{k\to\infty} D^{N_k}G(z) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$. Here D^jG denotes the j-th (complex) derivative of G.

Proof. □

References

[1] Walter Rudin. *Real and complex analysis*. McGraw-Hill Education, 3rd edition, 1987.