Elliptic PDEs of 2nd Order, Gilbarg and Trudinger
Chapter 8 Generalized Solutions and Regularity*

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1. *Proof.* I think it's done if one just need to check these conditions still imply the first ineqaulity in the proof of Theorem 8.1. □

2. *Proof.* Given $\Omega' \subset\subset \Omega$, construct the smooth cutoff function η such that $0 \le \eta \le 1, \eta \equiv 1$ on Ω', vanishes outside the set $\{x \in \Omega : \operatorname{dist}(x, \Omega') < d'/2\}$ and $|D\eta| \le 4/d'$, $(d' := \operatorname{dist}(\Omega', \partial\Omega))$. By density theorem, we know $\eta^2 u \in W_0^{1,2}(\Omega)$. From the weak fomulation, we know

$$\int_\Omega a^{ij}(x)D_i u(x)D_j(\eta^2 u)(x) + b^i u(x)D_i(\eta^2 u)(x) - (\eta^2 u)(x)\Big(c^i(x)D_i u(x) + d(x)u(x)\Big)\,dx$$
$$= \int_\Omega -g(x)(\eta^2 u)(x) + f^i(x)D_i(\eta^2 u)(x)\,dx.$$

That is,

$$\lambda \int_\Omega \eta^2(x)|Du(x)|^2\,dx \le \int_\Omega a^{ij}(x)D_i u(x)D_j u(x)\eta^2(x)\,dx$$
$$= \int_\Omega a^{ij}(x)D_i u(x)D_j(\eta^2 u)(x) - 2a^{ij}(x)D_i u(x)(\eta u)(x)D_j\eta(x)\,dx$$
$$= \int_\Omega -g(x)(\eta^2 u)(x) + f^i(x)D_i(\eta^2 u)(x) - b^i u(x)D_i(\eta^2 u)(x) + (\eta^2 u)(x)\Big(c^i(x)D_i u(x) + d(x)u(x)\Big)$$
$$- 2a^{ij}(x)D_i u(x)(\eta u)(x)D_j\eta(x)\,dx$$
$$\le \int_\Omega \frac{g^2+u^2}{2} + \frac{|\eta f^i|^2}{2\lambda/2} + \frac{\lambda/2}{2}|\eta Du|^2 + |f^i\eta D_i\eta|^2 + |u|^2 + \frac{\eta^2(|b^i|^2+|c^i|^2)u^2}{2\lambda/4} + \frac{\lambda}{4}|\eta Du|^2$$
$$+ 2|b^i\eta D_i\eta|u^2 + |d|u^2 + \frac{\lambda}{4\Lambda^2}\Lambda^2|\eta Du|^2 + \frac{n4/d'|u|^2}{\frac{\lambda}{4\Lambda^2}}\,dx.$$

Hence,

$$\int_{\Omega'} |Du(x)|^2\,dx \le \int_\Omega \eta^2(x)|Du(x)|^2\,dx$$
$$\le 4\Big(\frac{\|g\|_2^2}{2\lambda} + \frac{\|f^i\|_2^2}{\lambda^2} + \frac{4\|f^i\|_2^2}{\lambda d'} + \|u\|_2^2\big(\frac{3}{2\lambda} + 2\nu^2 + \frac{8\nu}{d'} + \frac{16n\Lambda^2}{\lambda^2 d'}\big)\Big)$$

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Since we may assume $\Lambda \geq 1 \geq \lambda$, $M := \Lambda/\lambda \geq 1/\lambda$, we have

$$\|Du\|_{2,\Omega'}^2 \leq 2M\|g\|_2^2 + 4(M^2 + \frac{4M}{d'})\|f\|_2^2 + 4\Big(\frac{3M}{2} + 2\nu^2 + \frac{8\nu}{d'} + \frac{16nM^2}{d'}\Big)\|u\|_2^2,$$

that is,

$$\|Du\|_{2,\Omega'} \leq C(n, M, \nu, d')(\|u\|_2 + \|f\|_2 + \|g\|_2).$$

□

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □