Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 2 Laplace Equation

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1. Mimic the proof for Theorem 3.1.

2. *Proof.* I think we should assume $u \in C^2(\Omega \cup \Gamma)$. Let W be an open ball containing some interior part Γ' of the given portion Γ. Extend $u(x)$ on W to be 0 if $x \in W \setminus \Omega$. This extended function, still call it u, is in $C^2(\overline{W})$ due to the zero Dirichlet and Newmann boundary condition.

   We further consider a smaller portion $\Gamma'' \subset \Gamma'$ and a smaller ball $W' \subset W$ containing Γ'' Given $x'' \in \Gamma''$ and then given open ball $B \subset W$ with center at x'', then

$$\int_B \Delta u \, dx = \int_{\partial B} \frac{\partial u}{\partial n} \, ds = \int_{\partial B \cap \Omega} \frac{\partial u}{\partial n} \, ds = \int_{\partial(B \cap \Omega)} \frac{\partial u}{\partial n} \, ds = \int_{B \cap \Omega} \Delta u \, dx = 0.$$

   This implies $\Delta u = 0$ on Γ'' and hence implies $\Delta u = 0$ on W'.

   By analyticity of u in W', and the vansishing property on $w' \setminus \Omega$ which contains an open set by the regularity of boundary portion Γ, we see that $u \equiv 0$ on W' and hence on $W' \cup \Omega$. By analyticity again, $u \equiv 0$ on Ω.

   $\square$

3. **Let G be the Green's function for a bounded domain Ω. Prove that**

   (a) $G(x, y) = G(y, x)$ **for all** $x, y \in \Omega, x \neq y$;

   (b) $G(x, y) < 0$ **for all** $x, y \in \Omega, x \neq y$;

   (c) $\int_\Omega G(x, y) f(y) \, dy \to 0$ **as** $x \to \partial\Omega$, **if** f **is bounded and integrable on** Ω.

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Proof. (a) The idea is to apply the Green's second identity to $u(z) = G(x,z)$ and $v(z) = G(y,z)$ on $\Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y))$ and then let $\epsilon \to 0$. More precisely, we note that

$$\Big| \int_{\partial B_\epsilon(x)} G(x,z) \frac{\partial G(y,\cdot)}{\partial n}(z)\, dz \Big| \leqslant C\epsilon^{d-1}(1+\epsilon^{2-d})$$
$$\Big| \int_{\partial B_\epsilon(y)} G(x,z) \frac{\partial G(y,\cdot)}{\partial n}(z)\, dz \Big| \leqslant C\epsilon^{d-1}(1+\epsilon^{2-d}),$$

so both are tending to zero as $\epsilon \to 0$. On the other hand,

$$\int_{\partial B_\epsilon(x)} G(y,z) \frac{\partial G(x,\cdot)}{\partial n}(z)\, dz = \int_{\partial B_\epsilon(x)} G(y,z) \frac{\partial}{\partial n}(h + \Gamma(|x-z|))\, dz,$$

where the first term tends to zero by Stokes' theorem and the second term tends to $G(y,x)$ as the proof for (2.16). Similar for showing $\int_{\partial B_\epsilon(y)} G(x,z)\frac{\partial G(y,\cdot)}{\partial n}(z)\, dz \to G(x,y)$ as $\epsilon \to 0$.
(b) Given $x \in \Omega$. By the fact that the harmonic function h defined in (2.19) is bounded and $\Gamma(|x-y|) \to -\infty$ as $y \to x$, it's known that $G(x,y) < 0$ if $|y-x| < \delta$ for some small $\delta > 0$. The proof is now completed by applying the strong maximum principle to $G(x,\cdot)$ on $\Omega \setminus B_\delta(x)$.
(c) The key is that the unknown harmonic function $h_x(y)$ is positive by minimum principle. So $|G(x,y)| = -G(x,y) = -\Gamma(|x-y|) - h_x(y) < |\Gamma(|x-y|)|$. Now, given $\epsilon > 0$, we note that for any $x_0 \in \Omega$ and any $x \in B_\epsilon(x_0)$, we always have $B_{2\epsilon}(x_0) \subset B_{3\epsilon}(x)$

$$\begin{aligned}\Big| \int_\Omega G(x,y) f(y)\, dy \Big| &= \Big| \Big(\int_{\Omega \setminus B_{2\epsilon}(x_0)} + \int_{B_{2\epsilon}(x_0)} \Big) G(x,y) f(y)\, dy \Big| \\ &\leqslant \int_{\Omega \setminus B_{2\epsilon}(x_0)} |G(x,y) f(y)|\, dy + \int_{B_{3\epsilon}(x)} |G(x,y)| \|f\|_\infty\, dy \\ &\leqslant \int_{\Omega \setminus B_{2\epsilon}(x_0)} |G(x,y) f(y)|\, dy + M_d \epsilon^2 \|f\|_\infty.\end{aligned}$$

Now for the first term, we note that for fixed $y \in \Omega \setminus B_{2\epsilon}(x_0)$, $G(x,y) = G(y,x) = \Gamma(|x-y|) + h_y(x) \to 0$ as $x \to x_0$. Moreover, for $x \in B_\epsilon(x_0)$ and $y \in B_{2\epsilon}(x_0)$, we always have

$$|G(x,y)f(y)| \leqslant |\Gamma(|x-y|)||f(y)| \leqslant \epsilon^{2-d}|f| \in L^1(\Omega).$$

So Lebesgue dominated convergence theorem implies that $\int_{\Omega \setminus B_{2\epsilon}(x_0)} |G(x,y)f(y)|\, dy \to 0$ as $x \to x_0$. So the proof is completed.

Remark 1. It seems that the convergence in (c) is uniform in x. But I don't know how to prove it.

□

4. *Proof.* It's clear that U is continuous on $\Omega := \Omega^+ \cup T \cup \Omega^-$ and harmonic on Ω. Given $(x_0, 0) \in T$, let B be a ball centered at x_0 and compactly contained in Ω. By translating and

dilating the coordinates, we may assume that $x_0 = 0$ and B is the unit ball. Since u is continous on $\overline{B}$, we can find a harmonic function $v \in C^2(B) \cap C(\overline{B})$ such that $v = u$ on ∂B. By the Poisson formula and the fact $u(x, -t) = -u(x, t)$, $v(x, -t) = -v(x, t)$. In particular $v = 0$ on T and hence $v = u$ on $\partial B^+ \cup T$ and $\partial B^- \cup T$. Apply the maximum principle twice, we see $v = u$ on B and hence u is harmonic on B. □

Remark 2. If we assume $u \in C^2(\overline{\Omega^+})$, we can show $\Delta U = 0$ in Ω directly.

5. **Determine the Green's function for the annular region bounded by two concentric spheres in $\mathbb{R}^d$.**

Proof. By dilating and translating the coordinates, we may consider $\Omega = B_1(0) \setminus \overline{B_\lambda(0)}$ for some $0 < \lambda < 1$. We also assume $d \geqslant 3$ first. Let $\Gamma(r) = c_d r^{2-d}$ $(r = |x|)$ be the fundamental solution of Laplace operator with singularity at the origin, $\overline{x} = \frac{x}{|x|^2}$ be the reflection of x with respect to ∂B_1 and $\widetilde{x} = \frac{\lambda^2 x}{|x|^2}$.

Then $u_1 = \Gamma(|x - y|) - \Gamma(|x||\overline{x} - y|)$ is x-harmonic in $\Omega \setminus \{y\}$, equals to zero if $y \in \partial B_1$ and equals to $u_2 = \Gamma(\frac{|x|}{\lambda}|\widetilde{x} - y|) - \Gamma(\lambda^{-1}|\lambda^2 x - y|)$ if $y \in \partial B_\lambda$. Note that $u_1 - u_2$ is x-harmonic in $\Omega \setminus \{y\}$, equals to zero if $y \in \partial B_\lambda$ and equals to $-u_3 = -\Gamma(\lambda|\lambda^{-2}x - y|) + \Gamma(\lambda|x||\lambda^{-2}\overline{x} - y|)$ if $y \in \partial B_1$.

Then we consider $u_1 - u_2 + u_3$ which is x-harmonic in $\Omega \setminus \{y\}$, equals to zero if $y \in \partial B_1$ and equals to $u_4 = \Gamma(\frac{|x|}{\lambda^2}|\lambda^2\widetilde{x} - y|) - \Gamma(\lambda^{-2}|\lambda^4 x - y|)$ if $y \in \partial B_\lambda$.

Fix $y \in \overline{\Omega}$. Define the function $u(x)$ on $\Omega \setminus \{y\}$ by

$$-\Gamma(|x-y|)+\sum_{k=0}^{\infty}\Gamma(\lambda^{-k}|\lambda^{2k}x-y|)+\Gamma(\lambda^{k}|\lambda^{-2k}x-y|)-\Gamma(\lambda^{k}|x||\lambda^{-2k}\overline{x}-y|)+\Gamma(\lambda^{-k-1}|x||\lambda^{2k}\widetilde{x}-y|).$$

Note that each term converges uniformly in x on $\Omega \setminus \{y\}$ by M-test (comparing to geometric series). By the previous argument, we know u is harmonic on $\Omega \setminus \{y\}$, $u = 0$ for all $y \in \partial B_1$ since $u_{2k-1} = \Gamma(\lambda^{k-1}|\lambda^{-2k+2}x - y|) - \Gamma(\lambda^{k-1}|x||\lambda^{-2k+2}\overline{x} - y|) \to 0$, and $u = 0$ for all $y \in \partial B_\lambda$ since $u_{2k} = \Gamma(\frac{|x|}{\lambda^k}|\lambda^{2k-2}\widetilde{x} - y|) - \Gamma(\lambda^{-k}|\lambda^{2k}x - y|) \to 0$.

For planar case ($d = 2$), the boundary behavior is used the same argument. But I don't see the pointwise convergence for the series,

$$c_2^{-1}u(x) = -\log|x - y| + \sum_{k=0}^{\infty}\log\Big(\frac{\lambda^{-k}|\lambda^{2k}x - y|}{\lambda^{-k-1}|x||\lambda^{2k}\widetilde{x} - y|}\frac{\lambda^{k}|\lambda^{-2k}x - y|}{\lambda^{k}|x||\lambda^{-2k}\overline{x} - y|}\Big),$$

since the tails does NOT tend to zero. □

Two-Dimensional Case. incomplete ! □

6. Estimate the denominator in the integrand and then apply mean value theorem.

7. *Proof.* If u is subharmonic, then we derive the local sub mean-value inequality by taking the harmonic function h on B_δ with $h = u$ on ∂B_δ.

   On the other hand, if u satisfies the local sub mean-value inequality, then given $B \subset\subset \Omega$ and any harmonic function h in B with $u - h \leqslant 0$ on ∂B. Since $u - h$ satisfies the sub mean-value inequality in B, by the same proof of maximum principle, $u \leqslant h$ in B. □

8. *Proof.* Let ϕ_ϵ be the standard mollifiers. Then $\Delta(\phi_\epsilon * u) \geqslant 0$ on $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$. For $0 < \epsilon < 1$, since $\phi_\epsilon * u \in C^2(\Omega_\epsilon)$, we see for each $B_r(x) \subset\subset \Omega_\epsilon \subset \Omega$,

$$(\phi_\epsilon * u)(x) \leqslant \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} \phi_\epsilon * u \, ds.$$

   Since u is continuous in Ω, $\phi_\epsilon * u \to u$ locally uniform, and hence

$$u(x) = \lim_{\epsilon\to 0} (\phi_\epsilon * u)(x) \leqslant \lim_{\epsilon \to 0} \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} \phi_\epsilon * u \, ds = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} u \, ds.$$

   So u is subharmonic in Ω by Exercise 2.7. □

   Remark 3. This is the special case of Weyl's lemma, or so-called elliptic regularity theory.

9. *Proof.* Obviously, (i) implies (iii). (iii) implies (ii) by Exercise 2.8. Now we assume u is subharmonic in Ω. Suppose $\Delta u < 0$ at some point x, then $\Delta u < 0$ in some open ball $B_r(x)$ since $u \in C^2$. But this will contradict to the sub-mean inequality we obtained in Exercise 2.7 by the same argument as the proof of Theorem 2.1. □

10. *Proof.* This problem is done if we know the sequence of the harmonic lifting V_n is monotone increasing. But this is proved if v_n is monotone increasing, for example, we can modify the definition of v_n to be $\max\{v_1, v_2, \cdots, v_n, \inf \phi\}$, which is similar to the construction of Problem 2.16. □

11. *Proof.* □

12. This exercise is the special case of Exercise 6.3, and sometimes called the Zaremba's criterion. One of the counterparts of exterior cone condition for parabolic equations is the exterior tusk condition. See Lieberman [2, Exercise 3.11] and Lorenz [3, Section 3.11.4].

    Proof. Since the Laplace equation is invariant under translation and rotation, we may assume $\xi = 0$ and the exterior cone C is a right cone, that is, $C = \{x : \frac{x_n}{|x|} =: \cos\theta > \cos\alpha\}$ for some $\alpha \in (0, \frac{\pi}{2})$. Let $V = B_2(0) \cap C^c$. By truncating the cone properly and scaling, we may assume

$\Omega \cap B_2(0) \subset B_2(0) \cap C^c = V$. Consider the Dirichlet problem on V with continuous boundary value $g(x) = |x|$, then we have a harmonic function $u \in C^2(V) \cap C^0(\overline{V} \setminus \{0\})$ such that $u = g$ on $\partial V \setminus \{0\}$. Since g is subharmonic in V, and with the same boundary value as u, $u(x) \geqslant |x| > 0$ in V. (In this proof, the geometry property of cone condition is only used here ?)

If we can prove that $u(x) \to 0$ as $x \to 0$, then u is a local barrier at 0 and hence 0 is regular.

Let $v(x) := u(2x)$ defined on the set $\overline{V_2} \setminus \{0\}$, where $V_2 := \{\frac{1}{2}x : x \in V\}$. Since the constant function 2 and -2 are superfunction and subfunction to g on V respectively, $0 < u < 2$ on V by strong maximum principle and hence on $\partial V_2 \cap \partial B_1(0)$. Since $v = 2$ on $\partial V_2 \cap \partial B_1(0)$ and both u and v are continuous on $\partial V_2 \cap \partial B_1(0)$, there is some $\frac{1}{2} < \alpha < 1$ such that $u < \alpha v$ on $\partial V_2 \cap \partial B_1(0)$. On the remainder of $\partial V_2 \setminus \{0\}, u = \frac{1}{2}v$. Therefore, $u < \alpha v$ on $\partial V_2 \setminus \{0\}$. Since u and v are continuous on $\overline{V_2} \setminus \{0\}$, for $y \in \partial V_2 \setminus \{0\}, \lim_{z \to y, z \in V_2} \alpha v(z) - u(z) \geqslant 0$. Consider the function $\alpha v - u + \epsilon \Gamma$, where $\Gamma(x) = |x|^{2-n}$ for $n \geqslant 3$ and $\Gamma(x) = -\log|x|$ for $n = 2$. Then for each $\epsilon > 0$, $\liminf_{x \to y, x \in V_2} \alpha v(x) - u(x) + \epsilon \Gamma(x) \geqslant 0$ if $y \in \partial V_2 \setminus \{0\}$ and $\liminf_{x \to 0, x \in V_2} \alpha v(x) - u(x) + \epsilon \Gamma(x) = \infty$. Then weak minimum principle implies that $\alpha v - u + \epsilon \Gamma \geqslant 0$ on V_2. For each $x \in V_2$, since $0 \leqslant \alpha v(x) - u(x) + \epsilon \Gamma(x) \to \alpha v(x) - u(x)$ as $\epsilon \to 0$. Therefore,

$$0 \leqslant c := \limsup_{x \to 0, x \in V_2} u(x) \leqslant \alpha \limsup_{x \to 0, x \in V_2} u(2x) = \alpha c.$$

Since $\alpha < 1, 0 = c = \limsup_{x \to 0, x \in V_2} u(x) \geqslant \liminf_{x \to 0, x \in V_2} u(x) \geqslant 0$. □

Remark 4. Counterexamples of Lebesgue and Osgood's theorem on $\mathbb{R}^2$

Second Proof. incomplete ! We try to follow the hint given there. WLOG, we may assume $\xi = 0$ and the exterior cone C is a right cone. Consider the function $w(r, \theta) = r^\lambda f(\theta)$ □

13. **Let u be harmonic in $\Omega \subset \mathbb{R}^n$. Use the argument leading to (2.31) to prove the interior gradient bound,**

$$|D_i u(x_0)| \leqslant \frac{n}{d_0}[\sup_\Omega u - u(x_0)], d_0 = \textbf{dist}\,(x_0, \partial\Omega), \ \forall\, i.$$

If $u \geqslant 0$ in Ω, infer that

$$|D_i u(x_0)| \leqslant \frac{n}{d_0} u(x_0), \ \forall\, i.$$

Proof. (The first appearance of this proof in the internet seems to be [here])

If $\sup u = \infty$, then it's done. Otherwise, consider $v = u - u(x_0)$ and apply the mean-value theorem for $D_i v$ in ball $B := B_{d_{x_0}}(x_0)$, we have

$$D_i u(x_0) = D_i v(x_0) = \frac{1}{w_n d_0^n} \int_B D_i v(y)\,dy = \frac{1}{w_n d_0^n} \int_{\partial B} v(y)\nu_i\,dy \leqslant \frac{n}{d_0} \sup_{\Omega} v.$$

If $D_i u(x_0) \geq 0$, then we are done. If $D_i u(x_0) < 0$, it seems that we need to take advantages of the mean-value property of harmonic functions, that is,

$$\begin{aligned} 0 < -D_i u(x_0) &= \frac{1}{w_n d_0^n} \int_{\partial B} u(y)(-\nu_i)\,dy \\ &= \frac{1}{w_n d_0^n} \int_{\partial B} u(y)(1-\nu_i)\,dy - \frac{n}{d_0} u(x_0) \\ &\leq \frac{n}{d_0} \sup_{\Omega} u - \frac{n}{d_0} u(x_0). \end{aligned}$$

For the second one,

$$|D_i u(x_0)| = |\frac{1}{w_n d_0^n} \int_B D_i u(y)\,dy| = \frac{1}{w_n d_0^n} |\int_{\partial B} u(y)\nu_i\,dy| \leqslant \frac{1}{w_n d_0^n} \int_{\partial B} u(y)\,dy = \frac{n}{d_0} u(x_0).$$

□

14. **(a) Prove Liouville's theorem: A harmonic function defined over $\mathbb{R}^d$ and bounded above is constant. Therefore, a positive harmonic function on $\mathbb{R}^d$ is a constant function.**

 (b) If $d = 2$ prove that the Liouville theorem in part (a) is valid for subharmonic functions.

 (c) If $d > 2$ show that a bounded subharmonic function defined over $\mathbb{R}^d$ need not be constant.

Proof. (a) Apply Exercise 2.13 with $d_0 \to \infty$ or apply the Harnack inequality derived in Exercise 2.6 to $v(x) = -u(x) + \sup_{\mathbb{R}^d} u$. (c) Consider $v = \max\{-|x|^{2-d}, -1\}$.

(b) We may assume $\max_{\partial B_1} u = 0$ on by adding some constant. Since u is bounded above, there is a sequence of radii $\{M_n\} \to \infty$ as $n \to \infty$ such that $u(x) < \frac{1}{n} \log|x|$ for all $|x| \geqslant M_n$. For each n, apply the Maximum principle to $u(x) - \frac{1}{n}\log|x|$ on $B_{M_n}(0) \setminus \overline{B_1}(0)$, we see that $u \leqslant 0$ on $\mathbb{R}^d \setminus B_1(0)$. Then we see that, for each $r > 1$, $\max_{\overline{B_r}} u$ attains at the interior since $\max_{\partial B_1} u = 0$. Hence the strong maximum principle implies u is a constant function on B_r for each $r > 1$, that is, on $\mathbb{R}^d$. □

Remark 5. First, note the unboundedness at infinity of two-dimensional fundamental solution for Laplace equation. Second, in the above proof, we see that the assumption can be weaken to assume that $\limsup_{|x|\to\infty} \frac{u(x)}{\log|x|} \leqslant 0$.

Remark 6. If one wants a smooth counterexample, one can take a smooth compactly supported $\rho : \mathbb{R}^d \to \mathbb{R}$ with $0 \leqslant \rho \leqslant 1$ and $\int_{\mathbb{R}^d} \rho = 1$ and use the fundamental solution to construct a solution to $\Delta u = \rho$ given by

$$u(y) = \int_{\mathbb{R}^d} \frac{\rho(x)}{|x-y|^{d-2}}\, dy.$$

You can check directly that this is bounded.

Remark 7. For more structure theorems about positive harmonic functions, see Chapter 3 of Axler's book *"Harmonic Function Theory"*. In particular, check Bocher's Theorem therein.

Remark 8. The followings are other Liouville-type theorems in higher dimension:

Theorem 9. *Let $p \in (1, \infty)$ if $N = 1, 2$ and $p \in (1, \frac{N}{N-2}]$ if $N \geqslant 3$. If $u \in L^p(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ satisfies $-\Delta u \leqslant 0$ in $\mathbb{R}^N$, then $u \equiv 0$.*

Remark 10. The counterexample we construct in (c) only stays in $L^p(\mathbb{R}^N)$ with $p > \frac{N}{N-2}$.

The proof is taken from M. Damlakhi's paper "L^p-subharmonic functions in $\mathbb{R}^n$" Electronic Journal of Differential Equations Vol. (2016), No. 322, pp. 1-4.

Proof. The key is Riesz Decomposition Theorem for super-sub harmonic functions. We divide the proof into three steps:

Step I: When $u \geqslant 0$, sub mean-inequality implies $u \equiv 0$.

Step II: For general, L^p subharmonic function u, one must have $u \leqslant 0$. (This is proved by considering u^+.)

Step III: Completion the proof:

For $d = 2$, this is done in this Exercise. For $d \geqslant 3$, we use the Riesz decomposition theorem (see Helmes' book "*Potential Theory (2nd Ed.) Theorem 4.5.11*" or Armitage's book "*Classical Potential Theory* Page xv and Theorem 4.4.1")

Since $-u$ a is nonnegative supharmonic function, by Riesz decomposition $-u = v + h$ where v is a potential and $h \geqslant 0$ is harmonic and hence constant.

Since $0 \leqslant v \leqslant -u$, $v \in L^p(\mathbb{R}^d)$ and hence $h = 0$. That is,

$$-u(y) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}}\, d\mu(x)$$

for a unique nonnegative measure μ. Suppose $u \not\equiv 0$, then μ is nontrivial. By Fatou's lemma, we know

$$\liminf_{|y|\to\infty} -u(y)|y|^{d-2} \geqslant \mu(\mathbb{R}^d) > 0,$$

which leads to the L^p integrability of $|y|^{2-d}$ near the infinity. But this is impossible since $p(2-d)+d-1>-1$. So we know $u\equiv 0$.

$\square$

15. $\int |Du|^2 = \int \operatorname{div}(uDu) - \int u\Delta u\, dx$ and then use the boundary condition and Young's inequality.

16. *Proof.* Given $B_\rho(x)\subset\subset\Omega$, there exists a dense subset $\{x_k\}$ of $B_\rho(x)$. Note that for each k, there is $\{v_k^j\}\subset S_\varphi$ such that $v_k^j(x_k)\to u(x_k)$ as $j\to\infty$. Define $v^j=\max\{v_1^j, v_2^j\cdots v_j^j\}$, then for each k, $u(x_k)\geqslant v^j(x_k)\geqslant v_k^j(x_k)$ and hence $v^j(x_k)\to u(x_k)$ for all k. Let $V^j:=\max\{v^j,\inf_{\partial\Omega}\varphi\}$, then $\inf_{\partial\Omega}\varphi\leqslant V^j\leqslant\sup_{\partial\Omega}\varphi$ by weak maximum principle. Consider the harmonic lifing $\widetilde{V^j}$ of V^j on $B_\rho(x)$, then by weak maximum principle $v^j\leqslant V^j\leqslant\widetilde{V^j}$, and hence $\widetilde{V^j}$ converges to $u(x_k)$ for each k. Moreover, by Theorem 2.9 (Harnack convergence theorem, see Problem 2.10) or Theorem 2.11 (interior gradient estimate and Arzela-Ascoli), we see that for each $0<r<\rho$ we can find a subsequence of $\{\widetilde{V^j}\}$ converges uniformly on $B_r(x)$ to v_r which is hence harmonic in $B_r(x)$ and we can define $v(y)=v_r(y)$ if $y\in B_r(x)$ and hence v is harmonic in B_ρ and $u(x_k)=v(x_k)$ for each k.

    Given $x_0\in B_r(x)$, we can choose a subsequence x_{k_j} of x_k converging to x_0. By continuity of v and lower semi-continuity of u, we see

$$v(x_0)=\lim_{j\to\infty}v(x_{k_j})=\liminf_{j\to\infty}u(x_{k_j})\geqslant u(x_0).$$

    If $v(y)>u(y)$ for some $y\in B_r(x)$, then there is some $\widetilde{V^j}\in S_\varphi$ such that $\widetilde{V^j}(y)>u(y)$ which contradicts to the definition of u.

    Hence u is harmonic in $B_\rho(x)$ since $u=v$. Since $B_\rho(x)\subset\subset\Omega$ is arbitrary chosen, u is harmonic in Ω. $\square$

17. *Proof.* $\square$

18. **Let u be harmonic in (open, connected) $\Omega\subset\mathbb{R}^n$, and suppose $B_c(x_0)\Subset\Omega$. If $a\le b\le c$, where $b^2=ac$, show that**

$$\int_{|w|=1}u(x_0+aw)u(x_0+cw)\,dw=\int_{|w|=1}u^2(x_0+bw)\,dw.$$

    Hence, conclude that if u is constant in a neighborhood it is identically constant

    Proof. Since the Laplace equation is isotropic everywhere, we may assume $x_0=0$. Note that for each $s,w\in S^n$, $|cs-bw|=|bs-cw|$. By this fact, $a=\frac{b^2}{c}$, Fubini's theorem and Poisson integral formula, we have

$$
\begin{aligned}
\int_{S^n} u(aw)u(cw)\,dw &= \int_{S^n} u(cw)\frac{b^{n-1}}{nw_n b}\int_{S^n}\frac{b^2-a^2}{|bs-aw|^n}u(bs)\,ds\,dw \\
&= \int_{S^n} u(bs)\frac{c^{n-1}}{nw_n c}\int_{S^n}\frac{c^2-b^2}{|cs-bw|^n}u(cw)\,dw\,ds \\
&= \int_{S^n} u(bs)\frac{c^{n-1}}{nw_n c}\int_{S^n}\frac{c^2-b^2}{|bs-cw|^n}u(cw)\,dw\,ds \\
&= \int_{S^n} u(bs)u(bs)\,ds.
\end{aligned}
$$

If u is constant C (we may assume $C = 0$) on some open subset of Ω, then the open subset $W := (\{u = 0\})^{\circ}$ is non-empty. If we can prove that W is closed in Ω, then by the connectedness of Ω, $\Omega = W$.

Let $x_0 \in \Omega \setminus W$ be a limit point of W, then we can find a radius c such that $B(x_0, 2c) \subset\subset \Omega$ and $q \in W \cap B(x_0, c/2)$.

Let a be the largest radius such that $B(q, a) \subset W$, then $a \leqslant c/2$ and $B(q, c) \subset B(x_0, 2c) \subset\subset \Omega$. According to the integral identity we just proved, for each $0 < \epsilon < a$,

$$
\int_{S^n} u(q + \sqrt{(a-\epsilon)c}w)^2\,dw = 0.
$$

By weak maximum and minimum principle, we conclude that $u = 0$ on $B(q, \sqrt{(a-\epsilon)c})$. But this contradicts to the definition of a since $\sqrt{(a-\epsilon)c} > a$ if ϵ is small.

Therefore, if $x_0 \in \Omega$ is a limit point of W, then $x_0 \in W$. This is equivalent that $W = \overline{W} \cap \Omega$ and thus W is closed in Ω. $\square$

Another interesting result is the following generalization of Liouville's theorem:

Theorem 11. *[1, Theorem 6] Suppose u is harmonic in $\mathbb{R}^n$ and vanishes at the origin. If a constant A exists such that for every $\rho > 0$*

$$
\int_{S^n} u^+(\rho w)\,dw \leqslant A,
$$

then $u \equiv 0$. (This includes the case that u is bounded above.)

Proof. By mean value theorem and continuity at the origin, for each $\epsilon > 0$, we can find a radius ρ such that $|u(\rho w)| \leqslant \epsilon$ for all $w \in S^n$ and hence

$$
\begin{aligned}
\int_{S^n} u(w)^2\,dw &= \int_{S^n} u(\rho w)u(\rho^{-1}w) = \int_{S^n}[u(\rho w)+\epsilon]u(\rho^{-1}w)\,dw \leqslant \int_{S^n}[u(\rho w)+\epsilon]u^+(\rho^{-1}w)\,dw \\
&\leqslant 2\epsilon\int_{S^n} u^+(\rho^{-1}w)\,dw \leqslant 2\epsilon\mu.
\end{aligned}
$$

So $u = 0$ on S^n and hence on $B_1(0)$ by weak maximum and minimum principle. Therefore, $u \equiv 0$ by the exercise we just proved. $\square$

Remark 12. The usual way to prove the uniqueness theorem for analytic functions are through power series expansion to show the closed subset $K = \{x \in \Omega : D^\alpha f(x) = 0,\ \forall \alpha\}$ of domain Ω is open. we also show the following theorem as an application of uniqueness theorem which distinguishes how different the real analyticity and complex holomorphy are.

Theorem 13 (Open Mapping Theorem). *Let Ω be a domain in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ is holomorphic and non-constant, then for every open subset V of Ω, $f(V)$ is an open subset of $\mathbb{C}$. Note that this is not true for $g(x) = x^2$ defined on $\mathbb{R}$ since $g((-1, 1)) = [0, 1)$.*

Proof. We may assume $n > 1$ by assuming the one-dimensional open mapping theorem in complex analysis is known.

Given open subset V and $a \in V$. By uniqueness theorem, there is some $b \in V$ such that $f(a) \neq f(b)$. Let $D = \{s \in \mathbb{C} : a + s(b - a) \in V\} \neq \emptyset$, D is open in $\mathbb{C}$ since V is open.

Let $g(s) = f(a + s(b - a))$, then g is holomorphic on D and non-constant since $g(0) = f(a) \neq f(b) = g(1)$. Then $g(D)$ is an open set containing $g(0) = f(a)$. Therefore $f(V)$ is open. $\square$

Note that this Open Mapping Theorem implies the strong maximum principle easily.

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(2019.06.05) The proof for Problem 13 I rewrite last time is still wrong since I forgot to consider the case $D_i u(x_0) < 0$.

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