Real Analysis, E.M.Stein-R.Shakarchi

Chapter 5 Hilbert Spaces Several Examples *

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Exercises

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

11. *Proof.* □

12. *Proof.* □

13. *Proof.* Let $L = \sum_{|\alpha| \leq k} a_\alpha D^\alpha$, where $k \in \mathbb{N}, \alpha \in (\mathbb{Z}_{\geq 0})^d, d \geq 2$ and $a_\alpha \in C$ for each α. Denote the symbol of L by $P(\xi) = \sum_{|\alpha| \leq k} a_\alpha \xi^\alpha$

*Last Modified: 2017/01/15

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Let $r = (r^1, \cdots, r_d) \in \mathbb{C}^d$ and $u_r(x_1, \cdots, x_d) = \exp(r^1 x_1 + \cdots + r^d x_d)$, then $Lu_r = 0$ if and only if $P(r) = 0$.

Given finite many zeros $r_k \in \mathbb{C}^d$ of P, if $0 \equiv \sum_{k=1}^N c_k u_{r_k}$, we partition $\{1, \cdots, N\}$ into $\{J_1^1, J_2^1, \cdots J_N^1(1)\}$ where each J_k^1 has the property that for each $r_q, r_s \in J_k^1, r_q^1 = r_s^1$. we further partition each J_k^1, into $\{J_m^2\}_{m=1}^{N(2)}$ where each J_k^2 has the property that for each $r_q, r_s \in J_k^2, r_q^2 = r_s^2$ and of course $r_q^1 = r_s^1$, furthermore we denote $J_q^2 = \emptyset$ if $\cup_{m=1}^{q-1} J_m^2 = J_k^1$ for convenience. Inductively, we have a family of $\{J_m^k\}_{m=1}^{N(k)}, k = 1, \cdots, d-1$, which has the properties that for each $r_q, r_s \in J_m^k, r_q^i = r_s^i$ for every $i = 1, \cdots, k$ and for each k, m, $J_m^k \subseteq J_n^{k-1}$ for some n between 1 and $N(k-1)$. Hence

$$0 \equiv \sum_{k=1}^{N} c_k u_{r_k} = \sum_{k^1=1}^{N(1)} \sum_{j \in J_{k^1}^1} c_j u_{r_j} = \sum_{k^1=1}^{N(1)} \Big(\sum_{j \in J_{k^1}^1} c_j \exp(r_j^2 x_2 + \cdots + r_j^d x_d) \Big) \exp(r_{k^1}^1 x_1),$$

where $r_{k^1}^1$ is the common value for vectors in $J_{k^1}^1$. This implies that for each $k^1 = 1, \cdots, N(1)$,

$$\sum_{j \in J_{k^1}^1} c_j \exp(r_j^2 x_2 + \cdots + r_j^d x_d) = 0.$$

Inductively, we have for each $m = 1, \cdots, d-1$, each $k^m = 1, \cdots, N(m)$,

$$\sum_{j \in J_{k^{d-1}}^{d-1}} c_j \exp(r_j^d x_d) = 0.$$

Hence $c_j = 0$ for all $j = 1, \cdots, N$, which means $\{u_j\}_1^N$ is linearly independent.

The desired result follows from the fact that a polynomial of d variables ($d \geq 2$) has infinitely many zeros. $\square$

14. *Proof.* Define $H(t) = \displaystyle\int_0^t G(s)\, ds$, then H is absolutely continuous. Given $\phi \in C_c^\infty$,

$$\begin{aligned}
\int_0^1 [H - F]\phi'\, dx &= \int_0^1 \int_0^x G(t)\, dt \phi'(x)\, dx - \int_0^1 F(x)\phi'(x)\, dx \\
&= \int_0^1 \int_t^1 \phi'(x)\, dx G(t)\, dt + \int_0^1 G(x)\phi(x)\, dx \\
&= -\int_0^1 \phi(t)G(t)\, dt + \int_0^1 G(x)\phi(x)\, dx = 0.
\end{aligned}$$

(This argument avoids the use of the Lebesgue Differentiation theorem if we only want to show the 1-d weakly differentiable function $F = H$ a.e., an absolutely continuous function.) We are done if we apply the following theorem from Distribution theory with a partial proof, the whole proof can be found in Lieb and Loss [1, Section 6.9-6.11].

Theorem 1. *Let T be a distribution on an open connected set Ω in $\mathbb{R}^n$. If all the first distributional derivatives of T are 0, then T is a constant function.*

Proof. (Taken from Wheeden and Zygmund [3, Page 464], we assume $T \in L^1_{loc}(\Omega)$.)

Let $k_\epsilon(x)$ be the standard mollifier. Given B be a small ball contained in Ω. For small ϵ, by hypothesis, we see that for each $y \in B$,

$$\begin{aligned} 0 &= \int_\Omega T(x)\frac{\partial}{\partial x_i}[k_\epsilon(y-x)]dx \\ &= -\frac{\partial}{\partial y_i}\int_\Omega T(x)[k_\epsilon(y-x)]dx, \quad \textit{by LDCT} \end{aligned}$$

Hence, for all small ϵ, there is a constant c_ϵ such that (we omit some irrelevant details)

$$\int_\Omega T(x)k_\epsilon(y-x)dx = c_\epsilon \quad \textit{for all } y \in B.$$

As $\epsilon \to 0$, this integral converges to T in $L^1(B)$, hence it converges to $T(y)$ for a.e. y (up to subsequence), and consequently c_ϵ converges to some constant c, and $T(y) = c$ a.e. in B. Since Ω is path connected, T is constant a.e. in Ω. □

Remark 2. I think the reader should try to prove the general case that T is a distribution by modifying the above proof.

□

15. *Proof.* If for each $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$(-1)^{|\alpha|}\int_{\mathbb{R}^d} f(x)\Big[\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi\Big](x)\,dx = \int_{\mathbb{R}^d} g(x)\varphi(x)\,dx,$$

then the left hand side

$$\begin{aligned} L.H.S &= \lim_{R\to\infty}(-1)^{|\alpha|}\int_{\mathbb{R}^d} f(x)\chi_{\{|x|\le R\}}\Big[\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi\Big](x)\,dx \\ &= \lim_{R\to\infty}(-1)^{|\alpha|}\int_{\mathbb{R}^d} \mathscr{F}(f(x)\chi_{\{|x|\le R\}})(\xi)\cdot\overline{\mathscr{F}\Big[\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi\Big](\xi)}\,d\xi \\ &= \int_{\mathbb{R}^d} \mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha\overline{\mathscr{F}(\varphi)(\xi)}\,d\xi, \end{aligned}$$

and the right hand side

$$\begin{aligned} R.H.S &= \lim_{R\to\infty}\int_{\mathbb{R}^d} g(x)\chi_{\{|x|\le R\}}\varphi(x)\,dx = \lim_{R\to\infty}\int_{\mathbb{R}^d}\mathscr{F}(g(x)\chi_{\{|x|\le R\}})(\xi)\cdot\overline{\mathscr{F}(\varphi)(\xi)}\,d\xi \\ &= \int_{\mathbb{R}^d}\mathscr{F}(g)(\xi)\overline{\mathscr{F}(\varphi)(\xi)}\,d\xi. \end{aligned}$$

Since $C_c^\infty(\mathbb{R}^d)$ is dense in L^2 and the surjective map $\mathscr{F}$ preserves the L^2 norm, for each $\Phi \in L^2$,

$$|\int_{\mathbb{R}^d} \mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha \overline{\Phi(\xi)}\, d\xi| < \infty.$$

By an well-known characterization of L^p function, $\mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha \in L^2(\mathbb{R}^d)$. (See the following Theorem 3)

Hence for each $\Phi \in L^2$, $\int_{\mathbb{R}^d} \Big(\mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha - \mathscr{F}(g)(\xi)\Big)\overline{\Phi(\xi)}\, d\xi = 0$.

In particular, we take $\Phi(\xi) = \mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha - \mathscr{F}(g)(\xi) \in L^2(\mathbb{R}^d)$ to conclude that

$$\mathscr{F}(f)(\xi)(2\pi i\xi)^\alpha \equiv \mathscr{F}(g)(\xi).$$

□

Theorem 3. *(Rudin, [2, Exercise 6.4])*

Suppose $1 \le p \le \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure (this assumption is for the case $p = 1$ in the Riesz representation theorem for $L^p(\mu)$) and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$, then $g \in L^q(\mu)$.

Proof. Suppose there is no $C > 0$, such that $\|fg\|_1 \le C\|f\|_p$. Then there exist f_n such that $\|f_n\|_p = 1$ and $\int |f_n g| > 3^n$. Let $f = \sum 2^{-n}|f_n|$, then $\|f\|_p \le 1$ and for each n

$$\int |fg| > \int |f_n g| 2^{-n} > (\frac{3}{2})^n$$

This contradicts to $fg \in L^1$ and therefore the map $f \mapsto \int fg$ from $L^p \to \mathbb{R}$ or $\mathbb{C}$ is bounded. if $1 \le p < \infty$, then by Riesz Representation theorem, $g = \tilde{g} \in L^q$ a.e.. For $p = \infty$, if g is not integrable, then it contradicts to the hypothesis by taking $f \equiv 1$. □

16. *Proof.* □

17. *Proof.* □

18. *Proof.* □

19. *Proof.* □

20. **(Remark on the Dirichlet principle, the Hadamard example)**

We state a general result first: Let F be a continuous function on $\overline{B_1(0)} \subset \mathbb{R}^2$, C^1 in $B_1(0)$ and $\int_{B_1} |\nabla F|^2 < \infty$. Let $f(e^{i\theta})$ denote the restriction of F to ∂B_1 and write

$$f(e^{i\theta}) \sim \sum_{n\in\mathbb{Z}} a_n e^{in\theta}.$$

(a) Prove that $\sum_{n\in\mathbb{Z}} |n||a_n|^2 < \infty$.

[Hint: Write $F(re^{i\theta}) \sim \sum_{n\in\mathbb{Z}} F_n(r)e^{in\theta}$, **with** $F_n(1) = a_n$**. Express** $\int_{B_1} |\nabla F|^2$ **in the polar coordinates, and use the fact that**

$$\frac{1}{2}|F(1)|^2 \leq L^{-1}\int_{1/2}^{1} |F'(r)|^2\, dr + \int_{1/2}^{1} |F(r)|^2\, dr,$$

for $L \geq 2$**; apply to** $F = F_n, L = |n|$**.]**

(b) For each $\alpha > 0$**, consider the function**

$$f(\theta) = f_\alpha(\theta) = \sum_{n=0}^{\infty} 2^{-n\alpha}e^{i2^n\theta}$$

Which is easy to see that f **has a holomorphic extension to** B_1**:**

$$u(r,\theta) = \sum_{n=0}^{\infty} 2^{-n\alpha}r^{2^n}e^{i2^n\theta}.$$

Use (a) to show that if $\alpha \leq 1/2$**, then the energy functional**

$$\int_{B_1} |\nabla u|^2 = \infty$$

Proof. □

Problems

1. *Proof.* □
2. *Proof.* □
3. *Proof.* □
4. *Proof.* □
5. *Proof.* □
6. *Proof.* □

References

[1] Elliott H Lieb and Michael Loss. *Analysis*, volume 14. American Mathematical Society, Providence, RI,, 2nd edition, 2001.

[2] Walter Rudin. *Real and Complex Analysis.* Tata McGraw-Hill Education, 3rd edition, 1987.

[3] Richard L Wheeden. *Measure and Integral: An Introduction to Real Analysis*, volume 308. CRC Press, 2nd edition, 2015.