Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 12 Nonlinear Wave Equations

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1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. **Assume u solves**

$$u_{tt} - \Delta u + du_t = 0 \ \textbf{ in } \mathbb{R}^n \times (0, \infty),$$

which for $d > 0$ is a damped wave equation. Find a simple exponential term that, when multiplied by u, gives a solution v of

$$v_{tt} - \Delta v + cv = 0$$

for a constant $c < 0$. (This is the opposite of the sign for the Klein-Gordon equation.)

Proof. $v(x,t) = e^{\frac{dt}{2}} u(x,t)$ solves the Klein-Gordon like equation with $c = -\frac{d^2}{4} = (i\frac{d}{2})^2$. □

5. **Check that for each given $y \in \mathbb{R}^n, y \neq 0$, the function $u = e^{i(x \cdot y - \sigma t)}$ solves the Klein-Gordon equation**

$$u_{tt} - \Delta u + m^2 u = 0$$

provided $\sigma = (|y|^2 + m^2)^{\frac{1}{2}}$. The phase velocity of this plane wave solution is $\frac{\sigma}{|y|} > 1$. Why does this not contradict the assertions in §12.1 that the speed of propagation for solutions is less than or equal to one?

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Proof. ☐

The following two problems show that the decay rates for solutions of wave equation in 3-dimension and 2-dimension are different.

***Remark* 1.** In general, the time decay rate is $t^{\frac{1-n}{2}}$ (where n is the spatial dimension). This can be shown by modifying the proof given here for $n = 2, 3$.

***Remark* 2.** Strichartz estimates are another important estimates for wave equations. See [1] or [2, Chapter IV] for the proofs and applications to semilinear wave equations.

6. **Let u be the solution of**

$$\begin{cases} u_{tt} - \Delta u = 0 \ \text{ in } \ \mathbb{R}^3 \times (0, \infty) \\ u(x, 0) = f(x), u_t(x, 0) = g(x) \ \text{ on } \ \mathbb{R}^3 \end{cases}$$

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ have compact support. Show there exists a constant $C > 0$ such that for each $x \in \mathbb{R}^3, t > 0$,

$$|u(x, t)| \le C/t$$

Proof. We suppose both the supports of f and g are included in $B_R(0)$.

From uniqueness theorem (proved by the conservation law of energy) and the solution formula given in the textbook and Problem 4, that is,

$$\begin{aligned} u(x,t) &= \partial_t \Big(\frac{t}{4\pi} \int_{S^2} f(x - t\gamma)\, d\sigma(\gamma) \Big) + \frac{t}{4\pi} \int_{S^2} g(x - t\gamma)\, d\sigma(\gamma) \\ &= \frac{1}{4\pi} \int_{S^2} f(x - t\gamma)\, d\sigma(\gamma) + \frac{t}{4\pi} \Big(\int_{S^2} g(x - t\gamma)\, d\sigma(\gamma) - \int_{S^2} (\nabla f)(x - t\gamma) \cdot \gamma\, d\sigma(\gamma) \Big). \end{aligned}$$

We use the divergence theorem as follows:

$$\begin{aligned} \int_{S^2} f(x - t\gamma)\gamma \cdot \gamma\, d\sigma(\gamma) &= \frac{1}{t^3} \int_{\partial B_t(x)} f(y)(y - x) \cdot \frac{y - x}{|y - x|}\, d\sigma(y) = \frac{1}{t^3} \int_{B_t(x)} \operatorname{div}_y (f(y)(y - x))\, dy \\ &= \frac{1}{t^3} \int_{B_t(x)} \nabla f(y) \cdot (y - x) + 3f(y)\, dy \le \frac{1}{t^2} \|\nabla f\|_{L^1} + \frac{3}{t^3} \|f\|_{L^1}. \end{aligned}$$

Similarly, $t \int_{S^2} g(x - t\gamma)\, d\sigma(\gamma) \le \frac{1}{t} \|\nabla g\|_{L^1} + \frac{3}{t^2} \|g\|_{L^1}$. To estimate the rest term, we take advantage that f have compact support:

$$t \int_{S^2} (\nabla f)(x - t\gamma) \cdot \gamma\, d\sigma(\gamma) = \frac{1}{t} \int_{\partial B_t(x)} \nabla f(y) \cdot \frac{y - x}{|y - x|}\, d\sigma(y) = \frac{1}{t} \|\Delta f\|_{L^1(B_R)},$$

(Or bounded by $\frac{4\pi R^2}{t} \|\nabla f\|_\infty$ since the intersection of $\partial B_t(x)$ and $B_R(0)$ has area at most $4\pi R^2$).

(We remark that for small time, we can find a better estimate that u is obviously dominated by $(\|\nabla f\|_\infty + \|g\|_\infty)t + \|f\|_\infty$). ☐

$6\frac{1}{2}$. **(b) Let u be the unique solution of**

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times (0, \infty) \\ u(x,0) = f(x), u_t(x,0) = g(x) \text{ on } \mathbb{R}^2 \end{cases}$$

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ have compact support. Prove that for each $x \in \mathbb{R}^2$, there exists a constant C_x, such that for each $t > 0$

$$|u(x,t)| \le C_x/t$$

Proof. We suppose both the supports of f and g are included in $B_R(0)$.

The uniqueness theorem implies

$$\begin{aligned} u(x,t) &= \partial_t\Big(\frac{t}{2\pi}\int_{B_1(0)} \frac{f(x-ty)}{\sqrt{1-|y|^2}}\,dy\Big) + \frac{t}{2\pi}\int_{B_1(0)} \frac{g(x-ty)}{\sqrt{1-|y|^2}}\,dy \\ &= \frac{1}{2\pi}\int_{B_1(0)} \frac{f(x-ty)}{\sqrt{1-|y|^2}}\,dy + \frac{t}{2\pi}\int_{B_1(0)} \frac{(\nabla f)(x-ty)\cdot(-y) + g(x-ty)}{\sqrt{1-|y|^2}}\,dy \\ &= \frac{1}{2\pi t}\int_{B_t(x)} \frac{f(z)}{\sqrt{t^2-|x-z|^2}}\,dz + \frac{1}{2\pi}\int_{B_t(x)} \frac{(\nabla f)(z)\cdot\frac{x-z}{t} + g(z)}{\sqrt{t^2-|x-z|^2}}\,dz \end{aligned}$$

One notes that the integral $\int_0^t \frac{r}{\sqrt{t^2-r^2}}\,dr = t$, so we need to consider the large time and small time cases separately. Because it is a little delicate issue how to determine a suitable threshold for t, we don't describe how to find it here.

If $|x| + 2R < t$, then $B_R(0) \subset B_t(x)$

$$\int_{B_t(x)} \frac{|f(z)|}{\sqrt{t^2-|x-z|^2}}\,dz = \frac{1}{t}\int_{B_R(0)} \frac{|f(z)|}{\sqrt{1-\frac{|x-z|^2}{t^2}}}\,dz \le \frac{\|f\|_\infty}{t}\frac{\pi R^2}{\sqrt{1-\frac{(R+|x|)^2}{(2R+|x|)^2}}}$$

If $0 < t < |x| + 2R$, then

$$\int_{B_t(x)} \frac{|f(z)|}{\sqrt{t^2-|x-z|^2}}\,dz \le 2\pi\|f\|_\infty \int_0^t \frac{r}{\sqrt{t^2-r^2}}\,dr = 2\pi\|f\|_\infty t \le 2\pi\|f\|_\infty \frac{(|x|+2R)^2}{t} \tag{1}$$

Similar for the terms involving g and $|\nabla f|$. □

7. **Let u be the same function as Problem $6\frac{1}{2}$. Prove that there exists a constant $C > 0$ such that for each $x \in \mathbb{R}^2, t > 0$,**

$$|u(x,t)| \le C/t^{1/2}$$

Proof. We suppose both the supports of f and g are included in $B_R(0)$.

To obtain the L^∞_x bound, we use the divergence theorem as Problem 6 and separate the ball $B_t(x)$ into the inner ball $B_{t-1}(x)$ of radius $t-1$ and the outer annulus $A = B_t(x) \setminus \overline{B_{t-1}(x)}$.

Note that for $t \geq 2$ ($\Leftrightarrow t - 1 \geq \frac{t}{2}$) ,

$$\int_{B_{t-1}(x)} \frac{|f(z)|}{\sqrt{t^2 - |x-z|^2}}\, dz \leq \int_{B_{t-1}(x)} \frac{|f(z)|}{\sqrt{t^2 - (t-1)^2}}\, dz \leq \frac{\|f\|_{L^1}}{\sqrt{2t-1}} \leq \frac{\|f\|_{L^1}}{\sqrt{t}}.$$

For the integral over A, one observes that $\nabla_y \sqrt{t^2 - y^2} = -\frac{y}{\sqrt{t^2-y^2}}$ and $\nabla_z \frac{x-z}{|x-z|^2} \equiv 0$ on $\mathbb{R}^2$, so

$$\begin{aligned}
&\int_A \frac{f(z)}{\sqrt{t^2 - |x-z|^2}}\, dz = -\int_A f(z) \frac{x-z}{|x-z|^2} \cdot \nabla_z \sqrt{t^2 - |x-z|^2}\, dz \\
&= \int_A \operatorname{div}(f(z)\frac{x-z}{|x-z|^2})\sqrt{t^2 - |x-z|^2} - \int_{\partial A} f(z)\sqrt{t^2 - |x-z|^2}\frac{x-z}{|x-z|^2} \cdot n\, d\sigma(z) \\
&= \int_A (\nabla f)(z) \cdot \frac{x-z}{|x-z|^2}\sqrt{t^2 - |x-z|^2} - \int_{\partial B_{t-1}(x)} f(z)\sqrt{2t-1}\frac{1}{t-1}\, d\sigma(z).
\end{aligned}$$

Note that the first term is dominated by $\frac{\sqrt{2t-1}}{t-1}\|\nabla f\|_{L^1(A)} \leq \frac{\sqrt{2t}}{t/2}\|\nabla f\|_{L^1(\mathbb{R}^2)}$. For the second term, we find

$$\begin{aligned}
&\Big|\int_{\partial B_{t-1}(x)} f(z)\frac{(x-z)}{t-1} \cdot \frac{(x-z)}{t-1}\, d\sigma(z)\Big| = \Big|\frac{1}{t-1}\int_{B_{t-1}(x)} \operatorname{div}_z[f(z)(x-z)]\, dz\Big| \\
&= \frac{1}{t-1}\Big|\int_{B_{t-1}(x)} \nabla f(z) \cdot (x-z) - 2f(z)\, dz\Big| \leq \int_{\mathbb{R}^2} |\nabla f(z)|\, dz + \frac{2}{t-1}\int_{\mathbb{R}^2} |f(z)|\, dz.
\end{aligned}$$

So for $t \geq 2$,

$$\Big|\int_{B_t(x)} \frac{f(z)}{\sqrt{t^2 - |x-z|^2}}\, dz\Big| \leq \frac{1}{\sqrt{t}}\Big(4\sqrt{2}\|\nabla f\|_{L^1(\mathbb{R}^2)} + (1 + \frac{2}{t-1})\|f\|_{L^1}\Big).$$

For $t \in (0,2)$, we see $\Big|\int_{B_t(x)} \frac{f(z)}{\sqrt{t^2-|x-z|^2}}\, dz\Big| \leq 2\pi\|f\|_\infty t$ from (1).

Similar for the terms involving g and $|\nabla f|$. □

8. *Proof.* □

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11. *Proof.* □

12. *Proof.* □

13. *Proof.* □

14. *Proof.* □

15. *Proof.* □

16. *Proof.* □

17. *Proof.* □

18. *Proof.* □

References

[1] Keel, Markus, and Terence Tao. "Endpoint Strichartz estimates." American Journal of Mathematics 120.5 (1998): 955-980.

[2] Sogge, Christopher Donald: "Lectures on non-linear wave equations." 2nd Edition. Vol. 2. Boston, MA: International Press, 2008.