Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 7 Sobolev Spaces*

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1. *Proof.* ☐

2. *Proof.* ☐

3. *Proof.* ☐

4. **Prove that the product formula**

$$D(uv) = vDu + uDv$$

holds for all $u, v \in W^1(\Omega)$ **such that** $uv, uDv + vDu \in L^1_{loc}(\Omega)$**.**

***Remark* 1.** The key idea to define U_n comes from `http://math.stackexchange.com/questions/443425/product-rule-of-weak-derivatives`

Proof. Step 1 is to prove the theorem under the additional assumption $v \in C^1$. This is standard, so we just state it without proof.

Step 2 is to prove the theorem under the additional assumption $u \in W^1(\Omega) \cap L^\infty(\Omega)$. This is proved by Step 1 and Theorem 7.4, a characterization of weakly differentiabilty. This is also standard.

Step 3 is consider the problem under the assumption $u, v \geq 0$: we first assume $v \geq 1$ and define $U_n := \min\{u, \frac{n}{v}\}$. By Lemma 7.6, $U_n \in W^1(\Omega) \cap L^\infty(\Omega)$, and $D(U_n v) = Du\chi_{\{uv<n\}} + (-\frac{nDv}{v^2})\chi_{\{uv\geq n\}}$. Hence $U_n Dv + vDU_n = (uDv + vDu)\chi_{\{uv<n\}} \in L^1_{loc}(\Omega)$. Step 2 implies that, for each $\phi \in C^1_0(\Omega)$,

$$\int_\Omega U_n v D\phi = -\int_\Omega [U_n Dv + vDU_n]\phi.$$

*Last Modified: 2017/07/15.

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Note that $U_n v = \min\{uv, n\} \nearrow uv, U_n Dv + vDU_n \to uDv + vDu$ a.e. and $|U_n Dv + vDU_n| \leq |uDv + vDu| \in L^1(supp\phi)$, MCT and LDCT imply that

$$\int_\Omega uvD\phi = \Big(\int_{\{D\phi\geq 0\}} + \int_{\{D\phi<0\}}\Big) uvD\phi = \lim_{n\to\infty} \Big(\int_{\{D\phi\geq 0\}} + \int_{\{D\phi<0\}}\Big) U_n vD\phi$$
$$= \lim_{n\to\infty} \int_\Omega U_n vD\phi = \lim_{n\to\infty} -\int_\Omega [U_n Dv + vDU_n]\phi = -\int_\Omega (uDv + vDu)\phi.$$

The above formula is also true for general $v \geq 0$, since

$$\int_\Omega uvD\phi + \int_\Omega uD\phi = \int_\Omega u(v+1)D\phi = -\int_\Omega (uD(v+1)+(v+1)Du)\phi = -\int_\Omega (uDv+vDu)\phi - \int_\Omega Du\phi.$$

Finally, we decompose $u = u^+ - u^-, v = v^+ - v^-$. Since $u^+v^+ = uv\chi_{\{u>0,v>0\}}$ and $u^+Dv^+ + v^+Du^+ = (uDv + vDu)\chi_{\{u>0,v>0\}}$, they are in $L^1_{loc}(\Omega)$. Other similar functions also in $L^1_{loc}(\Omega)$ without proof. Then by Step 3, for each $\phi \in C^1_0(\Omega)$,

$$\int_\Omega uvD\phi = \int_\Omega (u^+v^+ - u^+v^- - u^-v^+ + u^-v^-)D\phi$$
$$= -\int_\Omega \Big((u^+Dv^+ + v^+Du^+) - (u^+Dv^- + v^-Du^+) - (u^-Dv^+ + v^+Du^-) + (u^-Dv^- + v^-Du^-)\Big)\phi$$
$$= -\int_\Omega (uDv + vDu)\phi.$$

□

Remark 2. In the same webcite, Mateusz Kwanicki (from Wrocaw University of Science and Technology?) informed me that this problem may be solved by the characterization of absolutely continuity along any line. He said:

"How about using the ACL characterisation of differentiability? I mean, u has weak partial derivatives if and only if it is absolutely continuous on almost every line in any cardinal direction, and the classical partial derivatives (defined thus almost everywhere) are integrable.

Since the product of absolutely continuous functions is absolutely continuous, with $(fg)' = f'g + fg'$, it follows that uv has the ACL property, and the classical partial derivatives of uv obey the product rule almost everywhere. In order that uv has weak partial derivatives it is therefore sufficient to assume that the classical partial derivatives of uv are locally integrable."

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* $\square$

9. *Proof.* $\square$

10. *Proof.* $\square$

11. *Proof.* $\square$

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16. *Proof.* $\square$

17. *Proof.* $\square$

18. *Proof.* $\square$