Real and Complex Analysis, 3rd Edition, W.Rudin
Chapter 7 Differentiation *

Yung-Hsiang Huang†

1. *Proof.* Let x be a Lebesgue point of $f \in L^1_{\text{loc}}(\mathbb{R}^k)$. For each $r > 0$, we have

$$
\begin{aligned}
|f(x)| &\le \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y) - f(x)|\, dm(y) + \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\, dm(y) \\
&\le \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y) - f(x)|\, dm(y) + Mf(x).
\end{aligned}
$$

   The proof is completed by letting $r \to 0$. □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. Construct a continuous monotonic function f on $\mathbb{R}^1$ so that f is not constant on any segment although $f'(x) = 0$.

   Proof. □

   Remark 1. A weaken result that a strictly increasing on any finite interval, say $[0, 1]$, with vanishing derivative on a subset of positive measure can be constructed easily by letting $f(x) = \int_0^x \chi_K(t)\, dt$, where $K \subset [0, 1]$ is a Cantor-like set with positive measure.

   Remark 2. I learn this special case of [3, Example 18.8] from Prof. Kai-Seng Chou. See also [2, Example 8.30] and [4, Theorem 1.47].

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†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

8. *Proof.* $\square$

9. *Proof.* $\square$

10. *Proof.* $\square$

11. Use Theorem 7.20 and Hölder's inequality.

12. *Proof.* $\square$

13. *Proof.* $\square$

14. *Proof.* $\square$

15. *Proof.* $\square$

16. *Proof.* $\square$

17. *Proof.* To show μ is countably additive on the Borel σ-algebra, we note that

$$\mu(\cup_k E_k) = \lim_{N\to\infty} \mu^N(\cup_k E_k) := \lim_{N\to\infty} \sum_{i=1}^N \mu_i(\cup_k E_k) = \lim_{N\to\infty} \sum_k \sum_{i=1}^N \mu_i(E_k) \le \sum_k \mu(E_k),$$

and

$$\sum_k \mu(E_k) = \lim_{M\to\infty} \sum_{k=1}^M \sum_i \mu_i(E_k) = \lim_{M\to\infty} \sum_i \sum_{k=1}^M \mu_i(E_k) \le \lim_{M\to\infty} \sum_i \mu_i(\cup_k E_k) = \mu(\cup_k E_k),$$

where the existence of those limits follows from monotonicity (due to $\mu_i \ge 0$).

Since $\mu(\mathbb{R}^k) < \infty$, the Lebesgue-Radon-Nikodym Theorem can be applied to μ, μ_i. Let $\mu_i = \mu_{i,a} + \mu_{i,s}$ be the Lebesgue decomposition of μ_i with respect to the Lebesgue measure m (or any other positive measure). We claim that $\sum_i \mu_{i,a}$ and $\sum_i \mu_{i,s}$ are the absolutely continuous part and singular part of μ respectively. Then $\mu_a = \sum_i \mu_{i,a}$ and $\mu_s = \sum_i \mu_{i,s}$ by the uniqueness of Lebesgue decomposition.

By hypothesis, for each n, $\mathbb{R}^k = A_n \cup B_n$, where $A_n \cap B_n = \emptyset$, $\mu_{n,s}(E) = 0$ for every Borel set $E \subset B_n$ and $m(F) = 0$ for every Borel set $F \subset A_n$. So $\mathbb{R}^k = (\cup_n A_n) \cup (\cap_n B_n)$, $(\cup_n A_n) \cap (\cap_n B_n) = \emptyset$, $\sum_n \mu_{n,s}(E') = \sum_n 0 = 0$ for every Borel set $E' \subset \cap_n B_n$ and $m(F') = \sum_n m(F' \cap A_n) = 0$ for every Borel set $F' \subset \cup_n A_n$. This proves that $\mu_s = \sum_i \mu_{i,s}$. The proof for absolutely continuous part is trivial.

Since $\mu_a = \sum_{i=1}^k \mu_{i,a} + \sum_{i=k+1}^\infty \mu_{i,a} =: \mu_a^k + \tilde{\mu}_a^k$, $D\mu_a = D\mu_a^k + D\tilde{\mu}_a^k = \sum_{i=1}^k D\tilde{\mu}_{i,a} + D\tilde{\mu}_a^k$ (their existence are guaranteed by Theorem 7.8). We are going to show $D\mu_a^k(x) \to D\mu_a(x)$ (up to a subsequence) for $x \notin F$ as $k \to \infty$ for some Lebesgue measurable zero set F by showing

$D\tilde{\mu}_a^k(x) \to 0$. Then by the definition, we have $\sum_{i=1}^{\infty} D\tilde{\mu}_{i,a} = D\mu_a$ on F^c. The method is adapted from [5, Lemma 7.24].

According to Theorem 7.8, we have

$$\mu_a(\mathbb{R}^d) = \int_{\mathbb{R}^d} D\mu_a(y)\,dm(y) = \int_{\mathbb{R}^d} D\mu_a^k(y) + D\tilde{\mu}_a^k(y)\,dm(y) = \mu_a^k(\mathbb{R}^d) + \int_{\mathbb{R}^d} D\tilde{\mu}_a^k(y)\,dm(y).$$

Then $0 \le \int_{\mathbb{R}^d} D\tilde{\mu}_a^k(y)\,dm(y) = \mu_a(\mathbb{R}^d) - \mu_a^k(\mathbb{R}^d) \to 0$ as $k \to \infty$. So we can choose a subsequence $k_j \nearrow \infty$ such that

$$0 \le \sum_j \int_{\mathbb{R}^d} D\tilde{\mu}_a^{k_j}(y)\,dm(y) \le \sum_j \frac{1}{2^j}.$$

Since $D\tilde{\mu}_a^k \ge 0$ a.e., we apply the monotone convergence theorem to conclude that

$$0 \le \int_{\mathbb{R}^d} \sum_j D\tilde{\mu}_a^{k_j}(y)\,dm(y) \le \sum_j \frac{1}{2^j}.$$

and hence there is a Lebesgue measure zero set F such that for each $x \notin F$

$$D\tilde{\mu}_a^{k_j}(x) \to 0$$

On the other hand, by applying Theorem 7.13 to each $i = 0, 1, \cdots$, we know $D\mu_{i,s}(x) = 0$ for every $x \notin N^i$ where N^i is a Lebesgue measure zero set. So $N \cup F := \cup_i N^i \cup F$ is a Lebesgue measure zero set and for $x \notin N \cup F$ we have

$$D\mu(x) = D\mu_a(x) + D\mu_s(x) = \sum_i D\mu_{i,a}(x) + 0 = \sum_i D\mu_{i,a}(x) + D\mu_{i,s}(x) = \sum_i D\mu_i(x).$$

To the final part, my conclusion is $(\sum f_n)' = \sum f_n'$ a.e $[m]$, which is a special case of a theorem of Fubini (without assuming $f_n > 0$, cf: [5, Lemma 7.24]). However, I don't know how to prove without knowing a theorem (cf: [5, Theorem 7.21]) corresponding to Theorem 7.8. So I only sketch my idea here.

1.) If we further assume each f_n is right continuous, then we apply the previous result for the Lebesgue-Stieljes measure on each $[M, M+1]$, $M \in \mathbb{Z}$. (cf: Exercise 13(d) and [1, Section 1.5]) defined by

$$\mu_n((a,b)) := f_n(b) - f_n(a)$$

to conclude that $(\sum f_n)' = \sum f_n'$ a.e $[m]$ on $[M, M+1]$ and hence a.e. $[m]$ on $\mathbb{R}$. ($f_n > 0$ is not assumed here.)

2.) For the general case, one is referred to read the proof for [5, Lemma 7.24 and 7.21]. ☐

18. *Proof.* ☐

19. *Proof.* $\square$

20. *Proof.* $\square$

21. *Proof.* $\square$

22. *Proof.* $\square$

23. We define $SF(x)$ to be $f(x)$, then every claim can be proved easily.

References

[1] Gerald B Folland. *Real analysis: Modern Techniques and Their Applications.* John Wiley & Sons, 2nd edition, 1999.

[2] Bernard R Gelbaum and John MH Olmsted. *Counterexamples in analysis.* Dover Publications, corrected reprint of the second (1965) edition edition, 2003.

[3] Edwin Hewitt and Karl Stromberg. *Real and abstract analysis: a modern treatment of the theory of functions of a real variable.* Springer-Verlag, 1975.

[4] Giovanni Leoni. *A first course in Sobolev spaces*, volume 105. American Mathematical Society Providence, RI, 2009.

[5] Richard L Wheeden. *Measure and Integral: An Introduction to Real Analysis*, volume 308. CRC Press, 2nd edition, 2015.