Real and Complex Analysis, 3rd Edition, W.Rudin

Chapter 3 L^p Spaces *

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1. *Proof.* (i) Given the collection $\mathscr{F}$ of convex functions on (a,b). Given $x,y \in (a,b), t \in (0,1), \epsilon > 0$, there exists $g \in \mathscr{F}$ such that $\sup\{f(tx+(1-t)y) : f \in \mathscr{F}\} - \epsilon < g(tx+(1-t)y) \leq tg(x) + (1-t)g(y) \leq t\sup\{f(x) : f \in \mathscr{F}\} + (1-t)\sup\{f(y) : f \in \mathscr{F}\}$. Letting $\epsilon \to 0$, then we can conclude that $\sup\{f : f \in \mathscr{F}\}$ is convex on (a,b).

   (ii)Given a sequence of convex functions f_n with pointwise limit f. Given $x,y \in (a,b), t \in (0,1), \epsilon > 0$, there exists $K = K(x,y,t,\epsilon)$ such that $|f - f_K| < \epsilon$ at x,y and $tx+(1-t)y$, then

   $$f(tx+(1-t)y) < f_K(tx+(1-t)y) + \epsilon \leq tf_K(x) + (1-t)f_K(y) + \epsilon < tf(x) + (1-t)f(y) + 3\epsilon.$$

   Letting $\epsilon \to 0$, then we can conclude that f is convex on (a,b).

   (iii) For upper limit, it's still convex by (i) and (ii). For lower limit, it may be concave function. For example $f_n(x) = (-1)^n|x|$ on $(-1,1)$. □

2. *Proof.* Since φ is continuous, its range under (a,b) is also connected, that is, an interval. So it's customary to consider the convexity of ψ. The rest of the proof is step by step. The counterexample is $\varphi(x) = x$. □

3. *Proof.* Iterate the hypothesis, we know $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for each $x,y \in (a,b)$ and for every $t = \frac{k}{2^n}$ $(n \in \mathbb{N}, 2^n > k \in \mathbb{N})$. Using continuity assumption, we know this is true for every $t \in (0,1)$. □

   Remark 0.1. Without continuity assumption, we have the following counterexample from Donoghue [1, p.11]:

   By Axiom of Choice, there exists $\{x_\lambda\}$ be the Hamel basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$. Let $\mathbb{R} \ni x = \sum c_\lambda(x)x_\lambda$. Then for each function $c_\lambda : \mathbb{R} \to \mathbb{R}$ is mid convex but not continuous since it attains rational numbers only and not a constant function, and hence it's not convex.

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4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* By theorem 3.12, a subsequence $f_{n_j} \to f = g$ a.e. □

11. *Proof.*

$$1 \le \Big(\int \sqrt{g}\sqrt{f}\, d\mu \Big)^2 \le \int g\, d\mu \int f\, d\mu.$$

□

12. *Proof.* Since $\sqrt{1+x^2}$ has positive second derivative everywhere, it's a convex function. So the first inequality follows from the Jensen's inequality. Its geometric interpretation for $d\mu = dx, h = f'$ is the arc-length of the graph of f is always longer than the straight line from $(0, f(0))$ to $(1, f(1))$, that is longer than the hypotenuse of the corresponding right triangle. On the other hand, since $(1+h^2)^{1/2} \le 1+h$, $\int_\Omega \sqrt{1+h^2} \le \mu(\Omega) + A$. The geometric interpretation of this inequalities is the graph of increasing f is never longer than the total length of the legs of right triangle.

    The intuition for the condition of the first equality from the geometric view is $h \equiv$ constant. To prove it's true, we go back to the proof of Jensen's inequality (Theorem 3.3) and observe that the equality holds iff the one in (2) holds. But since $\varphi(x) = \sqrt{1+x^2}$ is strictly convex, (2) is a strict inequality either for all $s > t := \int h$ or for all $s < t$. If h is not a constant, then $h(x) - t$ takes on both positive and negative values on sets of positive measure, and so the Jensen's inequality is strict.

    The intuition for the condition of the second equality from the geometric view is $h \equiv 0$ a.e. It's true since $\sqrt{1+h^2} = 1+h$ a.e. iff $2h = (1+h)^2 - (1+h^2) = 0$ a.e. □

13. *Proof.* □

14. *Proof.* (a) Follow the steps provided by Suggestions and density argument.

    (b) I can't prove this problem with the method Rudin used here. The way I prove this is based on the method used in Exercise 8.14, that is, with Fubini's theorem: note that the equality holds iff the one holds in the Hölder's inequality used there, that is, for a.e. t, $at^{\alpha p}f(t)^p = bt^{-\alpha p'}$ for some constants a, b. But any power of t is not integrable on $(0,\infty)$), except zero function.

(c) Instead of following the hint, we consider $f(x) = x^{-\alpha}\chi_{[1,\infty)}$, $1 > \alpha > \frac{1}{p}$, wee see

$$F(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ \frac{1}{1-\alpha}(x^{-\alpha} - \frac{1}{x}) & \text{if } x \in [1,\infty), \end{cases} \tag{1}$$

Since for arbitrary $b > 1$,

$$\int_1^\infty (x^{-\alpha})^p\,dx > \int_1^\infty (x^{-\alpha} - \frac{1}{x})^p\,dx > \int_b^\infty (x^{-\alpha} - \frac{1}{x})^p\,dx > \int_b^\infty (x^{-\alpha} - b^{\alpha-1}x^{-\alpha})^p\,dx$$
$$= (1-b^{\alpha-1})^p b^{1-\alpha p}\int_1^\infty (t^{-\alpha})^p\,dt = (b^{\frac{1}{p}-\alpha} - b^{\frac{1}{p}-1})^p\int_1^\infty (t^{-\alpha})^p\,dt.$$

Choosing $b = 2^{1/c}$, where $c = \sqrt{\frac{1}{p} - \alpha}$ and then we yields

$$\int_1^\infty (x^{-\alpha})^p\,dx > \int_1^\infty (x^{-\alpha} - \frac{1}{x})^p\,dx > (2^{-c} - 2^{(\frac{1}{p}-1)/c})^p\int_1^\infty (t^{-\alpha})^p\,dt.$$

So $(1-p^{-1})^p$ is the best constant in this Hardy's inequality since

$$\limsup_{\alpha\searrow\frac{1}{p}}(\frac{1}{1-\alpha})^p\int_1^\infty f(x)^p\,dx \ge \limsup_{\alpha\searrow\frac{1}{p}}\int_1^\infty F(x)^p\,dx \ge \limsup_{\alpha\searrow\frac{1}{p}}(\frac{2^{-c}-2^{(\frac{1}{p}-1)/c}}{1-\alpha})^p\int_1^\infty f(x)^p\,dx.$$

(d) Since $f > 0$, there exists $x_0 > 0$ such that $\int_0^{x_0} f(t)\,dt =: a > 0$. Then

$$\int_{x_0}^\infty F(x)\,dx = \int_{x_0}^\infty \frac{1}{x}\int_0^x f(t)\,dt\,dx \ge \int_{x_0}^\infty \frac{1}{x}a\,dx = \infty.$$

□

Remark 0.2. This is a special case of weighted fractional integral operators, see Folland [3, Exercise 6.28-29].

15. *Proof.* We may assume $\{a_n\} \in l^p$. First, we suppose the sequence a_n is decreasing. Consider $f(x) = a_n$ if $x \in (n-1, n]$, then for every $N \in \mathbb{N}$ and every $x \in (N-1, N]$,

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt = \frac{a_1 + \cdots a_{N-1} + (x-N+1)a_N}{x} = \frac{Na_1 + \cdots Na_{N-1} + N(x-N+1)a_N}{Nx}$$
$$= \frac{xa_1 + \cdots xa_{N-1} + (N-x)a_1 + \cdots (N-x)a_{N-1} + N(x-N+1)a_N}{Nx}$$
$$\ge \frac{xa_1 + \cdots xa_{N-1} + Na_N}{Nx} \ge \frac{xa_1 + \cdots xa_{N-1} + xa_N}{Nx} = \frac{a_1 + \cdots a_{N-1} + a_N}{N},$$

where the first inequality is due to $\{a_n\}$ is decreasing.

Therefore, $\sum_N \left(\frac{1}{N}\sum_{n=1}^N a_n\right)^p \le \int_0^\infty F(x)^p\,dx \le (\frac{p}{p-1})^p\int_0^\infty f(x)^p\,dx = (\frac{p}{p-1})^p\sum_{n=1}^\infty a_n^p$.

For the general case, since $\{a_n\} \in l^p$, for each $k \in \mathbb{N}$, there are finitely many $\frac{1}{k} \ge a_n > \frac{1}{k+1}$ and $a_n > 1$. Therefore we can rearrange $\{a_n\}$ into a decreasing sequence $\{b_n\}$ and note that $\sum_N \left(\frac{1}{N}\sum_{n=1}^N a_n\right)^p \le \sum_N \left(\frac{1}{N}\sum_{n=1}^N b_n\right)^p \le (\frac{p}{p-1})^p\sum_{n=1}^\infty b_n^p = (\frac{p}{p-1})^p\sum_{n=1}^\infty a_n^p$. □

16. *Proof.* (a) The counterexample for σ-finite space is on $\mathbb{R}$, $f_n = \chi_{(n,n+1)}$.

(b) The problem for continuous parameter is the same as Folland [3, Exercise 2.43]. The technical difficulty is to show the set $E_{\epsilon,M} = \{x : |f_t(x) - f(x)| \le \epsilon \text{ for all } t > M\}$ is measurable for every $\epsilon, M > 0$. Without the continuity assumption (ii), a counterexample is given by Weston [5], a much simpler construction by Walter [4] is mentioned in the end Notes and Comments of Rudin's book. See my additional exercises 2.4.1-2 for Folland's Real analysis. □

17. *Proof.* (a) If $0 < p \le 1$, then $|\alpha - \beta|^p \le (|\alpha| + |\beta|)^p \le |\alpha|^p + |\beta|^p$ since $1 \le t^p + (1-t)^p$ with $0 \le t = \frac{|\alpha|}{|\alpha|+|\beta|} \le 1$. If $p > 1$, then by convexity of x^p, with the same $\frac{|\alpha|}{|\alpha|+|\beta|} = t \in [0,1]$, we see $\frac{1}{2}t^p + \frac{1}{2}(1-t)^p \ge (\frac{1}{2})^p$, that is, $2^{p-1}(t^p + (1-t)^p) \ge 1$. (b) As hints. (c) $f_n = \chi_{(0,\frac{1}{n})}$. □

Remark 0.3. (b) gives a different proof for Theorem 17.11(c), F.Riesz's theorem on mean convergence of H^p functions to their boundary function (see Duren [2, Theorem 2.6 and 2.2]). Also see my further remarks for Folland [3, Exercise 10,20-22 of chapter 6].

18. This is standard, it can be found in Folland [3, Section 2.4], for example.

(b)(c) are always true even if $\mu(X) = \infty$. (a) is a weak form of Egoroff's theorem. If $\mu(X) = \infty$, then (a) is not true by considering $f_k = \chi_{\{|x|<k\}}$.

19. *Proof.* (a)Given $z_n \to z$ in $\mathbb{C}$ with all $z_n \in R_f$. Then given $\epsilon > 0$, since $|f(x) - z| \le |f(x) - z_n| + |z_n - z|$, by choosing n large such that $|z_n - z| < \epsilon$, we have

$$\{x : |f(x) - z_n| < \epsilon\} \subset \{|f(x) - z| < 2\epsilon\}.$$

Therefore $0 < \mu(\{x : |f(x) - z_n| < \epsilon\}) \le \mu(\{|f(x) - z| < 2\epsilon\})$ and hence $z \in R_f$.

(b) By definition and proof by contradiction, $\|f\|_\infty \ge \sup\{|w| : w \in R_f\}$. The compactness follows from (a) and Bolzano-Weierstrauss. So there is $z \in R_f$, such that $|z| = \sup\{|w| : w \in R_f\}$. If $|z| < \|f\|_\infty$, then $|z| < \frac{1}{2}(|z| + \|f\|_\infty)$.

Hence the set $\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \epsilon_0\}$ has zero measure for some $\epsilon_0 > 0$, but this implies $\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \epsilon\}$ has zero measure for all $0 < \epsilon < \epsilon_0$. In particular, choosing $1 > \delta > 0$ such that $\frac{\delta}{2}(|z| + \|f\|_\infty) < \epsilon_0$. Since

$$\{x : |f(x)| > \frac{1-\delta}{2}(|z| + \|f\|_\infty)\} \subset \{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \frac{\delta}{2}(|z| + \|f\|_\infty)\},$$

$0 = \mu(\{x : |f(x) - \frac{1}{2}(|z| + \|f\|_\infty)| < \frac{\delta}{2}(|z| + \|f\|_\infty)\}) \ge \mu(\{x : |f(x)| > \frac{1-\delta}{2}(|z| + \|f\|_\infty)\}) = 0$, which contradicts the definition of $\|f\|_\infty$ since $0 < \frac{1-\delta}{2}(|z| + \|f\|_\infty) < \|f\|_\infty$. Therefore, $\max\{|w| : w \in R_f\} = |z| = \|f\|_\infty$.

(c) ????? □

20. *Proof.* Given $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$, take $f(t) = x\chi_{(0,\lambda)} + y\chi_{(\lambda,1)} \in L^\infty$. Then by hypothesis, $\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y)$. $\square$

21. *Proof.* $\square$

22. The answer is yes and the proof is an application of triangle inequality.

23. *Proof.* Note that

$$\limsup_{n\to\infty} \frac{\int_X |f|^{n+1}\,d\mu}{\int_X |f|^n\,d\mu} \le \limsup_{n\to\infty} \frac{\int_X |f|^n \|f\|_\infty\,d\mu}{\int_X |f|^n\,d\mu} = \|f\|_\infty.$$

On the other hand, by Hölder's inequality,

$$\int_X |f|^n \le (\int_X |f|^{n+1}\,d\mu)^{n/(n+1)}\mu(X)^{1/(n+1)} = \int_X |f|^{n+1}\,d\mu \Big(\frac{1}{\frac{1}{\mu(X)}\int_X |f|^{n+1}}\Big)^{1/(n+1)}$$

Then by Exercise 4(e),

$$1 \le \liminf \frac{\int_X |f|^{n+1}\,d\mu}{\int_X |f|^n}\Big(\frac{1}{\frac{1}{\mu(X)}\int_X |f|^{n+1}}\Big)^{1/(n+1)} = \liminf \frac{\int_X |f|^{n+1}\,d\mu}{\int_X |f|^n}\frac{1}{\|f\|_\infty}.$$

$\square$

24. *Proof.* (a) The inequality is proved by the same method as Exercise 17 (a). The completeness is the same proof as Theorme 3.11.

(b) Prove the inequality in the hint by mean-value theorem and then use Hölder's inequality. $\square$

25. *Proof.* If ϕ is a concave increasing function, then by Jensen's inequality and f is positive,

$$\mu(E)\frac{1}{\mu(E)}\int_E \phi(f)\,dx \le \mu(E)\phi(\frac{1}{\mu(E)}\int_E f\,dx) \le \mu(E)\phi(\frac{1}{\mu(E)}\int_X f\,dx) = \mu(E)\phi(\frac{1}{\mu(E)}).$$

Now take $\phi(x) = x^p$ $(0 < p < 1)$ and $\log(x)$ respectively. $\square$

26. *Proof.* Since $x \to x\log x$ is convex (due to positive second derivative) and $x \to \log x$ is concave,

$$\int f(x)\log f(x)\,dx \ge \int f(x)\,dx \log\Big(\int f(x)\,dx\Big) \ge \int f(x)\,dx\Big(\int \log f(x)\,dx\Big).$$

$\square$

References

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