Fourier Analysis, Stein and Shakarchi

Chapter 2 Basic Properties of Fourier Series

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Abstract

Notation: $\mathbb{T} := [-\pi, \pi]$. Note that we provide a proof for Big-O Tauberian theorem for Cesáro sum in Exercise 14 which is much easier than the one for Abel sum.

Exercises

1. Trivial.

2. Easy.

3. Easy to see the Fourier series for the initial condition f in plucked string problem converge absolutely and uniformly by Weierstrass M-test. So the uniqueness theorem implies the series equal to f everywhere (check page 17).

4. By computing the Fourier coefficients of the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = x(\pi - x)$ and using the convergence and uniqueness theorem, one can show that
$$f(x) = \frac{8}{\pi} \sum_{k \geq 1,\, \text{odd}} \frac{\sin kx}{k^3}.$$
Note that this result and the Parseval's identity can be applied to compute the Riemann zeta function $\zeta(z)$ at $z = 6$. See Exercise 3.8(b).

5. As Exercise 4, one can show that the tent map $f(x) = (1 - |x|/\delta)^+$ on $[-\pi, \pi]$ equals to its the Fourier series, that is,
$$f() = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos n\delta}{n^2\pi\delta} \cos nx.$$

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6. Similarly, one can show the following identities by considering the Fourier series of $f(x) = |x|$

$$\sum_{n\geq 1,\,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7. Discrete summation by parts formula and the proofs for Dirichlet's and Abel's test are standard.

8. **Verify that $\frac{1}{2i}\sum_{n\neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic sawtooth function illustrated in Figure 6, defined by $f(0) = 0$, $f(x) = -\frac{\pi}{2} - \frac{x}{2}$ if $-\pi < x < 0$, and $f(x) = \frac{\pi}{2} - \frac{x}{2}$ if $0 < x < \pi$. Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.**

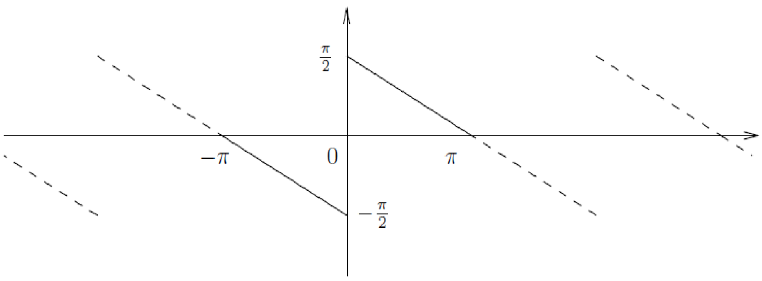


Figure 6. The sawtooth function

Proof. It's easy to compute $\widehat{f}(n) = \frac{1}{n}$ if $n \neq 0$ and $\widehat{f}(0) = 0$. Then we can apply the Abel-Dirichlet test to show the series converges for every $x \neq 0$. Note that for $x = 0$, the symmetric partial sum is always zero. □

***Remark* 1.** There are two ways to compute the limit of series, see Problem 3 and its remark. Also take a look at Problem 3.1 and its remark.

9. **Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subseteq [-\pi, \pi]$.**

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}.$$

(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x.

(c) However, prove that the Fourier series converges at every point x. What happens if $a = -\pi$ and $b = \pi$?

Proof. (a) is standard, we omit it. (b) Standard computations shows that

$$\sum_{n\neq 0} \left|\frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}\right| = \frac{2}{\pi} \sum_{n>0} \left|\frac{\sin(n\frac{b-a}{2})}{n}\right|,$$

where $\theta_0 := \frac{b-a}{2} < \pi$ by the assumption. So we can find $c > 0$ such that $\pi - \theta_0 > 2\sin^{-1} c$. Note that for each $k \in \mathbb{N}$, we can find a $n_k \in \mathbb{N}$ such that $n_k\theta_0 \in ((k-1)\pi + \sin^{-1} c, k\pi - \sin^{-1} c) =: I_k$ since if $m\theta_0 \in ((k-1)\pi - \sin^{-1} c, (k-1)\pi + \sin^{-1} c)$, then $(m+1)\theta_0 \in I_k$, and if $m\theta_0 \in (k\pi - \sin^{-1} c, k\pi + \sin^{-1} c)$, then $(m-1)\theta_0 \in I_k$. (This is why we take $2\sin^{-1} c$ instead of $\sin^{-1} c$ in the beginning.)

Also note that (1) $I_k \cap I_j = \emptyset$, so $n_k \neq n_j$ if $k \neq j$; (2) $|\sin n_k\theta_0| \geq c$ for all k; (3) $n_k \leq \frac{k\pi}{\theta_0}$. Hence $\sum_{n\neq 0} |\frac{e^{-ina}-e^{-inb}}{2\pi in} e^{inx}| = \infty$ for all x since

$$\sum_{n>0} \frac{|\sin(n\theta_0)|}{n} \geq \sum_{k=1}^{\infty} \frac{|\sin(n_k\theta_0)|}{n_k} \geq \sum_{k=1}^{\infty} \frac{c}{\frac{\pi k}{\theta_0}} = \frac{\theta_0}{c\pi} \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

(c) Note that the series is $\frac{1}{2\pi i} \sum_{n\neq 0} \frac{e^{in(x-a)} - e^{in(x-b)}}{n}$. If $x \notin \{a, b\}$, then both $\sum_{n\neq 0} \frac{e^{in(x-a)}}{n}$ and $\sum_{n\neq 0} \frac{e^{in(x-b)}}{n}$ are convergent by Abel-Dirichlet Test. If $x = a$, then the "symmetric" partial sum is $\frac{1}{2\pi i} \sum_{0<|n|\leq N} \frac{-e^{in(a-b)}}{n}$, which converges as $N \to \infty$ by Abel-Dirichlet Test since $a \neq b$. Similar argument can be applied to the case $x = b$.

Finally, in this case $a = -\pi$ and $b = \pi$, the coefficients in series are 0. □

10. When f is smoother, $\hat{f}$ decays faster. (The big-o can be improved to little-o by Riemann-Lebesgue Lemma, see Exercise 3.13).

11. $L^1(\mathbb{T})$ convergence of f_k implies uniform convergence of $\hat{f}_k$ (i.e., $C(\mathbb{Z})$)

12. The proof for convergent series is Cesàro summable is standard.

13. **It's intuitive to see that Abel summability is stronger than the standard or Cesàro methods of summation because of the regularization effects of multiplying r^n is much stronger than taking the average of those partial sums. The purpose of this exercise is to prove this intuition.**

    (a) Show that $\sum c_n = \sigma$ implies that $\sum c_n = \sigma$ (Abel).

(b) However, show that there exist series which are Abel summable, but that do not converge.

(c) Argue similarly to prove that if a series $\sum c_n = \sigma$ (Cesàro), then $\sum c_n = \sigma$ (Abel) [Hint: Note that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1} n\sigma_n r^n$$

and assume $\sigma = 0$.]

(d) Give an example of a series that is Abel summable but not Cesàro summable. [Hint: Try $c_n = (-1)^{n-1}n$, which is highly oscillated. Note that if $\{c_n\}$ is Cesàro summable, then $\frac{c_n}{n}$ tends to 0.]

Proof. Although (a) is a consequence of Exercise 12 and (c), we provide a direct proof here. May assume $\sigma = 0$. Note that for each $0 < r < 1$ and $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = (1-r) \sum_{j=1}^{N} s_j r^j + s_N r^{N+1}$$

Passing $N \to \infty$, we have

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{j=1}^{\infty} s_j r^j$$

Given $\epsilon > 0$. Since $\sigma = 0$, the series in the right-hand side converges absolutely and hence

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{j=1}^{M} s_j r^j + (1-r) \sum_{M+1}^{\infty} s_j r^j,$$

where M is chosen so that $|s_j| < \epsilon$ for all $j > M$. Note that the absolute value of second term is less than $\epsilon r^{M+1} < \epsilon$. And the first term is less than ϵ provided r is close to 1.

(b) The example is $\sum_{n=0}^{\infty} (-1)^n$ which is not convergent but converges to $\frac{1}{2}$ in Abel's sense.

(c) May assume $\sigma = 0$. Note that for each $0 < r < 1$ and $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{N} c_n r^n &= \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = \sum_{n=1}^{N} (n\sigma_n - 2(n-1)\sigma_{n-1} + (n-2)\sigma_{n-2}) r^n \\ &= (1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n - N\sigma_N r^{N+2} + (2N\sigma_N - (N-1)\sigma_{N-1}) r^{N+1}\end{aligned}$$

Passing $N \to \infty$, we have (by L'Hôpital's rule)

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{j=1}^{\infty} j\sigma_j r^j$$

Given $\epsilon > 0$. Since $\sigma = 0$, the series in the right-hand side converges absolutely and hence

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{j=1}^{M} j\sigma_j r^j + (1-r)^2 \sum_{M+1}^{\infty} j\sigma_j r^j,$$

where M is chosen so that $|\sigma_j| < \epsilon$ for all $j > M$. Note that the absolute value of second term is less than $\epsilon r^{M+1} < \epsilon$. And the first term is less than ϵ provided r is close to 1.

(d) It's standard to show that $\sum_{n=1}^{\infty} n(-1)^{n-1}x^n = \frac{x}{(1+x)^2}$. We also note that if $\{c_n\}$ is Cesáro summable, then we have a contradiction that

$$(-1)^n = \frac{c_n}{n} = \frac{n\sigma_n - (n-1)\sigma_{n-1}}{n} = \sigma_n - \sigma_{n-1} + \frac{\sigma_{n-1}}{n} \to 0, \text{ as } n \to \infty.$$

□

Exercise 13 says that "Convergent $\Rightarrow_{\not\Leftarrow}$ Cesàro summable $\Rightarrow_{\not\Leftarrow}$ Abel summable."

14. **This exercise deals with a theorem of Tauber which says that under an additional condition on c_n, the above arrows can be reversed.**

(a) If $\sum c_n = \sigma$ (Cesàro) and $c_n = o(1/n)$, then $\sum c_n = \sigma$.

(b) The above statement holds if we replace Cesàro summable by Abel summable.

[Hint: Estimate the difference between $\sum_{n=1}^{N} c_n$ and $\sum_{n=1}^{N} c_n r^n$ where $r = 1 - \frac{1}{N}$.]

(c)-(d) are from Rudin [4, Exercise 3.14]. Recall that $s_n = \sum_{j=1}^{n} c_j$ and $\sigma_n = \frac{s_1 + \cdots + s_n}{n}$.

(c) Can it $\limsup_{N\to\infty} s_N = \infty$ but $\sum c_j = 0$ (Cesàro)?

(d) Prove (a) under a weaker assumption $c_n = O(1/n)$. Also see Problem 4.5.

[Hint: Say, $|nc_n| \le M$. If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{j=m+1}^{n}(s_n - s_j).$$

For these i,

$$|s_n - s_i| \le \frac{(n-i)M}{i+1} \le \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n the integer m that satisfies

$$m \le \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Then $(m+1)/(n-m) \le 1/\epsilon$ and $|s_n - s_i| < M\epsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \le M\epsilon.$$

Since ϵ is arbitrary, $\lim s_n = \sigma$.]

(e) We will also prove (b) under a weaker assumption $c_n = O(1/n)$ in Problem 3.

Proof. (a) $n(s_n - \sigma_n) = ns_n - \sum_{j=1}^n s_j = ns_n - \sum_{j=1}^n (j-(j-1))s_j = \sum_{j=1}^n (j-1)s_j - \sum_{j=1}^{n-1} js_j = \sum_{k=1}^{n-1} k(s_{k+1}-s_k) = \sum_{k=1}^{n-1} kc_k$. Therefore given $\epsilon > 0$, there is N_1 such that $|ka_k| < \epsilon$ if $k > N_1$. So there is some $N_2(\epsilon)$ such that

$$|s_n - \sigma_n| = |\frac{1}{n}\sum_{k=1}^{n-1} kc_k| \leq \frac{1}{n}\sum_{k=1}^{N_1}|kc_k| + \frac{1}{n}\sum_{k=N_1+1}^{n-1}|kc_k| \leq 2\epsilon,$$

whenever $n > N_2(\epsilon)$.

(b) Given $\epsilon > 0$, there is $N = N(\epsilon)$ such that $n|c_n| \leq \epsilon$ whenever $n > N$. Consider $r_n = 1 - \frac{1}{n} \to 1^-$ as $n \to \infty$. Note that $\sup_{n\in\mathbb{N}} n|c_n| =: M < \infty$, so for n sufficently large,

$$\begin{aligned} |s_n - A_{r_n}| &\leq \sum_{j=1}^N |c_j|(1-r_n^j) + \sum_{j=N+1}^\infty r_n^j|c_j| \leq \sum_{j=1}^n |c_j|(1-(1-\frac{1}{n})^j) + \epsilon\sum_{j=N+1}^\infty \frac{r_n^j}{j} \\ &\leq \sum_{j=1}^N |c_j|\frac{j}{n} + \frac{\epsilon}{n}\frac{(1-\frac{1}{n})^{N+1}}{1-(1-\frac{1}{n})} \leq \frac{MN}{n} + \epsilon \leq 2\epsilon. \end{aligned}$$

(c) An example is $s_n = 2^{-n}$ if $n \notin 2^{\mathbb{N}}$ and $s_{2^m} = m$ whenever $m \in \mathbb{N}$. Then $\limsup_{n\to\infty} s_n = \infty$. However $0 \leq \sigma_n < 2^{-m}(\frac{(m+1)m}{2} + 2)$ if $2^m \leq n < 2^{m+1}$. So $\sigma_n \to 0$.

(d) We first prove an identity similar to the hint: for $n > m$,

$$\begin{aligned} &n\sigma_n - m\sigma_m - \sum_{j=m+1}^n (n-j+1)a_j \\ &= \sum_{j=1}^n (n-j+1)a_j - \sum_{j=1}^m (m-j+1)a_j - \sum_{j=m+1}^n (n-j+1)a_j \\ &= (n-m)\sum_{j=1}^m a_j = (n-m)s_m \end{aligned}$$

So $s_m - \sigma_m = \frac{n}{n-m}\Big(\sigma_n - \sigma_m\Big) - \frac{1}{n-m}\sum_{j=m+1}^n (n-j+1)a_j =: I - II$.

Now we estimate II as integral test, that is, (with $\sup_{n\in\mathbb{N}} n|c_n| =: M < \infty$)

$$\begin{aligned} |II| &\leq \frac{M}{n-m}\sum_{j=m+1}^n (n+1-j)\frac{1}{j} = \frac{M(n+1)}{n-m}\sum_{j=m+1}^n \frac{1}{j} - \frac{1}{n+1} \\ &\leq \frac{M(n+1)}{n-m}\int_m^n \frac{dt}{t} - M. \end{aligned}$$

So $|s_m - \sigma_m| \leq \frac{n}{n-m}|\sigma_n - \sigma_m| + M\Big(\frac{n+1}{n-m}\log\frac{n}{m} - 1\Big)$.

Given $\epsilon > 0$, by Cesáro summability, there is N_ϵ such that $|\sigma_n - \sigma_m| < \epsilon^2$ if $n, m > N_\epsilon$.

Note that

$$(1-\epsilon)\epsilon \leq \frac{n-m}{m} \leq \epsilon \Leftrightarrow (1+\epsilon-\epsilon^2)m \leq n \leq (1+\epsilon)m$$

So $\frac{n}{n-m} < \frac{n+1}{n-m} = 1 + \frac{m}{n-m}\frac{m+1}{m} \leq 1 + \frac{1}{(1-\epsilon)\epsilon}(1+\frac{1}{m}) < 1 + \frac{1+\epsilon}{(1-\epsilon)\epsilon}$ if $m > \frac{1}{\epsilon}$ and hence

$$\begin{aligned}|s_m - \sigma_m| &\leq (1 + \frac{1+\epsilon}{(1-\epsilon)\epsilon})\epsilon^2 + M\Big((1+\frac{1+\epsilon}{(1-\epsilon)\epsilon})\log(1+\epsilon) - 1\Big) \\ &\leq \epsilon^2 + 2\epsilon + M(\epsilon + \frac{1+\epsilon}{1-\epsilon} - 1) = \epsilon^2 + (2 + M + \frac{2}{1-\epsilon})\epsilon,\end{aligned}$$

whenever $m > \max(\frac{1}{\epsilon}, N_\epsilon)$. □

Remark 2. Another proof for (d), via delay means, is given in Problem 4.5.

15. We omit the basic computation for showing the Fejér kernel equals to

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

This result can also be predict from the graph of $\widehat{F_N}$, that is, $\widehat{F_N} = c\widehat{D_M} * \widehat{D_M}$ for some c, M. I learned this perspective from [5, page 9].

16. **Prove the Weierstrauss approximation theorem by Fejér's theorem (Corollary 5.4).**

Proof. Extend f from $[a,b]$ to $[a,c]$ continuously so that $f(a) = f(c)$. So there is $Q(x) = \sum_{k=M}^{N} a_k e^{ikx}$ such that $\|Q - f\| < \frac{\epsilon}{2}$. Note that those finitely many e^{ikx} can be approximate by polynomial uniformly. □

17. **In Section 5.4 we proved that the Abel means of f converge to f at all points of continuity, that is,**

$$\lim_{r\to 1^-} Ar(f)(\theta) = \lim_{r\to 1^-}(Pr * f)(\theta) = f(\theta)$$

whenever f is continuous at θ. In this exercise, we will study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity. An integrable function is said to have a jump discontinuity at θ if the two limits

$$\lim_{h\to 0^+} f(\theta+h) = f(\theta+) \text{ and } \lim_{h\to 0^-} f(\theta+h) = f(\theta-)$$

exist.

(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1^-} A_r(f)(\theta) = \frac{f(\theta+) + f(\theta-)}{2}.$$

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesáro summable to $\frac{f(\theta+)+f(\theta-)}{2}$.

Proof. (a) Since $P_r(t) = P_r(-t) \geq 0$ and $\int_{-\pi}^{\pi} P_r = 1$, we have $\int_0^{\pi} P_r = \int_{-\pi}^{0} P_r = \frac{1}{2}$ for all $r < 1$. Given $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $|f(\theta - t) - f(\theta -)| < \epsilon$ and $|f(\theta + t) - f(\theta +)| < \epsilon$ for all $0 < t < \delta$. So

$$
\begin{aligned}
& A_r(f)(\theta) - \frac{f(\theta+) + f(\theta-)}{2} = \int_{-\pi}^{\pi} f(\theta - t) P_r(t)\,dt - \frac{f(\theta+) + f(\theta-)}{2} \\
&= \int_{-\pi}^{\pi} f(\theta - t) P_r(t)\,dt - \frac{f(\theta+) + f(\theta-)}{2} \\
&= \int_0^{\pi} [f(\theta + t) - f(\theta+)] P_r(t)\,dt + \int_{-\pi}^{0} [f(\theta - t) - f(\theta-)] P_r(t)\,dt \\
&= (\int_{\delta}^{\pi} + \int_0^{\delta})[f(\theta + t) - f(\theta+)] P_r(t)\,dt + (\int_{-\pi}^{-\delta} + \int_{-\delta}^{0})[f(\theta - t) - f(\theta-)] P_r(t)\,dt \\
&=: I + II + III + IV
\end{aligned}
$$

It's easy to see $|II|, |IV| < \epsilon$ for all $r < 1$. Since Poisson kernel on the unit disc is a good kernel, there is $R = R(\delta) = R(\epsilon) < 1$ such that $|I|, |III| < \epsilon$ if $r > R$. So $|A_r(f)(\theta) - \frac{f(\theta+)+f(\theta-)}{2}| < 4\epsilon$ if $r > R(\epsilon)$.

(b) Same proof as (a) except P_r is replaced by F_N. □

Remark 3. See Problem 3 and its remark to derive the Fourier series converges to $\frac{f(\theta+)+f(\theta-)}{2}$.

18. **This is an example mentioned in Remark of Theorem 5.7.**

    If $P_r(\theta)$ denotes the Poisson kernel, show that the function

    $$u(r, \theta) = \frac{\partial P_r}{\partial \theta},$$

    defined for $0 \leq r < 1$ and $\theta \in \mathbb{R}$, satisfies:

    (i) $\Delta u = 0$ in the disc. (ii) $\lim_{r \to 1} u(r, \theta) = 0$ for each θ. (iii) u does not converges to 0 uniformly as $r \to 1$.

    However, u is not identically zero.

Proof. Direct computation. Note $u(x, y) = \frac{2y(x^2+y^2-1)}{((1-x)^2+y^2)^2}$. So $u(1 - \epsilon, \epsilon) \to -\infty$ as $\epsilon \to 0^+$. □

19. **Solve Laplace's equation $\Delta u = 0$ in the semi infinite strip**

    $$S = \{(x, y) : 0 < x < 1, 0 < y\},$$

    subject to the following boundary conditions

    $$u(0, y) = 0 = u(1, y) \ \textbf{when} \ 0 \leq y, \ \ u(x, 0) = f(x) \ \textbf{when} \ 0 \leq x \leq 1,$$

where f is a given C^α function ($\alpha > \frac{1}{2}$, see Exercise 3.16), with $f(0) = f(1) = 0$. Write

$$f(x)\text{"} = \text{"}\sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y}\sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula.

Proof. Note that for $y > 0$ and $x \in [0,1]$, the series $\sum_{n=1}^{\infty} e^{-n\pi y}\sin(n\pi x)\sin(n\pi z)$ converges uniformly on $z \in [0,1]$ by Weierstrass M-Test. So

$$\begin{aligned} u(x,y) :&= \sum_{n=1}^{\infty} a_n e^{-n\pi y}\sin(n\pi x) = \sum_{n=1}^{\infty} e^{-n\pi y}\sin(n\pi x) 2\int_0^1 f(z)\sin(n\pi z)\,dz \\ &= \int_0^1 f(z)\Big(2\sum_{n=1}^{\infty} e^{-n\pi y}\sin(n\pi x)\sin(n\pi z)\Big)\,dz =: \int_0^1 f(z)\Phi_y(x,z)\,dz \end{aligned}$$

First we note that $u(x,y)$ did solve the problem. Second we have

$$\begin{aligned} \Phi_y(x,z) &= \frac{2}{-4}\sum_{n=1}^{\infty} e^{-n\pi y}\Big\{\big(e^{in\pi(x+z)} + e^{-in\pi(x+z)}\big) - \big(e^{in\pi(x-z)} + e^{-in\pi(x-z)}\big)\Big\} \\ &= \frac{-1}{2}\Big\{\frac{1}{1-e^{\pi[-y+i(x+z)]}} + \frac{1}{1-e^{\pi[-y-i(x+z)]}} - \frac{1}{1-e^{\pi[-y+i(x-z)]}} - \frac{1}{1-e^{\pi[-y-i(x-z)]}}\Big\} \\ &= \frac{-1}{2}\Big\{\frac{2-e^{-\pi y}\big(e^{i\pi(x+z)} + e^{-i\pi(x+z)}\big)}{1-e^{-\pi y}\big(e^{i\pi(x+z)} + e^{-i\pi(x+z)}\big) + e^{-2\pi y}} - \frac{2-e^{-\pi y}\big(e^{i\pi(x-z)} + e^{-i\pi(x-z)}\big)}{1-e^{-\pi y}\big(e^{i\pi(x-z)} + e^{-i\pi(x-z)}\big) + e^{-2\pi y}}\Big\} \\ &= \frac{1-e^{-\pi y}\cos\pi(x-z)}{1-2e^{-\pi y}\cos\pi(x-z) + e^{-2\pi y}} - \frac{1-e^{-\pi y}\cos\pi(x+z)}{1-2e^{-\pi y}\cos\pi(x+z) + e^{-2\pi y}}. \end{aligned}$$

□

***Remark* 4.** Note that this computation is the same as the one for computing $P_r(\theta)$ if one use odd extension from $(0,1)$ to $(-1,1)$. Also note there is no zero-th order $\omega^0 = 1$ in this result which exists in P_r, this is why the denomiator $1 - e^{-\pi y}\cos t$.

20. **Consider the Dirichlet problem in the annulus defined by $\{(r,\theta) : \rho < r < 1\}$, where $0 < \rho < 1$ is the inner radius. The problem is to solve**

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions $u(1,\theta) = f(\theta), u(\rho,\theta) = g(\theta)$, where f and g are given continuous functions.

Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

$$u(r,\theta) = \sum c_n(r)e^{in\theta}$$

with $c_n(r) = A_n r^n + B_n r^{-n}, n \neq 0$**. Set**

$$f(\theta) \sim \sum a_n e^{in\theta} \text{ and } g(\theta) \sim \sum b_n e^{in\theta}.$$

We want $c_n(1) = a_n$ **and** $c_n(\rho) = b_n$**. This leads to the solution**

$$u(r,\theta) := \sum_{n\neq 0} \Big(\frac{1}{\rho^n - \rho^{-n}}\Big)[((\rho/r)^n - (r/\rho)^n)a_n + (r^n - r^{-n})b_n]e^{in\theta} + a_0 + (b_0 - a_0)\frac{\log r}{\log \rho}.$$

Show that as a result we have

$$u(r,\theta) - (P_r * f)(\theta) \to 0 \text{ as } r \to 1 \text{ uniformly in } \theta,$$

and

$$u(r,\theta) - (P_{\rho/r} * g)(\theta) \to 0 \text{ as } r \to \rho \text{ uniformly in } \theta.$$

Proof. First we confirm the absolute convergence of the series

$$u_1(r,\theta) := \sum_{n\neq 0} \Big(\frac{(\frac{\rho}{r})^n - (\frac{\rho}{r})^{-n}}{\rho^n - \rho^{-n}}\Big)a_n e^{in\theta}$$

and

$$u_\rho(r,\theta) := \sum_{n\neq 0} \Big(\frac{r^n - r^{-n}}{\rho^n - \rho^{-n}}\Big)b_n e^{in\theta}$$

so that $u =: u_1 + u_\rho + u_0$ is confirmed to be well-defined. This is a consequence that

$$\sup_n \Big\{|a_n|, |b_n|\Big\} \leq \frac{1}{2\pi} \max\Big(\|f\|_{L^1(\mathbb{T})}, \|g\|_{L^1(\mathbb{T})}\Big)$$

and for each $n \neq 0$

$$\Big|\frac{(\frac{\rho}{r})^n - (\frac{\rho}{r})^{-n}}{\rho^n - \rho^{-n}}\Big| = r^{|n|}\Big(\frac{(\frac{\rho}{r})^{2|n|} - 1}{\rho^{2|n|} - 1}\Big) = \frac{r^{|n|}(1 - \frac{\rho^2}{r^2})}{1 - \rho^{2|n|}}\Big(1 + (\frac{\rho}{r})^2 + \cdots + (\frac{\rho}{r})^{2|n|-2}\Big) \leq (1 - \frac{\rho^2}{r^2})\frac{|n|}{1 - r^2}r^{|n|} \tag{1}$$

$$\Big|\frac{r^n - r^{-n}}{\rho^n - \rho^{-n}}\Big| = \Big(\frac{\rho}{r}\Big)^{|n|}\Big(\frac{1 - r^{2|n|}}{1 - \rho^{2|n|}}\Big) = \Big(\frac{\rho}{r}\Big)^{|n|}\frac{(1 - r^2)(1 + r^2 + r^4 + \cdots + r^{2|n|-2})}{1 - \rho^2} \leq |n|\Big(\frac{\rho}{r}\Big)^{|n|}\frac{1 - r^2}{1 - \rho^2}. \tag{2}$$

Note that the above method to derive the estimates also imply u is harmonic on the annual. We omit the details. To check it did satisfy the boundary condition, it's enough to show the rest assertions.

To arrive this, we first show $u_\rho \to 0$ and $u_1 + u_0 - P_r * f \to 0$ as $r \to 1$ uniformly in θ. Note that the u_ρ part is a consequence of (2). For $1-\delta < r < 1$ and $n \neq 0$,

$$\begin{aligned}\left|\frac{(\frac{\rho}{r})^n - (\frac{\rho}{r})^{-n}}{\rho^n - \rho^{-n}} - r^{|n|}\right| &= r^{|n|}\frac{\rho^{|n|}[r^{-2|n|}-1]}{\rho^{-|n|}-\rho^{|n|}} = \frac{r^{|n|}\rho^{2|n|}}{1-\rho^{2|n|}}\left(\frac{1}{r^2}-1\right)\left[1+\frac{1}{r^2}+\frac{1}{r^4}+\cdots\frac{1}{r^{2|n|-2}}\right] \\ &\leq \frac{r^{-2}-1}{1-\rho^2}|n|\frac{\rho^{2|n|}}{(1-\delta)^{2|n|-2}}.\end{aligned}$$

In particular we pick $1-\delta > \rho$ so that the majorant series of $u_1 + u_0 - P_r * f$ converges (by ratio test) and hence bounded by a multiple (independent of θ) of $r^{-2}-1$. This proves that $u_1 + u_0 - P_r * f \to 0$ as $r \to 1$ uniformly in θ.

Finally, we show that $u_1 \to 0$ and $u_\rho + u_0 - P_{\rho/r} * f \to 0$ as $r \to \rho$ uniformly in θ. Note that the u_1 part is a consequence of (1). For $\rho < r < 1$ and $n \neq 0$,

$$\begin{aligned}\left|\frac{r^n - r^{-n}}{\rho^n - \rho^{-n}} - (\frac{\rho}{r})^{|n|}\right| &= \left(\frac{\rho}{r}\right)^{|n|}\frac{r^{2|n|}-\rho^{2|n|}}{1-\rho^{2|n|}} = \frac{\rho^{|n|}r^{|n|}}{1-\rho^{2|n|}}(1-\frac{\rho^2}{r^2})\left[1+(\frac{\rho}{r})^2+(\frac{\rho}{r})^4+\cdots(\frac{\rho}{r})^{2|n|-2}\right] \\ &\leq \frac{1-\frac{\rho^2}{r^2}}{1-\rho^2}\rho^{|n|}|n|.\end{aligned}$$

So the majorant series of $u_1 + u_0 - P_{\rho/r} * f$ converges (by ratio test) and hence bounded by a multiple (independent of θ) of $1-\rho^2 r^{-2}$. This proves that $u_1 + u_0 - P_{\rho/r} * f \to 0$ as $r \to \rho$ uniformly in θ.

$\square$

Problems

1. **One can construct Riemann integrable functions on $[0,1]$ that have a dense set of discontinuities as follows.**

   (a) Let $f(x) = 0$ when $x < 0$ and $f(x) = 1$ if $x \geq 0$. Choose a countable dense sequence $\{r_n\}$ in $[0,1]$. Then, show that the function

   $$F(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}f(x-r_n)$$

   is integrable and discontinuous precisely at each r_n.

   [Hint: F is monotonic and bounded.]

(b) Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n)$$

where $g(x) = \sin(1/x)$ when $x \neq 0$, and $g(0) = 0$. Then F is integrable, discontinuous precisely at each $x = r_n$, and fails to be monotonic in any subinterval of $[0, 1]$. [Hint: Use the fact that $3^{-k} > \sum_{n>k} 3^{-n}$.]

(c) The original example of Riemann is the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

where $(x) = x$ for $x \in (-1/2, 1/2]$ and extended to $\mathbb{R}$ by periodicity, that is, $(x+1) = (x)$. It can be shown that F is discontinuous whenever $x \in D := \{\frac{2k+1}{2m} | k, m \in \mathbb{Z}, m \neq 0\}$ and continuous at $\mathbb{R} - D$. Moreover, F is Riemann-integrable on every bounded interval.

(d) If the above (nx) is replace by $\{nx\}$, where $\{nx\}$ is the fractional part of nx (that is, $\{z\} := z$ if $z \in [0, 1)$ and then extend periodically), then we have F is discontinuous precisely at $\mathbb{Q}$.

Remark 5. Are the functions F in (c)(d) monotone?

Proof. We may assume all r_n are distinct (why?)

(a) For every $x \in [0, 1]$, the series converges and is less than $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so F is well-defined and bounded. The integrability can be easily proved from the obvious monotonicity of F. The continuity on $[0, 1] \setminus \{r_n\}_{n \in \mathbb{N}}$ and discontinuity on $\{r_n\}_{n \in \mathbb{N}}$ can be proved similarly to the cases for the popcorn function (note that $F(r_n+) - F(r_n-) \geq \frac{1}{n^2}$).

Remark 6. Another proof for discontinuity at r_n is a contradiction proof, that is, suppose F is continuous at r_n, then with the advantage of uniform convergence we know the function $F_n = \sum_{j \neq n} \frac{1}{j^2} f(x - r_j)$ is continuous at r_n. So $\frac{1}{n^2} f(x - r_n) = F(x) - F_n(x)$ is continuous at r_n, which is a contradiction. This method can be applied to (b) too.

(b) Since the Riemann integrability, continuity and discontinuity of F can be proved as (a), we only prove the last assertion here.

Given $I \subseteq [0, 1]$, by the density of $\{r_j\}_j$ there is some r_k and an interval I'_k such that $r_k \in I'_k \subseteq I_k$ and $r_1, r_2, \cdots, r_{k-1} \notin I'_k$. So $F_k(x) = \sum_{n=1}^{k-1} 3^{-n} g(x - r_n)$ is continuous on I'_k, and hence $|F_k| \leq \frac{3^{-k}}{100}$ in some small neighboorhood J_k of r_k.

Note that $3^{-k} = 2\sum_{j=k+1}^{\infty} 3^{-j}$ and

$$F(x) = F_k(x) + 3^{-k}g(x - r_k) + \sum_{j=k+1}^{\infty} 3^{-j}g(x - r_j).$$

So for any neighborhood $J \subseteq J_k$ of r_k, we can find $z < w < z'$ such that $g(z-r_k) = 1 = g(z'-r_k)$ and $-1 = g(w - r_k)$ so $F(z) \geq -\frac{3^{-k}}{100} + 3^{-k} - \sum_{j=k+1} 3^{-j} > \frac{3^{-k}}{100} - 3^{-k} + \sum_{j=k+1} 3^{-j} \geq F(w)$ and similarly $F(z') > F(w)$.

We prove (d) first to explain our idea clearly. Since $F_N(x) = \sum_{n=1}^{N} \frac{\{nx\}}{n^2}$ is Riemann integrable for each $N \in \mathbb{N}$ and converges to $F(x)$ uniformly on every bounded interval (by Weierstrass M-test), F is Riemann-integrable on every bounded interval. Also, the continuity of F on $\mathbb{R} - \mathbb{Q}$ will be a consequence of the continuity of F_N and hence of $\frac{\{nx\}}{n^2}$ on $\mathbb{R} - \mathbb{Q}$ for each $n \in \mathbb{N}$ which can be easily proved.

Now we examine that F is discontinuous at any $z \in \mathbb{Q}$, say $z = \frac{p}{q}$, where $\gcd(p, q) = 1$. Note that the convergence of the series $F(x)$ and $f_r(x) := \sum_{k=0}^{\infty} \frac{\{(kq+r)x\}}{(kq+r)^2}$ for each $1 \leq r \leq q$ imply

$$F(x) = \sum_{r=1}^{q} f_r(x).$$

We are done if we prove f_r is continuous at $\frac{p}{q}$ if $1 \leq r < q$ and f_q has a discontinuity at $\frac{p}{q}$.

For every $x \in (\frac{p}{q} - \frac{1}{4q}, \frac{p}{q})$, that is, $qx \in (p - \frac{1}{4}, p)$, we have $\{qx\} > \frac{3}{4}$ and hence

$$f_q(x) = \sum_{j=1}^{\infty} \frac{\{jqx\}}{j^2q^2} > \frac{3}{4q^2} - \frac{1}{q^2}\sum_{j=2}^{\infty} \frac{1}{j^2} = \frac{1}{q^2}(\frac{7}{4} - \frac{\pi^2}{6}) > 0 = f_q(\frac{p}{q}).$$

Therefore f_q is discontinuous at $\frac{p}{q}$.

For $1 \leq r < q$, one notes that for each $k \geq 0$ the map $x \mapsto \{(kq + r)x\}$ is continuous at $\frac{p}{q}$ since $(kq + r)\frac{p}{q} = kp + \frac{rp}{q} \notin \mathbb{Z}$. Hence f_r is continuous at $\frac{p}{q}$ by the uniform convergence of its series.

Finally, in (c), we change the notations $(\cdot)$ to $[\cdot]$ and $F(x)$ to $G(x)$ for notational convenience. For $G(x) := \sum_{n=1}^{\infty} \frac{[nx]}{n^2}$ the Riemann-integrability and continuity are proved as (d).

Now we examine that G is discontinuous at any $z \in D$, say $z = \frac{p}{2q}$, where $\gcd(p, q) = 1$, p is odd. Note that the absolute convergence of the series $G(x)$ and $g_r(x) := \sum_{k=0}^{\infty} \frac{[(kq+r)x]}{(kq+r)^2}$ imply

$$G(x) = \sum_{r=1}^{q} g_r(x).$$

We are done if we prove g_r is continuous at $\frac{p}{2q}$ if $1 \leq r < q$ and g_q has a discontinuity at $\frac{p}{2q}$.

For every $x \in (\frac{p}{2q}, \frac{1}{2q} + \frac{p}{2q})$, that is, $qx \in (\frac{p}{2}, \frac{p+1}{2})$, we have $[qx] < 0$ since p is odd. Hence

$$g_q(x) = \sum_{j=1}^{\infty} \frac{[jqx]}{j^2q^2} < \frac{1}{2q^2}\sum_{j=2}^{\infty} \frac{1}{j^2} = \frac{1}{2q^2}(\frac{\pi^2}{6} - 1) < \frac{1}{2q^2}(\frac{\pi^2}{6} - \frac{1}{2^2}\frac{\pi^2}{6}) = \frac{1}{2q^2}(\sum_{j=1}^{\infty} \frac{1}{j^2} - \sum_{j=1}^{\infty} \frac{1}{(2j)^2}) = g_q(\frac{p}{2q}).$$

Therefore g_q is discontinuous at $\frac{p}{q}$.

For $1 \leq r < q$, one notes that for each $k \geq 0$ the map $x \mapsto [(kq+r)x]$ is continuous at $\frac{p}{2q}$ since $(kq+r)\frac{p}{2q} \notin \mathbb{Z}+\frac{1}{2}$ (if $(kq+r)\frac{p}{2q} \in \mathbb{Z}+\frac{1}{2}$, then $kp+\frac{rp}{q} = (kq+r)\frac{p}{q} \in \mathbb{Z}$ which is a contradiction). Hence g_r is continuous at $\frac{p}{2q}$ by the uniform convergence of its series. □

Remark 7. Another construction us the so-called popcorn function. More interestingly, there is **no** functions which are **precisely** continuous at the rational. One can prove this by using Baire Category Theory, see Exercise 4.6 of Book IV.

2. **Let D_N denote the Dirichlet kernel**

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N+\frac{1}{2})\theta}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta.$$

(a) Prove that

$$L_N \geq c\log N$$

for some constant $c > 0$. A more careful estimate gives

$$L_N = \frac{4}{\pi^2}\log N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $S_n(f_n)(0) \geq c' \log n$.

Proof. (a)

$$\begin{aligned}
L_N =& \frac{1}{\pi}\int_0^{\pi} |\frac{\sin(N+\frac{1}{2})\theta}{\sin(\theta/2)}|\, d\theta = \frac{1}{\pi}\int_0^{\pi} |\frac{\sin(N+\frac{1}{2})\theta}{\theta/2}|\, d\theta + \frac{1}{\pi}\int_0^{\pi} |\sin(N+\frac{1}{2})\theta|\Big(\frac{1}{\sin(\theta/2)} - \frac{1}{\theta/2}\Big)\, d\theta \\
=& \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin z|}{z}\, dz + O(1) = \frac{2}{\pi}\sum_{j=1}^{N}\int_{(j-\frac{1}{2})\pi}^{(j+\frac{1}{2})\pi} |\sin z|\Big(\frac{1}{j\pi} + \frac{1}{z} - \frac{1}{j\pi}\Big)\, dz + O(1) \\
=& \frac{4}{\pi^2}\sum_{j=1}^{N}\frac{1}{j} + O(1) = \frac{4}{\pi^2}\log N + O(1).
\end{aligned}$$

(b) Since D_n has finite zeros, $F_n(t) := \mathrm{sgn} D_n(t)$ has finitely many discontinuities. Note that $S_n(F_n)(0) = L_n$ and we can approximate F_n by continuous function in L^1 norm since $[0,1]$ has finite Lebesgue measure. The desired result follows by noting $D_n(0) \neq 0$ for all $n \in \mathbb{N}$ so $|D_n|$ have an uniform bound in every neighborhood of discontinuities of every F_n. □

Remark 8. In Book IV, Chapter 4, one can combine this problem with Uniform Boundedness Principle to show the existence of continuous function whose Fourier series diverges at a given point, a result will also be proved by an explicit construction of P. du Bois-Reymond (1873) in Section 3.2.2. On the other hand, one can combine this with Open Mapping Theorem to show the mapping $f \in L^1[-\pi,\pi] \mapsto \{\hat{f}(n)\} \in C_0(\mathbb{Z})$ is not surjective.

3. **Littlewood provided a refinement of Tauber's theorem:**

   (a) If $\sum c_n$ is Abel summable to s and $nc_n \geq -M$ for all n, then $\sum c_n$ converges to s.

   (b) As a consequence of Exercise 13(c), we know that if $\sum c_n$ is Cesàro summable to s and $nc_n \geq -M$ for all n, then $\sum c_n$ converges to s. Note that another easier proof is also given in Exercise 14(c).

   These results may be applied to Fourier series. By Exercise 17, they imply that if f is an integrable function that satisfies $\hat{f}(\nu) = O(1/|\nu|)$, then:

   (i) If f is continuous at θ, then

$$S_N(f)(\theta) \to f(\theta) \text{ as } N \to \infty$$

   (ii) If f has a jump discontinuity at θ, then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \text{ as } N \to \infty$$

   (iii) If f is continuous on $[-\pi,\pi]$, then $S_N(f) \to f$ uniformly.

   Other proofs for the simpler assertion (b), hence proofs of (i),(ii),and (iii), can be found in my Exercise 2.14 and Problem 4.5

Remark 9. The proof by Wielandt's method is taken from [3, Section 1.12], I think this book is a great survey of Tauber's theorem.

Remark 10. Another similar result (weaker hypothesis and weaker conclusion) is the Jordan's test: $\lim_{N\to\infty} S_N(f)(\theta_0) = \frac{f(\theta_0+)+f(\theta_0-)}{2}$ for f is only monotone (and hence for f is of bounded variation) on a neighborhood of θ_0, which can be proved by Riemann-Lebesgue lemma and second mean-value theorem. See [1, Theorem 1.1.2]. Note that the control of Dirichlet kernel is much difficult than the Fejér kernel, cf. [2, Theorem 3.4.1].

Proof. For every polynomial $P(x) = \sum_{k=0}^m b_k x^k$,

$$\sum_{n=0}^{\infty} c_n P(x^n) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^{m} b_k x^{nk} = \sum_{k=0}^{m} b_k \sum_{n=0}^{\infty} c_n x^{nk} \to \sum_{k=0}^{m} b_k \cdot s = P(1)s \tag{3}$$

as $x \nearrow 1$. (The reader should be able to prove the second identity in (3) by $\epsilon - N$ argument.)

Define $g(x) = \chi_{[e^{-1},1]}(x)$, that is, the characteristic function of $[e^{-1},1]$ so that for $s_N = \sum_{n\le N} c_n = \sum_{n=0}^{\infty} c_n g(e^{-n/N})$; To complete the proof that $s_N \to s$, we will show below that the conclusion in (3) is also valid with g instead of P:

$$s \le \liminf_{x\nearrow 1} \sum c_n g(x^n) \le \limsup_{x\nearrow 1} \sum c_n g(x^n) \le s.$$

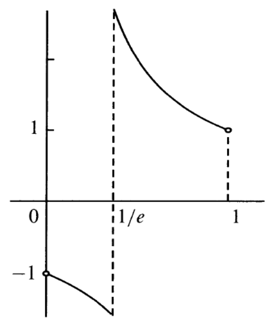
To the last inequality of (3), we look for a suitable polynomial majorant

$$P \ge g \text{ for which } P(0) = g(0) = 0,\ P(1) = g(1) = 1.$$

Equivalently,

$$\frac{P(t)-t}{t(1-t)} \text{ must be a polynomial } Q(t) \ge h(t) = \frac{g(t)-t}{t(1-t)}. \tag{4}$$

The piecewise continuous function h on $[0,1]$ looks like the following graph.



For $0 < x < 1$ it follows from (4) and the condition $nc_n \ge -M$ that

$$\sum_{n=0}^{\infty} c_n g(x^n) - \sum_{n=0}^{\infty} c_n P(x^n) = -\sum_{n=1} c_n\{P(x^n) - g(x^n)\} \le M \sum_{n=1}^{\infty} \frac{1}{n}\{P(x^n) - g(x^n)\}$$
$$\le M \sum_{n=1}^{\infty} \frac{1-x}{1-x^n}\{P(x^n) - g(x^n)\} = M(1-x)\sum_{n=1}^{\infty} \phi(x^n), \quad \text{where } \phi(t) = \frac{P(t)-g(t)}{1-t}. \tag{5}$$

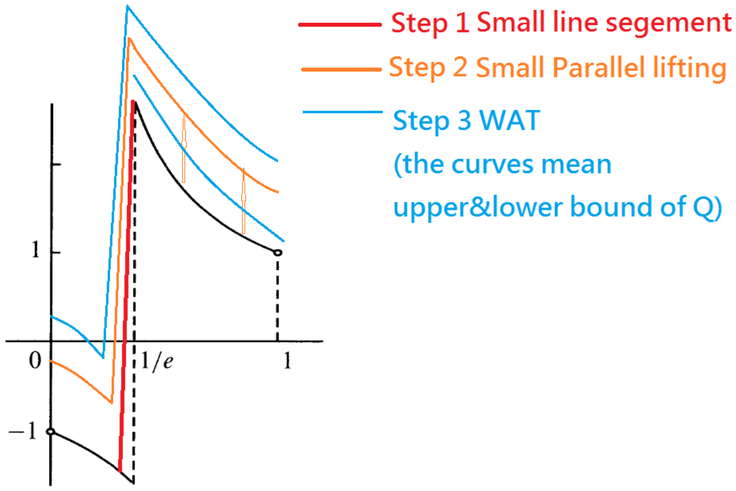
By (3), the second term in the first member of (5) has limit s as $x \nearrow 1$.

Note that the final member in (5) has limit (6) since it can serves as a Riemann sum of that integral under the partition $P_x^{\infty} = \{1, x, x^2, \cdots\}$. (The integrability of $\frac{\phi(t)}{t}$ is easy to see from $Q(t) - h(t)$. On the other hand, the detail for reducing P_x^{∞} into a finite set as a regular partition is left to the reader, this is not trivial, you need to observe some uniformity or monotonicity.)

$$M\int_0^1 \phi(t)\frac{dt}{t} = M\int_0^1 \frac{P(t) - t - (g(t) - t)}{t(1-t)}\,dt = M\int_0^1 \{Q(t) - h(t)\}\,dt. \tag{6}$$

Now for given $\epsilon > 0$, Weierstrass' approximation theorem (WAT) makes it possible to construct a polynomial $Q \ge h$ such that $\int_0^1 (Q-h) \le \epsilon$. A construction is given in the figure below.

(Note that the graph is not accurate due to the author's poor drawing technique and softwares)



For such a Q and the corresponding P determined by (4), it follows from (5), (3) and (6) that

$$\begin{aligned}\limsup_{x\nearrow 1}\sum_{n=0}^{\infty} c_n g(x^n) &\le \lim_{x\nearrow 1}\sum_{n=0}^{\infty} c_n P(x^n) + \lim_{x\nearrow 1} M(l-x)\sum_{n=0}^{\infty}\phi(x^n)\\ &= P(1)s + M\int_0^1 \{\phi(t)/t\}\,dt \le s + M\epsilon.\end{aligned}$$

Since ϵ was arbitrary, our lim sup does not exceed s.

To the first inequality of (3), the method is the same with looking some polynomial $\widetilde{P} \le g$, $\widetilde{P}(0) = g(0) = 0$ and $\widetilde{P}(1) = g(1) = 1$. We omit it. $\square$

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