Real and Complex Analysis, 3rd Edition, W.Rudin
Chapter 6 Complex Measures

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1. Let ν be a complex measure on $(X, \mathscr{M})$. If $E \in \mathscr{M}$, define

$$\mu_1(E) = \sup\Big\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, E_2, \cdots E_n \text{ disjoint}, E = \cup_1^n E_j\Big\}$$
$$|\nu|(E) = \sup\Big\{\sum_1^\infty |\nu(E_j)| : E_1, E_2, \cdots \text{ disjoint}, E = \cup_1^\infty E_j\Big\}$$
$$\mu_3(E) = \sup\Big\{|\int_E f\, d\nu| : |f| \le 1\Big\}$$

Show that $\mu_1 = |\nu| = \mu_3$.

Remark 1. We extend the problem slightly, which comes from [3, Exercise 3.21]. To prove the original problem, we can simplify the proof given here, that is, without μ_3.

Proof. We are going to show $\mu_1 \le |\nu| \le \mu_3 \le \mu_1$. The first inequality is trivial. For the second one, since there exists $w_j \in \mathbb{C}, |w_j| = 1$ such that $|\nu(E_j)| = w_j \nu(E_j)$. Consider the function $f = \sum_j w_j \chi_{E_j}$. Since $\{E_j\}$ is mutually disjoint, $|f| \le 1$. Take $f_n = \sum_1^n w_j \chi_{E_j}$. We then have, by dominate convergence theorem,

$$|\int f_n - \int f\, d\nu| \le \int |f_n - f|\, d|\nu| \to 0,$$

We then have

$$\int_E f\, d\nu = \lim_{n\to\infty} \int_E f_n\, d\nu = \lim_{n\to\infty} \sum_1^n w_j \nu(E_j) = \sum_1^\infty |\nu(E_j)|.$$

Then $\sum_1^\infty |\nu(E_j)| = |\int_E f\, d\nu| \le \mu_3(E)$. Since $\{E_j\}$ is arbitrary, $|\nu| \le \mu_3$. For the third inequality, given $\epsilon > 0$, then we can find some f with $|f| \le 1$ such that

$$\mu_3(E) < |\int_E f\, d\nu| + \epsilon$$

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We approximate f by a simple function as follows. Let $D \subset \mathbb{C}$ be the closed unit disc. The compactness of D implies that there are finite many $z_j \in D$ such that $B_\epsilon(z_j)$ covers D. Define $B_j = f^{-1}(B_\epsilon(z_j)) \subset X$, which is measurable since f is. The union of B_j is X. Let

$$A_1 = B_1,\ A_j = B_j \setminus \cup_{i=1}^{j-1} B_i$$

be the disjoint sets with $A_j \subset B_j$ and $\cup A_j = X$. Define the simple function $\phi = \sum_1^m z_j \chi_{A_j}$, then $|\phi| \leq 1$ and $|f(x) - \phi(x)| < \epsilon$ for all x by the construction. Then

$$|\int_E f\, d\nu| - |\int_E \phi\, d\nu| \leq |\int_E f - \phi\, d\nu| < |\nu|(E).$$

$$\mu_3(E) < |\int_E f\, d\nu| + \epsilon < |\int_E \phi\, d\nu| + \epsilon + \epsilon|\nu|(E)$$

Now we define $F_j = A_j \cap E$, then F_j is a finite partition of E and

$$|\int_E \phi\, d\nu| = |\int \sum_j z_j \chi_{A_j \cap E}\, d\nu| = |\sum_j z_j \nu(F_j)| \leq \sum_j |\nu(F_j)| \leq \mu_1(E).$$

Since ϵ is arbitrary, we have $\mu_3(E) \leq \mu_1(E)$. □

2. Find two measures μ, λ that are σ-finite positive measure on some $(X, \mathscr{M})$, but the function h in the absolutely continuous part of the Lebesgue-Radon-Nikodym decomposition of λ with respect to μ is not μ-integrable.

   Proof. Let μ be the Lebesgue measure on $((0,1), \mathscr{L})$. The desired λ is defined by

   $$\lambda(E) = \int_E \frac{1}{x}\, d\mu(x).$$

   which is easy to see it's positive, σ-finite, absolutely continuous with respect to μ. □

   Remark 2. This example also provide us a reason why we need to assume the measure λ is finite in Theorem 6.11.

3. *Proof.* Any multiple of complex regular Borel measure is still a complex regular Borel measure.

   Given two complex regular Borel measures μ_1, μ_2. Given a Borel set E and $k \in \mathbb{N}$, then there exist open sets O_1, O_2 containing E such that $|\mu_i|(O_i \setminus E) < \frac{1}{2k}$ $(i = 1, 2)$, then

   $|\mu_1 + \mu_2|(O_1 \cap O_2) = |\mu_1 + \mu_2|(E) + |\mu_1 + \mu_2|((O_1 \cap O_2) \setminus E) \leq |\mu_1 + \mu_2|(E) + |\mu_1|(O_1 \setminus E) + |\mu_2|(O_2 \setminus E) \leq |\mu_1 + \mu_2|(E) + \frac{1}{k}$, which implies the outer regularity of $|\mu_1 + \mu_2|$.

   Given any open set E or Borel set E with finite measure, we see for each $k \in \mathbb{N}$, then there exist open sets K_1, K_2 contained in E such that $|\mu_i|(E \setminus K_i) < \frac{1}{2k}$ $(i = 1, 2)$, then $|\mu_1 + \mu_2|(K_1 \cup K_2) =$

$|\mu_1+\mu_2|(E) - |\mu_1+\mu_2|(E\setminus(K_1\cup K_2)) \geq |\mu_1+\mu_2|(E) - \frac{1}{k}$, which implies the inner regularity of $|\mu_1+\mu_2|$. Hence $\mu_1+\mu_2$ is a complex regular Borel measure.

Therefore $M(X)$ is a vector space over $\mathbb{C}$. For each $\nu \in M(X)$, by Radon-Nikodym theorem, there is a integrable function f with respect to $\mu = |\nu_r| + |\nu_i|$ such that $d\nu = f d\mu$ and then $d|\nu| = |f| d\mu$. So $|\nu|(X) = \|\nu\| = 0$ iff f is zero μ-a.e. iff $\nu = 0$. For each $a \in \mathbb{C}$, $\|a\nu\| = |a\nu|(X) = \int_X |af|\, d\mu = |a||\nu|(X) = |a|\|\nu\|$. The triangle inequality is proved by the same method with $\mu = \mu_1 + \mu_2 := |\nu_{1,r}| + |\nu_{1,i}| + |\nu_{2,r}| + |\nu_{2,i}|$. Therefore, $\|\cdot\|$ is a norm on $M(X)$.

Given $\sum \nu_n$ be a absolutely convergent series in $M(X)$. Since for each $A \in \Sigma$, $\sum |\nu_n(A)| \leq \sum |\nu_n|(A) \leq \sum |\nu_n|(X) = \sum \|\nu_n\| < \infty$, we may define $\nu : \Sigma \to \mathbb{C}$ by $\nu(A) = \sum_n \nu_n(A)$. Clearly, $\nu(\emptyset) = 0$. If A_k is a sequence of disjoint measureable sets, since by Tonelli's theorem

$$\sum_{n,k} |\nu_n|(A_k) = \sum_n \sum_k |\nu_n|(A_k) = \sum_n |\nu_n|(\cup_k A_k) \leq \sum \|\nu_n\| < \infty$$

and therefore

$$\nu(\cup_k A_k) = \sum_n \nu_n(\cup_k A_k) = \sum_n \sum_k \nu_n(A_k) = \sum_k \sum_n \nu_n(A_k) = \sum_k \nu(A_k).$$

Hence, ν is a complex measure. $\nu \in M(X)$. Given $X_1 \cdots X_m$ covers X,

$$\sum_{j=1}^m |\Big(\nu - \sum_{n=1}^N \nu_n\Big)(X_j)| = \sum_{j=1}^m |\sum_{n=N+1}^\infty \nu_n(X_j)| \leq \sum_{j=1}^m \sum_{n=N+1}^\infty |\nu_n|(X_j) = \sum_{n=N+1}^\infty |\nu_n|(X) = \sum_{n=N+1}^\infty \|\nu_n\|.$$

By Exercise 1, $\|\nu - \sum_{n=1}^N \nu_n\| \leq \sum_{n=N+1}^\infty \|\nu_n\| \to 0$ as $N \to 0$. So $M(X)$ is complete. □

4. A special case was discussed in Exercise 4.7 earlier.

1st proof. Suppose there is no $C > 0$, such that $\|fg\|_1 \leq C\|f\|_p$. Then there exist f_n such that $\|f_n\|_p = 1$ and $\int |f_n g| > 3^n$. Let $f = \sum 2^{-n}|f_n|$, then $\|f\|_p \leq 1$ and for each n

$$\int |fg| > \int |f_n g| 2^{-n} > (\frac{3}{2})^n$$

This contradicts to $fg \in L^1$ and therefore the map $f \mapsto \int fg$ from $L^p \to \mathbb{R}$ or $\mathbb{C}$ is bounded. if $1 \leq p < \infty$, then by Riesz Representation theorem, $g = \tilde{g} \in L^q$ a.e.. For $p = \infty$, if g is not integrable, then it contradicts to the hypothesis by taking $f \equiv 1$. □

2nd proof. This is inspired by Exercise 5.10, which is an application of Banach-Steinhaus Theorem. I think this method is easier than the first proof since we don't need duality theorem.

This time, we define a linear map $\Lambda_n : L^p \to \mathbb{C}$ by $\Lambda_n(f) = \int fg_n$, where $g_n = g\chi_{\{|x|\le n : |g(x)|\le n\}}$ (this is the place we use the σ-finiteness of μ). Note that $\|\Lambda_n\| = \|g_n\|_q$ is bounded for all n and $\{\Lambda_n(f)\}$ is bounded by $\int |fg|$ for each $f \in L^p$. Therefore $g \in L^q$ by Banach-Steinhaus Theorem and monotone convergence theorem. $\square$

There is a theorem so called the converse Hölder's inequality:

Theorem 3. *Let (X, Σ, μ) be a measure space, $1 \le q < \infty$. If $g \in L^q$, then*

$$\|g\|_q = \sup\{|\int fg\, d\mu| : \|f\|_p = 1\}.$$

If $q = \infty$, this result holds if μ is semi-finite.

This exercise seems to be the converse of the theorem (but it's NOT) if we assume

$$M_q(g) = \sup\{|\int fg\, d\mu| : f \text{ is simple and its support has finite measure, } \|f\|_p = 1\} < \infty.$$

I see this result in Folland [3, Theorem 6.14]. Compare the differences between their proofs. I think they can NOT be replaced by each other. (Reason: Exercise 6.4 do NOT permit $M_q(g) < \infty$. On the other hand, I think we can't use approximate argument to see $M_q(g) < \infty$ implies $fg \in L^1$ for all $f \in L^p$.)

5. *Proof.* No, since $L^1(\mu)$ is one dimensional, so does its dual space. But $L^\infty(\mu)$ is two dimensional. $\square$

Remark 4. This exercise shows that the map from $L^\infty \to (L^1)^*$ defined by $g \mapsto \phi_g$ is not injective if μ is not semifinite. This problem, however, can be remedied by redefining L^∞. See Folland [3, Exercise 6.23-25].

6. *Proof.* For each σ-finite set $E \subset X$ there is an a.e.-unique $g_E \in L^q(E)$ such that $\Phi(f) = \int fg_E$ for all $f \in L^p(E)$ and $\|g_E\|_q \le \|\Phi\|$. If F is σ-finite and $E \subset F$, then $g_F = g_E$ a.e. on E, so $\|g_F\|_q \ge \|g_E\|_q$. Let M be the supremum of $\|g_E\|_q$ as E ranges over all σ-finite sets, noting that $M \le \|\Phi\|$. Choose a sequence $\{E_n\}$ so that $\|g_{E_n}\|_q \to M$, and set $F = \cup_1^\infty E_n$. Then F is σ-finite and $\|g_F\|_q \ge \|g_{E_n}\|_q$ for all n, whence $\|g_F\|_q = M$. Now, if A is a σ-finite set containing F, we have
$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \le M^q = \int |g_F|^q,$$
and thus $g_{A\setminus F} = 0$ and $g_A = g_F$ a.e. (Here we use the fact $q < \infty$.) But if $f \in L^p$, then $A_f = F \cup \{x : f(x) \ne 0\}$ is σ-finite, so $\Phi(f) = \int fg_{A_f} = \int fg_F$. Thus we may take $g = g_F$, and the proof is complete. $\square$

7. *Proof.* As Hint, the hypothesis holds for any $f\,d\mu$ with f is a trigonometric polynomial, and hence with f is continuous function by Weierstrauss' theorem, and therefore with f is any bounded Borel function by Lusin's theorem. Since by Theorem 6.12, $d\mu = h\,d|\mu|$ for some $|h| = 1$ a.e., $\widehat{|\mu|}(n) \to 0$ as $n \to \infty$. Since $|\widehat{|\mu|}(n)| = |\overline{\widehat{|\mu|}(n)}| = |\widehat{|\mu|}(-n)|$, we know $\widehat{|\mu|}(m) \to 0$ as $m \to -\infty$. By the same argument, it holds with $f\,d|\mu|$ for any f is bounded. In particular, we pick $hd|\mu| = d\mu$. $\square$

8. *Proof.* Assume that there is some $k \in \mathbb{N}$ such that for all n,

$$\int e^{-int}\,d\nu := \int e^{-int}(e^{-ikt} - 1)\,d\mu = \widehat{\mu}(n+k) - \widehat{\mu}(n) = 0$$

Since trigonometric polynomials are dense in $C[0, 2\pi]$, $\int f\,d\nu = 0$ for any continuous f. Given open set U, there is an increasing sequence of continuous function $f_n = 1$, if $\operatorname{dist}(x, \partial U) > \frac{1}{n}$ and $f_n = n \cdot \operatorname{dist}(x, \partial U)$ otherwise, that approximate to the charactertic function χ_U pointwise. By monotone convergence theorem, we see $\nu(U) = 0$. So $\nu_r(U) = \nu_i(U) = 0$, that is $\nu_r^+ = \nu_r^-, \nu_i^+ = \nu_i^-$ for all open sets. Since they are outer regular (Theorem 2.17), $\nu_r^+ = \nu_r^-, \nu_i^+ = \nu_i^-$, that is, $\nu_r = \nu_i = 0 = \nu$. Since $(e^{-ikt} - 1) \neq 0$ iff $t \neq \frac{2\pi j}{k} (j \in \mathbb{N})$, $\mu = \sum_{j=0}^{k-1} a_j \delta_{\frac{2\pi j}{k}}$ for some $a_j \in \mathbb{C}$. $\square$

9. *Proof.* We are going to construct $h_n \geq 0$ such that the conditions (i)(ii) are satisfied and for each $f \in C(I)$, $\int f h_n\,dm \to \int f\,dm$, more precisely, the Riemann integral of f. The desired $g_n = (1 + \frac{1}{n})^{-1}(h_n + \frac{1}{n}) > 0$ and satisfies conditions (i)-(iii).

   Let $\delta_n = (2n^3)^{-1}$. Consider $h_n = \sum_{k=1}^{n} h_n^k$ where

$$h_n^k(x) = \begin{cases} 2n^2 & \text{for } |x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n, \\ \text{linear} & \text{for } \frac{n-1}{2n}\delta_n < |x - \frac{2k-1}{2n}| \leq \frac{n+1}{2n}\delta_n, \\ 0 & \text{for } \frac{1}{2}\delta_n < |x - \frac{2k-1}{2n}|. \end{cases} \tag{1}$$

   Now observe that it's easy to check they satisfies (i)(ii). Moreover,

$$\begin{aligned} &|\int f h_n\,dm - \frac{1}{n}\sum_{k=1}^{n} f(\frac{2k-1}{2n})| \\ &\leq \sum_{k=1}^{n} \int_{|x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n} 2n^2 |f(x) - f(\frac{2k-1}{2n})|\,dx + \Big(2n^2 \|f\|_\infty \frac{1}{n}\delta_n + \|f\|_\infty \frac{1}{n^2}\Big) n \\ &\leq \sup_{k=1,\cdots,n} \sup\{|f(\frac{2k-1}{2n}) - f(x)| : |x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n\} + \|f\|_\infty \frac{2}{n} \end{aligned}$$

   which tends to zero as $n \to 0$ by the uniform continuity of f. This is what we want to see. $\square$

10. (a) it's obvious.

(b) The Vitali's Convergence Theorem is easy to prove if we know Egoroff's theorem, Fatou's lemma, and the equivalence between the uses of $|\int f|$ and $\int |f|$ in the definition of uniform integrability.

(c) Show that we can not omit the tightness condition (iii): for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all n, even if $\{\|f_n\|_1\}$ is bounded.

Proof. Consider the Lebesgue measure on $(-\infty, \infty)$ with $f_n = \chi_{(n,n+1)}$. □

(d) To apply Vitali's theorem in finite measure space, sometimes we see $|f(x)| < \infty$ a.e. is automatically true, but sometimes it's not. Give examples.

Proof. (i) Consider the Lebesgue measure on $[0,1]$, by the uniform integrability we know there exist nonoverlapping closed intervals $I_1, \cdots I_k$ whose union is $[0,1]$, and for all $j = 1, \cdots k$ and $n \in \mathbb{N}$, $\int_{I_j} |f_n| < 1$. (It's easy to show it's equivalent to use $|\int f|$ and $\int |f|$ in the definition of uniform integrability.) By Fatou's Lemma,

$$\int_{[0,1]} |f| \leq \liminf_n \int_{[0,1]} |f_n| < k.$$

(ii) We need to find out a finite measure space $(X, \mathscr{M}, \mu)$ and an uniformly integrable sequence of L^1 functions f_n with $f_n \to f$ a.e., $f(x)$ is not finite a.e. and $f_n \nrightarrow f$ in L^1.

On $\mathbb{R}$, let $\mathscr{M}$ is the σ-algebra of countable or co-countable sets. $\mu(E) := 0$ if E is countable, $\mu(E) := 1$ if E is co-countable, which is easy to check μ is a measure on $\mathscr{M}$. Consider $f_n \equiv n$, which is the desired example. □

(e) It's easy to see Vitali's Theorem implies Lebesgue Dominated Convergence Theorem in finite measure space. The sequence $f_n(x) = \frac{1}{x}\chi_{(\frac{1}{n+1}, \frac{1}{n})}(x)$ is an example in which Vitali's theorem applies although the hypothesis of Lebesgue's theorem do not hold.

(f) The sequence $f_n = n\chi_{(0,1/n)} - n\chi_{(1-\frac{1}{n},1)}$ on [0,1] shows the assumption that $f_n \geq 0$ is sometimes important in some applications. Note that $f_n(x) \to 0$ for every $x \in [0,1]$, $\int f_n(x)\,dx = 0$, but f_n is not uniformly integrable.

(g) However, the following converse of Vitali's theorem is true:

Theorem 5. *If $\mu(X) < \infty, f_n \in L^1(\mu)$, and $\lim_{n\to\infty} \int_E f_n\,d\mu$ exists for every $E \in \mathscr{M}$, then $\{f_n\}$ is uniformly integrable.*

Proof. As hint, we define $\rho(A,B) = \int |\chi_A - \chi_B|\, d\mu$. Then $(\mathscr{M}, \rho)$ is a complete metric space (modulo sets of measure zero), and $E \mapsto \int_E f_n\, d\mu$ is continuous for each n, (denote this map by F_n.) If $\epsilon > 0$, consider $A_N = \{E : |F_n(E) - F_m(E)| < \epsilon, \text{ if } n, m \geq N\}$. Since $X = \cup A_N$ by hypothesis, Baire Category theorem implies that some A_N has nonempty interior, that is, there exist $E_0 \in \mathscr{M}, \delta > 0, N \in \mathbb{N}$ so that

$$|\int_E (f_n - f_N)\, d\mu| < \epsilon \text{ if } \rho(E, E_0) < \delta,\ n > N. \tag{2}$$

If $\mu(A) < \delta$, (2) holds with $B = E_0 \setminus A$ and $C = E_0 \cup A$ in place of E. Thus,

$$|\int_A (f_n - f_N)\, d\mu| = |\int_C - \int_B (f_n - f_N)\, d\mu| < 2\epsilon.$$

By considering $\{f_1, \cdots f_N\}$, there exists $\delta' > 0$, such that

$$|\int_A f_n\, d\mu| < 3\epsilon \text{ if } \mu(A) < \delta',\ n = 1, 2, 3, \cdots$$

□

Remark 6. The Dunford-Pettis theorem: [2, p.466 and 472]

11. *Proof.* By Fatou's lemma $\int_X |f|^p < C$ and hence f is finite a.e.. Given $\epsilon > 0$, by Egoroff's theorem, there is a measurable set F with $C^{1/p}\mu(F)^{1-1/p} < \epsilon$, such that $f_n \to f$ uniformly on $X \setminus F$. Then there is an $N(\epsilon) \in \mathbb{N}$ such that for $n > N(\epsilon)$,

$$\int_X |f_n - f| \leq \int_{X \setminus F} |f_n - f| + \int_F |f_n| + \int_F |f| < \epsilon + 2C^{1/p}\mu(F)^{1-1/p} < 3\epsilon.$$

□

12. The assertions in this problem are easy to prove, we omit the proof and remark several things: First, this problem shows that the map $g \mapsto \Phi_g$ from $L^\infty \to (L^1)^*$ is not surjective if the measure space is not σ-finite; second, that map is also not injective if μ is not semifinite, this can be seen from the example I learned from Folland [3, p.191-192]: Let $E \subset X$ be a set of infinite measure that contains no subset of positive finite measure. Then for any $f \in L^1$, the set $\{x : f(x) \neq 0\}$ intersects E in a null set. It follows that $\Phi_{\chi_E} = 0$ although $\chi_E \not\equiv 0$ in L^∞.

13. Apply Hahn-Banach Theorem, see Theorem 5.19, p.107.

Additional Results

We present the proof that the constant $\frac{1}{\pi}$ in Sec 6.3 is sharpest. (Due to Bledsoe [1] which is a simpler treatment of Kaufman-Rickert [4]).

Theorem 7. *Let C_0 be the supremum of the number C for which*

$$|\sum_{z\in S}| \geq C \sum_{z\in T} |z|$$

where T is any nonempty finite set of complex numbers and S is any subset of T. Then $C_0 = \frac{1}{\pi}$.

Proof. By Theorem 6.3, $C_0 \geq \frac{1}{\pi}$. On the other hand, consider $z_j^n = \exp[i(2\pi/n)j],\ j = 1, \cdots, n$. For this choice of points we have for each θ, let S_θ^n be the set of those z_j^n such that $|\theta - \arg(z_j^n)| \leq \frac{\pi}{2}$

$$\begin{aligned}\lim_{n\to\infty}\Big(|\sum_{z\in S_\theta^n} z| \Big/ \sum_{j=1}^{n} |z_j|\Big) &= \lim_{n\to\infty} \frac{1}{n} |\sum_{z\in S_\theta^n} z| \\ &= \lim_{n\to\infty} \frac{1}{2\pi} |\frac{\pi}{n/2} \sum_{z\in S_\theta^n} z| \\ &= \frac{1}{2\pi} |\int_{\theta-\pi/2}^{\theta+\pi/2} e^{i\phi}\, d\phi| = \frac{1}{\pi}.\end{aligned}$$

Thus, $C_0 \leq \frac{1}{\pi}$. □

References

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