Partial Differential Equations, 2nd Edition, L.C.Evans
Chapter 9 Nonvariational Techniques

Yung-Hsiang Huang*

2017.04.17

1. *Proof.* $\square$

2. *Proof.* Given $\alpha, \beta \in \mathbb{R}$. Consider the 1-periodic function $f(x) = \alpha\chi_{(0,\frac{1}{2})}(x) + \beta\chi_{(\frac{1}{2},1)}(x)$ and $f_n(x) = f(nx)$. Then $f_n \rightharpoonup \frac{1}{2}(\alpha+\beta)$ as $n \to \infty$, see Exercise 8.1(b).

   Similarly, $a(f(x)) = a(\alpha)\chi_{(0,\frac{1}{2})}(x) + a(\beta)\chi_{(\frac{1}{2},1)}(x)$ and hence $a(f_n) \rightharpoonup \frac{1}{2}(a(\alpha)+a(\beta))$. By assumption, $\frac{1}{2}(a(\alpha)+a(\beta)) = a(\frac{\alpha+\beta}{2})$ for any α, β. In particular, $a(0) = \frac{1}{2}(a(x)+a(-x))$ and hence $b(x) := a(x) - a(0)$ is a continuous odd function. Note that $b(0) = 0$ and $b(x+y) = \frac{1}{2}(b(2x)+b(2y))$ for all x, y. Therefore, $b(m) = \frac{1}{2}mb(2)$ for all $m \in \mathbb{N}\cup\{0\}$, and by the oddity, the above is true for all $m \in \mathbb{Z}$. Thus, $b(r) = \frac{b(2)}{2}r$ for all $r \in \mathbb{Q}$. By continuity, $b(z) = \frac{b(2)}{2}z$ for all $z \in \mathbb{R}$, that is, $a(z) = \frac{a(2)-a(0)}{2}z + a(0)$. $\square$

   Remark 1. I see the following result from Brezis [2, Exercise 4.20]:

   Assume $|\Omega| < \infty$. Let $p, q \in [1,\infty)$ amd $a : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

   $$|a(t)| \leq C\{|t|^{p/q} + 1\}.$$

   Consider the map $A : L^p(\Omega) \to L^q(\Omega)$ defined by $A(u)(x) = a(u(x))$. Then A is continuous from $L^p(\Omega)$ strong into $L^q(\Omega)$ strong.

   Proof. By the Growth condition and $|\Omega| < \infty$, the map A is well-defined.

   Given $u \in L^p$. Let $\{u_n\}$ be a sequence that converges to u in L^p. Given a subsequence $\{u_{n_k}\}$ of $\{u_n\}$. Then there is a further subsequence $\{u_{n_{k_m}}\}$ converges to u a.e. and $|u_{n_{k_m}}| \leq v$ a.e. for some $v \in L^p$. So $|u| \leq v$ a.e.,too.

*Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw

Hence $|a(u_{n_{k_m}}) - a(u)|^q \le C^q(|u_{n_{k_m}}|^{p/q} + |u|^{p/q} + 2)^q \le C^q 3^q(2v^p + 2^q) \in L^1(\Omega)$ for all m. By LDCT and continuity of a, $A(u_{n_{k_m}}) \to A(u)$ in L^q. Since every subsequence of $A(u_n)$ contains a subsubsequence that converges to $A(u)$, $A(u_n) \to A(u)$ in L^q. □

3. (Penalty method, related to Exercise 14 in Chapter 8.)

   Proof. □

   Remark 2. See also, *D.Kinderlehrer and G.Stampacchia, An Introduction to Variational Inequalities and Their Applications, Chapter 4.*

4. *Proof.* □

5. *Proof.* Define a map T on $H_0^1(\Omega)$ by $T(v) = w$, where $w \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies $-\Delta w = f - b(Dv)$ in Ω with zero Dirichlet boundary condition. Note that such w exists by Riesz's representation theorem, the fact that $b(Dv) \in L^2(\Omega)$ (since $|b(Dv)| \le |b(0)| + \text{Lip}(b)|Dv|$ and Ω is bounded,) and global regularity theorem. To complete the proof, it suffices to show T is a contraction map if $\mathbf{Lip}(b)$ is small enough.

   Given $v_1, v_2 \in H_0^1(\Omega)$. We note that, by Poincaré inequality,

   $$\|T(v_1) - T(v_2)\|_{H^1}^2 \le C(\Omega)^2 \|D[T(v_1) - T(v_2)]\|_{L^2}^2 = C(\Omega)^2 \int_\Omega (\Delta[T(v_2) - T(v_1)])(T(v_1) - T(v_2))$$
   $$= C(\Omega)^2 \int_\Omega (b(Dv_2) - b(Dv_1))(T(v_1) - T(v_2)) \le C(\Omega)^2 \|b(Dv_1) - b(Dv_2)\|_2 \|T(v_1) - T(v_2)\|_2$$
   $$\le C(\Omega)^2 \mathbf{Lip}(b) \|v_1 - v_2\|_{H^1} \|T(v_1) - T(v_2)\|_{H^1}.$$

   Therefore, $T : H_0^1 \to H_0^1$ is a contraction map if $\mathbf{Lip}(b)$ is small enough. □

6. *Proof.* □

7. *Proof.* □

8. (Noncompact families of solutions) **(a) Assume $n \ge 3$. Find a constant c such that**

   $$u(x) = c(1 + |x|^2)^{\frac{2-n}{2}}$$

   solves Yamabe's equation

   $$-\Delta u = cu^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n$$

   (b) Check that for each $\lambda > 0$,

   $$u_\lambda(x) := \Big(\frac{\lambda}{\lambda^2 + |x|^2}\Big)^{\frac{n-2}{2}}$$

is also a solution.

(c) Show that

$$\|u_\lambda\|_{L^{\frac{n+2}{n-2}}(\mathbb{R}^n)} = \|u\|_{L^{\frac{n+2}{n-2}}(\mathbb{R}^n)}, \ \|Du_\lambda\|_{L^2(\mathbb{R}^n)} = \|Du\|_{L^2(\mathbb{R}^n)}$$

for each λ and thus that $\{u_\lambda\}_{\lambda>0}$ is not precompact in $L^{\frac{n+2}{n-2}}(\mathbb{R}^n)$.

Remark 3. Compare with Exercise 6 of Chapter 4. Also see Ni-Ding's and Ambrosetti-Azorero-Peral's papers [4, 3, 1].

Proof. (a) $c = n(n-2)$. (b) Directly. (c) (note that $u \in L^2$ iff $n > 3$) The norm equalities are obviously. The non-precompactness in L^{2^*} is due to $u_\lambda(x) \to 0$ as $\lambda \to \infty$ for each fixed x. □

9. I think (b) is correct if we change $n-2 \to n$.

Proof. (a) Direct differentiation and multiply the PDE by u_t.

(b)The left hand side =

$$\int_{\mathbb{R}^n} -|x|^2 u_t^2 - 2(x \cdot Du)u_t \, dx = \int_{\mathbb{R}^n} -|x|^2 u_t^2 - 2(x \cdot Du)(\Delta u + f(u)) \, dx$$
$$= \int_{\mathbb{R}^n} -|x|^2 u_t^2 + \left(\operatorname{div}(x|Du|^2) - n|Du|^2\right) - 2x \cdot D(F(u)) \, dx = \int_{\mathbb{R}^n} -|x|^2 u_t^2 - n|Du|^2 + 2nF(u) \, dx.$$

□

10. *Proof.* □

11. *Proof.* □

12. *Proof.* □

13. *Proof.* □

14. *Proof.* □

References

[1] A Ambrosetti, J Garcia Azorero, and I Peral. Perturbation of $\Delta u + u^{(n+2)/(n-2)} = 0$, the scalar curvature problem in R^n, and related topics. *Journal of Functional Analysis*, 165(1):117–149, 1999.

[2] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science & Business Media, 2010.

[3] Wei-Yue Ding, Wei-Ming Ni, et al. On the elliptic equation $\Delta u + ku^{(n+2)/(n-2)} = 0$ and related topics. *Duke mathematical journal*, 52(2):485–506, 1985.

[4] W-M Ni. On the elliptic equation $\Delta u + k(x)u^{(n+2)/(n-2)} = 0$, its generalization, and applications in geometry. *Indiana Univ. Math. J.*, 31:493–529, 1982.