Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 3 The Classical Maximum Principle

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1. In (b), I interpret the term $u(\beta \cdot \nu)$ as $u(x_0)(\beta(x_0) \cdot \nu(x_0))$.

Proof. (a) Suppose u is not a constant. Since either u or $-u$ is a solution of $Lv = 0$ that attains a positive maximum at $x_0 \in \overline{\Omega}$, we may assume u has a positive maximum $u(x_0)$. By strong maximum principle and hypothesis, $x_0 \in S_2$. Since the translated and rotated transformation does not change the ellipticity constants λ, Λ, we may assume $x_0 = 0$ and $\nu(x_0) = -e_n$. Let $B = B_R(Re_n)$ be the interior ball at the origin. Note that $s\beta(x_0) \in \partial B$ for $s = 2\frac{\beta_n(x_0)R}{|\beta(x_0)|^2} \neq 0$, since $\beta_n(x_0) \neq 0$ and $|s\beta - Re_n| = R$. Then the proof of Hopf lemma implies that

$$\liminf_{B \ni x := x_0 + t\beta(x_0), t \to 0} \frac{u(x_0) - u(x)}{|x - x_0|} > 0.$$

Since $u \in C^1(\Omega \cup S_2), u \in C^1(B \cup \{x_0\})$. Hence the right hand side of the above expression equals to $\pm \frac{\beta(x_0)}{|\beta(x_0)|} \cdot Du(x_0) = 0$ (the sign depends on the sign of $\beta(x_0) \cdot \nu(x_0),$) which is a contradiction.

(b) If $u \equiv$ constant c, then the boundary condition becomes $\alpha(x_0)c \equiv 0$ on $\partial\Omega$. So the hypothesis implies that $c \equiv 0$.

If u is non-constant, since either u or $-u$ is a solution of $Lv = 0$ that attains a nonnegative maximum at $x_0 \in \overline{\Omega}$, we may assume u has a nonnegative maximum $u(x_0)$, then $x_0 \in \partial\Omega$ by strong maximum principle. Under the same setting as (a), we note if $\beta(x_0) \cdot \nu(x_0) > 0$, then

$$0 < \liminf_{B \ni x := x_0 + t\beta(x_0), t \to 0} \frac{u(x_0) - u(x)}{|x - x_0|} = \frac{\beta(x_0)}{|\beta(x_0)|} \cdot Du(x_0).$$

The hypothesis implies $\alpha(x_0) > 0$. So we see a contradiction that

$$0 = \alpha(x_0)u(x_0) + \beta(x_0) \cdot Du(x_0) > 0.$$

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If $\beta(x_0)\cdot\nu(x_0) < 0$, then

$$0 < \liminf_{B\ni x:=x_0+t\beta(x_0),t\to 0} \frac{u(x_0)-u(x)}{|x-x_0|} = -\frac{\beta(x_0)}{|\beta(x_0)|}\cdot Du(x_0).$$

The hypothesis implies $\alpha(x_0) < 0$. So we see a contradiction that

$$0 = -\alpha(x_0)u(x_0) - \beta(x_0)\cdot Du(x_0) > 0.$$

□

2. *Proof.* (a) If u attains the positive maximum at $x_0 \in \Omega$, then $Lu(x_0) < 0$ which is impossible.

   (b) Since Ω is bounded, the supremum of u in $\overline{\Omega}$ is bounded. If u attains its maximum at $x_0 \in \Omega$, then $f(x_0) = Lu(x_0) \leq c(x_0)u(x_0)$. Since $c < 0$, this is equivalent to $\sup|f/c| \geq f(x_0)/c(x_0) \geq u(x_0) \geq \sup u$. Apply the same method to $-u$ and consider the non-existence of interior maximum case, we complete the proof. □

3. See Gilbarg-Serrin [1].

   Proof. WLOG, we may assume $u(x_1,x_2) \geq 0$ and $r_0 = 1$. Let $\liminf_{|x|\to\infty} u(x) = c \geq 0$. If $c = \infty$, then we are done.

   Otherwise, for each $m \in \mathbb{N}$, we define $v_m = u - c + \frac{1}{m}$. Since $\liminf_{|x|\to\infty} u = c, v_m(x) \geq 0$ on $|x| \geq R_m$ for some $R_m > 100$ and we can find $z_m \in \mathbb{R}^2$ with $R_m \leq |z_m| \nearrow \infty$ such that $0 \leq u(z_m) < c + \frac{1}{m}$, that is, $0 \leq v_m(z_m) < \frac{2}{m}$.

   Now we consider the change of variables $\xi = s^{-1}x$ $(s > 100)$ and define $w(\xi) = v(s\xi) \geq 0$. On $\mathbb{R}^2_\xi$, since the uniform ellipticity constant μ for differential operators $L_\xi = s^{-2}a\partial_{11} + 2s^{-2}b\partial_{12} + s^{-2}c\partial_{22}$ is the same as the one for L_x, the constant $K > 1$ in the Harnack inequality (Theorem 3.10) is the same. Since the unit sphere in $\mathbb{R}^2_\xi$ is compact, we can find $\xi_1, \cdots \xi_d$ from the unit sphere such that $\{B_{1/4}(\xi_i)\}_{i=1}^d$ covers the unit sphere. Note that those balls transform back to $\mathbb{R}^2_x$ are $B_{\frac{1}{4}s}(x_i)$, where $|x_i| = s > 100$, and hence are contained in the domain of definition $\{|x| \geq 1\}$.

   In particular, we take $s = |z_m|$ and denote $\xi_m = z_m/|z_m|$. Then for each $|\xi| = 1, w(\xi) \leq K^d w(\xi_m)$. Therefore, given $m \in \mathbb{N}$, for each $M > m$ and each $|x| = |z_M|, 0 \leq v_m(x) \leq K^d v(z_M) < 2K^d/M$. Apply the maximum principle, we know on $\{|z_M| \leq |x| \leq |z_{M+1}|\}, v_m(x) < \frac{2K^d}{M}$. Since $\frac{2K^d}{M} \searrow 0, v_m(x) < \frac{2K^d}{M}$ on $\{|z_M| \leq |x|\}$. In particular, we take $M = 2m$ and hence for each $m \in \mathbb{N}, u(x) < c + \frac{K^d - 1}{m}$ on $\{|z_{2m}| \leq |x|\}$. Equivalently, we have $\limsup_{|x|\to\infty} u(x) \leq c$. □

4. **Let u be a non-negative solution of**

$$Lu \equiv a^{ij}D_{ij}u + b^iD_iu + cu = 0, \quad c \le 0, \quad i,j = 1,2$$

with the coefficients of L satisfy the inequalities

$$\Lambda/\lambda \le \mu, |b^i|/\lambda, |c|/\lambda \le \nu, \quad (\mu, \nu = \textbf{const}.).$$

Prove the Harnack inequality (3.21) with $K = K(\mu,\nu)$, that is, at all points $z = (x,y) \in D_{R/4}$, we have the inequality

$$K^{-1}u(z) \le u(0) \le Ku(z).$$

Also prove Corollary 3.11 with $\kappa = \kappa(\mu,\nu,\Omega,\Omega')$. That is,

$$\sup_{\Omega'} u \le \kappa \inf_{\Omega'} u.$$

Remark 1. Sometimes the Harnack inequality holds for $c > 0$. For example, it's known there is a modified mean value property for Helmholtz equation $(\Delta + k^2)u = 0$, see Fritz John's PDE (4th edition) Page 101. So Harnack's inequality holds for Helmholtz operator.

Remark 2. It's based on Serrin [5]. For dimension ≥ 3, Serrin's method ceaced to be useful without an unnatural restriction on a^{ij}. To remove such restriction, we need Moser's iteration technique, see [2, 3] and Chapter 8.

Proof. □

5. The idea is the same as Exercise 3. Also see Gilbarg-Serrin [1].

Proof. WLOG, we may assume $u(x_1,x_2) \ge 0$ and $r_0 = 1$. Let $\liminf_{|x|\to 0} u(x) = c \ge 0$. If $c = \infty$, then we are done.

Otherwise, for each $m \in \mathbb{N}$, we define $v_m = u - c + \frac{1}{m}$. Since $\liminf_{|x|\to 0} u = c$, $v_m(x) \ge 0$ on $0 < |x| \le R_m$ for some $R_m < \frac{1}{100}$ and we can find $z_m \in \mathbb{R}^2$ with $R_m \ge |z_m| \searrow 0$ such that $0 \le u(z_m) < c + \frac{1}{m}$, that is, $0 \le v_m(z_m) < \frac{2}{m}$.

Now we consider the change of variables $\xi = s^{-1}x$ $(s < 100^{-1})$ and define $w(\xi) = v(s\xi) \ge 0$. On $\mathbb{R}^2_\xi$, for differential operators $L_\xi = s^{-2}a\partial_{11} + 2s^{-2}b\partial_{12} + c^{-2}\partial_{22} + s^{-1}d\partial_1 + s^{-1}e\partial_2 + f$, the uniform ellipticity constant $\mu = \sup \Lambda/\lambda = \sup s^{-2}\Lambda/s^{-2}\lambda$ is the same as the one for L_x and the boundedness constant ν is unchanged since $|s^{-1}b^i/s^{-2}\lambda| \le |b^i/\lambda|$ and $|c/s^{-2}\lambda| \le |c/\lambda|$. Hence the constant $K > 1$ in the Harnack inequality (Exercise 4) is the same. Since the unit sphere in $\mathbb{R}^2_\xi$ is compact, we can find $\xi_1, \cdots \xi_d$ from the unit sphere such that $\{B_{1/4}(\xi_i)\}_{i=1}^d$ covers the unit sphere. Note that those balls transform back to $\mathbb{R}^2_x$ are $B_{\frac{1}{4}s}(x_i)$, where $|x_i| = s < 100^{-1}$, and hence are contained in the domain of definition $\{|x| \leq 1\}$.

In particular, we take $s = |z_m|$ and denote $\xi_m = z_m/|z_m|$. Then for each $|\xi| = 1, w(\xi) \leq K^d w(\xi_m)$. Therefore, given $m \in \mathbb{N}$, for each $M > m$ and each $|x| = |z_M|, 0 \leq v_m(x) \leq K^d v(z_M) < 2K^d/M$. Apply the maximum principle, we know on $\{|z_{M+1}| \leq |x| \leq |z_M|\}, v_m(x) < \frac{2K^d}{M}$. Since $\frac{2K^d}{M} \searrow 0, v_m(x) < \frac{2K^d}{M}$ on $\{|z_M| \geq |x|\}$. In particular, we take $M = 2m$ and hence for each $m \in \mathbb{N}, u(x) < c + \frac{K^d-1}{m}$ on $\{|z_{2m}| \geq |x|\}$. Equivalently, we have $\limsup_{|x|\to 0} u(x) \leq c$. □

6. *Proof.* □

7. See Gilbarg-Serrin [1].

   Proof. □

8. The same function is used in Section 10.3 to give a non-uniqueness result in the theory of quasilinear elliptic equations. Also see Gilbarg-Serrin [1].

   Proof. First, direct computation tells us $L_n u(x) = (1+g)u''(r) + \frac{n-1}{r}u'(r)$ if u is radial.

   (a)

   (b)

   □

9. It's based on Pucci-Serrin [4, Section 2.7].

   Proof. We omit all the computations. $r := \sqrt{x^2+y^2}$, B is the ball centered at $(1,0)$ with radius 1. Consider the function
$$u(x,y) = xe^{-\sqrt{\log\frac{4}{r}}},$$
   then $u \in C^1(\overline{B})$, $u > 0 = u(0,0)$ in B and $\frac{\partial u}{\partial n}(0,0) = -u_x(0,0) = 0$.

   Let $\mu = 1 + \frac{1}{2\sqrt{\log\frac{4}{r}}}$ and
$$a = \frac{1}{\mu} + \frac{\mu^2-1}{r^2\mu}y^2,\ b = \frac{1-\mu^2}{r^2\mu}xy,\ c = \frac{1}{\mu} + \frac{\mu^2-1}{r^2\mu}x^2.$$
   Then $a, b, c \in C^\infty(B) \cap C(\overline{B})$ and all the limit behaviors of a, b, c as $r \to 0$ are satisfied. □

10. *Proof.* I think we should assume v is $C^2(\overline{\Omega})$.

Let $w = u/v$. A direct and reasonable computation shows that

$$a^{ij} D_{ij} w + B^i D_i w + \frac{Lv}{v} w \geq 0,$$

where $B^i = b^i + \frac{2}{v} a^{ij} D_j v$ is bounded by assumptions (some are stated in Problem 3.6.) Also note that $\frac{Lv}{v}$ is bounded.

The proof is completed by strong maximum principle (the strict and uniform ellipticity are assumed as Problem 3.6.) □

References

[1] David Gilbarg and James Serrin. On isolated singularities of solutions of second order elliptic differential equations. *Journal d'Analyse Mathématique*, 4(1):309–340, 1954.

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[5] James Serrin. On the harnack inequality for linear elliptic equations. *Journal d'Analyse Mathématique*, 4(1):292–308, 1954.