Fourier Analysis, Stein and Shakarchi
Chapter 5 The Fourier Transform on $\mathbb{R}$

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Abstract

Problem 3(c) need the theory of residues from complex analysis. Here we assume it and modify the computations given in page 83 of Book II.

After Problem 6, we contain a proof for Widder's uniqueness theorem in the class of nonnegative solutions of heat equation in the strip $S_T := \{(x,t) \in \mathbb{R}^d \times (0,T)\}$. It's taken from [4, Section 5.14]. In this proof, We need an interior gradient estimate for heat equation, that is, Proposition 25 and refer its proof to [4, Section 5.12 and 5.12c].

We also combine Problem 7 and Book III's Problem 4.11 on Hermite functions here.

I finish this solution file when I am a teaching assistant of the course "Analysis II" in NTU 2018 Spring. The following students contribute some answers:

Problem 3(b)(e): Ge-Cheng Cheng.

Problem 3(c): Sam Wang.

Problem 8 is discussed with Shuang-Yan Li and Yi-Heng Tsai.

Exercises

1. **Corollary 2.3 in Chapter 2 leads to the following simplified version of the Fourier inversion formula. Suppose f is a continuous function supported on $[-M, M]$, whose Fourier transform $\hat{f}$ is of moderate decrease.**

   (a) Fix L with $L/2 > M$, and show that $f(x) = \sum a_n(L)e^{2\pi inx/L}$ where

$$a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-2\pi inx/L}\,dx = \frac{1}{L}\hat{f}(n/L).$$

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Alternatively, we may write $f(x) = \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta) e^{2\pi n \delta x}$ **with** $\delta = 1/L$**.**

(b) Prove that if F **is continuous and of moderate decrease, then**

$$\int_{-\infty}^{\infty} F(\xi)\, d\xi = \lim_{\delta \to 0^+} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

(c) Conclude that $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi x \xi}\, d\xi$**.**

Proof. (c) is proved by taking $F(\xi) = \widehat{f}(\xi) e^{2\pi i \xi x}$ in (b) and then apply (a).

For (a), we note that the Fourier series of f on $[-\frac{L}{2}, \frac{L}{2}]$ is $\sum a_n(L) e^{2\pi i n x/L}$. which converges absolutely and hence uniformly since we have, by the moderate decreasing property of $\widehat{f}$,

$$|a_n(L)| = |\frac{1}{L}\hat{f}(n/L)| \le \frac{L}{L^2 + n^2}.$$

The uniqueness theorem of Fourier series and continuity of f imply the desired equality at every point x.

(b) Given $\epsilon > 0$, we have $N = N(\epsilon)$ such that $\left| \int_{-n}^{n} F(\xi)\, d\xi - \int_{-\infty}^{\infty} F(\xi)\, d\xi \right| < \epsilon$ whenever $n \ge N$. Note that there is a $L_\epsilon > 0$ such that for each $L \ge L_\epsilon$ and for each $0 < \delta < 1$,

$$|\delta \sum_{|n| > \frac{L+1}{\delta}} F(\delta n)| \le A\delta \sum_{|n| > \frac{L}{\delta}+1} \frac{1}{1+\delta^2 n^2} \le 2A \int_{\frac{L}{\delta}}^{\infty} \frac{\delta}{1+\delta^2 x^2}\, dx = 2A(\frac{\pi}{2} - \arctan L) < \epsilon.$$

Therefore, for $M =: \max\{N, L_\epsilon + 1\}$ we have

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} F(\xi)\, d\xi - \delta \sum_{n\in\mathbb{Z}} F(\delta n) \right| \le & \left| \int_{-\infty}^{\infty} F(\xi)\, d\xi - \int_{-M}^{M} F(\xi)\, d\xi \right| + \left| \int_{-M}^{M} F(\xi)\, d\xi - \delta \sum_{|n| \le \frac{M}{\delta}} F(\delta n) \right| \\
& + \left| \delta \sum_{|n| \le \frac{M}{\delta}} F(\delta n) - \delta \sum_{n\in\mathbb{Z}} F(\delta n) \right| \\
& \le 2\epsilon + \left| \int_{-M}^{M} F(\xi)\, d\xi - \delta \sum_{|n| \le \frac{M}{\delta}} F(\delta n) \right|.
\end{aligned}$$

Then by the uniform continuity of F on $[-M, M]$, there is a $\delta_\epsilon > 0$ such that the last term is less than ϵ whenever $\delta \in (0, \delta_\epsilon)$. $\square$

2. Let f and g be the functions defined by

$$f(x) = \chi_{[-1,1]}(x) \;\text{ and }\; g(x) = \chi_{[-1,1]}(x)\big(1 - |x|\big).$$

Although f is not continuous, the integral defining its Fourier transform still make sense. It's easy to see that

$$\widehat{f}(\xi) = \frac{\sin 2\pi\xi}{\pi\xi} \;\text{ and }\; \widehat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2$$

with the understanding that $\widehat{f}(0) = 2$ and $\widehat{g}(0) = 1$. Note that one can interpret g as a convolution of cf_δ with itself, for some $c, \delta \in \mathbb{R}$ and $f_\delta(x) := f(\delta x)$.

3. **The following exercise illustrates the principle that the decay of $\widehat{f}$ is related to the continuity properties of f.**

(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\widehat{f}$ is continuous and satisfies

$$\widehat{f}(\xi) = O\Big(\frac{1}{|\xi|^{1+\alpha}}\Big) \quad \textbf{as} \quad |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α.

(b) Let f be a continuous function on R which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\widehat{f}$ is not of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\widehat{f}(\xi) = O(1/|\xi|^{1+\epsilon})$ as $|\xi| \to \infty$.

Proof. (a) Since $\widehat{f} \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$, we have, for each $|h| > 0$

$$\begin{aligned}
&|f(x+h) - f(x)| = \Big| \int_{\mathbb{R}} \widehat{f}(\xi)[e^{2\pi i(x+h)\xi} - e^{2\pi i x\xi}]\, d\xi \Big| \\
&\leq \Big| \int_{|\xi|>|h|^{-1}} \widehat{f}(\xi)[e^{2\pi i(x+h)\xi} - e^{2\pi i x\xi}]\, d\xi \Big| + \Big| \int_{|\xi|\leq|h|^{-1}} \widehat{f}(\xi)[e^{2\pi i(x+h)\xi} - e^{2\pi i x\xi}]\, d\xi \Big| \\
&\leq 2\int_{|h|^{-1}}^{\infty} 2C|\xi|^{-1-\alpha}\, d\xi + \int_{|\xi|\leq|h|^{-1}} |\widehat{f}(\xi)| 2\sin(\pi|h\xi|)\, d\xi.
\end{aligned}$$

By the hypotheses, there is for some $M > 0$ such that $|\widehat{f}| \leq M/(1 + |\xi|^{1+\alpha})$ for all $\xi \in \mathbb{R}$

$$\begin{aligned}
|f(x+h) - f(x)| &\leq 4C\alpha|h|^{\alpha} + 2M\int_0^{|h|^{-1}} \frac{\sin(\pi|h\xi|)}{1+|\xi|^{1+\alpha}}\, d\xi \\
&\leq 4C\alpha|h|^{\alpha} + 2M|h|^{\alpha}\int_0^1 \frac{\sin(\pi u)}{u^{1+\alpha}}\, du \\
&\leq 4C\alpha|h|^{\alpha} + 2M|h|^{\alpha}M'\int_0^1 \frac{1}{u^{\alpha}}\, du = K|h|^{\alpha},
\end{aligned}$$

where K is independent of h.

(b) The continuity of $\widehat{f}$ is a general fact. Suppose $\widehat{f}$ satisfies the decay condition for some $1 > \epsilon > 0$, then f is ϵ-Hölder, especially near the origin. However one find that as $|h| \to 0$,

$$\Big|\frac{f(h) - f(0)}{h^{\epsilon}}\Big| = \frac{|h|^{-\epsilon}}{\log|h|^{-1}} \to \infty.$$

This is a contradiction.

$\square$

4. **Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis. Some examples are:**

(a) Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \leq a$ or $x \geq b$ and $f(x) = e^{-1/(x-a)} e^{-1/(b-x)}$ if $a < x < b$. Show that f is indefinitely differentiable on $\mathbb{R}$.

(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \leq a$, $F(x) = 1$ if $x \geq b$, and F is strictly increasing on $[a, b]$.

(c) Let $\delta > 0$ be so small that $a + \delta < b - \delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \leq a$ or $x \geq b$, g is 1 on $[a + \delta, b - \delta]$, and g is strictly monotonic on $[a, a + \delta]$ and $[b - \delta, b]$.

Proof. We may assume $a > 0$ (why?). (a) is easy (by mathematical induction).

For (b) consider $F(x) = c \int_{-\infty}^{x} f(t)\, dt$ where c is the reciprocal of $\int_{-\infty}^{\infty} f(t)\, dt$.

For (c) we first observe that $\widetilde{h}(x) = 1 - F(x^2)$ is a function that equals to 1 on $[-\sqrt{a}, \sqrt{a}]$, decreasing on $\{\sqrt{a} \leq |x| \leq \sqrt{b}\}$ and vanishes outside $\{|x| \geq \sqrt{b}\}$. After a suitable scaling and translation, we have a desired one. $\square$

5. **Suppose f is continuous on $\mathbb{R}$ and of moderate decrease.**

   (a) Prove that $\widehat{f}$ is uniformly continuous and vanishes at infinity.

   (b) Show that if $\widehat{f}(\xi) = 0$ for all ξ, then f is identically 0.

Proof. (a) (1st proof, without LDCT) Given $\epsilon > 0$, pick large N such that $2A \int_{|x|>N} \frac{1}{1+x^2}\, dx < \epsilon/2$ where $A > 0$ is the moderate constant. Then

$$\begin{aligned}|\widehat{f}(\xi + h) - \widehat{f}(\xi)| &\leq \int_{\mathbb{R}} |f(x)||e^{-2\pi i x h} - 1|\, dx \leq 2A \int_{\mathbb{R}} \frac{|\sin \pi h x|}{1 + x^2}\, dx \\ &\leq 2A\Big(\int_{-N}^{N} \frac{|\sin \pi h x|}{1 + x^2} + \int_{|x|>N} \frac{1}{1 + x^2}\, dx\Big) < \epsilon\end{aligned}$$

provided h is small enough such that $|\sin \pi h x| < \frac{\epsilon}{8AN}$ for all $x \in [-\pi, \pi]$. Note that the smallness for h is independent of ξ, so the continuity is uniform.

(2nd proof, with LDCT.) The continuity is directly from LDCT. Note that

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx = \int_{\mathbb{R}} -f(y - \frac{1}{2\xi}) e^{-2\pi i y \xi}\, dy.$$

So $\widehat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}} \big[f(x) - f(x - \frac{1}{2\xi})\big] e^{-2\pi i x \xi}\, dx$ and hence the LDCT implies the decay at ∞.

(b) Given $g \in \mathcal{S}(\mathbb{R})$, then by Fubini's theorem

$$\int_{\mathbb{R}} f(x) \widehat{g}(x)\, dx = \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(y) e^{-2\pi i y x}\, dy\, dx = \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x) e^{-2\pi i y x}\, dx\, dy = \int_{\mathbb{R}} g(y) \widehat{f}(y)\, dy = 0.$$

In particular, for each $y \in \mathbb{R}$, we take $g_y(\xi) = e^{-\pi\xi^2\delta}e^{2\pi i\xi y}$ whose Fourier transform is $\widehat{g_y}(x) = e^{-\pi(x-y)^2/\delta}\delta^{-1/2}$. Letting $\delta \to 0$, we have $f(y) = 0$.

$\square$

Remark 1. See Exercise 2.22 (Riemann-Lebesgue lemma) and 2.25 (the decay rate of $\widehat{f}$ may be slow as $|\xi|^{-\epsilon}$ for any ϵ) in Book III. Also consult [18, Section I.4.1] or [15, Exercise 1.6] for showing the Fourier transform map from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not surjective. For $L^1(\mathbb{T})$ case, see Section 3.1 of Chapter 4 in Book IV.

Remark 2. For functions on $\mathbb{T}$, one can construct functions with arbitrary slow decay for their Fourier transform, see [9, Section 3.3.1].

6. **The function $e^{-\pi x^2}$ is its own Fourier transform. Generate other functions that (up to a constant multiple) are their own Fourier transforms. What must the constant multiples be? To decide this, prove that $\mathcal{F}^4 = I$. Here $\mathcal{F}(f) = \widehat{f}$ is the Fourier transform, $\mathcal{F}^4 = \mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{F}$, and I is the identity operator $(If)(x) = f(x)$ (see also Problem 7).**

Proof. If $\mathcal{F}(g) = cg$, then $c^4 = 1$ since $\mathcal{F}^4 = I$. So $c \in \{1, -1, i, -i\}$. The case $c = 1$ is solved in the statement with $g(x) = e^{-\pi x^2}$. Note that $g' = -2\pi x g(x)$ and hence $\widehat{g'}(x) = 2\pi i x \widehat{g}(x) = 2\pi i x g(x) = -ig'(x)$, that is, the case $c = -i$ has an eigenfunction $g'(x) = -2\pi x e^{-\pi x^2}$.

Furthermore, we differentiate g' again and see that $g''(x) = -2\pi g(x) + 4\pi^2 x^2 g(x)$ and hence $\widehat{g''}(x) = -4\pi^2 x^2 \widehat{g}(x) = -4\pi^2 x^2 g(x) = -2\pi g(x) - g''(x)$, that is, the case $c = -1$ has an eigenfunction since $\widehat{g'' + \pi g}(x) = -(\pi g(x) + g''(x))$.

Finally, we differentiate the g'' once again and see that $g'''(x) = -2\pi g'(x) + 8\pi^2 x g(x) + 4\pi^2 x^2 g'(x) = -6\pi g' + 4\pi^2 x^2 g'(x)$. Taking Fourier transform to both sides, we have

$$\widehat{g''' + 3\pi g'}(\xi) = -3\pi\widehat{g'}(\xi) - \frac{d^2}{d\xi^2}\widehat{g'} = i(3\pi g' + g''')(\xi),$$

that is, $g''' + 3\pi g'$ is an eigenfunction for $c = i$. $\square$

Remark 3. Note that these four eigenspaces E_1, E_{-1}, E_i, E_{-i} span all of $L^2(\mathbb{R}^d)$, that is, given $f \in L^2(\mathbb{R}^d)$ we are searching $f_k \in E_k$ for $k = \pm 1, \pm i$ such that $f = f_1 + f_{-1} + f_i + f_{-i}$. To find these f_k, one can solve this equation with the following three equations through linear algebra (matrix): $\mathcal{F}f = f_1 - f_{-1} + if_i - if_{-i}$, $\mathcal{F}^2 f = f_1 + f_{-1} - f_i - f_{-i}$, $\mathcal{F}^3 f = f_1 - f_{-1} - if_i + if_{-i}$. This decomposition of L^2 can be used to give new examples of eigenfunctions for $\mathcal{F}$, e.g. the previous f_1 should be $(f + \mathcal{F}f + \mathcal{F}^2 f + \mathcal{F}^3 f)/4$ and now we take $f(x) = e^{-2\pi|x|}$ (so $\mathcal{F}f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$

and $\mathcal{F}^2 f = f$. Therefore $2f_1(x) = e^{-2\pi|x|} + \frac{1}{\pi}\frac{1}{1+x^2}$ is also an eigenfunction for $c = 1$ different from the one given in the exercise. I learned this perspective from [12, page 338-339].

7. **Prove that the convolution of two functions of moderate decrease is a function of moderate decrease.**

Proof. Let f, g be functions of moderate decrease. Then the continuity of $f * g$ can be proved by LDCT if one realize the integral in Lebesgue's sense, or using an standard decomposition of the integral as "local" part and "far-away" part like the following proof for moderate decrease if one realize the integral in Riemann's sense, we leave the details to the readers.

On the other hand, the moderate decreasing property can be proved as follows:

$$
\begin{aligned}
|f * g(x)| &\leq \int_{|t|\leq \frac{|x|}{2}} |f(t)||g(x-t)|\,dt + \int_{|t|>\frac{|x|}{2}} |f(t)||g(x-t)|\,dt \\
&\leq \int_{\mathbb{R}} |f(t)|\,dt \frac{c_g}{1+\frac{|x|^2}{4}} + \int_{\mathbb{R}} |g(x-t)|\,dt \frac{c_f}{1+\frac{|x|^2}{4}} \leq \frac{M}{1+|x|^2}.
\end{aligned}
$$

□

8. **Prove that if f is continuous, of moderate decrease, and $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \mathbb{R}$, then $f \equiv 0$.**

Proof. We know $[f * e^{-x^2}](z) = e^{-z^2}\int_{\mathbb{R}} f(y)e^{-y^2}e^{2zy}\,dy = 0$ for all $z \in \mathbb{R}$. So $\widehat{f} \equiv 0$ since $\widehat{f}(\xi)\widehat{e^{-x^2}}(\xi) = \widehat{f * e^{-x^2}}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Thus $f \equiv 0$ by Fourier inversion theorem. □

Remark 4. This is true for all $L^1(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$ functions.

9. **If f is of moderate decrease, then**

$$\int_{-R}^{R} \left(1 - \frac{|\xi|}{R}\right)\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi = (f * F_R)(x),$$

where the Fejér kernel on the real line is defined by $F_R(t) = R\left(\frac{\sin \pi tR}{\pi tR}\right)^2$ if $t \neq 0$ and $F_R(0) = R$. Show that $\{F_R\}$ is a family of good kernels as $R \to \infty$, and therefore $(f * F_R)$ tends uniformly to $f(x)$ as $R \to \infty$. This is the analogue of Fejér's theorem for Fourier series in the context of the Fourier transform.

Proof. Note that for all $R > 0$

$$\int_{\mathbb{R}} |F_R(t)|\,dt = \int_{\mathbb{R}} F_R(t)\,dt = \frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{2}{\pi}\int_0^\infty \frac{\sin z}{z}\,dz = 1$$

and hence for all $\delta > 0$

$$\int_{[-\delta,\delta]^c} F_R(t)\, dt = \frac{2}{\pi} \int_{\pi R\delta}^{\infty} \frac{\sin^2 x}{x^2}\, dx \to 0 \text{ as } R \to \infty$$

$\square$

10. **Below is an outline of a different proof of the Weierstrass approximation theorem. Dfine the Landau kernels by**

$$L_n(x) = \frac{(1-x^2)^n}{c_n} \chi_{|x|\le 1}.$$

where c_n is chosen so that $\inf_{-\infty}^{\infty} L_n(x)\, dx = 1$. Prove that $\{L_n\}_{n\ge 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

Proof. Note that $c_n = \int_{-1}^{1} (1-x^2)^n\, dx = 2\int_0^1 (1-x^2)\, dx \ge 2\int_0^1 x(1-x^2)^n\, dx = \frac{1}{n+1}$. So for each $\delta > 0$, $\int_{\delta \le |x|} L_n(x)\, dx \le (1-\delta^2)^n (n+1) \to 0$ as $n \to \infty$. $\square$

11. **Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x,0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is, $u(x,t) \to 0$ as $|x| + t \to \infty$.**

Remark 5. We should not only consider the tangential limit, i.e. $(x_0, t) \to (x_0, 0)$, but also the nontangential limit $(x,t) \to (x_0, 0)$. However, the proof is almost the same idea. Also, the condition on f can be weaken to $f \in C(\mathbb{R})$ and of moderate decreasing.

Proof. By the even weaker hypothesis stated in the remark, f is uniformly continuous. So for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(y) - f(x_0)| < \epsilon$ whenever $|y - x_0| < \delta$. Note that for each x with $|x - x_0| < \frac{\delta}{2}$, we have $\{y : |x - y| < \frac{\delta}{4}\} \subset \{|x_0 - y| < \delta\}$, and hence

$$\begin{aligned} |u(x,t) - f(x_0)| &= \left| \int_{\mathbb{R}} \Big[f(y) - f(x_0) \Big] \mathcal{H}_t(x-y)\, dy \right| \\ &\le \int_{|x-y|<\frac{\delta}{4}} 2\epsilon \mathcal{H}_t(x-y)\, dy + 2\|f\|_\infty \int_{|x-y|\ge\frac{\delta}{4}} \mathcal{H}_t(x-y)\, dy. \end{aligned}$$

And hence there is there is $t_\epsilon > 0$ independent of x such that $|u(x,t) - f(x_0)| \le 3\epsilon$ whenever $t \in (0, t_\epsilon)$ and $|x - x_0| < \frac{\delta}{4}$.

Note that $|u(x,t)| \le \|f\|_{L^1(\mathbb{R})}(4\pi t)^{-1}$ for all $x \in \mathbb{R}, t > 0$. So it tends to zero as $|x| + t \to \infty$ and $|x| \le 2t$. For the case $|x| > t$, we note that

$$
\begin{aligned}
|u(x,t)| &\le \int_{\mathbb{R}} |f(y)|(4\pi t)^{-\frac{1}{2}} e^{-|x-y|^2/4t}\,dy \\
&= \int_{|x-y|>\frac{|x|}{2}} |f(y)|(4\pi t)^{-\frac{1}{2}} e^{-|x-y|^2/4t}\,dy + \int_{|x-y|\le\frac{|x|}{2}} |f(y)|(4\pi t)^{-\frac{1}{2}} e^{-|x-y|^2/4t}\,dy \\
&\le (4\pi t)^{-\frac{1}{2}} e^{-|x|^2/16t} \int_{|x-y|>\frac{|x|}{2}} |f(y)|\,dy + \int_{|x-y|\le\frac{|x|}{2}} \frac{M}{1+|y|^2}(4\pi t)^{-\frac{1}{2}} e^{-|x-y|^2/4t}\,dy \\
&\le (4\pi t)^{-\frac{1}{2}} e^{-|x|^2/16t}\|f\|_{L^1(\mathbb{R})} + \frac{M}{1+\frac{|x|^2}{4}}.
\end{aligned}
$$

If $|x| + t \to \infty$ with $t \ge \epsilon$ for some $\epsilon > 0$, then $|x| \to \infty$ (since we assume $|x| > t$) and

$$|u(x,t)| \le (4\pi\epsilon)^{-\frac{1}{2}} e^{-|x|/16} + \frac{4M}{4+|x|^2} \to 0 \text{ as } |x| \to \infty.$$

If $|x| + t \to \infty$ with $t \to 0^+$, then $|x| \to \infty$ and

$$|u(x,t)| \le (4\pi t)^{-\frac{1}{2}} e^{-1/16t} + \frac{4M}{4+|x|^2} \to 0 \text{ as } |x| \to \infty,\ t \to 0^+.$$

□

12. **Show that the function**

$$u(x,t) = \frac{x}{t}\frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

(1) satisfies the 1-d heat equation for $t > 0$, (2) $\lim_{t\to 0} u(x,t) = 0$ for every $x \in \mathbb{R}$, (3) $\lim_{(x,t)\to(x_0,0)} u(x,t) = 0$ for every $x_0 \ne 0$ and (4) u is not continuous at the origin.

Remark 6. Hence, it can not serve as an example to the non-uniqueness phenomenon of the heat equation in $\mathbb{R} \times (0,T)$. An exact counterexample is given in Problem 4.

Proof. (1)(2) are trivial. (3) is a consequence of Lopital's rule and standard squeeze theorem. (4) $u(x,x^2) = \frac{1}{\sqrt{4\pi x^2}} e^{-1/4} \to \infty$ as $x \to 0$. □

13. **Prove the following uniqueness theorem for harmonic functions in the strip $S = \{(x,y) : 0 < y < 1, -\infty < x < \infty\}$: if u is harmonic in S, continuous on $\overline{S}$ with $u(x,0) = u(x,1) = 0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u = 0$.**

Proof. For each $\epsilon > 0$, one can use maximum modulus principle (Mean-Value theorem is proved in the textbook) to conclude $|u| \le \epsilon$ on S. So $u \equiv 0$.

We sketch another way which is similar to the proof for Theorem 2.7. Suppose not, we may assume $M = \sup u > 0$. Hence $u(x_1,y_1) = M$ for some $(x_1,y_1) \in S$. By the uniform continuity of u and mean-value property with ball $B_{r\min\{|y_1|,1-|y_1|\}}(x_1,y_1)$, we have a contradiction that $M \le M-\delta$ for some $\delta>0$ if r is close to 1. $\square$

14. **Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line (Exercise 9) is equal to the Fejér kernel for periodic functions of period 1. In other words,**

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \ge 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right)e^{2\pi i n x} = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

Proof. Since $\widehat{\mathcal{F}_R}(\xi) = (1-\frac{|\xi|}{R})\chi_{[-R,R]}(\xi)$ for all $R>0, \xi\in\mathbb{R}$. The desired result follows from Poisson summation formula. $\square$

15. **This exercise provides another example of periodization.**

(a) Apply the Poisson summation formula to the function g in Exercise 2 to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin\pi\alpha)^2}$$

whenever α is real, but not equal to an integer.

(b) Let $\alpha\in\mathbb{R}\setminus\mathbb{Z}$. Prove as a consequence that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan\pi\alpha}$$

where the series is defined through the symmetric partial sum about $\lfloor\alpha\rfloor$, the largest integer $\le\alpha$. (Note that this series is not absolutely convergent, so we need to assign the order of summation here.)

Remark 7. Other proofs are given in Exercise 3.9 and Book II's Exercise 3.12.

Proof. (a) Poisson summation formula applied to $f=\widehat{g}$ and $x=\alpha$ implies that

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2\pi\alpha}{\pi^2(n+\alpha)^2} = \sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi(n+\alpha))}{\pi^2(n+\alpha)^2} = \sum_{n=-\infty}^{\infty} g(-n)e^{2\pi i n\alpha} = 1.$$

(b) Assume that $0<\alpha<1$ first. Note that for $n\in\mathbb{N}$,

$$\frac{1}{n+\alpha}+\frac{1}{-n+\alpha} = -\int_0^{\alpha} \frac{1}{(n+x)^2}+\frac{1}{(-n+x)^2}\,dx$$

Note that $\sum_{n\neq 0}\frac{1}{(n+x)^2}$ converges uniformly on $(0,1)$ by M-test with $\sup_{x\in(0,1)}(n+x)^{-2}=n^{-2}$ and $\sup_{x\in(0,1)}(-n+x)^{-2}=(n-1)^{-2}$ for all $n\geq 2$. Therefore

$$\begin{aligned}\frac{1}{\alpha}+\sum_{n=1}^{\infty}\frac{1}{n+\alpha}+\frac{1}{-n+\alpha}&=\frac{1}{\alpha}-\int_0^\alpha\sum_{n\neq 0}\frac{1}{(n+x)^2}\,dx=\frac{1}{\alpha}-\lim_{\epsilon\to 0^+}\int_\epsilon^\alpha\frac{\pi^2}{\sin^2\pi x}-\frac{1}{x^2}\,dx\\&=\frac{1}{\alpha}-\left(-\frac{\pi}{\tan\pi\alpha}+\frac{1}{\alpha}\right)+\lim_{\epsilon\to 0^+}\left(-\frac{\pi}{\tan\pi\epsilon}+\frac{1}{\epsilon}\right)=\frac{\pi}{\tan\pi\alpha}.\end{aligned}$$

Finally, for arbitrary $\alpha\in\mathbb{R}$, $\alpha=\lfloor\alpha\rfloor+\alpha_1$, where $0<\alpha_1<1$. So by the definition of this series and previous result

$$\sum_{n=-\infty}^{\infty}\frac{1}{n+\alpha}=\sum_{n=-\infty}^{\infty}\frac{1}{n+\alpha_1}=\frac{\pi}{\tan\pi\alpha_1}=\frac{\pi}{\tan\pi\alpha}$$

□

16. **The Dirichlet kernel on the real line is defined by**

$$\int_{-R}^{R}\widehat{f}(\xi)e^{2\pi ix\xi}\,d\xi=(f*\mathcal{D}_R)(x)\ \textbf{ so that }\ \mathcal{D}_R(x)=\widehat{\chi_{[-R,R]}}(x)=\frac{\sin(2\pi Rx)}{\pi x}.$$

Also, the modified Dirichlet kernel for periodic functions of period 1 is defined by

$$D_N^*(x)=\sum_{|n|\leq N-1}e^{2\pi inx}+\frac{1}{2}(e^{-2\pi iNx}+e^{2\pi iNx}).$$

Show that the result in Exercise 15 gives

$$\sum_{n=-\infty}^{\infty}\mathcal{D}_N(x+n)=D_N^*(x),$$

where $N\geq 1$ is an integer, and the infinite series must be summed symmetrically. In other words, the periodization of $\mathcal{D}_N$ is the modified Dirichlet kernel D_N^*. Also show that if $R\in\mathbb{R}^+\setminus\mathbb{N}$, then

$$\sum_{n=-\infty}^{\infty}\mathcal{D}_R(x+n)=D_{[R]}(x),$$

where $[R]$ is the largest integer $\leq R$.

***Remark* 8.** Corresponding to Problem 3.1, one can define modified conjugate Dirichlet kernel $\widetilde{D}_N^*$ similarly ,that is,

$$\widetilde{D}_N^*(x)=\sum_{|n|\leq N-1}\operatorname{sign}(n)e^{2\pi inx}+\frac{1}{2}(-e^{-2\pi iNx}+e^{2\pi iNx}).$$

Proof. Note that we can't apply Poisson summation formula since $\mathcal{D}_R$ is not in $L^1(\mathbb{R})$ (so the inversion formula breaks down). So we go back to the symmetric partial sum:

$$\sum_{k=-L}^{L} D_R(x+k) = \sum_{k=-L}^{L}\int_{-R}^{R} e^{-2\pi i\xi(x+k)}\,d\xi = \int_{-R}^{R} e^{-2\pi i\xi x}\sum_{k=-L}^{L} e^{-2\pi i\xi k}\,d\xi = \int_{-R}^{R} e^{-2\pi i\xi x} D_L(\xi)\,d\xi$$

$$= \sum_{m\le[R]-1} e^{-2\pi i\xi m}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(0-\xi)D_L(\xi)\,d\xi + e^{-2\pi i[R]x}\int_{-\frac{1}{2}}^{R-[R]} f_x(0-\xi)D_L(\xi)\,d\xi$$

$$+ e^{2\pi i[R]x}\int_{-R+[R]}^{\frac{1}{2}} f_x(0-\xi)D_L(\xi)\,d\xi$$

where $f_x(s) := e^{2\pi i\xi s}$. So the Tauberian theorem for Cesaro or Abel sum (see Exercise 2.14 and Problem 2.3) of Fourier series of $f_x(s)$, $f_x(s)\chi_{(-\frac{1}{2},\delta)}(s)$ and $f_x(s)\chi_{(-\delta,\frac{1}{2})}(s)$ $(\delta \ge 0)$ implies that as $L \to \infty$,

$$\int_{\delta}^{\frac{1}{2}} f_x(0-\xi)D_L(\xi)\,d\xi \to f(0) = 1 \text{ if } \delta > 0 \text{ and } \int_{0}^{\frac{1}{2}} f_x(0-\xi)D_L(\xi)\,d\xi \to \frac{f(0+)+f(0-)}{2} = \frac{1}{2}.$$

Another case is similar, so we complete the proof. □

17. The gamma function is defined for $s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx.$$

(a) One can easily show that for $s > 0$ the above integral makes sense, that is, the following two limits exist:

$$\lim_{\delta\to 0^+}\int_{\delta}^{1} e^{-x}x^{s-1}\,dx \text{ and } \lim_{A\to\infty}\int_{1}^{A} e^{-x}x^{s-1}\,dx.$$

(b) One then can use integration by parts to prove $\Gamma(s+1) = s\Gamma(s)$ whenever $s > 0$, and conclude that (1) for every integer $n \ge 1$ we have $\Gamma(n+1) = n!$; (2) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{\pi}{2}$ easily.

18. **The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$. Verify the identity**

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}(\vartheta(t)-1)\,dt \text{ whenever } s > 1$$

where Γ and ϑ are the gamma and theta functions, respectively. $\vartheta(s) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$**.**

More about the zeta function and its relation to the prime number theorem can be found in Book II.

Proof. For $s > 1$

$$\int_0^\infty t^{\frac{s}{2}-1}(\vartheta(t)-1)\,dt = \int_0^\infty t^{\frac{s}{2}-1}(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}-1)\,dt = 2\sum_{n=1}^{\infty}\int_0^\infty e^{-\pi n^2 t}t^{\frac{s}{2}-1}\,dt$$

$$= 2\sum_{n=1}^{\infty}\pi^{-\frac{s}{2}}n^{-s}\int_0^\infty e^{-z}z^{\frac{s}{2}-1}\,dz = 2\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where the interchange of the order of sum and integration is permitted by Monotone convergence theorem (if one use Lebesgue integral) or an careful argument to uniform convergence and improper Riemann integral (interchange twice, one is for integral over $[0, M]$ and sum, another one is for $\lim_{M\to\infty}$ and sum) $\square$

19. **The following is a variant of the calculation of $\zeta(2m) = \sum_{n=1}^{\infty} 1/n^{2m}$ found in Problem 4, Chapter 3.**

(a) Applying the Poisson summation formula to $f(x) = t/(\pi(x^2+t^2))$ and $\widehat{f}(\xi) = e^{-2\pi t|\xi|}$ where $t > 0$, we get

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{t}{t^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi t|n|}.$$

(b) Prove the following identity valid for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

Also note the following trivial fact

$$\sum_{n=-\infty}^{\infty} e^{-2\pi t|n|} = \frac{2}{1-e^{-2\pi t}} - 1.$$

(c) Use the fact that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}z^{2m},$$

where B_k are the Bernoulli numbers to deduce from the above formula,

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

Proof. (b) For $0 < t < 1$,

$$\sum_{n=1}^{\infty}\frac{t}{t^2+n^2} = \sum_{n=1}^{\infty}\frac{t}{n^2}\frac{1}{(t/n)^2+1} = \sum_{n=1}^{\infty}\frac{t}{n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t}{n}\right)^{2m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{t^{2m-1}}{n^{2m}}$$
$$= \sum_{m=1}^{\infty}(-1)^{m+1}t^{2m-1}\sum_{n=1}\frac{1}{n^{2m}} = \sum_{m=1}^{\infty}(-1)^{m+1}t^{2m-1}\zeta(2m),$$

where the Fubini's theorem, that is, the interchange of the order of the sums $\sum_n$ and $\sum_m$ is permitted by the absolute convergence of the double series $\sum_{n,m}(-1)^{m+1}\frac{t^{2m-1}}{n^{2m}}$ which can be proved easily.

(c) Note that for $z = -2\pi t$, $(0 < t < 1)$

$$\sum_{m=1}^{\infty} \frac{B_{2m}(2\pi)^{2m}}{(2m)!} t^{2m} = \frac{-2\pi t}{e^{-2\pi t} - 1} - 1 - \pi t = -1 + \pi t \sum_{n=-\infty}^{\infty} e^{-2\pi t|n|}$$

$$= -1 + \pi t \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{t}{t^2 + n^2} = -1 + 1 + 2t \sum_{m=1}^{\infty} (-1)^{m+1} \zeta(2m) t^{2m-1} = \sum_{m=1}^{\infty} 2(-1)^{m+1} \zeta(2m) t^{2m}.$$

By the uniqueness of power series (see Baby Rudin's Theorem 8.5),

$$\frac{B_{2m}(2\pi)^{2m}}{(2m)!} = 2(-1)^{m+1} \zeta(2m).$$

□

20. **The following results are relevant in information theory when one tries to recover a signal from its samples.**

Suppose f is of moderate decrease and that its Fourier transform $\widehat{f}$ is supported in $I = [-1/2, 1/2]$. Then, f is entirely determined by its restriction to $\mathbb{Z}$. This means that if g is another function of moderate decrease whose Fourier transform is supported in I and $f(n) = g(n)$ for all $n \in \mathbb{Z}$, then $f = g$. More precisely:

(a) Prove that the following reconstruction formula holds:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) K(x-n) \quad \textbf{where} \quad K(y) = \frac{\sin \pi y}{\pi y}$$

Note that $K(y) = O(1/|y|)$ as $|y| \to \infty$.

(b) If $\lambda > 1$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f(\frac{n}{\lambda}) K_\lambda(x - \frac{n}{\lambda}) \quad \textbf{where} \quad K(y) = \frac{\cos \pi y - cos \pi \lambda y}{\pi^2 (\lambda - 1) y^2}.$$

Thus, if one samples f "more often," the series in the reconstruction formula converges faster since $K_\lambda(y) = O(1/|y|^2)$ as $|y| \to \infty$. Note that $K_\lambda(y) \to K(y)$ as $\lambda \to 1$.

(c) Prove that $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |f(n)|^2$.

Remark 9. Compare with Exercise 7.8.

Remark 10. This is the well-known Shannon sampling theorem. See the survey article [1]. Its relation to continuous wavelet transform can be found in [3, Chapter 2]

Proof. (a) Using Poisson summation formula for $g(x) = \widehat{f}(x)$, we have

$$\widehat{f}(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \sum_{n \in \mathbb{Z}} \widehat{f}(\xi + n) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i(-n)\xi}.$$

Hence

$$f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{f}(\xi)e^{2\pi i\xi x}\, d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} f(n)e^{2\pi i\xi(x-n)}\, d\xi = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin \pi\xi(x-n)}{\pi(x-n)},$$

where the interchange of sum and integration is promised by the uniform convergence with upper bound $|f(n)| \leq \frac{M}{1+n^2}$ from the moderate decreasing.

(b) It's the same idea: Using Poisson summation formula with $g(x) = \widehat{f}(\lambda x)$ and the characteristic function $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ there is replaced by the function $\phi(\xi)$ describe in the Figure 2. We omit the brutal computation for $\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \phi(\xi)e^{2\pi i\xi(x-\frac{n}{\lambda})}\, d\xi$.

(c) By Planchel's theorem and Poisson summation formula, we have

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widehat{f}(\xi)|^2\, d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} f(n)e^{2\pi i(-n)\xi} \sum_{m\in\mathbb{Z}} \overline{f(m)}e^{2\pi i m\xi}\, d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{(m,n)\in\mathbb{Z}^2} f(n)\overline{f(m)}e^{2\pi i(m-n)\xi}\, d\xi = \sum_{(m,n)\in\mathbb{Z}^2} f(n)\overline{f(m)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(m-n)\xi}\, d\xi = \sum_{n\in\mathbb{Z}} |f(n)|^2,$$

where the third and fourth equalities are due to the absolute (and hence uniform) convergence of the series with bound $|f(n)f(m)| \leq \frac{M^2}{(1+n^2)(1+m^2)}$. $\square$

21. **Suppose that f is continuous on $\mathbb{R}$. Show that f and $\widehat{f}$ cannot both be compactly supported unless $f = 0$. This can be viewed in the same spirit as the uncertainty principle.**

[Hint: Assume f is supported in $[0, 1/2]$. Expand f in a Fourier series in the interval $[0, 1]$, and note that as a result, f is a trigonometric polynomial.]

Proof. Suppose f is compactly supported in $[0, \frac{1}{2}]$ and $\widehat{f}$ is also compactly supported (say, in $[-M, M]$ for some $M > 0$.

As hint, $f(x) \sim \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx} = \sum_{|n|\leq M} \widehat{f}(n)e^{2\pi inx}$ where $\widehat{f}(n) = \int_{\mathbb{R}} f(t)e^{-2\pi int}\, dt = \int_0^1 f(t)e^{-2\pi int}$.

So the series is an trigonometric polynomial that converges on $\mathbb{R}$.

The uniqueness of Fourier series implies f equals to this polynomial. Unless $f \equiv 0$, f can have a finite number of roots in $[0, 1]$ and hence cannot be compactly supported in $[0, \frac{1}{2}]$. $\square$

Remark 11. The assumption on $widehatf$ can be weaken to the exponential decay, that is, $|widehatf(\xi)| \lesssim e^{-k|\xi|}$. The idea of the proof is: first, we use inversion formula to see the smoothness of f and convergence of its power series with a fix radius of convergence that is at least $\frac{k}{2\pi}$ everywhere. Thus, it is real analytic over $\mathbb{R}$, and cannot have compact support unless it is identically 0.

22. **The heuristic assertion stated before Theorem 4.1 can be made precise as follows. If F is a function on $\mathbb{R}$, then we say that the preponderance of its mass is contained in an interval I (centered at the origin) if**

$$\int_I x^2|F(x)|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}} x^2|F(x)|^2\,dx \tag{1}$$

Now suppose $f \in \mathcal{S}$, and (1) holds with $F = f$ and $I = I_1$; also with $F = \widehat{f}$ and $I = I_2$. Then if L_j denotes the length of I_j , we have

$$L_1L_2 \geq \frac{1}{2\pi}.$$

A similar conclusion holds if the intervals are not necessarily centered at the origin.

Proof. We may assume $\|f\|_2 = 1$ and hence $\|\widehat{f}\|_2 = 1$ by Planchel's theorem. So Theorem 1.4 implies the desired result as follows:

$$\begin{aligned}(L_1L_2)^2 &= 16\Big(\frac{L_1}{2}\Big)^2\Big(\frac{L_2}{2}\Big)^2\int_{\mathbb{R}}|f(x)|^2\,dx\int_{\mathbb{R}}|\widehat{f}(\xi)|^2\,d\xi \geq 16\int_{I_1}x^2|f(x)|^2\,dx\int_{I_2}|\xi|^2|\widehat{f}(\xi)|^2\,d\xi\\ &\geq 4\int_{\mathbb{R}}x^2|f(x)|^2\,dx\int_{\mathbb{R}}|\xi|^2|\widehat{f}(\xi)|^2 \geq \frac{1}{4\pi^2}.\end{aligned}$$

□

23. **The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$, which acts on Schwartz functions by the formula**

$$L(f) = -\frac{d^2f}{dx^2} + x^2f.$$

This operator, sometimes called the Hermite operator, is the quantum analogue of the harmonic oscillator. Consider the usual inner product on $\mathcal{S}$ given by

$$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \ \text{ whenever } \ f,g \in \mathcal{S}.$$

(a) Prove that the Heisenberg uncertainty principle implies

$$(Lf,f) \geq (f,f) \ \ \forall f \in \mathcal{S}.$$

This is usually denoted by $L \geq I$.

(b) Consider the operators A and A^* defined on $\mathcal{S}$ by

$$A(f) = \frac{df}{dx} + xf \quad \textbf{and} \quad A^*(f) = -\frac{df}{dx} + xf.$$

The operators A and A^* are sometimes called the annihilation and creation operators, respectively. Prove that for all $f, g \in \mathcal{S}$ we have

(i) $(Af, g) = (f, A^*g)$**, (ii)** $(A^*Af, f) = (Af, Af) \geq 0$**, (iii)** $A^*A = L - I$**.**

In particular, this again shows that $L \geq I$.

(c) Now for $t \in \mathbb{R}$, let

$$A_t(f) = \frac{df}{dx} + txf \quad \textbf{and} \quad A_t^*(f) = -\frac{df}{dx} + txf.$$

Use the fact that $(A_t^*A_tf, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $\int_{-\infty}^{\infty} |f(x)|^2\, dx = 1$ then

$$\Big(\int_{-\infty}^{\infty} x^2 |f(x)|^2\, dx\Big)\Big(\int_{-\infty}^{\infty} \Big|\frac{df}{dx}\Big|^2\, dx\Big) \geq \frac{1}{4}.$$

Remark 12. See Problem 7 and [6, Chapter 6].

Proof. (a) By Theorem 1.4 and arithmetic-geometric mean inequality

$$\begin{aligned}(Lf, f) &= \int_{\mathbb{R}} \Big|\frac{df}{dx}\Big|^2 + x^2|f(x)|^2 = \int_{\mathbb{R}} 4\pi^2 x^2 |\widehat{f}(x)|^2 + x^2|f(x)|^2 \\ &\geq 2\Big(\int_{\mathbb{R}} 4\pi^2x^2|\widehat{f}(x)|^2\Big)^{1/2}\Big(\int_{\mathbb{R}} x^2|f(x)|^2\Big)^{1/2} \geq 4\pi\frac{1}{4\pi}(f, f).\end{aligned}$$

(b) (i)(ii) Use the integration by parts, (iii) is obvious.

(c) Note that $0 \leq (A_tf, A_tf) = (A_t^*A_tf, f) = t^2 \int_{\mathbb{R}} x^2 |f(x)|^2\, dx - t\int_{\mathbb{R}} |f(x)|^2\, dx + \int_{\mathbb{R}} |\frac{df}{dx}(x)|^2\, dx$.

Then we complete the proof through the basic algebra that

$$\Big(-\int_{\mathbb{R}} |f(x)|^2\, dx\Big)^2 - 4\Big(\int_{\mathbb{R}} x^2|f(x)|^2\, dx\Big)\Big(\int_{\mathbb{R}} |\frac{df}{dx}(x)|^2\Big) \leq 0.$$

□

Problems

1. **The equation**

$$x^2\frac{\partial^2 u}{\partial x^2} + ax\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$$

with $u(x,0) = f(x)$ for $0 < x < \infty$ and $t > 0$ is a variant of the heat equation which occurs in a number of applications. To solve this equation, make the change of variables $x = e^{-y}$ so that $-\infty < y < \infty$. Set $U(y,t) = u(e^{-y},t)$ and $F(y) = f(e^{-y})$. Then the problem reduces to the equation

$$\frac{\partial^2 U}{\partial y^2} + (1-a)\frac{\partial U}{\partial y} = \frac{\partial U}{\partial t},$$

with $U(y,0) = F(y)$. This can be solved like the usual heat equation (the case $a = 1$) by taking the Fourier transform in the y variable. One must then compute the integral $\int_{-\infty}^{\infty} e^{(-4\pi^2\xi^2+(1-a)2\pi i\xi)t} e^{2\pi i\xi v}\, d\xi$. Show that the solution of the original problem is then given by

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_0^\infty e^{-(\log(v/x)+(1-a)t)^2/(4t)} f(v)\, \frac{dv}{v}.$$

Proof. We omit the easy proof for equivalence of these two equation. Taking Fourier transform in y to the equation, we have

$$(-4\pi^2\xi^2 + (1-a)2\pi i\xi)\widehat{U}(\xi,t) = \partial_t \widehat{U}(\xi,t)$$

So $\widehat{U}(\xi,t) = e^{(-4\pi^2\xi^2+2\pi i(1-a)\xi)t}\widehat{F}(\xi) =$ where $G(y,t)$ is determined by the inversion formula, that is,

$$G(y,t) = \int_{\mathbb{R}} e^{(-4\pi^2\xi^2+2\pi i(1-a)\xi)t} e^{2\pi i\xi y}\, d\xi = \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t} e^{2\pi i\xi(t(1-a)+y)}\, d\xi = \frac{1}{\sqrt{4\pi t}} e^{-(t(1-a)+y)^2/(4t)}.$$

Hence $U(y,t) = \int_{\mathbb{R}} F(s) \frac{1}{\sqrt{4\pi t}} e^{-(t(1-a)+y-s)/(4t)}\, ds$ and then (with $s = -\log v$ and $y = -\log x$)

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_0^\infty e^{-(\log(v/x)+(1-a)t)^2/(4t)} f(v)\, \frac{dv}{v}.$$

$\square$

2. The Black-Scholes equation from finance theory is

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0, \quad 0 < t < T,$$

subject to the "final" boundary condition $V(s,T) = F(s)$. An appropriate change of variables reduces this to the equation in Problem 1. Alternatively, the substitution $V(s,t) = e^{ax+b\tau}U(x,\tau)$ where $x = \log s, \tau = \frac{\sigma^2}{2}(T-t), a = \frac{1}{2} - \frac{r}{\sigma^2}$, and $b = -(\frac{1}{2} + \frac{r}{\sigma^2})^2$ reduces the equation to the one-dimensional heat equation with the initial condition $U(x,0) = e^{-ax}F(e^x)$. Thus a solution to the Black-Scholes equation is

$$V(s,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_0^\infty e^{-\frac{(\log(s/s^*)+(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}} F(s^*)\, ds^*.$$

3. **The Dirichlet problem in a strip. Consider the equation $\Delta u = 0$ in the horizontal strip**

$$\{(x, y) : 0 < y < 1; -\infty < x < \infty\}$$

with boundary conditions $u(x, 0) = f_0(x)$ and $u(x, 1) = f_1(x)$, where f_0 and f_1 are both in the Schwartz space.

(a) Show (formally) that if u is a solution to this problem, then

$$\widehat{u}(\xi, y) = A(\xi)e^{2\pi\xi y} + B(\xi)e^{-2\pi\xi y}.$$

Express A and B in terms of $\widehat{f_0}$ and $\widehat{f_1}$, and show that

$$\widehat{u}(\xi, y) = \frac{\sinh(2\pi(1-y)\xi)}{\sinh(2\pi\xi)}\widehat{f_0}(\xi) + \frac{\sinh(2\pi y\xi)}{\sinh(2\pi\xi)}\widehat{f_1(\xi)}.$$

(b) Prove as a result that we have L^2-convergence to the boundary conditions, that is,

$$\int_{-\infty}^{\infty} |u(x, y) - f_0(x)|^2\, dx \to 0 \quad \text{as} \quad y \to 0$$

and

$$\int_{-\infty}^{\infty} |u(x, y) - f_1(x)|^2\, dx \to 0 \quad \text{as} \quad y \to 1.$$

(c) If $\Phi_a(\xi) = (\sinh 2\pi a\xi)/(\sinh 2\pi\xi)$, with $0 \le a < 1$, then Φ_a is the Fourier transform of φ_a where

$$\varphi_a(x) = \frac{\sin \pi a}{2} \cdot \frac{1}{\cosh \pi x + \cos \pi a}.$$

This can be shown, for instance, by using contour integration and the residue formula from complex analysis (see Book II, Chapter 3).

(d) Use this result to express u in terms of Poisson-like integrals involving f_0 and f_1 as follows:

$$u(x, y) = \frac{\sin \pi y}{2}\left(\int_{-\infty}^{\infty} \frac{f_0(x-t)}{\cosh \pi t - \cos \pi y}\, dt + \int_{-\infty}^{\infty} \frac{f_1(x-t)}{\cosh \pi t + \cos \pi y}\, dt\right).$$

(e) Finally, one can check that the function $u(x, y)$ defined by the above expression is harmonic in the strip, and converges uniformly to $f_0(x)$ as $y \to 0$, and to $f_1(x)$ as $y \to 1$ which are improvements of (b). Moreover, one sees that $u(x, y)$ vanishes at infinity, that is, $\lim_{|x|\to\infty} u(x, y) = 0$, uniformly in y.

Remark 13. The identity in (c) can also be applied to prove the sum of two squares theorem, see Section 3.1 of Chapter 10 in Book II.

Proof. We omit (a)(d) since they are easy.

(b) By Planchel's theorem,

$$\int_{-\infty}^{\infty} |u(x,y) - f_1(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{u}(\xi, y) - \widehat{f}_0(\xi)|^2 \, d\xi$$
$$= \int_{-\infty}^{\infty} \left| \widehat{f}_1(\xi) \Big(\frac{\sinh 2\pi y \xi}{\sinh 2\pi \xi} - 1 \Big) + \widehat{f}_0(\xi) \frac{\sinh 2\pi(1-y)\xi}{\sinh 2\pi\xi} \right|^2 d\xi. \tag{2}$$

Note that, by using a simple geometrical argument, one can show the radial function $\xi \mapsto \frac{\sinh 2\pi a \xi}{\sinh 2\pi\xi}$ is decreasing in $|\xi|$ for each $0 < a < 1$ and hence bounded by a.

Hence the integrand in (2) is bounded by

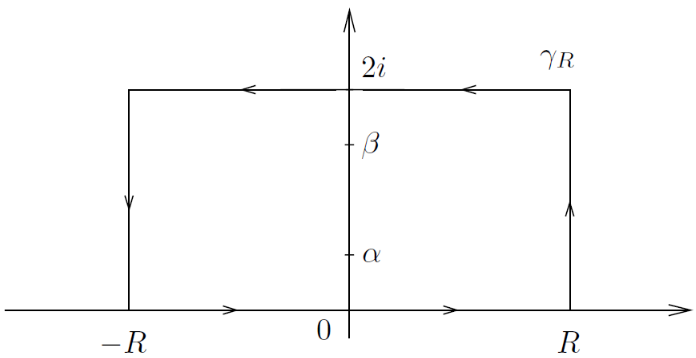
$$2\Big|\widehat{f}_1(\xi)(y+1)\Big|^2 + 2\Big|\widehat{f}_0(\xi)(1-y)\Big|^2 \le 2\Big|\widehat{f}_1(\xi) \cdot 2\Big|^2 + 2\Big|\widehat{f}_0(\xi)\Big|^2 \in L^1(\mathbb{R}).$$

Then the desired result follows from LDCT. Similar for $y \to 0$.

(c) Our goal is to show for fixed $\xi \in \mathbb{R}$,

$$I := \int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x + \cos \pi a} \, dx = \frac{2}{\sin \pi a} \frac{\sinh 2\pi a \xi}{\sinh 2\pi \xi}.$$

Let γ_R be as the following figure (copy from Book II) whose width goes to infinity but whose height is fixed.



Let $f(z) = \frac{e^{-2\pi i z \xi}}{\cosh \pi z + \cos \pi a}$ and note that the denominator of f vanishes precisely when $e^{\pi z} + e^{-\pi z} = -e^{\pi i a} - e^{-\pi i a}$. So the only poles of f inside γ_R are $\alpha = i(1-a)$ and $\beta = i(1+a)$. To find the residue of f at α, denoted by $\mathrm{Res}(f, \alpha)$, we note that, by L'Hôspital rule,

$$\mathrm{Res}(f,\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z \to \alpha} e^{-2\pi i z \xi} \frac{2(z-\alpha)}{e^{\pi z} + e^{-\pi z} + e^{\pi i a} + e^{-\pi i a}} = \frac{2e^{-2\pi i \alpha \xi}}{\pi(e^{\pi\alpha} - e^{-\pi\alpha})} = \frac{e^{2\pi(1-a)\xi}}{\pi i \sin \pi a}$$

Similar for $\mathrm{Res}(f,\beta) = \frac{e^{2\pi(1+a)\xi}}{-\pi i \sin \pi a}$

Then the residue theorem implies

$$\oint_{\gamma_R} f(z) = 2\pi i \Big(\mathrm{Res}(f,\alpha) + \mathrm{Res}(f,\beta) \Big) = \frac{2}{\sin \pi a} e^{2\pi\xi} (e^{-2\pi a \xi} - e^{2\pi a \xi}) \tag{3}$$

For the integrals of f on the vertical sides, one observes it tends to zero as $R \to \infty$. Indeed, if $z = R + iy$ with $0 \le y \le 2$, then

$$|e^{-2\pi i z\xi}| \le e^{4\pi|\xi|}$$

and

$$|\cosh \pi z + \cos \pi a| \ge |\frac{1}{2}(e^{\pi R} - e^{-\pi R}) - 1| \to \infty \text{ as } R \to \infty,$$

which shows that the integral over the vertical segment on the right goes to 0 as $R \to \infty$. A similar argument shows that the integral of f over the vertical segment on the left also goes to 0 as $R \to \infty$. So

$$\frac{2}{\sin \pi a} e^{2\pi\xi}(e^{-2\pi a\xi} - e^{2\pi a\xi}) = \lim_{R\to\infty} \oint_{\gamma_R} f(z) = (1 - e^{4\pi\xi})I$$

Hence $I = \frac{2}{\sin \pi a}(e^{-2\pi a\xi} - e^{2\pi a\xi})\frac{e^{2\pi\xi}}{1-e^{4\pi\xi}} = \frac{2}{\sin \pi a}\frac{\sinh 2\pi a\xi}{\sinh 2\pi\xi}$.

(e) We omit the proof for harmonicity of u in the strip. (For the interchange of the order of differentiation and integration, see [12, page 154]).

To show the unifrom convergence to the boundary conditions, we use the idea of approximate to the identity (good kernels).

Given $\epsilon > 0$. First we note that the kernel φ_a studied in (c) is positive and $\int_{\mathbb{R}} \varphi_a = \widehat{\varphi_a}(0) = \Phi_a(0) = a$. Second, by the uniform continuity of f_1 (due to decay at infinity), there is $\eta > 0$ such that $|f_1(x-t) - f_1(x)| < \epsilon$ for all $x \in \mathbb{R}$ and $|t| < \eta$. Therefore we obtain an upper bound estimate for $\|u(\cdot, y) - f_1(\cdot)\|_\infty$ as follows:

$$\begin{aligned}\Big|u(x,y) - f_1(x)\Big| &= \Big|\int_{\mathbb{R}} \varphi_{1-y}(t) f_0(x-t)\,dt + \int_{\mathbb{R}} \varphi_y(t)[f_1(x-t) - f_1(x)]\,dt - (1-y)f_1(x)\Big| \\ &\le (1-y)\|f_0\|_\infty + \int_{|t|\ge\eta} \varphi_y(t) 2\|f_1\|_\infty\,dt + \int_{|t|<\eta} \varphi_y(t)|f_1(x-t) - f_1(x)|\,dt + (1-y)\|f_1\|_\infty \\ &\le (1-y)(\|f_0\|_\infty + \|f_1\|_\infty) + 2\|f_1\|_\infty \int_{|t|\ge\eta} \varphi_y(t)\,dt + \epsilon y.\end{aligned}$$

The first and last are obviously less than ϵ whenever y is close enough to 1. For the second term, one note that the integrand is bounded by $\frac{\sin \pi y}{e^{\pi t} + e^{-\pi t} - 2}$. Although one can find $\int_\eta^\infty (e^{\pi t} + e^{-\pi t} - 2)^{-1}\,dt = \int_\eta^\infty \frac{e^{\pi t}}{(e^{\pi t} - 1)^2}\,dt$ can be calculated explicitly, we mention that for the situation without antiderivative, one can also observe prove this integral is finite by observing the exponential decay at infinity and a strict positive lower bound (due to $|t| \ge \eta$) for the denominator. Hence the target term is less than ϵ provided y is close enough to 1.

Finally, we prove that $u(x, y) \to 0$ as $|x| \to \infty$ uniform in y.

$$u(x,y) = \int_{-\infty}^{\infty} \varphi_{1-y}(t) f_0(x-t)\, dt + \int_{-\infty}^{\infty} \varphi_y(t) f_1(x-t)\, dt = \Big(\int_{|t|>\frac{|x|}{2}} + \int_{|t|\le\frac{|x|}{2}} \Big) \cdots$$

For $|t| > \frac{|x|}{2}$ part, we use the decreasing property of φ_a for each a to see the integral is dominated by

$$\varphi_{1-y}(\frac{|x|}{2}) \|f_0\|_{L^1(\mathbb{R})} + \varphi_y(\frac{|x|}{2}) \|f_1\|_{L^1(\mathbb{R})} \le \frac{\|f_0\|_{L^1(\mathbb{R})} + \|f_1\|_{L^1(\mathbb{R})}}{e^{\pi\frac{|x|}{2}} + e^{-\pi\frac{|x|}{2}} - 2}.$$

For $|t| \le \frac{|x|}{2}$ part, one can dominate the integral by

$$\Big(\sup_{|z-x|\le\frac{|x|}{2}} |f_0(z)| \Big) \|\varphi_{1-y}\|_{L^1(\mathbb{R})} + \Big(\sup_{|z-x|\le\frac{|x|}{2}} |f_1(z)| \Big) \|\varphi_y\|_{L^1(\mathbb{R})} \le \sup_{|z-x|\le\frac{|x|}{2}} |f_0(z)| + \sup_{|z-x|\le\frac{|x|}{2}} |f_1(z)|.$$

Note that all these upper bounds are independent of y and tends to zero as $|x| \to \infty$. So we complete the proof for (e).

$\square$

4. **(Tychonoff's counterexample) For $a > 1$, consider the function defined by**

$$g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ e^{-t^{-a}} & \text{if } t > 0 \end{cases} \tag{4}$$

(a) Show that there exists $0 < \theta < 1$ depending on a so that for $t > 0$,

$$|g^{(k)}(t)| \le \frac{k!}{(\theta t)^k} e^{-\frac{1}{2}t^{-a}}.$$

Moreover, show that g is in the Gevrey class $1 + \frac{1}{a}$ on $\mathbb{R}$, that is, there are M, R only depending on a, such that for all $k \in \mathbb{Z}_{\ge 0}$

$$|g^{(k)}(x)| \le (k!)^{1+\frac{1}{a}} M R^k.$$

Note that real analytic functions are of Gevrey class 1.

(b) Apply (a) to show that for each $x \in \mathbb{R}$ and $t > 0$, the series

$$u(x,t) := \sum_{n=0}^{\infty} g^{(n)}(t) \frac{x^{2n}}{(2n)!}$$

converges. Moreover, u solves the heat equation , vanishes for $t = 0$, continuous up to the initial $t = 0$ and satisfies the estimate

$$|u(x,t)| \le c_1 e^{c_2 |x|^{2a/(a-1)}}$$

for some constants $c_1, c_2 > 0$.

This example shows that the growth condition $|u(x,t)| \le Ae^{a|x|^2}$ we meet in the class is almost optimal, since for each $\epsilon > 0$, we can find $a > 0$ such that the growth exponent $2a/(a-1) = 2 + 2/(a-1) = 2+\epsilon$. (Also see Exercise 12 and Problem 6.)

Remark 14. The Gevrey part in (a) is taken from [11, Page 73 problem 3].

Proof. (a) The function $z \mapsto g(z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$. We identify the t-axis as the real axis of the complex plane. If $t > 0$ is fixed, the circle $\partial B_{\theta t}(t)$ does not meet the origin provided $0 < \theta < 1$ which will be determined later, and by the Cauchy integral formula (see Book II), we have for all $k \in \mathbb{Z}_{\ge 0}$

$$\frac{d^k g}{dt^k}(t) = \frac{k!}{2\pi i}\int_{\partial_{\theta t}(t)} \frac{g(z)}{(z-t)^{k+1}}\,dz$$

and hence

$$|\frac{d^k g}{dt^k}(t)| \le \frac{k!}{2\pi}(\frac{1}{\theta t})^k \int_0^{2\pi} e^{-\mathrm{Re}(z^{-a})}\,d\phi$$

Note that for $z \in \partial_{\theta t}(t)$, $z = t+\theta t e^{i\phi} = r_\phi e^{i\varphi_\phi}$ and hence $z^{-a} = r_\phi^{-a}e^{-ia\varphi_\phi}$. Since $(1-\theta)t \le r_\phi \le (1+\theta)t$ for all ϕ, $\mathrm{Re}(z^{-a}) \ge (\frac{1}{(1+\theta)t})^a > \frac{1}{2}t^{-a}$ if we pick θ small enough. So $|\frac{d^k g}{dt^k}(t)| \le \frac{k!}{(\theta t)^k}e^{-\frac{1}{2}t^{-a}}$. Note that this upper bound has maximum at $t^{-a} = \frac{2k}{a}$, that is,

$$\begin{aligned}|\frac{d^k g}{dt^k}(t)| &\le \frac{k!}{(\theta t)^k}e^{-\frac{1}{2}t^{-a}} \le k!(\frac{2k}{a\theta^a})^{\frac{k}{a}}e^{-\frac{k}{a}} = (k!)^{1+\frac{1}{a}}(\frac{k^k}{e^k k!})^{\frac{1}{a}}(\frac{2}{a\theta^a})^{\frac{k}{a}} \\ &\le (k!)^{1+\frac{1}{a}}\widetilde{M}^{\frac{1}{a}}\left(\frac{2^{\frac{1}{a}}}{a^{\frac{1}{a}}\theta}\right)^k,\end{aligned}$$

where the last inequality is a consequence of Stirling formula and $\widetilde{M}$ is independent of a.

(b) First, we use (a) to check the uniform convergence of $u(x,t)$.

$$\begin{aligned}|u(x,t)| &\le \sum_{n=0}^{\infty}|g^{(n)}(t)|\frac{x^{2n}}{(2n)!} \le \sum_{n=0}^{\infty}\frac{n!}{(\theta t)^n}e^{-\frac{1}{2}t^{-a}}\frac{x^{2n}}{(2n)!} = e^{-\frac{1}{2}t^{-a}}\sum_{n=0}^{\infty}\left(\frac{x^2}{\theta t}\right)^n\frac{n!}{(2n)!} \\ &\le e^{-\frac{1}{2}t^{-a}}\sum_{n=0}^{\infty}\left(\frac{x^2}{\theta t}\right)^n\frac{1}{n!} = e^{-\frac{1}{2}t^{-a}}e^{\frac{x^2}{\theta t}} \end{aligned} \tag{5}$$

which implies that $u(x,t)$ converges uniformly on $x \in [-M,M]$ and $t > t_0 > 0$ for each $M, t_0 > 0$ and $u(x_0,0) = 0$ since $a > 1$ and $\lim_{(x,t)\to(x_0,0)} e^{-\frac{1}{2}t^{-a}}e^{\frac{x^2}{\theta t}} = 0$. Similar for $\partial_t u, \partial_x u$ and $\partial_{xx}u$ with

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} g^{(n+1)}(t)\frac{x^{2n}}{(2n)!} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \sum_{j=1}^{\infty} g^{(j)}(t)\frac{x^{2j-2}}{(2j-2)!} = \sum_{n=0}^{\infty} g^{(n+1)}(t)\frac{x^{2n}}{(2n)!}.$$

So we see that $u \in C^2(\mathbb{R}\times(0,T)) \cap C(\mathbb{R}\times[0,T))$ solves the heat equation with zero initial condition.

Finally, for the optimal space-growth rate, we fix x and see the upper bound function obtained in (5) $f_x(t) := e^{-\frac{1}{2}t^{-a}} e^{\frac{x^2}{\theta t}}$ attains maximum at $t = (\frac{a\theta}{2x^2})^{\frac{1}{a-1}}$ with

$$f_x((\frac{a\theta}{2x^2})^{\frac{1}{a-1}}) = \exp(-\frac{1}{2}(\frac{2x^2}{a\theta})^{\frac{a}{a-1}} + \frac{x^2}{\theta}(\frac{2x^2}{a\theta})^{\frac{1}{a-1}}) = \exp\Big((\frac{2}{a\theta^a})^{\frac{1}{a-1}}(1-\frac{1}{a})|x|^{\frac{2a}{a-1}}\Big).$$

So we have $c_1 = 1$ and $c_2 = (\frac{2}{a\theta^a})^{\frac{1}{a-1}}(1-\frac{1}{a})$ here. □

$4\frac{1}{2}$ **Prove that the function**

$$u(x,t) = \int_0^\infty \Big[e^{xy}\cos(xy+2ty^2) + e^{-xy}\cos(xy-2ty^2)\Big] y e^{-y^{\frac{4}{3}}} \cos y^{\frac{4}{3}}\, dy$$

is another non-trivial solution of the Cauchy Problem in $\mathbb{R}\times\mathbb{R}^+$ with vanishing initial data. This was observed by Rosenbloom and Widder [16]. I saw it in [4, page 180].

Proof. Set $g(x,t) = e^x\cos(x+2t) + e^{-x}\cos(x-2t)$ and $a(y) = e^{-y^{\frac{4}{3}}} y \cos(\sqrt{3}y^{\frac{4}{3}})$. Hence

$$u(x,t) = \int_0^\infty a(y) g(xy, ty^2)\, dy.$$

First, we check $u(x,0) = \int_0^\infty a(y)(e^{xy}+e^{-xy})\cos(xy)\, dy \equiv 0$.

Note that

$$|e^{\pm xy \pm ixy}| = |\sum_0^\infty (\pm 1 \pm i)^n \frac{(xy)^n}{n!}| \le \sum_0^\infty \frac{2^n|xy|^n}{n!}$$

and hence

$$|(e^{xy}+e^{-xy})\cos(xy)| = |\sum_0^\infty \frac{a_n (xy)^n}{n!}| \le \sum_0^\infty \frac{2^{n+1}|xy|^n}{n!} = 2e^{2|xy|}$$

where $2a_n = (1+i)^n + (1-i)^n + (-1+i)^n + (-1-i)^n = 2^m 4(-1)^m$ if $n = 4m$ for some $m \in \mathbb{Z}_{\ge 0}$ and $2a_n = 0$ if n is not a multiple of 4.

Since $\int_0^\infty e^{-y^{\frac{4}{3}}} y e^{2|xy|}\, dy < \infty$ for each $x \in \mathbb{R}$, we have (from LDCT)

$$u(x,0) = \int_0^\infty a(y) \sum_{m=0}^\infty \frac{a_{4m}(xy)^{4m}}{(4m)!}\, dy = \sum_{m=0}^\infty \frac{a_{4m}x^{4m}}{(4m)!} \mathrm{Re} \int_0^\infty e^{-y^{\frac{4}{3}}(1-i\sqrt{3})} y^{4m+1}\, dy$$

The last integral vanishes for each $m \in \mathbb{Z}_{\ge 0}$ since for all Re $z > 0$

$$\int_0^\infty e^{-y^{\frac{4}{3}}z} y^{4m+1}\, dy = \frac{3}{4}\Gamma(3n+\frac{3}{2}) z^{-3n-\frac{3}{2}}$$

and the right hand side is purely imaginary when $z = 1 - i\sqrt{3}$. Hence we proved $u(\cdot, 0) \equiv 0$.

(This formula can be proved by residue theorem as problem 3(c) or just check it's true for $z \in \mathbb{R}^+$ and then use theory of analytical continuation to extend this formula to all Re $z > 0$.)

Second, we note that $u(0,t) = 2\int_0^\infty a(y)\cos(2ty^2)\,dy$ is not identically zero by the uniqueness theorem for Fourier-cosine transforms of functions $a(y) \in L^1(0,\infty)$.

Finally, we examine u is C^∞ and solves the heat equation. It's clear that $g(xy, ty^2)$ satisfies the heat equation for each fixed y. It remains to show we can interchange the order of integration in y and differentiations in x and in t.

There is a standard way by using the LDCT and mean-value theorem to achieve this (see e.g. [12, page 154]). Here we use that corollary and point out what the dominating function for derivatives is:

$$\left|\frac{\partial^{2n} g}{\partial x^{2n}}(x,t)\right| = \left|\frac{\partial^n g}{\partial t^n}(x,t)\right| \le 2^{n+1}\cosh x,$$

and then one consider $|x| \le R$ for each $R > 0$ (since differentiation is a local property) so that the further dominating integral is finite, that is,

$$2^{n+1}\int_0^\infty e^{-y^{\frac{4}{3}}} y^{2n+1}\cosh(Ry)\,dy < \infty.$$

□

5. **(Weak Maximum Principle for Heat Equation)**

Theorem 15. *Suppose that $u(x,t)$ is a real-valued solution of the heat equation in the upper half-plane, which is continuous on its closure. Let R denote the rectangle*

$$R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, 0 \le t \le c\}$$

and $\partial' R$ be the part of the boundary of R which consists of the two vertical sides and its base on the line $t = 0$. Then

$$\min_{(x,t)\in\partial' R} u(x,t) = \min_{(x,t)\in R} u(x,t) \quad \text{and} \quad \max_{(x,t)\in\partial' R} u(x,t) = \max_{(x,t)\in R} u(x,t).$$

The steps leading to a proof of this result are outlined below.

(a) Show that it suffices to prove that if $u \ge 0$ on $\partial' R$, then $u \ge 0$ in R.

(b) For $\epsilon > 0$, let $v(x,t) = u(x,t) + \epsilon t$. Then, v has a minimum on R, say at (x_1, t_1). Show that $x_1 = a$ or b, or else $t_1 = 0$. To do so, suppose on the contrary that $a < x_1 < b$ and $0 < t_1 \le c$, and prove that $v_{xx}(x_1,t_1) - v_t(x_1,t_1) \le -\epsilon$. However, show also that the left-hand side must be non-negative.

(c) Deduce from (b) that there is some $t_1 \in [0,c]$ such that $u(x,t) \ge \epsilon(t_1 - t)$ for any $(x,t) \in R$ and let $\epsilon \to 0$.

Remark 16. How about the boundary condition is changed to zero Neumann condition. A more precise statement is in my solution file to Evans' PDE Chapter 7.

Proof. (a) Due to this linear PDE has no zero-order term, we can consider $v := u - \min_{(x,t)\in\partial' R} u(x,t)$ and $V = \max_{(x,t)\in\partial' R} u(x,t) - u$ separately. Note that those extreme values are finite by the compactness of $\partial' R$ and continuity of u up to the closure. This is why we need the space domain to be bounded.

(b) Just note at the minimum point, $v_t \leq 0$ and $v_{xx} \geq 0$.

(c) From (b), there is $(x_1, t_1) \in \partial' R$ such that

$$u(x,t) + \epsilon t = v(x,t) \geq v(x_1,t_1) = u(x_1,t_1) + \epsilon t_1 \geq \epsilon t_1.$$

□

Remark 17. As explained in (a), one can replace the space domain R by any bounded domain Ω in any dimension $\mathbb{R}^d$. The proof is almost identical.

Remark 18. One can check [17, Chapter 9.B] and [14, Chapter 2] for strong maximum principle, that is, to exclude the possibility of existence of interior extrema.

6. **The examples in Problem 4 are optimal in the sense of the following uniqueness theorem due to Tychonoff.**

Theorem 19. *Suppose $u(x,t)$ satisfies the following conditions:*

(i) it is continuous on $\mathbb{R}^d \times [0,T]$ and solves the heat equation in $\mathbb{R}^d \times (0,T)$ with zero initial condition. (ii) $|u(x,t)| \leq M e^{a|x|^2}$ for some M, a, and all $x \in \mathbb{R}^d, 0 \leq t < T$.

Then $u \equiv 0$.

Remark 20. This growth condition (ii) can be relaxed to $|u(x,t)| \leq C \exp\left((\frac{a}{t})^\alpha + a|x|^2\right)$ for some $C > 0, a > 0$ and $0 < \alpha < 1$. It is proved in [2]. It's also shown the condition $0 < \alpha < 1$ is optimal, one can't relax to $\alpha = 1$ by constructing a non-trivial function u with $|u(x,t)| \leq M \exp(\frac{a}{t})$ for some constants $M, a > 0$ (note that in contrast to Problem 4 and $4\frac{1}{2}$, this u is bounded in x for fixed t).

Proof. One can find some discussions and proof in [4, Section 5.4], [5, Section 2.3] or [11, Section 7.1(b)]. □

$6\frac{1}{2}$. **There is another well-known uniqueness theorem for heat equation in $\mathbb{R}^d \times \mathbb{R}_+$ due to David Widder:**

Theorem 21. *Let $T > 0$ and $S_T := \mathbb{R}^d \times (0, T]$. Suppose $u \in C^{2,1}(S_T) := \{u_t, u_{x_i,x_j} \in C(S_T)\}$ is a nonnegative solution to heat equation in S_T and $u(\cdot, t) \to 0$ in $L^1_{loc}(\mathbb{R}^d)$ as $t \to 0^+$. Then $u \equiv 0$ in S_T.*

Remark 22. This theorem is very intuitive for physicists and useful in thermodynamics. I learned the proof from [4, Section 5.14] which is suitable for arbitrary dimension in contrast to Widder's original proof [11, Section 7.1(d)] only suitable for one dimensional case. Also note that Tychonoff's function defined in Problem 4 is sign-changing.

Proof. First, there is a more general result which will be proved in the end:

Theorem 23. *Let u be a solution for heat equation and satisfies all the regularity (smoothness) hypotheses and initial data describe above. If u satisfies*

$$\sup_{s\in[0,t)} \int_{\mathbb{R}^d} |u(y, s)|\mathcal{H}_{t-s}(x - y)\, dy := M_{x,t} < \infty \quad \forall (x, t) \in S_T,$$

then $u \equiv 0$ in S_T.

Using Theorem 23, Widder's theorem is proved provided that for each $(x_0, t_0) \in S_T$

$$\int_{\mathbb{R}^d} |u(y, s)|\mathcal{H}_{t_0-s}(x_0 - y)\, dy = \int_{\mathbb{R}^d} u(y, s)\mathcal{H}_{t_0-s}(x_0 - y)\, dy \le u(x_0, t_0) \quad \forall s \in [0, t_0). \tag{6}$$

Note that we may assume $T = 1$, $(x_0, t_0) = (0, 1)$ and $s = 0$ in (6) by introducing the change of variables

$$\tau = \frac{t - s}{t_0 - s}, \quad \eta = \frac{y - x_0}{\sqrt{t_0 - s}}$$

and the function

$$U(\eta, \tau) = u(x_0 + \sqrt{t_0 - s}\eta, s + (t_0 - s)\tau).$$

To prove (6), one fix $\rho > 0$ and consider the Cauchy problem that $v \in \mathcal{H}(\overline{S_1})$ solves heat equation in $\overline{S_1}$ and satisfies the initial condition $v(x, 0) = \zeta(x)u(x, 0)\chi_{B_{2\rho}(0)}(x)$, where χ_E is the characteristic function on E and $x \mapsto \zeta(x) \in C_0^\infty(B_{2\rho})$ is nonnegative and equals to 1 on B_ρ (it's called cut-off function or bump function, see Exercise 4). Since the initial datum is compactly supported in $B_{2\rho}$, the unique bounded solution v is given by

$$v(x, t) = \int_{|y|<2\rho} \zeta(y)u(y, 0)\mathcal{H}_t(x - y)\, dy.$$

(Uniqueness comes from energy method or weak maximum principle.)

Lemma 24. *$u \ge v$ in $\overline{S_1}$.*

By Lemma 24, we have

$$u(0,1) \geq v(0,1) = \int_{|y|<2\rho} \zeta(y) u(y,0) \mathcal{H}_1(-y)\, dy \geq \int_{|y|<\rho} u(y,0)\mathcal{H}_1(-y)$$

Then we prove (6) by letting $\rho \to \infty$ and hence complete the proof of Widder's theorem. □

Proof of Lemma 24. Let n_0 be a positive integer larger than 2ρ, and for $n \geq n_0$, consider the sequence of homogeneous Dirichlet problems (see Remark 26 for the existence.)

$$\begin{cases} \partial_t v_n - \Delta v_n = 0 \text{ in } B_n \times (0,1) \\ v_n = 0 \text{ on } \partial B_n \times (0,1) \\ v_n(x,0) = \zeta(x) u(x,0) \chi_{B_{2\rho}(0)}(x). \end{cases} \tag{7}$$

We regard the functions v_n as defined in the whole of S_1 by zero extension. By weak maximum principle (see Remark 17 with $\Omega = B_n$) and nonnegativity of u

$$0 \leq v_n \leq v_{n+1} \leq \|u(\cdot,0)\|_{L^\infty(B_{2\rho})} \text{ and } v_n \leq u \tag{8}$$

for all $n \geq n_0$. By the second of these, the proof of the lemma reduces to showing that the increasing sequence v_n converges to v. We decompose this into two steps:

Step 1: Use Arezla-Ascoli theorem to conclude, on every compact subset of S_1, $v_n \to w$ in C^∞.

Hence w solves the heat equation in S_1.

Step 2: Show $w(\cdot,t) \to \zeta(\cdot)u(\cdot,0)$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ as $t \to 0$.

Assume these steps are accomplished for the moment. Since w is bounded by $\|u(\cdot,0)\|_{L^\infty(B_{2\rho})}$, $w = v$ by the uniqueness theorem of bounded solutions of the Cauchy problem (Problem 6). Hence we complete the proof for Lemma 24.

We first assume Step 1 is true and prove Step 2 by testing the weak formulation of (7) with Heaviside function which is 1 on $\mathbb{R}^+$ and -1 on $\mathbb{R}^-$. However, this Heaviside function is not continuous, we need to approximate it by continuous functions so that one can use integration by parts.

Given $B_R \subset \mathbb{R}^d$ with $R \geq 2\rho$. First, we rewrite (7) as

$$\begin{cases} f_n(x,t) = v_n(x,t) - \zeta(x)u(x,0) \\ \partial_t f_n - \Delta f_n = \Delta[\zeta(\cdot)u(\cdot,0)] \text{ in } S_1 \\ f_n = 0 \text{ on } \partial B_n \times (0,1) \\ f_n(x,0) = 0. \end{cases} \tag{9}$$

For $\delta > 0$, let

$$h_\delta(u) = \begin{cases} [left]\ 1, & \text{if } u > \delta \\ \frac{u}{\delta}, & \text{if } |u| \le \delta \\ -1, & \text{if } u < -\delta. \end{cases} \tag{10}$$

(Think this as the approximation to the Heaviside function (as $\delta \to 0$) and note that although there is a corner at $\pm\delta$ we realize $h'_\delta(s) = \frac{1}{\delta}\chi_{|s|\le\delta}$ as the weak derivative of h_δ).

Multiply (9) by $h_\delta(f_n)$ and integrate over $B_n \times (0,t)$ for $t \in (0,1)$ with a well-known chain rule for weak derivative (see, e.g. [8, Theorem 7.8]) to obtain

$$\int_{B_n\times\{t\}} \Big(\int_0^{f_n} h_\delta(\xi)\,d\xi\Big)\,dy + \frac{1}{\delta}\int_0^t\int_{B_n} |Df_n|^2\chi_{\{|f_n|<\delta\}}(y)\,dyds = \int_0^t\int_{B_{2\rho}} \Delta[\zeta(y)u(y,0)]h_\delta(f_n)\,dyds.$$

Discard the second term on the left-hand side, which is nonnegative, and let $\delta \to 0$ to get (denote the Lebesgue measure of $\Omega \subset \mathbb{R}^d$ by $|\Omega|$) for $n \ge R$

$$\int_{B_R} |v_n(y,t) - \zeta(y)u(y,0)|\,dy \le \int_{B_n} |v_n(y,t) - \zeta(y)u(y,0)|\,dy \le t|B_{2\rho}|\|\Delta[\zeta u(\cdot,0)]\|_{L^\infty(B_{2\rho})}.$$

Letting $n \to \infty$, we have

$$\|w(\cdot,t) - \zeta(\cdot)u(\cdot,0)\|_{L^1(B_R)} \le t|B_{2\rho}|\|\Delta[\zeta u(\cdot,0)]\|_{L^\infty(B_{2\rho})} \to 0 \ \text{ as } \ t \to 0^+.$$

Since B_R is given, we complete the proof for Step 2.

To prove Step 1, we need the following interior gradient estimate whose proof can be found in [4, Section 5.12 and 5.12c].

Proposition 25. *Let u be a solution of the heat equation that satisfies all the regularity hypotheses as Widder's theorem. There exist constants γ and C depending only on space dimension d such that for every parabolic box $(x_0,t_0)+Q_{4\rho} := \Big\{(x,t) : |x-x_0| < 4\rho, t \in [t_0-(4\rho)^2, t_0]\Big\} \subset S_T$*

$$\sup_{(x_0,t_0)+Q_\rho} |D_x^\alpha u| \le \gamma\frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}}\fint_{(x_0,t_0)+Q_{4\rho}} |u|\,dy\,ds \tag{11}$$

for all multi-indices α, where $\fint_\Omega |u|\,dy\,ds = \frac{1}{|\Omega|}\int_\Omega |u|\,dy\,ds$ and $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^{d+1}$. Moreover

$$\sup_{(x_0,t_0)+Q_\rho} |\frac{\partial^k u}{\partial t^k}| \le \gamma\frac{C^{2k}(2k)!}{\rho^{2k}}\fint_{(x_0,t_0)+Q_{4\rho}} |u|\,dy\,ds$$

for all positive integers k.

Consider compact subsets of the type $K = \overline{B_R} \times [\epsilon, 1-\epsilon]$ for $\epsilon \in (0,\frac{1}{2})$ and $R \ge 2\rho$. By the uniform upper bound in (8) and the above interior gradient estimates (applied to v_n), we know for every multi-index α and every $k \in \mathbb{N}$, there exists a constant $M = M(d, \epsilon, R, |\alpha|, k)$ independent of n such that

$$\|D^\alpha v_n\|_{L^\infty(K)} + \|\frac{\partial^k}{\partial t^k} v_n\|_{L^\infty(K)} \leq M\|u(\cdot, 0)\|_{L^\infty(B_{2\rho})} \ \ \forall n \geq 2R.$$

By using the Ascoli-Arezla theorem with a diagonal process we have, up to a subsequence, $v_n \to w$ in $C^\infty(K)$. Using a further diagonal process (indexed by the domains) we have, up to a further subsequence, $v_n \to w$ in C^∞ on every compact subsets of S_1. So we complete the proof for Step 1.

□

Remark 26. The existence of (7) is a consequence of the existence of eigenpairs $\{(\lambda_k^n, w_k^n)\}_{k=0}^\infty$ for $-\Delta$ on each ball B_n, that is, $-\Delta w_k^n = \lambda_k^n w_k^n$ in B_n and $w_k^n = 0$ on ∂B_n. The essential tools for the proof are Fredholm alternatives and Sobolev compact embedding. Check [4, Chapter 4 and Section 5.10], [5, Section 6.2, 6.5 and 7.4] or [13, Section III.5] for the existence of eigenpairs, orthogonality of $\{w_k^n\}_k$, and convergence of the solution $v_n(x,t) = \sum_{k=0}^\infty c_k^n e^{-\lambda_k^n t} w_k^n(x)$ where c_k^n are chosen so that $v_n(x, 0) = \sum_{k=0}^\infty c_k^n w_k^n(x)$. Another method based on maximum principle can be found in [14, Section 3.3-3.4] (Perron's method).

Proof of Theorem 23. Let $(x,t) \in S_T$ be arbitrary but fixed. For $\rho > 2|x|$, we let $y \mapsto \zeta(t) \in C_0^2(B_{2\rho})$ be a non-negative cut-off (bump) function satisfying $\zeta = 1$ in B_ρ, $|D\zeta| \leq \frac{1}{\rho}$ and $|D^2\zeta| \leq (\frac{2}{\rho})^2$.

For $\delta > 0$, let $h_\delta(\cdot)$ be the approximation to the Heaviside function introduced in (10). Multiply the heat equation by $h_\delta(u(y))\zeta(y)\mathcal{H}_{t-s}(x-y)$, and integrate by parts in $dy\, ds$ over the cylindrical domain $B_{2\rho} \times (\tau, t-\epsilon)$ for $0 < \tau < t - \epsilon$ and $0 < \epsilon < t$. This gives

$$\begin{aligned}
&\int_{B_{2\rho}\times\{t-\epsilon\}} \Big(\int_0^u h_\delta(\xi)\, d\xi\Big)\mathcal{H}_\epsilon(x-y)\zeta(y)\, dy + \frac{1}{\delta}\int_\tau^{t-\epsilon}\int_{B_{2\rho}} |Du|^2 \mathcal{H}_{t-s}(x-y)\chi_{\{|u|<\delta\}}(y)\zeta(y)\, dy\, ds \\
&= \int_{B_{2\rho}\times\{\tau\}} \Big(\int_0^u h_\delta(\xi)\, d\xi\Big)\mathcal{H}_{t-\tau}(x-y)\zeta(y)\, dy \\
&+ \int_\tau^{t-\epsilon} \Big(\int_0^u h_\delta(\xi)\, d\xi\Big)\Big(\mathcal{H}_{t-s}(x-\cdot)\Delta\zeta - 2(D\mathcal{H}_{t-s})(x-\cdot)D\zeta\Big)(y)\, dy\, ds.
\end{aligned}$$

As $\tau \to 0$, the first integral on the right-hand side tends to 0 since it's majorized by

$$\text{const.} \int_{B_{2\rho}} |u(y,\tau)|\, dy \to 0 \text{ as } \tau \to 0 \ \text{ by the hypothesis.}$$

Discard the second term on the left-hand side since it's nonnegative and let first $\delta \to 0$ and then $\tau \to 0$ to obtain

$$\int_{B_\rho} |u(y,t-\epsilon)|\mathcal{H}_\epsilon(x-y)\,dy \le \int_{B_{2\rho}} |u(y,t-\epsilon)|\mathcal{H}_\epsilon(x-y)\zeta(y)\,dy$$
$$\le \frac{2}{\rho^2}\int_0^{t-\epsilon}\int_{B_{2\rho}} |u(y,s)|\mathcal{H}_{t-s}(x-y)\,dy\,ds + \frac{2}{\rho}\int_0^{t-\epsilon}\int_{B_{2\rho}\setminus B_\rho} |u(y,s)||D_y\mathcal{H}_{t-s}(x-y)|\,dy\,ds.$$

By the hypothesis, it's obvious that the first term of right-hand side of this inequality tends to 0 as $\rho \to \infty$. For the second term, one notes that it's majorized by (up to a constant multiple)

$$\frac{1}{\rho}\int_0^{t-\epsilon}\int_{\rho<|y|<2\rho} |u(y,s)|\mathcal{H}_{t-s}(x-y)\frac{|x-y|}{t-s}\,dy\,ds \le \frac{4}{\epsilon}\int_0^{t-\epsilon}\int_{\rho<|y|<2\rho} |u(y,s)|\mathcal{H}_{t-s}(x-y)\,dy\,ds,$$

which tends to 0 as $\rho \to \infty$ by the hypothesis.

Hence for all $r > 2|x|$ and $\epsilon \in (0,t)$

$$\int_{B_r} |u(y,t-\epsilon)|\mathcal{H}_\epsilon(x-y)\,dy = 0.$$

Finally, letting $\epsilon \to 0$, we expect the left-hand side tends to $|u(x,t)|$ and hence the proof is completed. This interpretation can be proved as follows:

Fix $r > 2|x|$. Write

$$\begin{aligned}\int_{B_r} |u(y,t-\epsilon)|\mathcal{H}_\epsilon(x-y) &= \int_{|y|<r, |y-x|>\epsilon^{\frac14}} |u(y,t-\epsilon)|(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\epsilon}}\,dy \\ &\quad + \int_{|y-x|\le\epsilon^{\frac14}} |u(y,t-\epsilon)|(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\epsilon}}\,dy \\ &=: I_1^\epsilon + I_2^\epsilon.\end{aligned}$$

Since the continuous function u is uniformly bounded on $\overline{B_r} \times [\frac{t}{2}, t]$, we have

$$I_1^\epsilon \le r^d M(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{\epsilon^{\frac12}}{4\epsilon}} \to 0 \text{ as } \epsilon \to 0.$$

We also note that

$$\begin{aligned}I_2^\epsilon &= \int_{|y-x|\le\epsilon^{\frac14}} \Big(|u(y,t-\epsilon)| - |u(x,t)|\Big)(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\epsilon}}\,dy + |u(x,t)|\int_{|y-x|\le\epsilon^{\frac14}} (4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\epsilon}}\,dy \\ &= \Big[o(\epsilon) + |u(x,t)|\Big]\Big(\int_{\mathbb{R}^d}(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|z|^2}{4\epsilon}}\,dz - \int_{|z|>\epsilon^{\frac14}}(4\pi\epsilon)^{-\frac{d}{2}}e^{-\frac{|z|^2}{4\epsilon}}\,dz\Big) \\ &= \Big[o(\epsilon) + |u(x,t)|\Big]\Big(1 - \int_{|z|>\frac{\epsilon^{\frac14}}{2\sqrt{\epsilon}}} \pi^{-\frac{d}{2}}e^{-|w|^2}\,dw\Big)\end{aligned}$$

where $o(\epsilon) \to 0$ as $\epsilon \to 0$ (due to the continuity of u). Note the last integral tends to 0 as $\epsilon \to 0$ too. So we are done. $\square$

7. **(Combined with Problem 4.11 in Book III)**

The Hermite function $h_k(x)$ are defined by the generating identity

$$\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} = e^{-(x^2/2-2xt+t^2)}$$

(a) Show that an alternate definition is given by the Rodrigues' formula

$$h_k(x) = (-1)^k e^{x^2/2}\Big(\frac{d}{dx}\Big)^k e^{-x^2}.$$

They also satisfy the "creation" and "annihilation" identities $(x - \frac{d}{dx})h_k(x) = h_{k+1}$ and $(x + \frac{d}{dx})h_k(x) = 2kh_{k-1}(x)$ for $k \geq 0$ where $h_{-1} = 0$.

Conclude from the above expression that each h_k is of the form $P_k(x)e^{-x^2/2}$, where P_k is a polynomial of degree k. In particular, the Hermite functions belong to the Schwartz space and $h_0(x) = e^{-x^2/2}, h_1(x) = 2xe^{-x^2/2}$. By (d), one can also show the linear span of $P_0, \cdots, P_m$ is the set of all polynomials of degree $\leq m$. (Folland [7, Exercise 8.23(f)]).

(b) Prove that the family $\{h_k\}_{k=0}^{\infty}$ is complete in L^2, that is if $f \in L^2(\mathbb{R})$, and for all $k \geq 0$,

$$(f, h_k) = \int_{\mathbb{R}} f(x)h_k(x) = 0,$$

then $f \equiv 0$. (An equivalent formulation of this fact is that the set of Hermite polynomials $\{P_k\}_0^{\infty}$ forms an orthogonal basis for $L^2(\mathbb{R}, e^{-x^2})$).

(c) Define $h_k^*(x) = h_k((2\pi)^{1/2}x)$. Then

$$\widehat{h_k^*}(\xi) = (-i)^k h_k^*(x).$$

Therefore, each h_k^* is an eigenfunction for the Fourier transform.

(d) Show that h_k is an eigenfunction for the operator $L = -d^2/dx^2 + x^2$, and in fact, prove that

$$Lh_k = (2k+1)h_k$$

In particular, we conclude that the functions h_k are mutually orthogonal in $L^2(\mathbb{R})$.

(e) Show that $\int_{\mathbb{R}}[h_k(x)]^2\,dx = \pi^{1/2}2^k k!$.

(f) Denote the normalized h_k by H_k, that is $\|H_k\|_{L^2} = 1$. Suppose that $K(x,y) = \sum_{k=0}^{\infty}\frac{H_k(x)H_k(y)}{\lambda_k}$, and also $F(x) = T(f)(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy$. Then T is a symmetric Hilbert-Schmidt operator, and if $f \sim \sum_{k=0}^{\infty} a_k H_k$, then

$$F \sim \sum_{k=0}^{\infty}\frac{a_k}{\lambda_k}H_k.$$

(g) One can show on the basis of (a) and (b) that whenever $f \in L^2(\mathbb{R})$, not only is $F \in L^2(\mathbb{R})$, but also $x^2F(x) \in L^2(\mathbb{R})$. Moreover, F can be corrected on a set of measure zero, so is C^1, F' is absolutely continuous, and $F'' \in L^2(\mathbb{R})$. Finally, the operator T is the inverse of L in the sense that for every $f \in L^2(\mathbb{R})$,

$$LT(f) = LF = -F'' + x^2F = f.$$

Proof. (a) Note that $e^{-(x^2/2-2xt+t^2)} = e^{x^2/2}e^{-(t-x)^2}$ and one can prove by induction that

$$[\left(\frac{d}{dz}\right)^k e^{-z^2}](x) = P_k(x)e^{-x^2}$$

where p_k is a polynomial of degree k which is even if k is even and is odd if k is odd.

From the definition, we know for each $x \in \mathbb{R}$,

$$\begin{aligned} h_k(x) &= e^{x^2/2}\left(\frac{d}{dt}\right)^k e^{-(t-x)^2}|_{t=0} = e^{x^2/2}P_k(-x)e^{-x^2} = (-1)^k e^{x^2/2}P_k(x)e^{-x^2} \\ &= (-1)^k e^{x^2/2}[\left(\frac{d}{dz}\right)^k e^{-z^2}](x). \end{aligned}$$

The "creation" identity can be proved by this formula easily, we omit it. For the "annihilation" identity, one differentiate $e^{x^2/2}h_k(x) = (-1)^k e^{x^2}[\left(\frac{d}{dz}\right)^k e^{-z^2}](x)$ and apply the binomial theorem to the right-hand side.

To show $P_0, \cdots, P_m$ spans the set of polynomials of degree $\leq m$, it suffices to check $P_0, \cdots, P_m$ are linearly independent. This can be proved by the orthogonality of h_k easily, we omit it. (Another easy proof is by noting that each P_k is of exact degree k, cf. Folland [6, Lemma 6.1].)

(b) There are two equivalent ways (Folland and Stein) to prove this result, their principal idea both rely on the Fourier transform. (We present both of them for completeness).

(Method I) Given $t \in \mathbb{R}$, note that $\sum_{k=0}^K \frac{(2xt)^k}{k!}$ converges pointwisely to e^{2xt} as $K \to \infty$, $|\sum_{k=0}^K \frac{(2xt)^k}{k!}| \leq \sum_{k=0}^K \frac{|2xt|^k}{k!} \leq e^{|2xt|}$ for all $K \in \mathbb{N}$ and $f(x)e^{-x^2/2}e^{|2xt|} \in L_x^1(\mathbb{R})$. Hence by LDCT and (a)

$$\begin{aligned} \int_{\mathbb{R}} f(x)e^{-x^2/2+2xt}\,dx &= \int_{\mathbb{R}} f(x)e^{-x^2/2} \lim_{K\to\infty} \sum_{k=0}^K \frac{(2xt)^k}{k!}\,dx = \lim_{K\to\infty} \int_{\mathbb{R}} f(x)e^{-x^2/2} \sum_{k=0}^K \frac{(2xt)^k}{k!}\,dx \\ &= \lim_{K\to\infty} \sum_{k=0}^K \int_{\mathbb{R}} f(x)e^{-x^2/2}c_k(t)P_k(x)\,dx = \lim_{K\to\infty} \sum_{k=0}^K c_k(t) \int_{\mathbb{R}} f(x)h_k(x)\,dx = 0 \end{aligned}$$

The desired result follows from Exercise 8.

(Method II) Follow the hint of Folland [7, Exercise 8.23(g)], it is enough to show $\widehat{g} \equiv 0$, where $g(x) = f(x)e^{-x^2/2}$. Given $\xi \in \mathbb{R}$, note that $\sum_{k=0}^{K} \frac{(-2\pi i x\xi)^k}{k!}$ converges pointwisely to $e^{-2\pi i x\xi}$ as $K \to \infty$, $|\sum_{k=0}^{K} \frac{(-2\pi i x\xi)^k}{k!}| \leq \sum_{k=0}^{K} \frac{|-2\pi i x\xi|^k}{k!} \leq e^{2\pi|x\xi|}$ for all $K \in \mathbb{N}$ and $f(x)e^{-x^2/2}e^{2\pi|x\xi|} \in L^1_x(\mathbb{R})$. Hence by LDCT and (a),

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}} g(x)e^{-2\pi i x\xi}\,dx = \int_{\mathbb{R}} f(x)e^{-x^2/2} \lim_{K\to\infty} \sum_{k=0}^{K} \frac{(-2\pi i x\xi)^k}{k!}\,dx \\
&= \lim_{K\to\infty} \int_{\mathbb{R}} f(x)e^{-x^2/2} \sum_{k=0}^{K} \frac{(-2\pi i x\xi)^k}{k!}\,dx = \lim_{K\to\infty} \sum_{k=0}^{K} c_k(\xi) \int_{\mathbb{R}} f(x)e^{-x^2/2} P_k(x)\,dx \\
&= \lim_{K\to\infty} \sum_{k=0}^{K} c_k(\xi) \int_{\mathbb{R}} f(x)h_k(x)\,dx = 0.
\end{aligned}$$

(c) Using the induction argument and taking Fourier transform for both sides of creation (or annihilation) identity, we have $(-2\pi i)^{-1}\frac{d\widehat{h_{k-1}}}{d\xi} - 2\pi i\xi\widehat{h_{k-1}} = \widehat{h_k}(\xi)$. Note that the induction hypothesis is equivalent to $\frac{1}{\sqrt{2\pi}}\widehat{h_{k-1}}(\frac{\xi}{\sqrt{2\pi}}) = (-i)^{k-1}h_{k-1}(\sqrt{2\pi}\xi)$. Combining these, we have

$$\begin{aligned}
\widehat{h_k^*}(\xi) &= \frac{1}{\sqrt{2\pi}}\widehat{h_k}(\frac{\xi}{\sqrt{2\pi}}) = \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{-2\pi i}\frac{d\widehat{h_{k-1}}}{d\xi}(\frac{\xi}{\sqrt{2\pi}}) - \sqrt{2\pi}i\xi\widehat{h_{k-1}}(\frac{\xi}{\sqrt{2\pi}})\Big) \\
&= \frac{1}{-2\pi i}(-i)^{k-1}2\pi\frac{dh_{k-1}}{d\xi}(\sqrt{2\pi}\xi) - i\sqrt{2\pi}\xi(-i)^{k-1}h_{k-1}(\sqrt{2\pi}\xi) \\
&= (-i)^k\Big[-\frac{dh_{k-1}}{d\xi}(\sqrt{2\pi}\xi) + \sqrt{2\pi}\xi h_{k-1}(\sqrt{2\pi}\xi)\Big] \\
&= (-i)^k h_k(\sqrt{2\pi\xi}) = (-i)^k h_k^*(\xi).
\end{aligned}$$

Based on this, it remains to check the desired result holds for $k = 0$, which is well known result for Gaussian.

(d) This is proved by the creation and annihilation identities.

(e) Using induction argument and (d), we have

$$\begin{aligned}
\int_{\mathbb{R}} [h_k(x)]^2\,dx &= \frac{1}{2k}\int_{\mathbb{R}} h_k(x)(L-I)h_k(x) = \frac{1}{2k}\int_{\mathbb{R}} h_k(x)\Big[(x - \frac{d}{dx})(x + \frac{d}{dx})h_k\Big](x)\,dx \\
&= \frac{1}{2k}\int_{\mathbb{R}} \Big[(x + \frac{d}{dx})h_k(x)\Big]^2 + \frac{1}{2k}\lim_{|x|\to\pm\infty} h_k(x)(xh_k(x) + \frac{dh_k}{dx}(x)) \\
&= \frac{(2k)^2}{2k}\int_{\mathbb{R}} h_{k-1}(x)^2\,dx + \lim_{|x|\to\pm\infty} h_k(x)h_{k-1}(x) \\
&= 2k\pi^{1/2}2^{k-1}(k-1)! + 0 = \pi^{1/2}2^k k!,
\end{aligned}$$

where the boundary terms vanish since they are of products of polynomials and $e^{-x^2/2}$.

(f)

(g) □

8. **To refine the results in Chapter 4, and to prove that**

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{2\pi i 2^n x}$$

is nowhere differentiable even in the case $\alpha = 1$, we need to consider a variant of the delayed means Δ_N, which in turn will be analyzed by the Poisson summation formula. (A simplified proof can be found in Jones [12, Section 16.H]).

(a) Fix an indefinitely differentiable function Φ satisfying $\Phi \equiv 1$ on $B_1(0)$ and vanishes outside $B_2(0)$. By the Fourier inversion formula, there exists $\varphi \in S$ so that $\widehat{\varphi}(\xi) = \Phi(\xi)$. Let $\varphi_N(x) = N\varphi(Nx)$ so that $\widehat{\varphi_N}(\xi) = \Phi(\xi/N)$. Finally, set

$$\widetilde{\Delta}_N(x) = \sum_{n=-\infty}^{\infty} \varphi_N(x+n).$$

Observe by the Poisson summation formula that $\widetilde{\Delta}_N(x) = \sum_{-\infty}^{\infty} \Phi(n/N)e^{2\pi i n x}$, thus $\widetilde{\Delta}_N$ is a trigonometric polynomial of degree $\leq 2N$, with terms whose coefficients are 1 when $|n| \leq N$. Let

$$\widetilde{\Delta}_N(f) = f * \widetilde{\Delta}_N$$

Note that

$$S_N(f_\alpha) = \widetilde{\Delta}_{N'}(f_\alpha)$$

where N' is the largest integer of the form 2^k with $N' \leq N$.

(b) If we set $\widetilde{\Delta}_N(x) = \varphi_N(x) + E_N(x)$ where

$$E_N(x) = \sum_{|n|\geq 1} \varphi_N(x+n),$$

then one sees that:

(i) $\sup_{|x|\leq 1/2} |E_N'(x)| \to 0$ as $N \to \infty$. (ii) $|\widetilde{\Delta}_N'(x)| \leq cN^2$.

(iii) $|\widetilde{\Delta}_N'(x)| \leq c/(N|x|^3)$ for $|x| \leq 1/2$.

Moreover, $\int_{|x|\leq 1/2} \widetilde{\Delta}_N'(x)\,dx = 0$ and $-\int_{|x|\leq 1/2} x\widetilde{\Delta}_N'(x)\,dx \to 1$ as $N \to \infty$.

(c) The above estimates imply that if $f'(x_0)$ exists, then

$$(f * \widetilde{\Delta}_N')(x_0 + h_N) \to f'(x_0) \text{ as } N \to \infty,$$

whenever $|h_N| \leq C/N$. Then, conclude that both the real and imaginary parts of f_1 are nowhere differentiable, as in the proof given in Section 3, Chapter 4.

(d) Let $a > 1, |b| < 1$. Also prove the Weierstrauss function $W(x) = \sum_{n=1}^{\infty} b^n \cos(a^n x)$ is nowhere differentiable if $a|b| \geq 1$.

Remark 27. One can also check another characterization in [12, Section 16.H] that is different from the methods of delay means.

Proof. (a)

(b)

(c)

(d) □

Remark 28. For the differentiablity of Riemann function $\sum_{n=1}^{\infty} \frac{\sin \pi n^2 x}{n^2}$, see Jarnicki and Pflug [10, Chapter 13].

References

[1] Butzer, P. L., G. Schmeisser, and R. L. Stens: An introduction to sampling analysis. Nonuniform sampling. Springer, Boston, MA, 2001. 17-121.

[2] Chung, Soon-Yeong. "Uniqueness in the Cauchy problem for the heat equation." Proceedings of the Edinburgh Mathematical Society 42.3 (1999): 455-468.

[3] Daubechies, Ingrid: Ten lectures on wavelets. Vol. 61. SIAM, 1992.

[4] DiBenedetto, E: Partial Differential Equations. 2nd ed., Springer, 2010.

[5] Evans, L. C.: Partial Differential Equations, 2nd Edition. AMS, Providence, RI, 2010.

[6] Folland, Gerald B: Fourier analysis and its applications. Vol. 4. American Mathematical Soc., 1992.

[7] Folland, Gerald B: Real analysis: modern techniques and their applications. 2nd ed., John Wiley and Sons, 1999.

[8] Gilbarg, David, and Neil S. Trudinger: Elliptic partial differential equations of second order. Springer, 2001.

[9] Grafakos, Loukas: Classical Fourier Analysis. 3rd ed., Springer, 2014.

[10] Jarnicki, Marek, and Peter Pflug: Continuous Nowhere Differentiable Functions. Springer Monographs in Mathematics, New York (2015).

[11] John, Fritz: Partial Differential Equations. 4th ed., Springer, 1991.

[12] Jones, Frank: Lebesgue Integration on Euclidean Space. Revised Edition. Jones & Bartlett Learning, 2001.

[13] Ladyzhenskaia, Olga Aleksandrovna, Vsevolod Alekseevich Solonnikov, and Nina N. Ural'tseva. Linear and quasi-linear equations of parabolic type. Vol. 23. American Mathematical Soc., 1968.

[14] Lieberman, Gary M. Second order parabolic differential equations. Revised Edition. World scientific, 2005.

[15] Linares, Felipe, and Gustavo Ponce: Introduction to nonlinear dispersive equations. 2nd ed., Springer, 2015.

[16] Rosenbloom, P. C., and Widder, D.: A temperature function which vanishes initially. The American Mathematical Monthly 65.8 (1958): 607-609.

[17] Smoller, Joel. Shock waves and reaction-diffusion equations. 2nd Edition. Vol. 258. Springer, 1994.

[18] Stein, Elias M., and Guido L. Weiss: Introduction to Fourier Analysis on Euclidean Spaces. Vol. 1. Princeton University Press, 1971.