Flow Matching:

Generative Model

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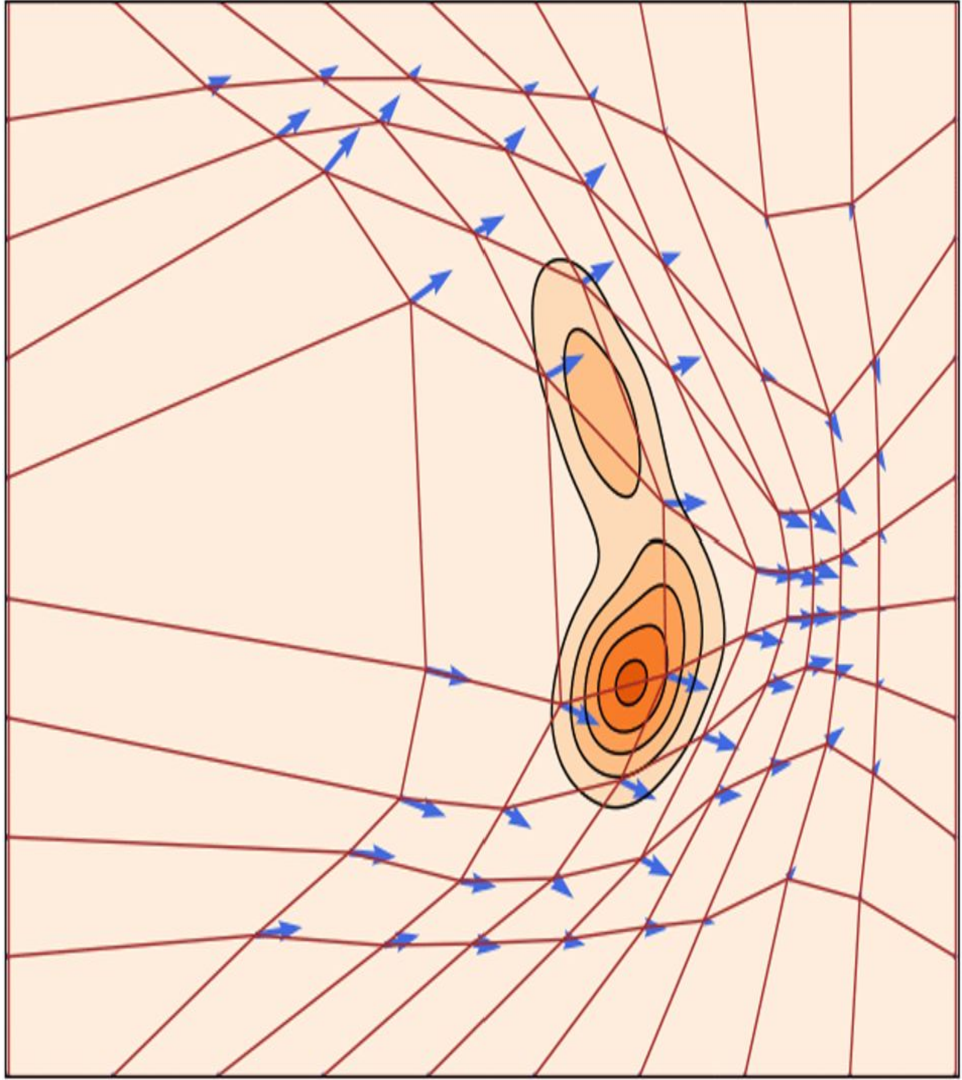


What’s Covered

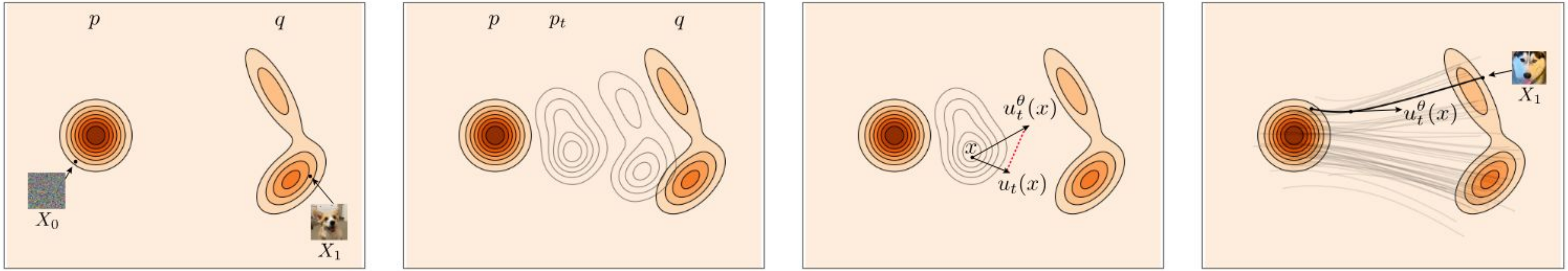
- Background
- Objective
- Key Concepts
- The Continuity Equation
- The Marginalization Trick
- Examples

Background

Flow Matching is a framework for generative modeling that is used in various fields and applications including generation of images, videos, speech, audio, proteins, and robotics. Figure (Lipman et al., 2024, p.1)

What is our Objective

- How can we propagate probability densities through a learned velocity field?
- Why this matters: Generative modeling, efficient sampling
- Goal: Transform a source distribution p to a target distribution q using a flow



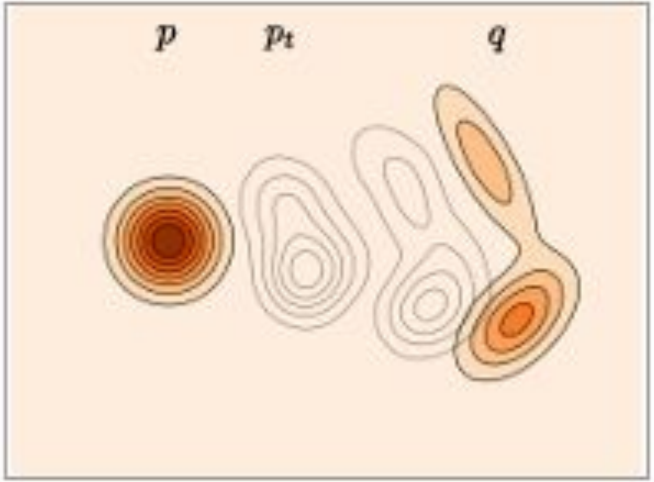
(a) Data. **(b)** Path design. **(c)** Training. **(d)** Sampling.

Figure (Lipman et al., 2024, p.3)

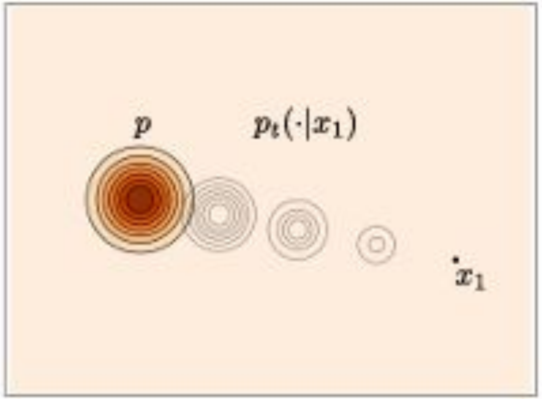
Key Concepts

- Probability Paths

$$p_t(x) = \int p_{t|1}(x|x_1)q(x_1)dx_1$$



(b) (Marginal) Probability path $p_t(x)$.



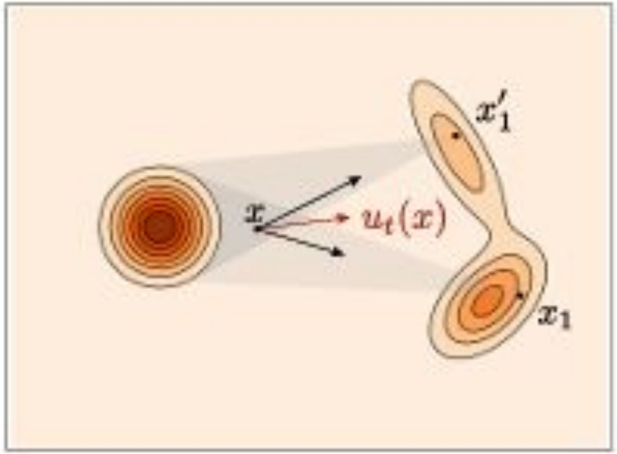
(a) Conditional probability path $p_t(x|x_1)$.

Figures a & b (Lipman et al., 2024, p.6) equation (4.4) Figure (Lipman et al., 2024, p.16)

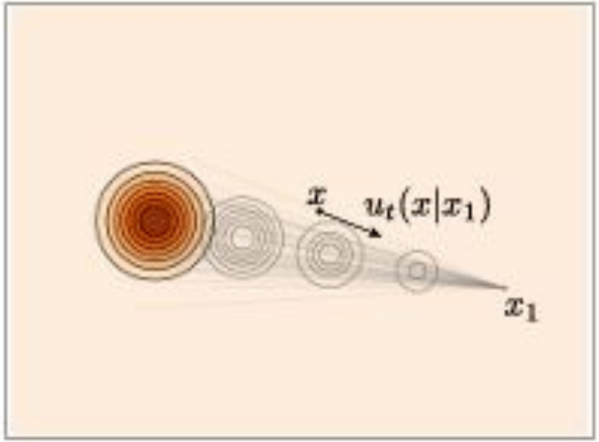
Key Concepts

- Velocity Fields

$$u_t(x) = \int u_t(x|x_1)p_{1|t}(x_1|x)dx_1$$



(d) (Marginal) Velocity field $u_t(x)$.



(c) Conditional velocity field $u_t(x|x_1)$.

Figures c & d (Lipman et al., 2024, p.6) equation (4.8) Figure (Lipman et al., 2024, p.17)

The Continuity Equation for Marginals

$$u_t \text{ generates } p_t \text{ if } X_t = \psi_t(X_0) \sim p_t \text{ for all } t \in [0, 1). \tag{3.24}$$

To verify that a velocity field u_t generates a probability path p_t, we can verify if the pair (u_t, p_t) satisfies a partial differential equation known as the continuity equation and satisfying the boundary conditions $p_0 = p, \quad p_1 = q,$

$$\frac{d}{dt} p_t(x) + div(p_t u_t)(x) = 0$$

Theorem 2 states the Mass Conservation Formula which states that a solution u_t to the Continuity Equation generates the probability path p_t

Theorem 2 (Mass Conservation). *Let p_t be a probability path and u_t a locally Lipchitz integrable vector field. Then, the following two statements are equivalent:*

1. *The Continuity Equation (3.25) holds for $t \in [0, 1)$.*
2. *u_t generates p_t, in the sense of (3.24).*

Equation 3.24/3.25 & Theorem 2 (Lipman et al., 2024, p.13) Boundary Conditions (Lipman et al., 2024, p.16)

The Marginalization Trick

Assumption 1. $p_{t|Z}(x|z)$ is $C^1([0,1) \times \mathbb{R}^d)$ and $u_t(x|z)$ is $C^1([0,1) \times \mathbb{R}^d, \mathbb{R}^d)$ as a function of (t, x). Furthermore, p_Z has bounded support, that is, $p_Z(x) = 0$ outside some bounded set in $\mathbb{R}^m$. Finally, $p_t(x) > 0$ for all $x \in \mathbb{R}^d$ and $t \in [0, 1)$.

Theorem 3 (Marginalization Trick). *Under assumption 1, if $u_t(x|z)$ is conditionally integrable and generates the conditional probability path $p_t(\cdot|z)$, then the marginal velocity field u_t generates the marginal probability path p_t, for all $t \in [0, 1)$.*

In the theorem above, being conditionally integrable refers to a conditional version of the integrability condition from the Mass Conservation Theorem

$$\int_0^1 \int \|u_t(x)\| \, p_t(x) \mathrm{d}x \mathrm{d}t < \infty.$$

Equation (3.26) (Lipman et al., 2024, p.13) Assumption 1 & Theorem 3 (Lipman et al., 2024, p.18)

Examples of Conditional Probability Paths and Conditional Velocity Fields

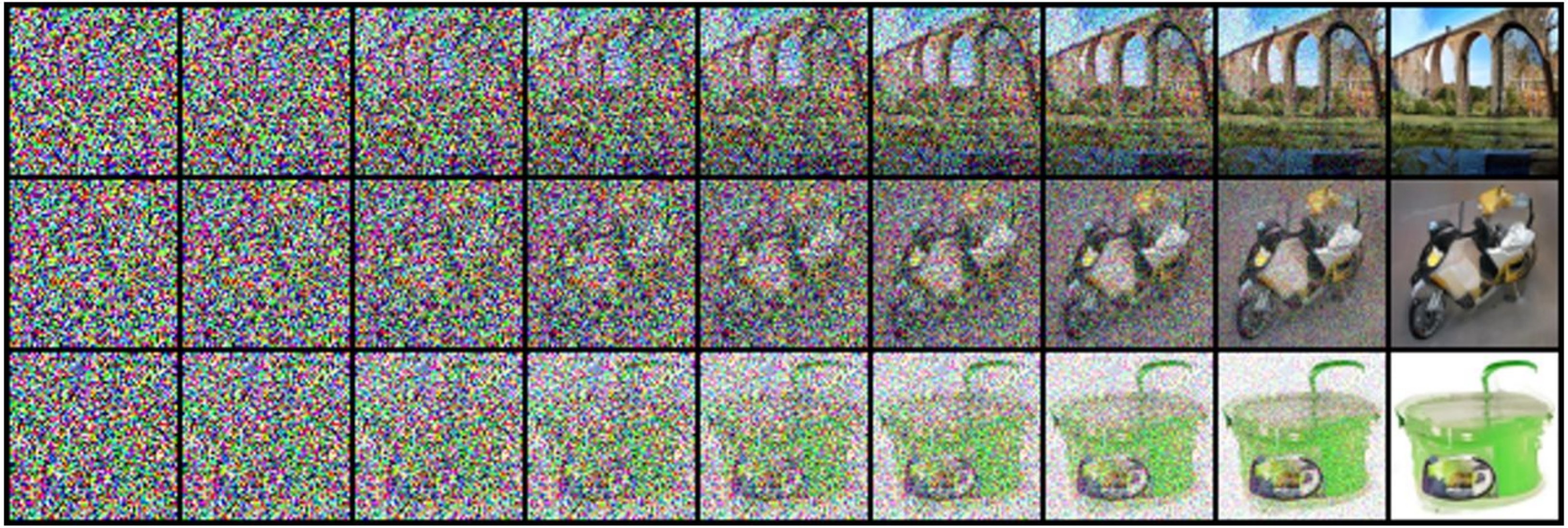
$$p_t(x|x_1) = \mathcal{N}(x \mid \mu_t(x_1), \sigma_t(x_1)^2 I),$$

$$\mu : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d \qquad \sigma : [0,1] \times \mathbb{R} \to \mathbb{R}_{>0}$$

$$u_t(x|x_1) = \frac{\sigma_t'(x_1)}{\sigma_t(x_1)}(x - \mu_t(x_1)) + \mu_t'(x_1).$$

Equation (10) (Lipman et al., 2023, p.4) Equation (15) (Lipman et al., 2023, p.5)

Example



Picture (Lipman et al., 2023, p.8)

Conclusions

Flow matching gives us an efficient way of sampling by using conditional velocity fields and conditional probability paths. This is just one of many examples of how techniques of generative modeling are becoming more efficient.

References/Citations

Lipman, Y., Havasi, M., Holderrieth, P., Shaul, N., Le, M., Karrer, B., Chen, R. T. Q., Lopez-Paz, D., Ben-Hamu, H., & Gat, I. (2024). *Flow Matching Guide and Code*. FAIR at Meta, MIT CSAIL, Weizmann Institute of Science. https://arxiv.org/abs/2412.06264

Lipman, Y., Chen, R. T. Q., Ben-Hamu, H., Nickel, M., & Le, M. (2023). *Flow Matching for Generative Modeling*. Meta AI (FAIR) & Weizmann Institute of Science. https://arxiv.org/abs/2210.02747

Questions

The Marginalization Trick

$$p_t(x) = \int p_{t|Z}(x|z)p_Z(z)dz$$

$$\frac{d}{dt}\,p_t(x) = \frac{d}{dt}\int p_{t|Z}(x|z)p_Z(z)dz$$

$$\frac{d}{dt}\,p_t(x) = \int \frac{d}{dt}p_{t|Z}(x|z)p_Z(z)dz$$

$$= \int -\nabla_{\mathbf{x}} u_t(x|z)p_{t|Z}(x|z)p_Z(z)dz \qquad , \frac{d}{dt}\,p_{t|Z}(x|z) = -\nabla u_t(x|z)p_{t|Z}(x|z)$$

$$= -\nabla_{\mathbf{x}}\int u_t(x|z)p_{t|Z}(x|z)p_Z(z)dz$$

$$= -\nabla_{\mathrm{x}} \int \; u_t(x|z) p_{t|Z}(x|z) p_Z(z) dz$$

$$= -\nabla_{\mathrm{x}} \frac{p_t(x)}{p_t(x)} \int \; u_t(x|z) p_{t|Z}(x|z) p_Z(z) dz$$

$$= -\nabla_{\mathrm{x}} p_t(x) \int \; u_t(x|z) \; \frac{p_{t|Z}(x|z) p_Z(z)}{p_t(x)} dz$$

$$\frac{d}{dt} \, p_t(x) = -\nabla_{\mathrm{x}} p_t(x) \int \; u_t(x|z) \, p_{Z|t}(z|x) dz$$

$$\frac{d}{dt} \, p_t(x) = -\nabla_{\mathrm{x}} p_t(x) u_t(x)$$