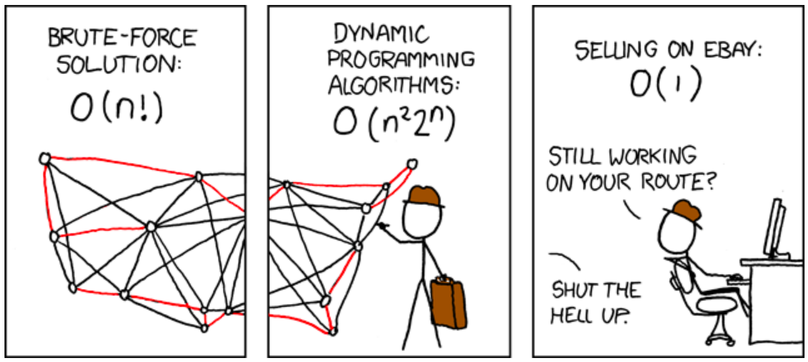
Algorithms: Dynamic Programming



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Overview – Dynamic Programming

- **Dynamic Programming** is a general algorithm design technique for solving problems defined by or formulated as recurrences with overlapping sub-instances.

- Invented by American mathematician Richard Bellman in the 1950s to solve optimization problems and later assimilated by CS

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- **Main idea:**
  - set up a recurrence relating a solution to a larger instance to solutions of some smaller instances
  - solve smaller instances once
  - record solutions in a table
  - extract solution to the initial instance from that table

An Example -- Fibonacci numbers

- Think about the Fibonacci number problem in Session 3:
- Let us change the problem to
  - "Giving an n > 1, return the nth Fibonacci number"

Generate the first N numbers of the Fibonacci sequence where

$$a_0 = 1$$

$$a_1 = 1$$

$$a_n = a_{n-1} + a_{n-2}$$

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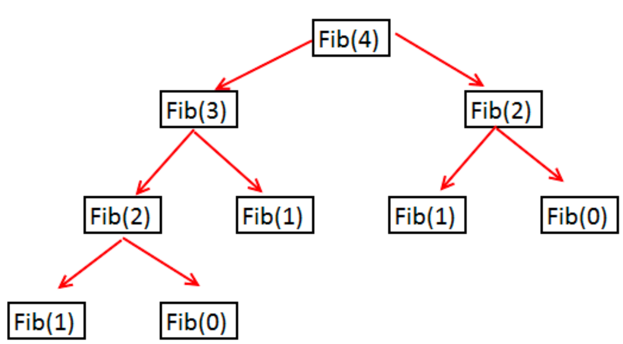
```
#Write your code here
N= 10

fibonacci=[1,1]

for i in range(N-2):
    nextTerm = fibonacci[-1] + fibonacci[-2]
    fibonacci+=[nextTerm]

fibonacci
```

An Example -- Fibonacci numbers

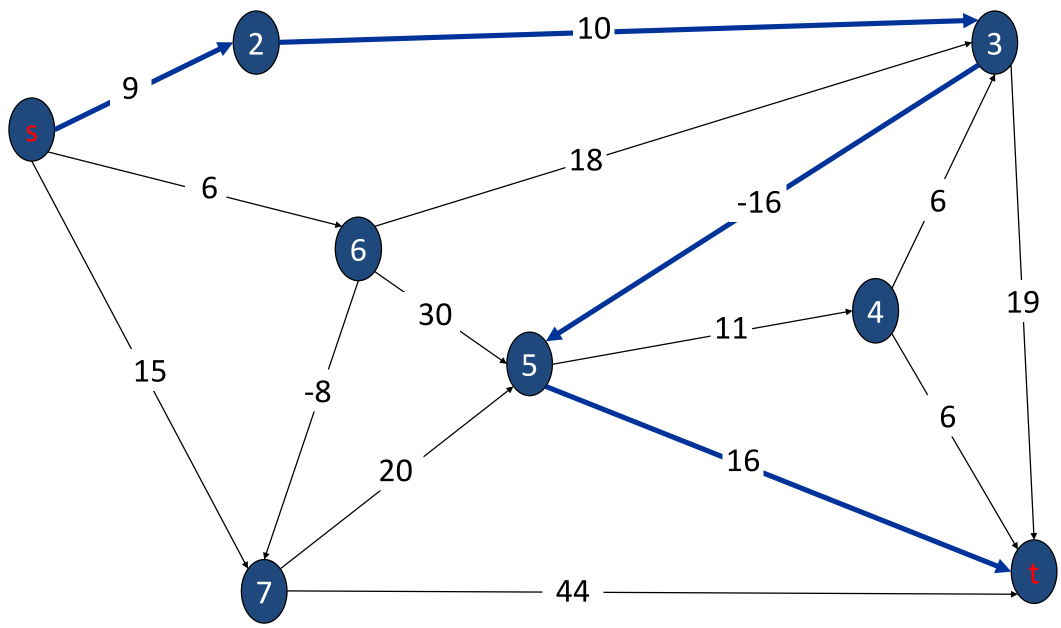
- Why is this dynamic programming?

- 1. Divide the problem into sub-problems:
  - We divide Fib problem of N into sub-problems of N-1, N-2, N-3, ..., 1
- 2. Represent the larger instance by smaller instances:
  - Fib(N) = Fib(N-1) + Fib(N-2)
- 3. Solve the smaller cases first:
  - Fib(0) = 1, Fib(1) = 1
- 4. Record solution in a list / list of lists, solve large problems using smaller instances.



Another Example -- Shortest Path

- **Shortest path problem.** Given a directed graph G = (V, E), with edge weights c(v,w), find shortest path from node s to node t.



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- Def. OPT(i, v) = length of shortest v-t path P using at most i edges.

- Case 1: **P uses at most i-1 edges.**
  - OPT(i, v) = OPT(i-1, v)

- Case 2: **P uses exactly i edges.**
  - if (v, w) is first edge, then OPT uses (v, w), and then selects best w-t path using at most i-1 edges

$$OPT(i,v) = \begin{cases} 0 \text{ if } v = t,\ \infty \text{ otherwise} & \text{if } i = 0 \\ \min\left\{ OPT(i-1,v), \min\limits_{(v,w)\in E} \left\{ OPT(i-1,w) + c_{vw} \right\} \right\} & \text{otherwise} \end{cases}$$

Another Example -- Shortest Path

```
Shortest-Path(G, t) {
   foreach node v ∈ V
      M[0, v] ← ∞
   M[0, t] ← 0

   for i = 1 to n-1
      foreach node v ∈ V
         M[i, v] ← M[i-1, v]
      foreach edge (v, w) ∈ E
         M[i, v] ← min { M[i, v], M[i-1, w] + c_vw }
}
```

Another Example -- Knapsack

- **Given a fixed Knapsack size, our midterm problem can be solved optimally with dynamic programming.**

- Here are some hints on how to think about the Knapsack problem as DP:
  - Define V(i, j) as the optimal value of knapsack problem for the first i items with capacity j.
  - i is an integer between 0 and 50 in our case.
  - j can be treated as an integer between 100 and 150, in our case.

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  - $V(0, j) = 0$ for all j, $V(i, 0) = 0$
  - Suppose the first item has weight 10 and value 5,
    - Then $v(1, j) = 0$ if $j < 10$ and 5 if $j >= 10$

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  - Question 1: how do you represent $v(i, j)$ as smaller instances (i.e., instances with smaller i and j)
  - Question 2: What information you should store along the way so that you can solve $V(50, 100)$ (assuming that knapsack size is fixed at 100)?