Exercise Set A-1

1)
a) ii) $f'(x) = 2x - 2$; iii) $x = 1$.
b) ii) $f'(x) = -6x + 12$; iii) $x = 2$.
c) ii) $f'(x) = 3x^2 - 6x = 3x(x - 2)$; iii) $x = 0$ and $x = 2$.

2) Notice that the function in (1a) is minimized at $x = 1$, the function in (1b) is maximized at $x = 2$, and the function in (1c) is locally maximized at $x = 0$ and locally minimized at $x = 2$. The second derivatives are a) $f''(x) = 2$, b) $f''(x) = -6$, and c) $f''(x) = 6x - 6$. Plugging in the appropriate *x*-coordinates suggests that if $f'(c) = 0$ and $f''(c)$ is positive, then $f(x)$ is locally minimized when $x = c$. Similarly, if $f'(c) = 0$ and $f''(c)$ is negative, then $f(x)$ is locally maximized when $x = c$. These conjectures turn out to be true in general. One way to think of this is that when the slope is zero but increasing (as indicated by the positive second derivative), the function is minimized.

3) a) By the definition of the derivative,

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

Because $g(x) = af(x)$,

$$g'(x) = \lim_{\Delta x \to 0} \frac{af(x + \Delta x) - af(x)}{\Delta x} = a \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = af'(x).$$

The first step comes from the definition of $g(x)$, the second step comes from the distributive property, and the third step comes from the definition of $f'(x)$.

b) Because $h(x) = f(x) + g(x)$,

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} =$$
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = f'(x) + g'(x).$$

4) To find the derivative of the function $f(x) = ax^n$, where n is a positive integer, we take an approach similar to the one we took to find the derivative of $f(x) = x^2$. The steps are:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a(x + \Delta x)^n - ax^n}{\Delta x}$$

Expanding using the binomial theorem gives

$$= \lim_{\Delta x \to 0} \frac{a(x^n + nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}(\Delta x)^2 + \binom{n}{3}x^{n-3}(\Delta x)^3 + \cdots + \Delta x^n) - ax^n}{\Delta x}.$$

Distributing the a across the expression in parentheses and rewriting all but the first two terms in parentheses as a sum gives

$$= \lim_{\Delta x \to 0} \frac{ax^n + anx^{n-1}\Delta x + a(\Delta x)^2 \sum_{k=2}^{n} \binom{n}{k} x^{n-k} (\Delta x)^{k-2} - ax^n}{\Delta x}.$$

The positive and negative ax^n terms in the numerator cancel to give

$$= \lim_{\Delta x \to 0} \frac{anx^{n-1}\Delta x + a(\Delta x)^2 \sum_{k=2}^{n} \binom{n}{k} x^{n-k} (\Delta x)^{k-2}}{\Delta x}.$$

We can divide out the Δx in the denominator:

$$= \lim_{\Delta x \to 0} anx^{n-1} + a\Delta x \sum_{k=2}^{n} \binom{n}{k} x^{n-k} (\Delta x)^{k-2}.$$

The first term does not have Δx in it, so it is unaffected by the limit. In contrast, the second term is multiplied by Δx, so it goes to zero as Δx does, giving the result

$$f'(x) = anx^{n-1}.$$

Exercise Set A-2

1) a) 1 b) 4 c) 8 d) 0 e) 8

2)

b	The definite integral of $f(x) = 2x$ from 0 to b (i.e., $\int_0^b 2x dx$)
1	1
2	4
3	9
4	16
5	25

Exercise Set A-3

1) To solve this problem, consider that the derivative of ax^n with respect to x is nax^{n-1} and that differentiation and integration are inverse processes, meaning that $\int f'(x)dx = f(x) + C$. Jointly, these two facts imply that $\int nax^{n-1}dx = ax^n + C$. That is, to integrate x raised to an exponent, we raise the power of the exponent by one (so $n - 1$ becomes n) and divide by the new value of the exponent (this gets rid of the n in front). Then we add C to get the indefinite integral. The same thing applies when we start with n in the exponent instead of $n - 1$: we add one to the exponent and divide by the new value of the exponent. Thus, $\int ax^n dx =$

$(ax^{n+1})/(n+1)+C$. My calculus teacher taught us to say "up and under" as a way to remember this. Remember that for differentiation of polynomials, the phrase to remember is "out in front and down by one." When we integrate, we are doing the opposite.

This rule does not apply when $n=-1$. When $n=-1$, the "up and under" rule would force us to divide by zero, which is not allowed. We will need the integral of ax^{-1} in chapter 9.