Exploring the Sierpiński Carpet

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Abstract

The Sierpiński Carpet is a classic fractal obtained by taking a square, dividing it into 9 equal squares, removing the center square, and then repeating this process on the remaining 8 squares ad infinitum. It has many interesting properties and is subject to topological, analytic, and physical investigations.

This poster aims to introduce the reader to several important properties of the Carpet, specifically that it is an iterated function system, a continuum, has a non-integer Hausdorff dimension, and that performing calculus on fractal spaces requires non-standard approaches.

Iterated Function System (IFS)

Given the following transformations:

$f_1(x) = x/3$ $\quad f_5(x) = x/3 + \langle 0, 2/3\rangle$

$f_2(x) = x/3 + \langle 1/3, 0\rangle$ $\quad f_6(x) = x/3 + \langle 2/3, 1/3\rangle$

$f_3(x) = x/3 + \langle 2/3, 0\rangle$ $\quad f_7(x) = x/3 + \langle 1/3, 2/3\rangle$

$f_4(x) = x/3 + \langle 0, 1/3\rangle$ $\quad f_8(x) = x/3 + \langle 2/3, 2/3\rangle$

and $X_0 = [0,1] \times [0,1]$, we have the Sierpiński Carpet X equal to

$$X = \bigcap_1^{\infty} X_i = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{8} f_n(X_{i-1}) \,.$$

The Sierpiński Carpet is the unique fixed set under this mapping and is called an attractor [1].

Continuum

A **continuum** is a non-empty compact connected metric space. Each iteration is a continuum, but we need to show that the end result is also a continuum.
The intersection of a nested sequence of continua is itself a continuum [2]. Using De Morgan's laws, we see that for each iteration we have
$\cup f_n(X) = ((\cup f_n(X))^c)^c = (\cap f_n(X)^c)^c = X_0 \setminus \cap f_n(X)^c$.
Then we show via induction that
$X_i = X_{i-1} \setminus \cap_{n=1}^{8} f_n(X_{i-1})^c$.
So clearly $X_i \supseteq X_{i+1}$ and thus the Sierpiński Carpet is a continuum. This is relevant to the study of calculus on this fractal.

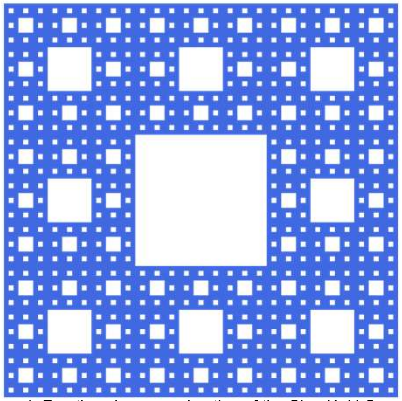

Figure 1. Fourth-order approximation of the Sierpiński Carpet

Dimensions $\quad D = \dfrac{\ln(8)}{\ln(3)} \approx 1.8928, d = 1$

One way to define a **fractal** is an object whose Hausdorff dimension (D) is greater than its topological dimension (d). D is the Hausdorff dimension and d is the topological, usual dimension [3].

Notice that when you shrink down the carpet by a factor of **3**, then you can fit **8** of those copies into the original. This is one way to measure non-integer dimension. There are many other kinds of dimension, all with important roles in characterizing fractals and performing calculus on these spaces.

Calculus

Most fractals are too 'rough' to be differentiable in the standard sense, so different approaches are needed. Some use fractional calculus like the below Laplace operator.

$$\Delta^{n\gamma} = \sum_i^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i - 1}{x_i}\frac{\partial}{\partial x_i}\right) = \sum_i^n \left[x_i^{1-\gamma_i}\frac{\partial}{\partial x_i}\left(x_i^{\gamma_i-1}\frac{\partial}{\partial x_i}\right)\right]$$

This formula has been used to describe physical phenomena in low dimensional semiconductors [5].

There is much math to be discovered, and that is the future of this research project.

Applications

The study of fractal dynamics and specifically the Sierpiński Carpet has applications in modelling self-similar structures and porous materials. There is literature on the transport and flow properties of Sierpiński Carpets, so there's direct application to fluid, heat, and electrical dynamics. Some antennas and electrical circuits are designed around Sierpiński Carpets.

There is a relationship between measures of fractal dimension and how diffusion occurs through similar physical media [4].

References

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