$\{x^2y : x, y \in C\} = [0, 1]$

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The Proof

Let C be the middle-third Cantor Set, and C_n be the nth step in the construction of C, with $C_0 = [0, 1], C_1 = [0, 1/3] \cup [2/3, 1]$, and so on, and let $s \in [0, 1]$. We will show that for all n, there exist $x_n, y_n \in C_n$ such that $x_n^2 y_n = s$. Once we do so, the rest of the proof is as follows: to ensure convergence, since (x_n) is bounded, there exists convergent subsequence (x_{n_k}) by Bolzano-Weierstrass Theorem. Let $x = \lim_{k\to\infty} x_{n_k}$, and $y_{n_k} = s/x_{n_k}^2$ implies $y = s/x^2$. x and y must be elements of C because for any natural N, when $n_k \geq N$, $x_{n_k} \in C_N$, and since C_n is closed, $x \in C_n$. This implies x is an element of every C_n, so $x \in \bigcap C_n = C$. Similar reasoning implies $y \in \bigcap C_n = C$. Hence we have that $x^2y = s$, where $s \in [0, 1]$ and $x, y \in C$.

Induction

Base Case: $C_0^2 C_0 = [0, 1]^2[0, 1] = [0, 1][0, 1] = [0, 1]$.
Inductive Hypothesis: There exists n such that $C_n^2 C_n = [0, 1]$.
Inductive Step: Consider $C_{n+1}^2 C_{n+1}$.
By the following Iterated Function System:

$$\begin{aligned} g_1(x) &= \frac{1}{3}x \\ g_2(x) &= \frac{1}{3}x + \frac{2}{3}, \end{aligned}$$

We have that $C_{n+1} = g_1(C_n) \cup g_2(C_n)$, and hence

$$\begin{aligned} C_{n+1}^2 C_{n+1} &= (g_1(C_n) \cup g_2(C_n))^2 (g_1(C_n) \cup g_2(C_n)) \\ &= g_1(C_n)^2 g_1(C_n) \cup g_1(C_n)^2 g_2(C_n) \\ &\cup g_2(C_n)^2 g_1(C_n) \cup g_2(C_n)^2 g_2(C_n). \end{aligned}$$

Define $f(x, y) = x^2y$, so

$$\begin{aligned} C_{n+1}^2 C_{n+1} &= f(g_1(C_n), g_1(C_n)) \cup f(g_1(C_n), g_2(C_n)) \\ &\cup f(g_2(C_n), g_1(C_n)) \cup f(g_2(C_n), g_2(C_n)) \\ &= f\left(\frac{1}{3}C_n, \frac{1}{3}C_n\right) \cup f\left(\frac{1}{3}C_n, \frac{1}{3}C_n + \frac{2}{3}\right) \\ &\cup f\left(\frac{1}{3}C_n + \frac{2}{3}, \frac{1}{3}C_n\right) \cup f\left(\frac{1}{3}C_n + \frac{2}{3}, \frac{1}{3}C_n + \frac{2}{3}\right). \end{aligned}$$

Further, define

$$\begin{aligned}
f_1(x,y) &= \left(\frac{1}{3}x\right)^2\left(\frac{1}{3}y\right) \\
f_2(x,y) &= \left(\frac{1}{3}x\right)^2\left(\frac{1}{3}y+\frac{2}{3}\right) \\
f_3(x,y) &= \left(\frac{1}{3}x+\frac{2}{3}\right)^2\left(\frac{1}{3}y\right) \\
f_4(x,y) &= \left(\frac{1}{3}x+\frac{2}{3}\right)^2\left(\frac{1}{3}y+\frac{2}{3}\right),
\end{aligned}$$

so we have

$$\begin{aligned}
C_{n+1}^2 C_{n+1} &= f_1(C_n,C_n)\cup f_2(C_n,C_n)\cup f_3(C_n,C_n)\cup f_4(C_n,C_n) \\
&= \bigcup_{i=1}^{4} f_i(C_n,C_n).
\end{aligned}$$

By observation, we claim that only $f_3(C_n,C_n)$ and $f_4(C_n,C_n)$ are needed. Specifically we claim $f_3(C_n,C_n)=[0,1/3]$ and $f_4(C_n,C_n)=[8/27,1]$.

$f_4(C_n,C_n)=[8/27,1]$

We proceed by Induction:
Base Case: $(1/3\cdot[0,1]+2/3)^2(1/3\cdot[0,1]+2/3)=[4/9,1]\cdot[2/3,1]=[8/27,1]$.
Inductive Hypothesis: There exists n such that $f_4(C_n,C_n)=[8/27,1]$.
Inductive Step: Consider $f_4(C_{n+1},C_{n+1})$.
Pick $s\in[8/27,1]$, and simplify $f_4(x,y)=(1/3\cdot x+2/3)^2(1/3\cdot y+2/3)=\frac{1}{27}(x+2)^2(y+2)$, and solve for y to make the level curve $s=\frac{1}{27}(x+2)^2(y+2)$ into $y=\frac{27\cdot s}{(x+2)^2}-2$. By inductive hypothesis, there must exist $x_n,y_n\in C_n$ such that $\frac{1}{27}(x_n+2)^2(y_n+2)=s$. Moreover, we can see this geometrically by doing the following. Enumerate the disjoint closed sets $I_i=[u_i,v_i]$ which constitute C_n from least to greatest. Then there must exist some square $A_{jk}:=I_j\times I_k=[u_j,v_j]\times[u_k,v_k]$, where $x_n\in I_j$ and $y_n\in I_k$, meaning the level curve intersected A_{jk}. Observe the possible s values that are associated with level curves intersecting A_{jk}:

$$\begin{aligned}
\frac{(u_j+2)^2(u_k+2)}{27} &\le s\le\frac{(v_j+2)^2(v_k+2)}{27} \\
\to s &\in\left[\frac{(u_j+2)^2(u_k+2)}{27},\frac{(v_j+2)^2(v_k+2)}{27}\right].
\end{aligned}$$

Considering the set $C_{n+1}\times C_{n+1}$, will the level curve intersect a square? Our A_{jk} is now split into four smaller squares with the following coordinates:

$$\begin{aligned}
&[u_j, u_j + 1/3^{n+1}] \times [u_k + 2/3^{n+1}, v_k] \text{ (top left)}\\
&[u_j + 2/3^{n+1}, v_j] \times [u_k + 2/3^{n+1}, v_k] \text{ (top right)}\\
&[u_j, u_j + 1/3^{n+1}] \times [u_k, u_k + 1/3^{n+1}] \text{ (bottom left)}\\
&[u_j + 2/3^{n+1}, v_j] \times [u_k, u_k + 1/3^{n+1}] \text{ (bottom right).}
\end{aligned}$$

Now calculating the associated s-ranges for the squares:

$$\begin{aligned}
\text{(bottom left): } s &\in \left[\frac{(u_j+2)^2(u_k+2)}{27}, \frac{(u_j+1/3^{n+1}+2)^2(u_k+1/3^{n+1}+2)}{27}\right]\\
\text{(top left): } s &\in \left[\frac{(u_j+2)^2(u_k+2/3^{n+1}+2)}{27}, \frac{(u_j+1/3^{n+1}+2)^2(v_k+2)}{27}\right]\\
\text{(bottom right): } s &\in \left[\frac{(u_j+2/3^{n+1}+2)^2(u_k+2)}{27}, \frac{(v_j+2)^2(u_k+1/3^{n+1}+2)}{27}\right]\\
\text{(top right): } s &\in \left[\frac{(u_j+2/3^{n+1}+2)^2(u_k+2/3^{n+1}+2)}{27}, \frac{(v_j+2)^2(v_k+2)}{27}\right].
\end{aligned}$$

The bottom left range connects into the top left range:

$$\begin{aligned}
&(u_j+1/3^{n+1}+2)^2(u_k+1/3^{n+1}+2) - (u_j+2)^2(u_k+2/3^{n+1}+2)\\
&= \frac{2}{3^{n+1}}u_ju_k + \frac{2}{3^{2n+2}}u_j + \frac{1}{3^{2n+2}}u_k + \frac{4}{3^{n+1}}u_k + \frac{1}{3^{3n+3}} + \frac{6}{3^{2n+2}} + \frac{4}{3^{n+1}} - \frac{1}{3^{n+1}}u_j^2\\
&> \frac{2}{3^{n+1}}u_ju_k + \frac{2}{3^{2n+2}}u_j + \frac{1}{3^{2n+2}}u_k + \frac{4}{3^{n+1}}u_k + \frac{1}{3^{3n+3}} + \frac{6}{3^{2n+2}} + \frac{4}{3^{n+1}} - \frac{1}{3^{n+1}}\\
&= \frac{2}{3^{n+1}}u_ju_k + \frac{2}{3^{2n+2}}u_j + \frac{1}{3^{2n+2}}u_k + \frac{4}{3^{n+1}}u_k + \frac{1}{3^{3n+3}} + \frac{6}{3^{2n+2}} + \frac{3}{3^{n+1}}\\
&> 0.
\end{aligned}$$

The top left range connects into the bottom right range:

$$\begin{aligned}
&(u_j+1/3^{n+1}+2)^2(v_k+2) - (u_j+2/3^{n+1}+2)^2(u_k+2)\\
&= \frac{1}{3^n}u_j^2 + \frac{2}{3^{2n+1}}u_j + \frac{8}{3^{n+1}}u_j + \frac{1}{3^{3n+2}} + \frac{6}{3^{2n+2}} + \frac{4}{3^{n+1}}\\
&\quad - \left(\frac{2}{3^{n+1}}u_ju_k + \frac{3}{3^{2n+2}}u_k + \frac{4}{3^{n+1}}u_k\right)\\
&\geq \frac{1}{3^n}u_j^2 + \frac{2}{3^{2n+1}}u_j + \frac{8}{3^{n+1}}u_j + \frac{1}{3^{3n+2}} + \frac{6}{3^{2n+2}} + \frac{4}{3^{n+1}}\\
&\quad - \left(\frac{2}{3^{n+1}}u_j(1-\frac{1}{3^n}) + \frac{3}{3^{2n+2}}(1-\frac{1}{3^n}) + \frac{4}{3^{n+1}}(1-\frac{1}{3^n})\right)\\
&= \frac{1}{3^n}u_j^2 + \frac{4}{3^{2n+1}}u_j + \frac{6}{3^{n+1}}u_j + \frac{4}{3^{3n+2}} + \frac{15}{3^{2n+2}}\\
&> 0.
\end{aligned}$$

The bottom right range connects into the top right range:

$$\begin{aligned}
&(v_j+2)^2(u_k+\frac{1}{3^{n+1}}+2)-(u_j+\frac{2}{3^{n+1}}+2)^2(u_k+\frac{2}{3^{n+1}}+2)\\
&=2u_ju_k+\frac{5}{3^{2n+2}}u_k+\frac{4}{3^{n+1}}u_k+\frac{1}{3^{3n+3}}+\frac{6}{3^{2n+2}}+\frac{4}{3^{n+1}}-(\frac{1}{3^{n+1}}u_j^2+\frac{2}{3^{2n+2}}u_j)\\
&>2u_ju_k+\frac{5}{3^{2n+2}}u_k+\frac{4}{3^{n+1}}u_k+\frac{1}{3^{3n+3}}+\frac{6}{3^{2n+2}}+\frac{4}{3^{n+1}}-(\frac{1}{3^{n+1}}+\frac{2}{3^{2n+2}})\\
&=2u_ju_k+\frac{5}{3^{2n+2}}u_k+\frac{4}{3^{n+1}}u_k+\frac{1}{3^{3n+3}}+\frac{4}{3^{2n+2}}+\frac{3}{3^{n+1}}\\
&>0.
\end{aligned}$$

And hence the level curve must intersect one of the four sub-squares in the inductive step, meaning there exist $x_{n+1}, y_{n+1} \in C_{n+1}$ such that $f_4(x_{n+1}, y_{n+1}) = s$. By induction, this is true for all natural n and $s \in [8/27, 1]$.

$f_3(C_n, C_n) = [0, 1/3]$

We proceed by Induction:
Base Case: $(1/3 \cdot [0,1] + 2/3)^2(1/3 \cdot [0,1]) = [4/9, 1] \cdot [0, 1/3] = [0, 1/3]$.
Inductive Hypothesis: There exists n such that $f_3(C_n, C_n) = [0, 1/3]$.
Inductive Step: Consider $f_3(C_{n+1}, C_{n+1})$.
Pick $s \in [0, 1/3]$, and simplify $f_3(x, y) = (1/3 \cdot x + 2/3)^2(1/3 \cdot y) = \frac{y}{27}(x+2)^2$, and solve for y to make the level curve $s = \frac{y}{27}(x+2)^2$ into $y = \frac{27 \cdot s}{(x+2)^2}$. By inductive hypothesis, the level curve must intersect some $A_{jk} = I_j \times I_k$, with $I_j, I_k \subset C_n$. s must be in the following range to intersect A_{jk}:

$$s \in \left[\frac{(u_j+2)^2 \cdot u_k}{27}, \frac{(v_j+2)^2 \cdot v_k}{27}\right].$$

In $C_{n+1} \times C_{n+1}$, A_{jk} splits into four smaller squares with associated s ranges:

$$\begin{aligned}
\text{(bottom left): } s &\in \left[\frac{(u_j+2)^2 \cdot u_k}{27}, \frac{(u_j+\frac{1}{3^{n+1}}+2)^2(u_k+\frac{1}{3^{n+1}})}{27}\right]\\
\text{(bottom right): } s &\in \left[\frac{(u_j+\frac{2}{3^{n+1}}+2)^2 \cdot u_k}{27}, \frac{(v_j+2)^2(u_k+\frac{1}{3^{n+1}})}{27}\right]\\
\text{(top left): } s &\in \left[\frac{(u_j+2)^2(u_k+\frac{2}{3^{n+1}})}{27}, \frac{(u_j+\frac{1}{3^{n+1}}+2)^2 \cdot v_k}{27}\right]\\
\text{(top right): } s &\in \left[\frac{(u_j+\frac{2}{3^{n+1}}+2)^2(u_k+\frac{2}{3^{n+1}})}{27}, \frac{(v_j+2)^2 \cdot v_k}{27}\right].
\end{aligned}$$

The bottom left range connects into the bottom right range:

$$\begin{aligned}
&(u_j + \frac{1}{3^{n+1}} + 2)^2(u_k + \frac{1}{3^{n+1}}) - (u_j + \frac{2}{3^{n+1}} + 2)^2(u_k) \\
&= \frac{1}{3^{n+1}}u_j^2 + \frac{2}{3^{2n+2}}u_j + \frac{4}{3^{n+1}}u_j + \frac{1}{3^{3n+3}} + \frac{4}{3^{2n+2}} + \frac{4}{3^{n+1}} \\
&\quad - \left(\frac{2}{3^{n+1}}u_j u_k + \frac{3}{3^{2n+2}}u_k + \frac{4}{3^{n+1}}u_k\right) \\
&\geq \frac{1}{3^{n+1}}u_j^2 + \frac{2}{3^{2n+2}}u_j + \frac{4}{3^{n+1}}u_j + \frac{1}{3^{3n+3}} + \frac{4}{3^{2n+2}} + \frac{4}{3^{n+1}} \\
&\quad - \left(\frac{2}{3^{n+1}}u_j(1 - \frac{1}{3^n}) + \frac{3}{3^{2n+2}}(1 - \frac{1}{3^n}) + \frac{4}{3^{n+1}}(1 - \frac{1}{3^n})\right) \\
&= \frac{1}{3^{n+1}}u_j^2 + \frac{2}{3^{2n+2}}u_j + \frac{2}{3^{n+1}}u_j + \frac{10}{3^{3n+3}} + \frac{19}{3^{2n+2}} \\
&> 0.
\end{aligned}$$

The top left range connects into the top right range:

$$\begin{aligned}
&(u_j + \frac{1}{3^{n+1}} + 2)^2 \cdot v_k - (u_j + \frac{2}{3^{n+2}} + 2)^2(u_k + \frac{2}{3^{n+1}}) \\
&= \frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{n+1}}u_j + \frac{4}{3^{n+1}} \\
&\quad - \left(\frac{2}{3^{n+1}}u_j u_k + \frac{2}{3^{2n+2}}u_j + \frac{1}{3^{2n+1}}u_k + \frac{4}{3^{n+1}}u_k + \frac{5}{3^{3n+3}} + \frac{4}{3^{2n+2}}\right) \\
&\geq \frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{n+1}}u_j + \frac{4}{3^{n+1}} \\
&\quad - \left(\frac{2}{3^{n+1}}u_j(1 - \frac{1}{3^n}) + \frac{2}{3^{2n+2}}u_j + \frac{1}{3^{2n+1}}(1 - \frac{1}{3^n}) + \frac{4}{3^{n+1}}(1 - \frac{1}{3^n}) + \frac{5}{3^{3n+3}} + \frac{4}{3^{2n+2}}\right) \\
&= \frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{2n+2}}u_j + \frac{2}{3^{n+1}}u_j + \frac{4}{3^{3n+3}} + \frac{5}{3^{2n+2}}. \\
&> 0.
\end{aligned}$$

The bottom right range does not always connect into the top left range, let's see when this occurs:

$$\begin{aligned}
&(v_j + 2)^2(u_k + \frac{1}{3^{n+1}}) - (u_j + 2)^2(u_k + \frac{2}{3^{n+1}}) \\
0 &= u_k\left(\frac{2}{3^n}u_j + \frac{1}{3^{2n}} + \frac{4}{3^n}\right) - \frac{1}{3^{n+1}}u_j^2 - \frac{4}{3^{n+1}}u_j - \frac{4}{3^{n+1}} + \frac{2}{3^{2n+1}}u_j + \frac{1}{3^{3n+1}} + \frac{4}{3^{2n+1}} \\
u_k &= \frac{\frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{n+1}}u_j + \frac{4}{3^{n+1}} - \frac{2}{3^{2n+1}}u_j - \frac{1}{3^{3n+1}} - \frac{4}{3^{2n+1}}}{\frac{2}{3^n}u_j + \frac{1}{3^{2n}} + \frac{4}{3^n}}
\end{aligned}$$

Thus we have a description for when all four corners cover the original A_{jk},

$$\frac{\frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{n+1}}u_j + \frac{4}{3^{n+1}} - \frac{2}{3^{2n+1}}u_j - \frac{1}{3^{3n+1}} - \frac{4}{3^{2n+1}}}{\frac{2}{3^n}u_j + \frac{1}{3^{2n}} + \frac{4}{3^n}} \leq u_k \leq 1 - \frac{1}{3^n}, \tag{1}$$

and a description for when the level curve can slip between the top left and bottom right:

$$0 \leq u_k < \frac{\frac{1}{3^{n+1}}u_j^2 + \frac{4}{3^{n+1}}u_j + \frac{4}{3^{n+1}} - \frac{2}{3^{2n+1}}u_j - \frac{1}{3^{3n+1}} - \frac{4}{3^{2n+1}}}{\frac{2}{3^n}u_j + \frac{1}{3^{2n}} + \frac{4}{3^n}}. \tag{2}$$

In the case where the level curve slips through the middle of A_{jk}, between the bottom right and the top left corner boxes, we find the following s range:

$$s \in \left(\frac{(v_j + 2)^2(u_k + \frac{1}{3^{n+1}})}{27}, \frac{(u_j + 2)^2(u_k + \frac{2}{3^{n+1}})}{27} \right)$$

To complete the proof, it is now our goal to cover the above range with associated s ranges of squares in $C_{n+1} \times C_{n+1}$.

References

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