Optimal Policy

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1 The unconstrained social planner's problem

Abstract away from different zones within a neighborhood here as they provide no new empirical content (apart from perfect labour mobility). We also set $\theta = \rho$ for simplicity. Case where $\theta \to \infty$ ignores re distribution motive because of idiosyncratic preference shocks.

The **Social planner's problem** for a set of weights $\{\alpha(z)\}_{z \in Z}$ and α^L is defined as choosing numeraire consumption allocations $g^C(i, z)$, $g^L(i, z)$, housing consumption allocations $A(i, z)$, and total capital inputs into housing production $g^A(i)$ such that

$$\max \sum_{z \in \boldsymbol{Z}} \alpha(z) \log \boldsymbol{W}(z) + \alpha^L \Pi \tag{1}$$

where $\log \boldsymbol{W}(z)$ is the average renter welfare and $\Pi := \sum_{i \in N} g^L(i)$ is the total numeraire consumption paid to landowners, subject to the following resource and free mobility constraints:

$$\sum_{i \in N} \left[\sum_{z \in Z} g^C(i, z) L(i, z) \right] + g^A(i) + g^L(i) \;=\; \underbrace{\sum_{c \in C} \left[\iota(c) \sum_{i \in N(c), z \in Z} z L(i, z) \right]}_{\text{Total production of numeraire}} \tag{2}$$

$$\forall i, \quad \sum_{z \in Z} A(i, z) \;=\; \underbrace{\tilde{\lambda}(i) g^A(i)^{\frac{\epsilon(i)}{1 + \epsilon(i)}} T(i)^{\frac{1}{1 + \epsilon(i)}}}_{\text{Local production of housing services}} \tag{3}$$

$$\forall i, z, \quad V(i, z) - \frac{1}{\theta} L(i, z) \;=\; \log \boldsymbol{W}(z) \tag{4}$$

$$\forall z, \quad \sum_{i \in N} L(i, z) \;=\; L(z) \tag{5}$$

where

$$V(i, z) := \underbrace{\kappa(z) \beta^{-\beta} (1 - \beta)^{-(1-\beta)} (A(i, z) - \bar{A})^\beta g^C(i, z)^{1-\beta}}_{\text{Consumption value}} + \underbrace{\Omega(z) \text{Inc}(i) + \log \nu(i, z)}_{\text{Amenity value}}$$

and $\text{Inc}(i)$ is neighborhood average income. In practice, we use our definition of the equivalent variation (expressed as a percentative of income relative to the baseline equilibrium that matches data) in lieu of $\log \boldsymbol{W}(z)$ above. This means I am are abstracting away from differences in the marginal utility of income across skill levels.

Let Λ^C be the lagrange multiplier for (2), $\Lambda^A(i)$ for (3), $\Lambda^{FM}(i, z)$ for (4) and $\Lambda^L(z)$ for (5).

2 First order conditions

(Objective; later find conditions that do not depend on welfare weights to put in the body of the paper).

The following important first order conditions hold in a socially optimal allocation when $\Omega(z) = 0$ for all z:

1. FOC w.r.t $g^C(i,z)$ for fixed z, i

$$- \underbrace{\Lambda^{FM}(i,z)\frac{\partial V(i,z)}{\partial g^C(i,z)}}_{\text{Weighted marginal utility of numeraire}} = \Lambda^C L(i,z) \tag{6}$$

2. FOC w.r.t. $L(i,z)$

$$\sum_{z' \in Z} -\Lambda^{FM}(i,z')\Omega(z')\frac{\partial \log \text{Inc}(i)}{\partial L(i,z)} + \Lambda^{FM}(i,z')\frac{1}{\theta}\frac{1}{L(i,z)} + \Lambda^C \iota(i)z - \Lambda^C - \Lambda^L(z) = 0 \tag{7}$$

3. FOC w.r.t $A(i,z)$ for fixed z, i

$$\underbrace{\Lambda^{FM}(i,z)\frac{\partial V(i,z)}{\partial A(i,z)}}_{\text{Weighted marginal utility of housing services}} = \Lambda^A(i) \tag{8}$$

4. FOC w.r.t. $g^A(i)$

$$\underbrace{\tilde{\lambda}(i)\frac{\epsilon(i)}{1+\epsilon(i)}\left[\frac{g^A(i)}{T(i)}\right]^{-\frac{1}{1+\epsilon(i)}}}_{\text{Marginal product of capital in } i\text{'s housing sector}} = \frac{\Lambda^C}{\Lambda^A(i)} \tag{9}$$

A condition for within-neighborhood production and consumption efficiency We now characterize conditions that are free of welfare weights $\alpha(z)$. We start with how a social planner trades off numeraire and housing consumption in each location. We can divide after rearranging equations (6) by (8), then substituting

$$\underbrace{\frac{\partial V(i,z)}{\partial g^C(i,z)}\bigg/\frac{\partial V(i,z)}{\partial A(i,z)}}_{\text{-MRS of housing for numeraire}} = \underbrace{\tilde{\lambda}(i)\frac{\epsilon(i)}{1+\epsilon(i)}\left[\frac{g^A(i)}{T(i)}\right]^{-\frac{1}{1+\epsilon(i)}}}_{\text{Marginal product of capital in } i\text{'s housing sector}} \tag{10}$$

This is the standard MRS = MRTS result in general equilbrium. This cannot be achieved with minimum lot size regulation because it necessarily distorts housing consumption for a given set of equilibrium prices.

A condition for an efficient spatial distribution Next, we can combine (6) and (7) to arrive at a spatial efficiency condition:

$$\sum_{z' \in Z} \Omega(z')\frac{\partial \log \text{Inc}(i)}{\partial L(i,z)}V_g(i,z')^{-1}L(i,z') + \iota(i)z - \frac{1}{\theta}V_g(i,z)^{-1} = \frac{\Lambda^L(z)}{\Lambda^C} + 1$$

This expression is informative. It says that a social planner needs to balance the benefits of redistributing labour across each neighborhood by skill level. These benefits are, respectively: 1) the total willingness to pay for all households in a neighborhood (measured in units of the numeraire good) for a marginal increase in neighborhood amenity value; 2) the marginal increase in output created by an additional resident and 3) distributional concerns arising from location preference shocks.

Constrained social planner's problem

The **Constrained social planner's problem** for a set of weights $\{\alpha(z), \alpha^L\}$ is defined as choosing the level of regulation $R(i)$ in every neighborhood to maximize utility subject to the equilibrium conditions outlined above. This is equivalent to choosing $R(i)$, $P(i)$ and $L(i, z)$ to solve

$$\max \sum_{z \in \mathbf{Z}} \alpha(z) \log \mathbf{W}(z) + \alpha^L \Pi \tag{11}$$

where welfare retains the same definition as before and Π is the sum of payments to landowners $\Pi = \sum_{i \in N} \frac{1}{1+\epsilon(i)} \lambda(i) P(i)^{1+\epsilon(i)} T(i)$; subject to equilibrium constraints

$$\forall i, \quad \underbrace{\sum_{z \in Z^c(i)} R(i) L(i, z) + \sum_{z \in Z \notin Z^c(i)} \left[\beta wz + (1-\beta)P(i)\bar{A}\right] L(i, z)}_{\text{Total spending on housing services}} = \underbrace{\lambda(i) P(i)^{1+\epsilon(i)} T(i)}_{\text{Value of supplied housing services}} \tag{12}$$

$$\forall i, z, \quad V(i, z) - \frac{1}{\theta} L(i, z) = \log \mathbf{W}(z) \tag{13}$$

$$\forall z, \quad \sum_{i \in N} L(i, z) = L(z) \tag{14}$$

where

$$V(i, z) := \underbrace{k(z) wz \left[\frac{1 - \frac{P(i)\bar{A}}{wz}}{P(i)^\beta}\right] s(i, z)}_{\text{Consumption value}} + \underbrace{\Omega(z)\text{Inc}(i) + \log \nu(i, z)}_{\text{Amenity value}}$$

and $s(i, z) := \left[\frac{\left(1 - \frac{R(i)}{wz}\right)\left(1 - \frac{P(i)\bar{A}}{wz}\right)^{-1}}{1-\beta}\right]^{1-\beta} \left[\frac{\left(R(i) - P(i)\bar{A}\right)\left(wz - P(i)\bar{A}\right)^{-1}}{\beta}\right]^\beta$ is the distortion factor, and wages are city specific – $w(i) = \iota(i)$, and $Z^C(i)$ is the set of constrained households in i.

How does this problem differ from the unconstrained social planner's? Minimum lot sizes are inherently distortionary, so too many housing services are created relative to what a social planner could achieve with place-based spatial transfers. However, this distortion may be efficient if it increases spatial efficiency in line with the above.

Note: SPP is not differentiable in $R(i)$; so FOC's should be interpreted with caution – they are taken at points where function is locally differentiable (i.e. not at the point where regulation is just binding for some type z). The FOC's are:

1. FOC w.r.t $R(i)$:

$$-\left[\sum_{z \in Z^C(i)} \Lambda^{FM}(i, z) \frac{\partial V(i, z)}{\partial R(i)} + \Lambda^A(i) L(i, z)\right] = 0$$

2. FOC w.r.t $L(i, z)$:

$$\sum_{z' \in Z} -\Lambda^{FM}(i, z')\Omega(z') \frac{\partial \log \text{Inc}(i)}{\partial L(i, z)} + \Lambda^{FM}(i, z) \frac{1}{\theta} \frac{1}{L(i, z)} - \Lambda^A(i) E(i, z) = 0$$

where $E(i, z)$ is the housing expenditure by an (i, z) household $(= R(i)$ if $z \in Z^C(i)$ or $\beta wz + (1-\beta)P(i)\bar{A}$ if $z \in Z \notin Z^C(i))$.

3. FOC w.r.t $P(i)$

$$\sum_{z \in Z} -\Lambda^{FM}(i, z) \frac{\partial V(i, z)}{\partial P(i)} - \Lambda^A(i) \sum_{z \in Z \notin Z^C(i)} (1-\beta)\bar{A}L(i, z) + \Lambda^A(i)(1+\epsilon(i))\lambda(i) P(i)^{\epsilon(i)} T(i) = 0$$

Can we solve this and get an efficient spatial distribution? Or should we just show consumption inefficiency? No need to show all of this, no additional insights.