# Optimal Policy

James Macek

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## 1 The unconstrained social planner's problem

Abstract away from different zones within a neighborhood here as they provide no new empirical content (apart from perfect labour mobility). We also set $\theta = \rho$ for simplicity. Case where $\theta \to \infty$ ignores re distribution motive because of idiosyncratic preference shocks.

The **Social planner's problem** for a set of weights $\{\alpha(z)\}_{z\in Z}$ and $\alpha^L$ is defined as choosing numeraire consumption allocations $g^C(i,z)$, $g^L(i,z)$, housing consumption allocations $A(i,z)$, and total capital inputs into housing production $g^A(i)$ such that

$$\max \sum_{z\in \boldsymbol{Z}} \alpha(z) \log \boldsymbol{W}(z) + \alpha^L \Pi \tag{1}$$

where $\log \boldsymbol{W}(z)$ is the average renter welfare and $\Pi := \sum_{i\in N} g^L(i)$ is the total numeraire consumption paid to landowners, subject to the following resource and free mobility constraints:

$$\begin{aligned}
\sum_{i\in N}\big[\sum_{z\in Z} g^C(i,z)L(i,z)\big] + g^A(i) + g^L(i) &= \underbrace{\sum_{c\in C}\left[\iota(c)\sum_{i\in N(c),z\in Z} zL(i,z)\right]}_{\text{Total production of numeraire}} && (2)\\
\forall i,\ \sum_{z\in Z} A(i,z) &= \underbrace{\tilde{\lambda}(i) g^A(i)^{\frac{\epsilon(i)}{1+\epsilon(i)}} T(i)^{\frac{1}{1+\epsilon(i)}}}_{\text{Local production of housing services}} && (3)\\
\forall i,z,\ V(i,z) - \frac{1}{\theta}L(i,z) &= \log \boldsymbol{W}(z) && (4)\\
\forall z,\ \sum_{i\in N} L(i,z) &= L(z) && (5)
\end{aligned}$$

where

$$V(i,z) := \underbrace{\kappa(z)\beta^{-\beta}(1-\beta)^{-(1-\beta)}(A(i,z)-\bar{A})^{\beta} g^C(i,z)^{1-\beta}}_{\text{Consumption value}} + \underbrace{\Omega(z)\text{Inc}(i) + \log \nu(i,z)}_{\text{Amenity value}}$$

and $\text{Inc}(i)$ is neighborhood average income. In practice, we use our definition of the equivalent variation (expressed as a percentative of income relative to the baseline equilibrium that matches data) in lieu of $\log \boldsymbol{W}(z)$ above. This means I am are abstracting away from differences in the marginal utility of income across skill levels.

Let $\Lambda^C$ be the lagrange multiplier for (2), $\Lambda^A(i)$ for (3), $\Lambda^{FM}(i,z)$ for (4) and $\Lambda^L(z)$ for (5).

## 2 First order conditions

(Objective; later find conditions that do not depend on welfare weights to put in the body of the paper).

The following important first order conditions hold in a socially optimal allocation when $\Omega(z) = 0$ for all $z$:

1. FOC w.r.t $g^C(i,z)$ for fixed $z, i$

$$-\underbrace{\Lambda^{FM}(i,z)\frac{\partial V(i,z)}{\partial g^C(i,z)}}_{\text{Weighted marginal utility of numeraire}} = \Lambda^C L(i,z) \tag{6}$$

2. FOC w.r.t. $L(i,z)$

$$\sum_{z' \in Z} -\Lambda^{FM}(i,z')\Omega(z')\frac{\partial \log \text{Inc}(i)}{\partial L(i,z)} + \Lambda^{FM}(i,z')\frac{1}{\theta}\frac{1}{L(i,z)} + \Lambda^C \iota(i) z - \Lambda^C - \Lambda^L(z) = 0 \tag{7}$$

3. FOC w.r.t $A(i,z)$ for fixed $z, i$

$$\underbrace{\Lambda^{FM}(i,z)\frac{\partial V(i,z)}{\partial A(i,z)}}_{\text{Weighted marginal utility of housing services}} = \Lambda^A(i) \tag{8}$$

4. FOC w.r.t. $g^A(i)$

$$\underbrace{\tilde{\lambda}(i)\frac{\epsilon(i)}{1+\epsilon(i)}\left[\frac{g^A(i)}{T(i)}\right]^{-\frac{1}{1+\epsilon(i)}}}_{\text{Marginal product of capital in } i\text{'s housing sector}} = \frac{\Lambda^C}{\Lambda^A(i)} \tag{9}$$

**A condition for within-neighborhood production and consumption efficiency** We now characterize conditions that are free of welfare weights $\alpha(z)$. We start with how a social planner trades off numeraire and housing consumption in each location. We can divide after rearranging equations (6) by (8), then substituting

$$\underbrace{\frac{\partial V(i,z)}{\partial g^C(i,z)} \Big/ \frac{\partial V(i,z)}{\partial A(i,z)}}_{\text{-MRS of housing for numeraire}} = \underbrace{\tilde{\lambda}(i)\frac{\epsilon(i)}{1+\epsilon(i)}\left[\frac{g^A(i)}{T(i)}\right]^{-\frac{1}{1+\epsilon(i)}}}_{\text{Marginal product of capital in } i\text{'s housing sector}} \tag{10}$$

This is the standard MRS = MRTS result in general equilbrium. This cannot be achieved with minimum lot size regulation because it necessarily distorts housing consumption for a given set of equilibrium prices.

**A condition for an efficient spatial distribution** Next, we can combine (6) and (7) to arrive at a spatial efficiency condition:

$$\sum_{z' \in Z} \Omega(z')\frac{\partial \log \text{Inc}(i)}{\partial L(i,z)} V_g(i,z')^{-1} L(i,z') + \iota(i)z - \frac{1}{\theta}V_g(i,z)^{-1} = \frac{\Lambda^L(z)}{\Lambda^C} + 1$$

This expression is informative. It says that a social planner needs to balance the benefits of redistributing labour across each neighborhood by skill level. These benefits are, respectively: 1) the total willingness to pay for all households in a neighborhood (measured in units of the numeraire good) for a marginal increase in neighborhood amenity value; 2) the marginal increase in output created by an additional resident and 3) distributional concerns arising from location preference shocks.

## Constrained social planner's problem

The **Constrained social planner's problem** for a set of weights $\{\alpha(z), \alpha^L\}$ is defined as choosing the level of regulation $R(i)$ in every neighborhood to maximize utility subject to the equilibrium conditions outlined above. This is equivalent to choosing $R(i)$, $P(i)$ and $L(i,z)$ to solve

$$\max \sum_{z \in \boldsymbol{Z}} \alpha(z) \log \boldsymbol{W}(z) + \alpha^L \Pi \tag{11}$$

where welfare retains the same definition as before and $\Pi$ is the sum of payments to landowners $\Pi = \sum_{i \in N} \frac{1}{1+\epsilon(i)} \lambda(i) P(i)^{1+\epsilon(i)} T(i)$; subject to equilibrium constraints

$$\forall i, \quad \underbrace{\sum_{z \in Z^c(i)} R(i) L(i,z) + \sum_{z \in Z \notin Z^c(i)} \left[\beta wz + (1-\beta) P(i) \bar{A}\right] L(i,z)}_{\text{Total spending on housing services}} = \underbrace{\lambda(i) P(i)^{1+\epsilon(i)} T(i)}_{\text{Value of supplied housing services}} \tag{12}$$

$$\forall i, z, \quad V(i,z) - \frac{1}{\theta} L(i,z) = \log \boldsymbol{W}(z) \tag{13}$$

$$\forall z, \quad \sum_{i \in N} L(i,z) = L(z) \tag{14}$$

where

$$V(i,z) := \underbrace{k(z) wz \left[\frac{1 - \frac{P(i)\bar{A}}{wz}}{P(i)^\beta}\right] s(i,z)}_{\text{Consumption value}} + \underbrace{\Omega(z) \text{Inc}(i) + \log \nu(i,z)}_{\text{Amenity value}}$$

and $s(i,z) := \left[\frac{\left(1 - \frac{R(i)}{wz}\right)\left(1 - \frac{P(i)\bar{A}}{wz}\right)^{-1}}{1-\beta}\right]^{1-\beta} \left[\frac{\left(R(i) - P(i)\bar{A}\right)\left(wz - P(i)\bar{A}\right)^{-1}}{\beta}\right]^{\beta}$ is the distortion factor, and wages are city specific – $w(i) = \iota(i)$, and $Z^C(i)$ is the set of constrained households in $i$.

How does this problem differ from the unconstrained social planner's? Minimum lot sizes are inherently distortionary, so too many housing services are created relative to what a social planner could achieve with place-based spatial transfers. However, this distortion may be efficient if it increases spatial efficiency in line with the above.

Note: SPP is not differentiable in $R(i)$; so FOC's should be interpreted with caution – they are taken at points where function is locally differentiable (i.e. not at the point where regulation is just binding for some type $z$). The FOC's are:

1. FOC w.r.t $R(i)$:

$$-\left[\sum_{z \in Z^C(i)} \Lambda^{FM}(i,z) \frac{\partial V(i,z)}{\partial R(i)} + \Lambda^A(i) L(i,z)\right] = 0$$

2. FOC w.r.t $L(i,z)$:

$$\sum_{z' \in Z} -\Lambda^{FM}(i,z') \Omega(z') \frac{\partial \log \text{Inc}(i)}{\partial L(i,z)} + \Lambda^{FM}(i,z) \frac{1}{\theta} \frac{1}{L(i,z)} - \Lambda^A(i) E(i,z) = 0$$

where $E(i,z)$ is the housing expenditure by an $(i,z)$ household ($= R(i)$ if $z \in Z^C(i)$ or $\beta wz + (1-\beta) P(i) \bar{A}$ if $z \in Z \notin Z^C(i)$).

3. FOC w.r.t $P(i)$

$$\sum_{z \in Z} -\Lambda^{FM}(i,z) \frac{\partial V(i,z)}{\partial P(i)} - \Lambda^A(i) \sum_{z \in Z \notin Z^C(i)} (1-\beta) \bar{A} L(i,z) + \Lambda^A(i)(1+\epsilon(i)) \lambda(i) P(i)^{\epsilon(i)} T(i) = 0$$

Can we solve this and get an efficient spatial distribution? Or should we just show consumption inefficiency? No need to show all of this, no additional insights.