

# Axisymmetrical equilibrium shapes of two-phase biological membranes

James Mathews 0813855

May 3, 2012

## Abstract

We investigate equilibrium shapes of vesicles formed by biomembranes, which are governed by an elastic energy. We first consider one phase membranes consisting of only one type of molecule, derive the equilibrium Shape Equations and then perform numerical simulations with these equations. We then consider membranes of two types of molecules which separate so that two phases emerge. We model the vesicle as a surface of revolution which is  $C^0$  at the phase boundary, as opposed to using  $C^1$  surfaces as in [JL96]. We derive the Shape Equations and matching conditions for the vesicle and then perform numerical simulations. We then go on to generalise the Shape Equations for membranes whose phases have not fully separated, before considering the existence of solutions to the Shape Equations for two phase membranes.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>One Phase Membranes</b>	<b>4</b>
2.1	Axisymmetric Setting . . . . .	4
2.2	Deriving the Shape Equations . . . . .	6
2.3	Example of using the Shape Equations . . . . .	7
2.4	Numerical Method . . . . .	8
2.5	Numerical Simulation . . . . .	9
2.5.1	Spherical Phase Portraits . . . . .	11
2.6	Conjectures on Existence and Uniqueness of solutions to the Shape Equations . .	12
<b>3</b>	<b>Two Phase Membranes</b>	<b>13</b>
3.1	Axisymmetric Setting . . . . .	13
3.2	Deriving the Shape Equations . . . . .	15
3.2.1	Bulk Terms . . . . .	16
3.2.2	Boundary Terms . . . . .	17
3.3	General Shape Equations . . . . .	20
3.4	Membranes with zero Neck . . . . .	20
3.5	Numerical Method . . . . .	21
3.6	Numerical Simulation . . . . .	22
<b>4</b>	<b>Multiple Phase Membranes</b>	<b>23</b>
<b>5</b>	<b>Existence of Solutions for Two Phase Membranes</b>	<b>24</b>
5.1	$\Gamma$ -convergence . . . . .	25
5.2	Results about surfaces of revolutions . . . . .	26
5.3	Approximate energy of the vesicle . . . . .	27
5.4	$\Gamma$ -Convergence of the approximate energy . . . . .	28
5.5	Extensions . . . . .	29

<b>A Mathematical Background</b>	<b>31</b>
A.1 Surface Curvature . . . . .	31
A.2 Gauss-Bonnet Theorem . . . . .	31
A.3 Direct Method of Calculus of Variations . . . . .	31
A.4 Generalisations of Sobolev Spaces to $\mathbb{R}^k$ . . . . .	32
<b>B Intermediate Steps for Shape Equations</b>	<b>32</b>
B.1 One Phase Membranes . . . . .	32
<b>C Matlab Code</b>	<b>34</b>
C.1 One Phase Membranes . . . . .	34
C.2 Two Phase Membranes . . . . .	37
<b>D Catalogue of Solutions</b>	<b>40</b>
D.1 One Phase Membranes . . . . .	40
D.2 Two Phase Membranes . . . . .	56
<b>E Classifying Shapes of Cells</b>	<b>57</b>

## 1 Introduction

Without cells, there would be no life; the cell is the building block of life. A cell consists of the essential biomolecules; water, protein, nucleic acids, carbohydrates and lipids. Depending on the geometry of the molecules, lipids can pack into a bilayer or a stack of bilayers [YL08], and we can see two stacks in Figure 1 below.

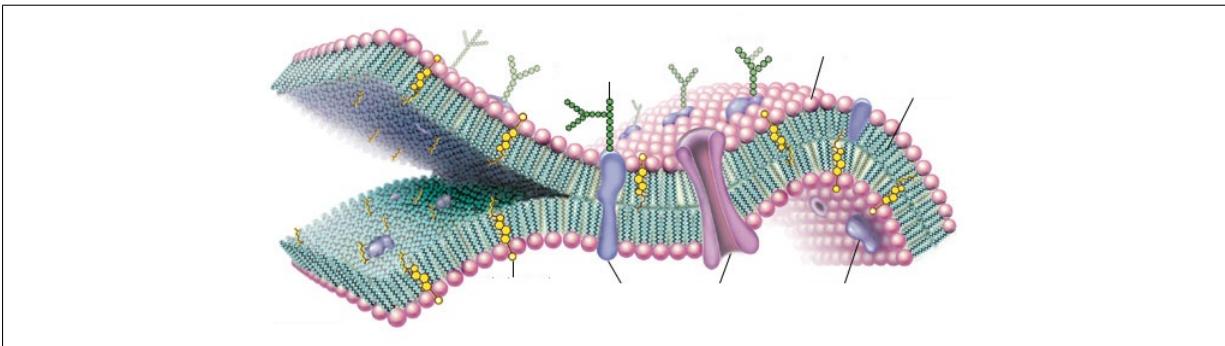


Figure 1: A typical cell membrane <http://media.web.britannica.com/eb-media/74/53074-004-9F65D813.jpg>.

The cell membranes of almost all living organisms (and viruses) are made up of this lipid bilayer, producing a barrier of only a few nanometres which separates living entities from the lifeless environment. The membrane has a crucial feature; selective permeability. This allows it to choose what atoms and molecules can cross it, depending on their chemical properties [W12a].

A vesicle is a bubble of liquid within another liquid, and it can be thought of as a small membrane enclosed sack that can store or transport substances [W12b]. They often form naturally as lipid bilayers bend and close up. Unilamellar vesicles have a single membrane, as opposed to multilamellar vesicles which have more than one. Traditionally giant unilamellar vesicles ( $\approx 10 - 30$  micrometers) have been studied [BHW03] since they are large enough to observe and manipulate. In Figure 2 we see such vesicles, made of a mixture of lipids.

In giant vesicles, the osmotic pressures in and out of the vesicle are equal, thus to control the vesicle volume you have to change the density or concentration of molecules in the lipid bilayer. While it is very easy to bend a membrane, stretching a membrane by only a small amount ruptures it [YL08]. Thus, we assume the stretching energy is constant and only consider the bending energy of the vesicle.

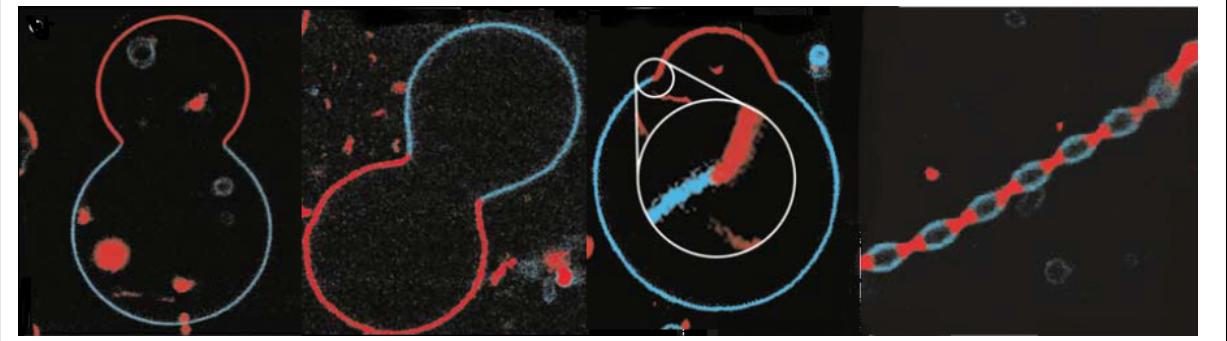


Figure 2: Photos taken from a microscope of a vesicle made up of the lipids sphingomyelin, cholesterol and DOPC. The first two combine in an ordered phase ( $L_o$ ) while DOPC is in a disordered phase ( $L_d$ ), and the photos show cross sectional images when both these phases exist, with chemicals used to label the ( $L_o$ ) phase blue and ( $L_d$ ) phase red. [BHW03]

Spontaneous curvature is introduced to account for different chemical structures and adjacent fluids to the vesicle, causing additional positive or negative curvature to the vesicle, and we assume it is constant. Through observations we can measure the spontaneous curvature of different lipids, for example for cholesterol it is  $-1/2.9 \text{ nm}^{-1}$  and for DOPC it is  $-1/20 \text{ nm}^{-1}$  [W12c]. Although we will be mainly working without dimensions, when we do need them we will use  $\text{nm}^{-1}$ . We also introduce the flip flop of a membrane, with a high flip flop allowing layers of molecules to slide over each other, and a low flip flop not allowing this. This concept will be used in section 3 when we consider phase separation.

In section 2 we consider axisymmetrical vesicles (vesicles which have rotational symmetry) consisting of just one type of lipid. We assume the area and volume of the vesicle is fixed, and proceed to parametrise the vesicle by arclength. We then derive the Shape Equations from the bending energy by using the Euler-Lagrange equations, Lagrangian multipliers and Lagrangian functions. We use these Shape Equations to find out under what conditions a sphere minimises the bending energy, and then we discuss how to implement a numerical method to find solutions. We then carry out numerical simulations in Matlab and analyse the results, and produce a catalogue of solutions in the appendix.

In section 3 we again consider axisymmetrical vesicles, but we now have two lipids  $\alpha$  and  $\beta$  mixing. The surface area of the membrane is fixed by the total number of molecules,  $N_\alpha + N_\beta$ . We assume a high flip flop, so phase separation fully occurs and we end up with two phases, one consisting solely of  $\alpha$  molecules ( $\alpha$  phase) and another consisting solely of  $\beta$  molecules ( $\beta$  phase) with corresponding surface areas  $A^{(\alpha)}$  and  $A^{(\beta)}$ . Since the number of each type of molecules and the density per unit area are fixed, we get that  $A^{(\alpha)}$ ,  $A^{(\beta)}$  and total volume are fixed. We introduce an additional energy term which depends on the line tension at the phase boundary, with a high line tension forcing the neck of the vesicle to have a small diameter. We parametrise both phases of the vesicle which then allows us to derive the Shape Equations similarly to the one phase membrane, although we now have additional matching conditions. We also consider when membranes have zero neck (diameter of the neck is zero), before discussing how to implement a numerical method to find solutions to the Shape Equation. Due to the rich dynamics of the vesicle, we only provide a small summary of numerical solutions.

In section 4 we generalise the results of section 3 to vesicles with three phases (three different lipids), and also place no conditions on the flip flop of the membrane so that the phases do not necessarily fully separate. We additionally allow for ghost interfaces where the vesicle has kinks not at phase boundaries.

In Section 5 we consider the notions of  $\Gamma$ -convergence and equi-coerciveness, and use these to establish when minimisers of the two phase total energy (defined in section 3) exist. We consider some results about surfaces of revolution before defining the approximate energy, and stating Helmers' theorem about convergence of this energy. We then consider extensions of this theorem to different number of phases and ghost interfaces.

## 2 One Phase Membranes

**Definition** (Bending energy of a vesicle). As noted in [SBL91], the bending energy of a vesicle (assuming a spontaneous curvature model) is given by

$$F_b := \frac{\kappa}{2} \oint (H - C_0)^2 dA + \kappa_G \oint K dA,$$

where  $H$  is the mean curvature,  $K$  is the Gauss curvature,  $C_0$  is the spontaneous curvature and  $\kappa, \kappa_G$  are constant bending rigidities.

### 2.1 Axisymmetric Setting

In order to minimise the bending energy, we first introduce a parametrisation for the axisymmetric vesicle [SBL91]. Let  $S$  be the arclength of the contour, measured from  $S = S_0$  to  $S_1$ . We can then introduce the  $x$  and  $z$  coordinates, as shown in Figure 3. Additionally, let  $\psi(S)$  be the angle between the tangent and the  $x$  axis.

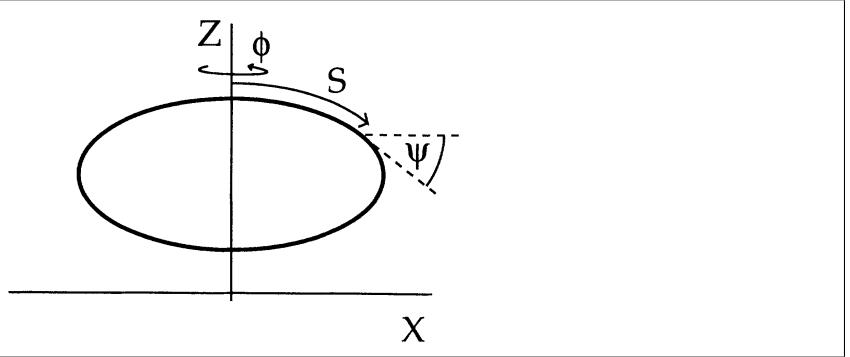


Figure 3: The parametrisation of the vesicle. [SBL91]

We note the geometric relations  $\dot{x} = \cos \psi$  and  $\dot{z} = -\sin \psi$  hold everywhere and that a parametrisation of the vesicle is given by  $\underline{r}: [S_0, S_1] \times [0, 2\pi] \mapsto \mathbb{R}^3$ ,

$$\underline{r}(S, \phi) = (x(S) \cos \phi, x(S) \sin \phi, z(S)).$$

**Proposition 2.1.** *In the above parametrisation above we have*

$$K = \frac{\dot{\psi} \sin \psi}{x}, \quad H = \frac{\sin \psi}{x} + \dot{\psi} \text{ and } dA = x d\phi dS.$$

*Proof.* Firstly, we note that we have

$$\begin{aligned} \underline{r}_s &= (\dot{x} \cos \phi, \dot{x} \sin \phi, \dot{z}), & \underline{r}_\phi &= (-x \sin \phi, x \cos \phi, 0), \\ \underline{r}_{\phi\phi} &= (-x \cos \phi, -x \sin \phi, 0), & \underline{r}_{s\phi} &= (-\dot{x} \sin \phi, \dot{x} \cos \phi, 0) \text{ and } \underline{r}_{ss} = (\ddot{x} \cos \phi, \ddot{x} \sin \phi, \ddot{z}). \end{aligned}$$

Additionally, we have  $\dot{x}^2 + \dot{z}^2 = 1$  from the geometric relations. As a result we get

$$\underline{n} = (\dot{z} \cos \phi, \dot{z} \sin \phi, -\dot{x}),$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} x \sin \psi & 0 \\ 0 & \dot{\psi} \end{pmatrix}.$$

Thus, we conclude

$$H = \text{Tr} \left[ \begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \sin \psi & 0 \\ 0 & \dot{\psi} \end{pmatrix} \right] = \text{Tr} \left( \begin{pmatrix} \frac{\sin \psi}{x} & 0 \\ 0 & \dot{\psi} \end{pmatrix} \right) = \frac{\sin \psi}{x} + \dot{\psi}$$

and

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\dot{\psi} \sin \psi}{x}.$$

We also can express the area element in terms of the new coordinates, giving

$$dA = \| \underline{r}_s \wedge \underline{r}_\phi \| d\phi dS = x d\phi dS.$$

□

**Lemma 2.2** (Gauss Bônnnet Theorem for surfaces of revolution). *For a surface of revolution  $\oint K dA = 4\pi$ .*

*Proof.* We have

$$\begin{aligned} \oint K dA &= 2\pi \int_0^{S_1} \frac{\dot{\psi} \sin \psi}{x} x dS = 2\pi \left[ -\cos(\psi(S)) \right]_0^{S_1} \\ &= 2\pi \left[ \cos(\psi(0)) - \cos(\psi(S_1)) \right] = 2\pi(1 + 1) = 4\pi. \end{aligned}$$

□

Hence from now on we can ignore the second term in the bending energy since it is constant, so we can just consider

$$E \equiv \frac{\kappa}{2} \oint (H - C_0)^2 dA,$$

and thus we need to minimise

$$E(\psi, x, z) = \kappa \int_0^{2\pi} \int_{S_0}^{S_1} \frac{x}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 dS d\phi = 2\pi \kappa \int_{S_0}^{S_1} \frac{x}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 dS.$$

Since we want to minimise this with respect to area and volume constraints we first need to express the area  $A$  and volume  $V$  as integrals with respect to  $S$ . Clearly we have

$$A = 2\pi \int_{S_0}^{S_1} x dS \text{ and } V = \pi \int_{S_0}^{S_1} x^2 \sin \psi dS.$$

We now introduce Lagrange multipliers  $\Sigma$  and  $P$ , then the Shape Equations are attained by solving  $\delta F \equiv \delta(E + \Sigma A + PV) = 0$ , where  $\delta$  denotes variation. We vary these Lagrangian multipliers in order to achieve the prescribed area and volume. Thus for a general vesicle, to minimise the bending energy we need to show that

$$F(\psi, x, z, \gamma, \eta) = E(\psi, x, z) + \Sigma A + PV + 2\pi \kappa \int_{S_0}^{S_1} \gamma(\dot{x} - \cos \psi) + \eta(\dot{z} + \sin \psi) dS = 2\pi \kappa \int_{S_0}^{S_1} L dS$$

has zero variation, since we have introduced the Lagrangian functions  $\gamma = \gamma(S)$  and  $\eta = \eta(S)$  to account for the earlier geometric relations  $\dot{x} = \cos \psi$  and  $\dot{z} = -\sin \psi$  holding everywhere. This then allows us to independently vary each variable when considering a variation. We see that

$$L(\psi, x, z, \gamma, \eta) = \frac{x}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 + \bar{\Sigma} x + \frac{\bar{P}}{2} x^2 \sin \psi + \gamma(\dot{x} - \cos \psi) + \eta(\dot{z} + \sin \psi)$$

and thus we have that

$$F = 2\pi \kappa \int_{S_0}^{S_1} \frac{x}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 + \bar{\Sigma} x + \frac{\bar{P}}{2} x^2 \sin \psi + \gamma(\dot{x} - \cos \psi) + \eta(\dot{z} + \sin \psi) dS,$$

needs to have zero variation, where we have rescaled the following parameters:

$$\bar{\Sigma} = \frac{\Sigma}{\kappa} \text{ and } \bar{P} = \frac{P}{\kappa}.$$

## 2.2 Deriving the Shape Equations

**Theorem 2.3** (Shape Equations for a general vesicle). *A general axisymmetric vesicle that minimises the bending energy  $E$  subject to volume and area constraints satisfies the following equations for  $S_0 < S < S_1$ :*

$$\ddot{\psi} = \frac{\cos \psi \sin \psi}{x^2} - \frac{\dot{\psi}}{x} \cos \psi + \frac{\bar{P}}{2} x \cos \psi + \frac{\gamma}{x} \sin \psi + \frac{\eta}{x} \cos \psi \quad (2.1)$$

$$\dot{\gamma} = \frac{1}{2}(\dot{\psi} - C_0)^2 - \frac{\sin^2 \psi}{2x^2} + \bar{\Sigma} + \bar{P}x \sin \psi \quad (2.2)$$

$$\dot{\eta} = 0 \quad (2.3)$$

$$\dot{x} = \cos \psi \quad (2.4)$$

$$\dot{z} = -\sin \psi. \quad (2.5)$$

*Proof.* We introduce a variation on each of the variables, so we have:

$$\begin{aligned} \psi(S) &= \psi_0 + \delta\psi(S), \\ x(S) &= x_0 + \delta x(S), \quad z(S) = z_0 + \delta z(S), \\ \gamma(S) &= \gamma_0 + \delta\gamma(S), \quad \eta(S) = \eta_0 + \delta\eta(S) \\ S_0 &= S_{0,0} + \delta S_0, \quad S_1 = S_{1,0} + \delta S_1. \end{aligned}$$

This leads to the variation of  $F$  being written as (noting that  $\partial L/\partial \dot{\gamma} = \partial L/\partial \dot{\eta} = 0$ ):

$$\delta F = \int_{S_0}^{S_1} \left[ \left( \delta\psi \frac{\partial L}{\partial \psi} + \delta\dot{\psi} \frac{\partial L}{\partial \dot{\psi}} \right) + \left( \delta x \frac{\partial L}{\partial x} + \delta\dot{x} \frac{\partial L}{\partial \dot{x}} \right) + \left( \delta z \frac{\partial L}{\partial z} + \delta\dot{z} \frac{\partial L}{\partial \dot{z}} \right) + \delta\gamma \frac{\partial L}{\partial \gamma} + \delta\eta \frac{\partial L}{\partial \eta} \right] dS.$$

Using integration by parts, this expression is more conveniently written as the following;

$$\begin{aligned} \delta F = \int_{S_0}^{S_1} &\left[ \left( \frac{\partial L}{\partial \psi} - \frac{d}{dS} \frac{\partial L}{\partial \dot{\psi}} \right) \delta\psi + \left( \frac{\partial L}{\partial x} - \frac{d}{dS} \frac{\partial L}{\partial \dot{x}} \right) \delta x + \left( \frac{\partial L}{\partial z} - \frac{d}{dS} \frac{\partial L}{\partial \dot{z}} \right) \delta z + \delta\gamma \frac{\partial L}{\partial \gamma} + \delta\eta \frac{\partial L}{\partial \eta} \right] dS \\ &- [\mathcal{H}\delta S]_{S_0}^{S_1} + \left[ \delta\psi \frac{\partial L}{\partial \dot{\psi}} \right]_{S_0}^{S_1} + \left[ \delta x \frac{\partial L}{\partial \dot{x}} \right]_{S_0}^{S_1} + \left[ \delta z \frac{\partial L}{\partial \dot{z}} \right]_{S_0}^{S_1} = 0, \end{aligned} \quad (2.6)$$

where the term with the Hamiltonian

$$\mathcal{H} := -L + \dot{\psi} \frac{\partial L}{\partial \dot{\psi}} + \dot{x} \frac{\partial L}{\partial \dot{x}} + \dot{z} \frac{\partial L}{\partial \dot{z}}$$

comes from considering variations in the length of the contour. This becomes more apparent if we switch back to a generalised parametrisation, and when we deal with this more rigorously for two phase membranes we will indeed do this. By considering the bulk terms in the integral, equations (2.1)-(2.5) follow from solving

$$\frac{\partial L}{\partial \psi} - \frac{d}{dS} \frac{\partial L}{\partial \dot{\psi}} = 0, \quad \frac{\partial L}{\partial x} - \frac{d}{dS} \frac{\partial L}{\partial \dot{x}} = 0, \quad \frac{\partial L}{\partial z} - \frac{d}{dS} \frac{\partial L}{\partial \dot{z}} = 0, \quad \frac{\partial L}{\partial \gamma} = 0, \quad \frac{\partial L}{\partial \eta} = 0$$

respectively (see Appendix B.1). □

Theorem 2.3 only gives us the differential equations that have to be satisfied for a general vesicle. The boundary terms in (2.6) also need to vanish, which give additional constraints that need to be satisfied. We examine these boundary terms specifically for a vesicle of spherical topology and now assume that  $S_0 = 0$  and  $S_1$  is the contour length between the two poles (although we could perform this analysis in other cases such as when it has toroidal topology [JS94]).

**Theorem 2.4** (Shape Equations for a vesicle of spherical topology). *A vesicle of spherical topology that minimises the bending energy  $E$  subject to volume and area constraints satisfies the following equations for  $0 < S < S_1$ :*

$$\ddot{\psi} = \frac{\cos \psi \sin \psi}{x^2} - \frac{\dot{\psi}}{x} \cos \psi + \frac{\bar{P}}{2} x \cos \psi + \frac{\gamma}{x} \sin \psi, \quad \psi(0) = 0, \quad \psi(S_1) = \pi, \quad (2.7)$$

$$\dot{\gamma} = \frac{1}{2}(\dot{\psi} - C_0)^2 - \frac{\sin^2 \psi}{2x^2} + \bar{\Sigma} + \bar{P}x \sin \psi, \quad \gamma(0) = 0, \quad \gamma(S_1) = 0, \quad (2.8)$$

$$\dot{x} = \cos \psi, \quad x(0) = 0, \quad x(S_1) = 0, \quad (2.9)$$

$$\dot{z} = -\sin \psi. \quad (2.10)$$

*Proof.* We have the boundary conditions  $\psi(0) = 0$ ,  $\psi(S_1) = \pi$  and  $x(0) = x(S_1) = 0$ . Hence we require the variations  $\delta\psi|_{S=0}$ ,  $\delta\psi|_{S=S_1}$ ,  $\delta x|_{S=0}$  and  $\delta x|_{S=S_1}$  all to be zero, so the second and third boundary terms disappear in (2.6). For the fourth boundary term to vanish we need that  $\partial L/\partial z = \eta$  is zero at both  $S = 0$  and  $S = S_1$  since we can vary  $\delta z|_{S=0}$  and  $\delta z|_{S=S_1}$  independently. Thus we require that  $\eta \equiv 0$  by (2.3). Finally we consider the first boundary term in (2.6), and since the contour length is not fixed we require that  $\mathcal{H}(0) = \mathcal{H}(S_1) = 0$  for this boundary term to be zero. As  $\partial L/\partial S = 0$  we get that the Hamiltonian is conserved ( $\dot{\mathcal{H}} = 0$ ), hence  $\mathcal{H} \equiv 0$ . Thus, we have

$$\mathcal{H} = \frac{x}{2} \left[ \dot{\psi}^2 - \left( \frac{\sin \psi}{x} - C_0 \right)^2 \right] - \bar{\Sigma}x - \frac{\bar{P}}{2} x^2 \sin \psi + \gamma \cos \psi \equiv 0, \quad (2.11)$$

giving  $\gamma(0) = \gamma(S_1) = 0$  from the boundary conditions of  $x$  and  $\psi$ .  $\square$

We note that the equation  $\dot{z} = -\sin \psi$  can be uncoupled from the system of differential equations since we can solve it once we determine  $\psi$ , and also that when  $\cos \psi \neq 0$  we can determine  $\gamma$  from (2.11):

$$\gamma = \frac{\bar{\Sigma}x}{\cos \psi} + \frac{\bar{P}x^2 \sin \psi}{2 \cos \psi} - \frac{x}{2 \cos \psi} \left[ \dot{\psi}^2 - \left( \frac{\sin \psi}{x} - C_0 \right)^2 \right].$$

Summarising this, we can restate Theorem 2.4 as the following:

**Corollary 2.5** (Alternative form of Shape Equations for a vesicle of spherical topology). *When  $\cos \psi \neq 0$  we can instead solve the following system of equations:*

$$\begin{aligned} \ddot{\psi} &= \frac{\cos \psi \sin \psi}{x^2} - \frac{\dot{\psi} \cos \psi}{x} + \frac{\bar{P}x \cos \psi}{2} + \frac{\bar{\Sigma} \sin \psi}{\cos \psi} + \frac{\bar{P}x \sin^2 \psi}{2 \cos \psi} \\ &\quad + \frac{\sin \psi}{2 \cos \psi} \left( \frac{\sin \psi}{x} - C_0 \right)^2 - \frac{\dot{\psi}^2 \sin \psi}{2 \cos \psi}, \quad \psi(0) = 0, \quad \psi(S_1) = \pi. \end{aligned} \quad (2.12)$$

$$\dot{x} = \cos \psi \quad x(0) = 0, \quad x(S_1) = 0. \quad (2.13)$$

### 2.3 Example of using the Shape Equations

We wish to use Corollary 2.5 to find the conditions for when a sphere of radius  $R$  minimises the bending energy. An arclength parametrisation of the sphere is given by

$$(x(S), z(S)) = \left( R \sin \left( \frac{S}{R} \right), R \cos \left( \frac{S}{R} \right) \right),$$

thus  $\dot{x}(S) = \cos(S/R)$ ,  $\dot{z}(S) = -\sin(S/R)$  and hence  $\psi(S) = S/R$  from (2.13). We also compute the derivatives  $\dot{\psi}(S) = 1/R$  and  $\ddot{\psi}(S) = 0$ . Substituting this into (2.12) gives us that the following equation needs to be satisfied:

$$\begin{aligned} 0 &= \frac{\sin(S)}{\cos(S)} \left\{ \frac{\bar{P}R}{2} \left[ \sin^2 \left( \frac{S}{R} \right) + \cos^2 \left( \frac{S}{R} \right) \right] + \bar{\Sigma} + \frac{1}{2} \left( \frac{1}{R^2} - \frac{2C_0}{R} + C_0^2 \right) - \frac{1}{2R^2} \right\} \\ &= \frac{\sin(S)}{2R \cos(S)} \left\{ \bar{P}R^2 + (2\bar{\Sigma} + C_0^2)R - 2C_0 \right\}. \end{aligned}$$

Hence for the Shape Equations to be satisfied everywhere for a sphere, we require

$$\bar{P}R^2 + (2\bar{\Sigma} + C_0^2)R - 2C_0 = 0 \Leftrightarrow R = \frac{-2\bar{\Sigma} - C_0^2 \pm \sqrt{(2\bar{\Sigma} + C_0^2)^2 + 8C_0\bar{P}}}{2\bar{P}}. \quad (2.14)$$

So setting  $C_0=0$  and discarding the case when  $R = 0$  for obvious reasons, we get the relation

$$R = -\frac{2\bar{\Sigma}}{\bar{P}}.$$

## 2.4 Numerical Method

I was unable to find a general explicit solution of equations (2.12) and (2.13) after experimenting with both Mathematica and Maple, although it may still have an explicit solution in some cases. However when we try to solve these equations numerically instead we encounter a significant problem as a result of dividing by  $\cos \psi$ . Since  $\psi(0) = 0$  and  $\psi(S_1) = \pi$ , then must exist  $e_\psi$  such that  $\psi(e_\psi) = \pi/2$  by the continuity of  $\psi$  and hence  $\cos(\psi(e_\psi)) = 0$ . Since this dividing by zero happens in the middle of the numerical computation, there is no solution to this problem. Instead, we return to Shape Equations (2.7)-(2.10). We introduce the variable  $U = \dot{\psi}$  in order to reduce the system to a system of five first order differential equations, with equation (2.7) transformed to

$$\dot{\psi} = U \quad (2.7a)$$

$$\dot{U} = \frac{\cos \psi \sin \psi}{x^2} - \frac{U}{x} \cos \psi + \frac{\bar{P}}{2}x \cos \psi + \frac{\gamma}{x} \sin \psi \quad (2.7b)$$

We now deal with the issue of the initial condition  $x(0) = 0$ , since we are dividing by  $x$  in (2.7b) and (2.8). To remedy this problem we want to replace this initial condition by  $x(0) = \varepsilon$ , where  $\varepsilon > 0$  and arbitrarily small. However, we cannot modify the initial condition by doing this, since having  $u(0) \neq 0$ ,  $\psi(0) = 0$  and  $x(0) = \varepsilon$  is inconsistent, as seen in Figure 4. Indeed, merely modifying the initial condition in this way results in Matlab unexpectedly halving the initial value of  $U$  within the first few steps of the numerical solver. Since the vesicle is locally spherical at the north pole, we have

$$x(0) = R \sin \psi(0) = \frac{1}{U(0)} \sin \psi(0).$$

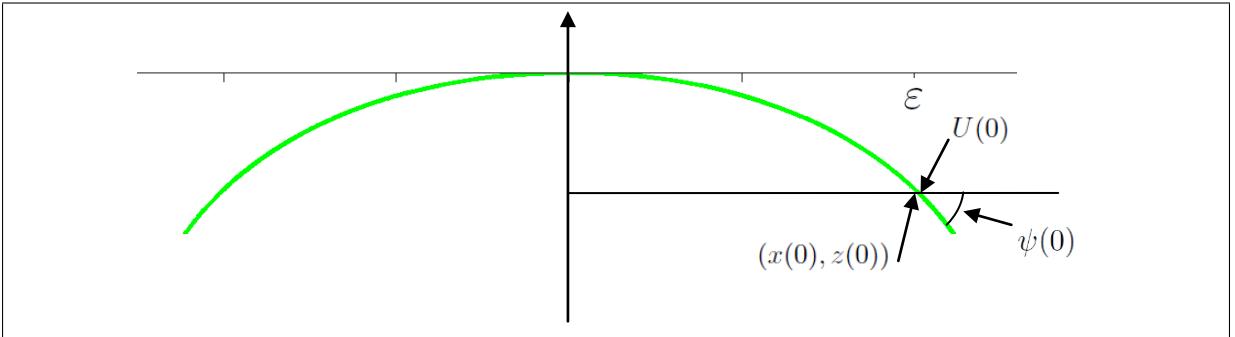


Figure 4: Inconsistent initial conditions.

Thus we choose  $\psi(0) = \varepsilon$  and  $x(0) = \sin \varepsilon / U(0)$ . We also check at this point that we have enough boundary conditions to solve the Shape Equations. We need to solve for the functions  $\psi$ ,  $U$ ,  $\gamma$ ,  $x$  and  $z$ , as well as the arclength  $S_1$ . We can use the condition  $x(S_1) = 0$  to determine  $S_1$ , and we can arbitrarily choose an initial condition  $z(0) = 0$  for (2.10), since we are only interested in the relative vertical position. Then we have five initial/boundary conditions ( $z(0) = 0$ ,  $\psi(0) = 0$ ,  $\psi(S_1) = \pi$ ,  $\gamma(0) = 0$ ,  $x(0) = 0$ ) for the three first order and one second order

differential equations and hence can specify a solution to the Shape Equations. We do not need the condition  $\gamma(S_1) = 0$  to determine the solution, and notice that it is automatically satisfied when  $x(S_1) = 0$  and  $\psi(S_1) = \pi$ , and we should check this still holds when we calculate the solution numerically. Hence, to solve the Shape Equations we solve the following system of differential equations by performing a shooting method to find the solution, choosing  $U(0) = u$  such that the vesicle also satisfies  $\psi(S_1) = \pi$  and  $\gamma(S_1) = 0$ , where  $S_1$  is defined as the solution of  $x(S_1) = 0$ .

$$\begin{aligned} \dot{\psi} &= U, & \psi(0) &= \varepsilon, \\ \dot{U} &= \frac{\cos \psi \sin \psi}{x^2} - \frac{U}{x} \cos \psi + \frac{\bar{P}}{2} x \cos \psi + \frac{\gamma}{x} \sin \psi, & U(0) &= u, \\ \dot{\gamma} &= \frac{1}{2}(\dot{\psi} - C_0)^2 - \frac{\sin^2 \psi}{2x^2} + \bar{\Sigma} + \bar{P}x \sin \psi, & \gamma(0) &= 0, \\ \dot{x} &= \cos \psi, & x(0) &= \frac{\sin \varepsilon}{u}, \\ \dot{z} &= -\sin \psi, & z(0) &= 0. \end{aligned}$$

So to determine the shape of the vesicle we have the parameters  $\bar{P}$ ,  $\bar{\Sigma}$ ,  $U(0)$  and  $C_0$ , which then determine  $S_1$ . We note the first three are determined by the shape of the vesicle, which can be described in terms of its reduced volume, that is

$$v \equiv \frac{V}{\frac{4\pi}{3}(\frac{A}{4\pi})^{3/2}} = 6\sqrt{\pi} \frac{V}{A^{\frac{3}{2}}},$$

so by specifying  $v$  and  $C_0$  we determine the shape of the vesicle. Also, the only parameters that have dimension are  $\bar{P}$ ,  $\bar{\Sigma}$ , and  $C_0$ , and by choosing a dimension of one of these we determine the dimensions of the whole system.

## 2.5 Numerical Simulation

We firstly fix  $\bar{P} = 5$ , which defines a length scale. We can in fact choose to fix  $\bar{\Sigma}$  or  $C_0$  instead, and we would proceed analogously. Once we fix a length scale, for the purpose of this section we also fix  $C_0 = -0.1$ . We use Matlab and the code we use is detailed in Appendix C.1, while the results of the simulations are detailed in Appendix D.1. We initially start with  $\bar{\Sigma} > 0$  but I am unable to find any solutions, so we instead consider  $\bar{\Sigma} < 0$ . Between  $0 > \bar{\Sigma} \geq -1$  we find phase portraits which are stomatocytes 46 and spherical, as shown in Figure 5.

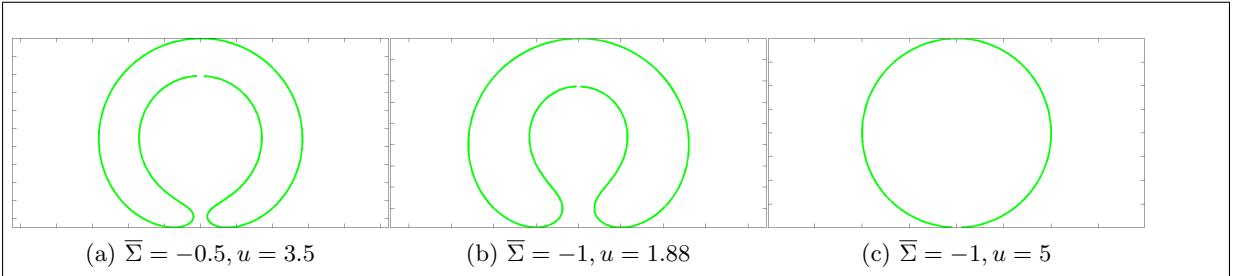


Figure 5: Various phase portraits when  $0 > \bar{\Sigma} \geq -1$  ( $\bar{P} = 5$ ,  $C_0 = -0.1$ ).

We notice these stomatocytes cells cease to exist when we reach  $\bar{\Sigma} = -2$ , after slowly getting fatter. This is a theme throughout solutions of the Shape Equations, with certain shapes only existing in a small region of  $(\bar{\Sigma}, \bar{P})$  space. We now make some more observations from the catalogue of solutions. Firstly, in general the more precise the value of  $u$ , the less stable the phase portrait is. This is because very small changes in  $u$  ( $\approx 0.0001$ ) can lead to large perturbations in some regions of  $(u, \bar{\Sigma})$  space, but in other regions it is relatively stable. However, this is clearly not an accurate measure of stability and for a more rigorous view see [SBL91], which considers

the second variation to determine stability. Thus, while some of the shapes might technically solve the system of differential equations, in nature they are too unstable to actually exist as cell membranes.

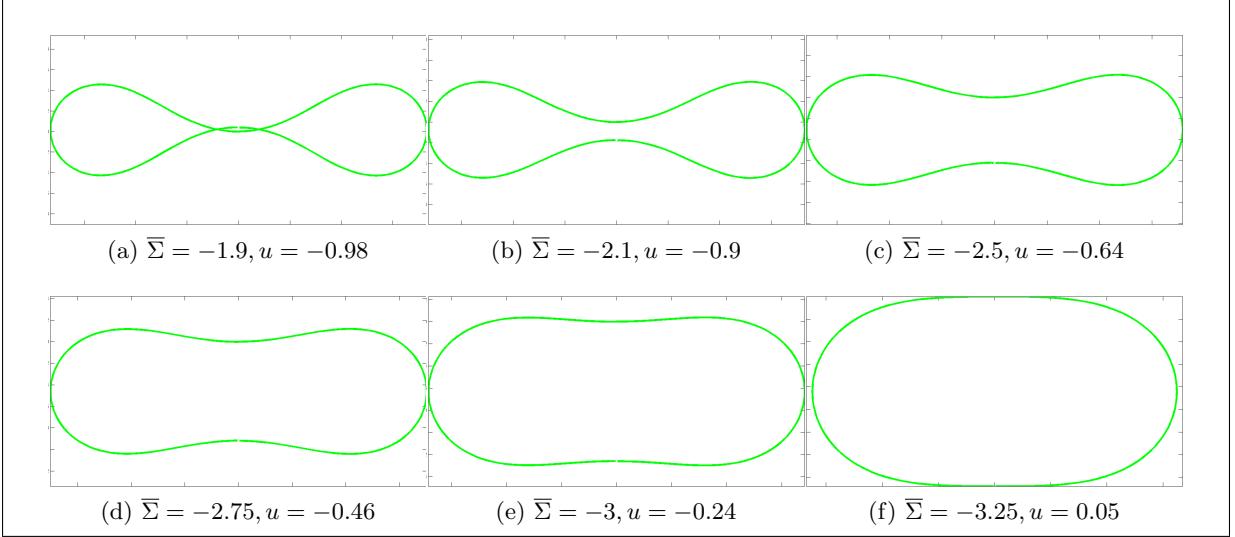


Figure 6: Dumbbell vesicles only exist for  $-1.9 > \bar{\Sigma} > -3.5$ . We see that at  $\bar{\Sigma} = -1.9$  we have intersection in the phase portrait, so it is not feasible. As  $\bar{\Sigma}$  decreases the dumbbell vesicles slowly get fatter and more prolate, until they no longer exist at  $\bar{\Sigma} = -3.5$ . We also see that the value of  $u$  slowly increases to  $\approx 0$  during this period ( $\bar{P} = 5, C_0 = -0.1$ ).

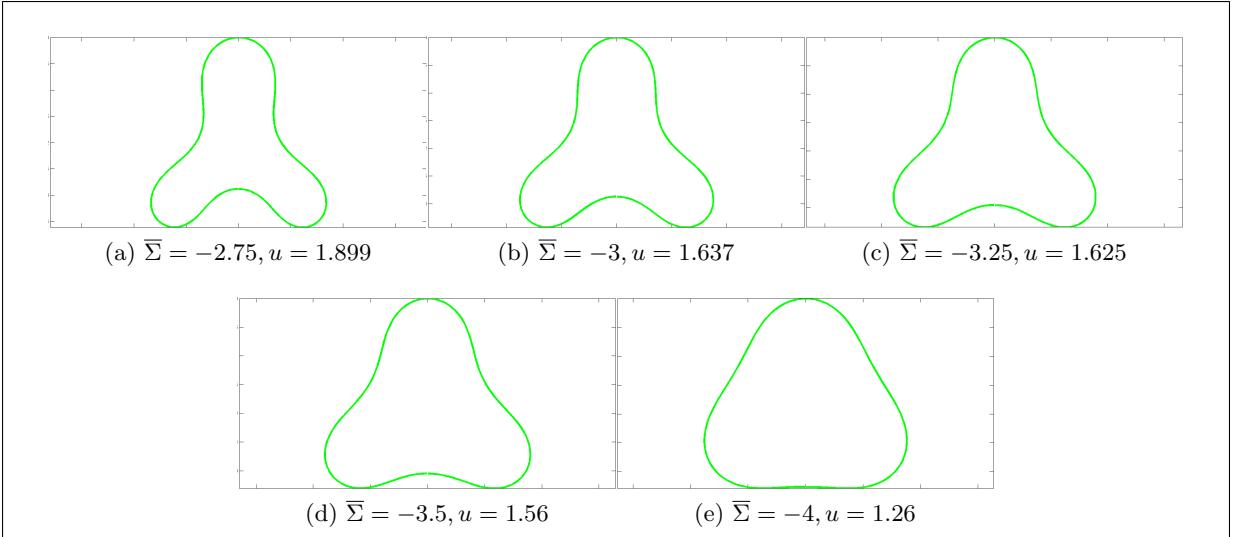


Figure 7: Type 3 echinocytes (echinocytes with three ‘protrusions’) only exist between  $-2.5 > \bar{\Sigma} > -4.5$ . The first point where we see them is at  $\bar{\Sigma} = -2.75$ , since although it gets very close to forming type 3 echinocytes at  $\bar{\Sigma} = -2.5$  they do not occur. As  $\bar{\Sigma}$  decreases the type 3 echinocytes slowly get fatter and more triangular, until they no longer exist after  $\bar{\Sigma} = -4$ . This happens whilst  $u$  slowly decreases to  $\approx 1.25$  ( $\bar{P} = 5, C_0 = -0.1$ ).

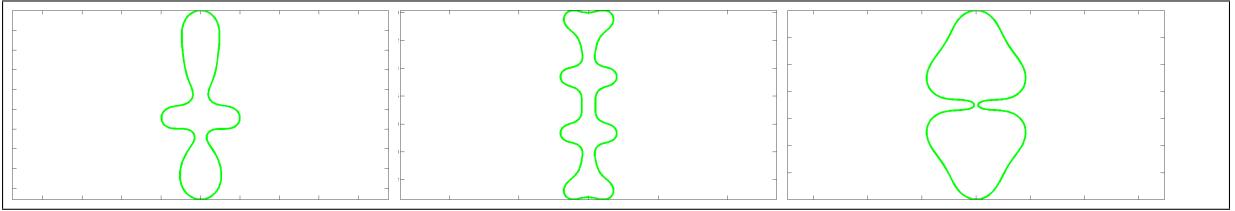


Figure 8: Some examples of unstable phase portraits. We can see that in the third example the limit of the shape would be better modelled by a two phase membrane.

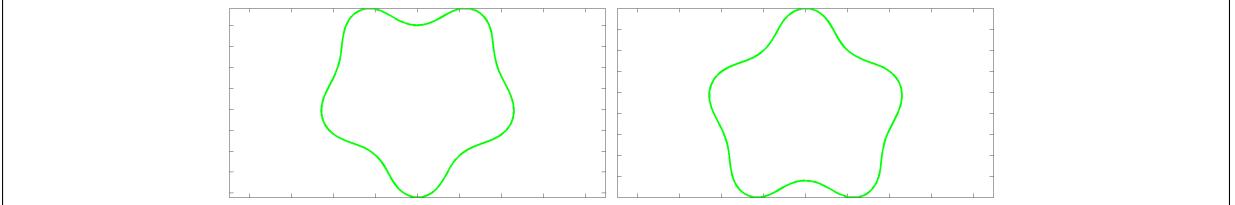


Figure 9: Phase portraits for  $u = -1.24$  and  $u = 1.76$  ( $\bar{P} = 5$ ,  $\bar{\Sigma} = 0.5$ ,  $C_0 = -0.1$ ). We note that vesicles which do not have horizontal symmetry (they have vertical symmetry since we are considering surfaces of revolution) often have their reflection as another possible solution, for a different value of  $u$ . In fact, if  $u_1 < 0 < u_2$  where  $u_1$  and  $u_2$  correspond to two solutions which are reflections of each other, then  $|u_1| \neq |u_2|$  for all the shapes in the catalogue. This is because  $u_2$  defines the initial curvature at the north pole, which has initial condition  $\psi(0) = 0$  whereas  $u_1$  defines the intial curvature at the south pole which has initial condition  $\psi(S_1) = \pi$ .

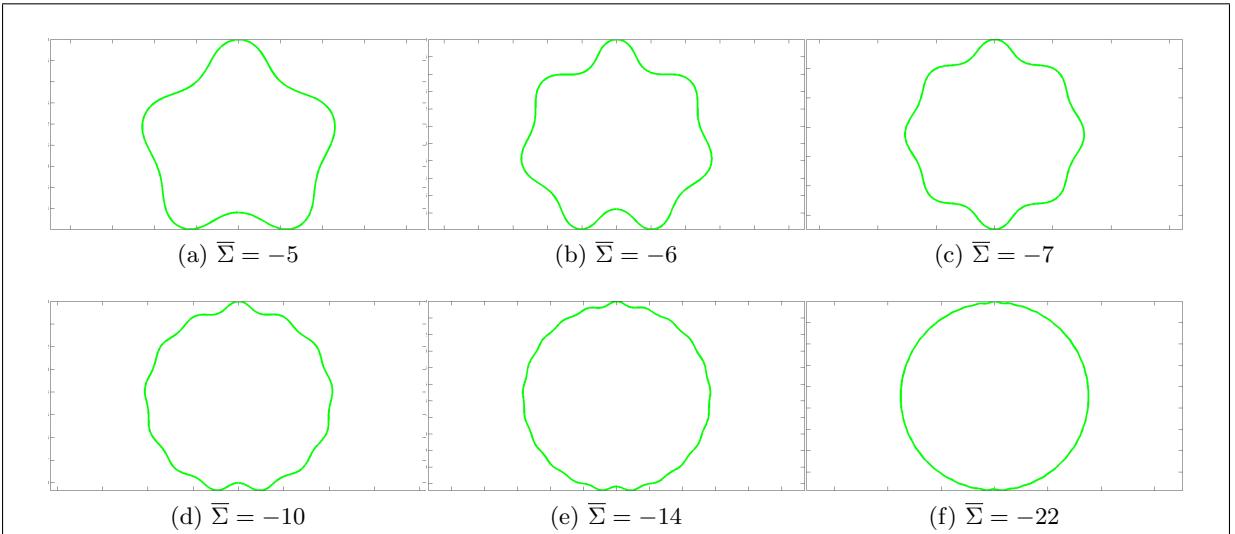


Figure 10: Finally, we note that as  $\bar{\Sigma} \rightarrow -\infty$  the shapes become more indistinguishable from circles. We see that the maximum number of ‘protrusions’ that the vesicle has increases as  $\bar{\Sigma}$  decreases, since when  $\bar{\Sigma} = -5$  we have five ‘protrusions’, and when  $\bar{\Sigma} = -6$  we have seven ‘protrusions’. By the time we reach  $\bar{\Sigma} = -10$  we have 13 ‘protrusions’, and after that it becomes too difficult to count how many we have. An interesting question would be to analyse the smoothness of these shapes as  $\bar{\Sigma}$  decreases.

### 2.5.1 Spherical Phase Portraits

By examining the catalogue of solutions, we note that for  $\bar{\Sigma} < 0$  we get a circle for precisely one value of  $u$ , and these values are noted in Table 1. We also see that the circles increase in area as  $\bar{\Sigma}$  decreases, since  $u$  decreases and hence the radius of the circle increases.

Referring back to equation (2.14) and substituting in the values for  $\bar{P}$ ,  $C_0$  and  $u = 1/R$  we get

Table 1: Values of  $u$  and  $\bar{\Sigma}$  such that the phase portrait is a circle

$\bar{\Sigma}$	-1	-1.9	-2.1	-2.5	-2.75	-3	-3.25	-3.5
$u$	5	1.425	1.27	1.045	0.93	0.86	0.79	0.725
$\bar{\Sigma}$	-4	-4.5	-5	-6	-7	-10	-14	-22
$u$	0.635	0.565	0.5	0.425	0.36	0.25	0.175	0.11

that  $0.2u^2 + (2\bar{\Sigma} + 0.01)u + 5 = 0$ . Solving for  $\bar{\Sigma}$  gives us

$$\bar{\Sigma} = \frac{-20u^2 - u - 500}{200u}, \quad (2.15)$$

which is satisfied for the values in the table. Since we require the curvature  $u$  at the north pole to be positive then from equation (2.15)  $\Rightarrow \bar{\Sigma} < -1/200$ , showing why we get no spherical phase portraits for  $\bar{\Sigma} > 0$ . From 2.15 we also see that as  $\bar{\Sigma} \rightarrow -\infty$  then  $u \rightarrow 0$ , which seems intuitive.

## 2.6 Conjectures on Existence and Uniqueness of solutions to the Shape Equations

Having studied the Shape Equations for a sphere in section 2.3 and numerical simulations specifically for  $\bar{P} = 5$  and  $C_0 = -0.1$  in section 2.5, we now make conjectures about the existence and uniqueness of solutions. We firstly consider existence, and deal with the cases  $\bar{P} > 0$  and  $\bar{P} = 0$  (since the case  $\bar{P} < 0$  is similar), and consider what values of  $C_0$  and  $\bar{\Sigma}$  are required for the Shape Equations in Theorem 2.4 to have a solution. For now we only consider existence and uniqueness in terms of the Lagrangian multipliers.

**Conjecture 2.6** (Existence). *Sufficient conditions for the existence of solutions to the Shape Equations are that either*

1.  $\bar{P} = 0$  and either

- (a)  $C_0 > 0$  and  $\bar{\Sigma} > -C_0^2/2$
- (b)  $C_0 < 0$  and  $\bar{\Sigma} < -C_0^2/2$

2.  $\bar{P} > 0$  and either

- (a)  $C_0 > 0$
- (b)  $C_0 < 0$  and  $\bar{\Sigma} < (-\sqrt{-8C_0\bar{P}} - C_0^2)/2$ .

*Proof.* The sufficient conditions come from considering when (2.14) has solutions for positive  $R$ , and since we can find a sphere of some radius which satisfies the Shape Equations, we have a solution.  $\square$

However these conditions are not necessary, since for example in the numerical simulations in Appendix D.1 we have a solution for  $\bar{\Sigma} = -0.5$ , whereas these sufficient conditions suggest that we would be only able to find solutions for  $\bar{\Sigma} < -1.005$ . Instead, a conjecture for necessary and sufficient conditions for existence based on the numerical simulations would be the conditions above but with (2b) replaced by  $C_0 < 0$  and  $\bar{\Sigma} < -C_0^2/2$ .

**Conjecture 2.7** (Uniqueness). *If a solution to the Shape Equations exists then it is non unique.*

*Proof.* We saw in the numerical simulations that whenever we had solutions to the Shape Equations in the specific case  $\bar{P} = 5$  and  $C_0 = -0.1$  then there was always at least two values of  $u$  which resulted in a solution for each value of  $\bar{\Sigma}$ , and hence the solution is not unique.  $\square$

### 3 Two Phase Membranes

**Definition** (Energy of a two phase vesicle). The total energy of the vesicle can be expressed as (assuming a spontaneous curvature model)

$$F = F_b + F_G + F_l,$$

where

$$F_b := \frac{\kappa^{(\alpha)}}{2} \int_{\alpha} (H - C_0^{(\alpha)})^2 dA + \frac{\kappa^{(\beta)}}{2} \int_{\beta} (H - C_0^{(\beta)})^2 dA,$$

and

$$F_G := \kappa_G^{(\alpha)} \int_{\alpha} K dA + \kappa_G^{(\beta)} \int_{\beta} K dA, \quad F_l := \int_{\partial\beta} \sigma dl \equiv \int_{\partial\alpha} \sigma dl.$$

In terms of what this represents,  $F_b + F_G$  is the bending energy we considered before while  $F_l$  is the energy arising due to the line tension at the phase transition. Compared to [JL96] we have omitted the stretching energy. We assume the bending rigidities  $\kappa^{(\alpha)}, \kappa^{(\beta)}, \kappa_G^{(\alpha)}$  and  $\kappa_G^{(\beta)}$ , spontaneous curvatures  $C_0^{(\alpha)}$  and  $C_0^{(\beta)}$  and line tension  $\sigma$  are all constant.

#### 3.1 Axisymmetric Setting

Again, we assume the vesicle is axisymmetric and that we want to parametrise it. We use a similar method to the one phase membrane, but now the  $\alpha$  phase is represented by  $0 < S < S_1$  and the  $\beta$  phase is represented by  $S_1 < S < S_2$ . In this approach the phase boundary is at  $S = S_1$ , which we allow to vary, and the total arclength is  $S_1 + S_2$ . We will use a generalised parametrisation to perform variational calculus and then reparametrise by arclength, to avoid any difficulties in trying to vary  $x(S_1)$ ,  $\psi(S_1)$  and  $S_1$  independently.

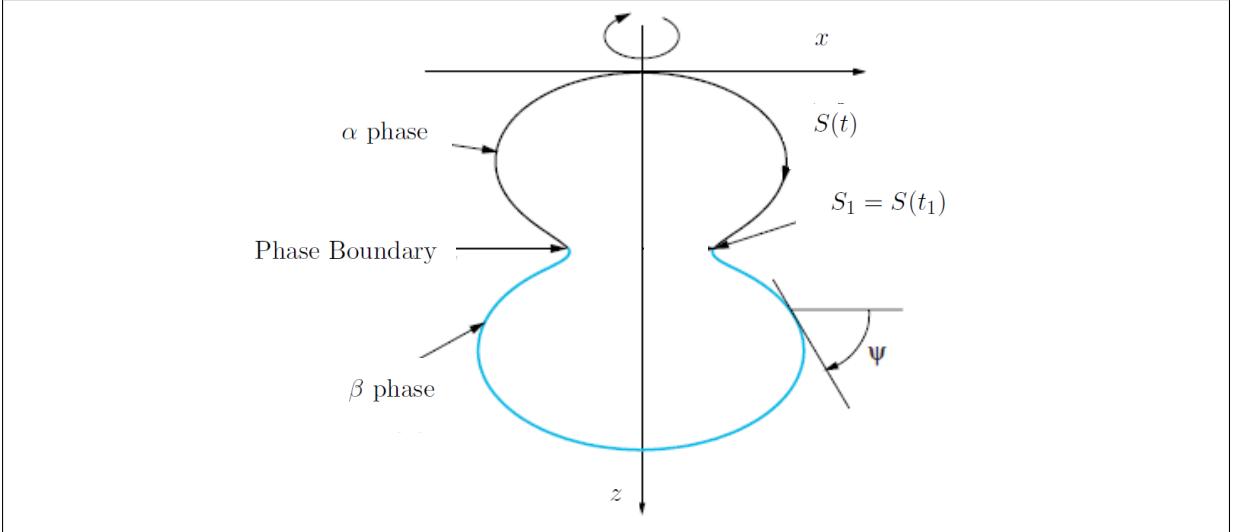


Figure 11: Parametrisation of a two phase membrane. [DJB09]

We introduce  $t : [0, 1] \mapsto [0, t_2]$ , and then  $S : [0, t_2] \mapsto [0, S_2]$  satisfying  $S(0) = 0, S(t_1) = S_1$  and  $S(t_2) = S_2$ , and we have  $\psi, x$  and  $z$  as functions of  $t$  instead of  $S$ . Unlike [JL96] we do not assume that  $\psi$  is continuous at  $t_1$ , indeed if this was the case then due to the geometric relations between  $\dot{x}, \dot{z}$  and  $\psi$  then  $x$  and  $z$  would be differentiable at  $t_1$ . Thus to model vesicles seen by [BHW03] we make no assumption on the continuity of  $\psi$  at  $t_1$  but we do assume that  $\psi, x, z$  are sufficiently smooth elsewhere to ensure that we can perform variational calculus. The parametrisation of the vesicle is given by  $\underline{r} : [0, 1] \times [0, 2\pi] \mapsto \mathbb{R}^3$ ,

$$\underline{r}(t, \phi) = (x(t) \cos \phi, x(t) \sin \phi, z(t)).$$

**Definition.** We define  $t_1^{(\alpha)} := 0$ ,  $t_1^{(\beta)} := t_1$ ,  $t_2^{(\alpha)} := t_1$  and  $t_2^{(\beta)} := t_2$ , and similarly for  $S_j^{(i)}$ .

**Lemma 3.1.** In the above parametrisation we have for  $t_1^{(i)} < t < t_2^{(i)}$ ;

$$K = \frac{\dot{\psi} \sin \psi}{x \dot{S}}, \quad H = \frac{\sin \psi}{x} + \frac{\dot{\psi}}{\dot{S}} \text{ and } dA = x \dot{S} d\phi dS.$$

*Proof.* We have (all arguments from now on are  $t$  unless otherwise stated):

$$\underline{n} = \frac{\underline{r}_s \wedge \underline{r}_\phi}{\|\underline{r}_s \wedge \underline{r}_\phi\|} = \frac{(\dot{z}x \cos \phi, \dot{z}x \sin \phi, -\dot{x}x)}{\sqrt{x^2 \dot{z}^2 + x^2 \dot{x}^2}} = \frac{1}{\sqrt{\dot{z}^2 + \dot{x}^2}} (\dot{z} \cos \phi, \dot{z} \sin \phi, -\dot{x}),$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & \dot{z}^2 + \dot{x}^2 \end{pmatrix} \text{ and } \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{\sqrt{\dot{z}^2 + \dot{x}^2}} \begin{pmatrix} -x\dot{z} & 0 \\ 0 & \ddot{x}\dot{z} - \dot{x}\ddot{z} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{\sqrt{\dot{z}^2 + \dot{x}^2}} \begin{pmatrix} -\frac{\dot{z}}{x} & 0 \\ 0 & \frac{\ddot{x}\dot{z} - \dot{x}\ddot{z}}{\dot{z}^2 + \dot{x}^2} \end{pmatrix}.$$

From the relations

$$\frac{dx}{dS} = \cos \psi(S) \text{ and } \frac{dz}{dS} = -\sin \psi(S)$$

we get that

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{dS} \frac{dS}{dt} = \dot{S}(t) \cos \psi(t) \text{ and } \dot{z} = \frac{dz}{dt} = \frac{dz}{dS} \frac{dS}{dt} = -\dot{S}(t) \sin \psi(t).$$

Thus  $\sqrt{\dot{z}^2 + \dot{x}^2} = \dot{S}$  and differentiating the relations above we get

$$\ddot{x}\dot{z} - \dot{x}\ddot{z} = (-\dot{S} \sin \psi)(\ddot{S} \cos \psi - \dot{S} \dot{\psi} \sin \psi) - (\dot{S} \cos \psi)(-\ddot{S} \sin \psi - \dot{S} \dot{\psi} \cos \psi) = \dot{S}^2 \dot{\psi}.$$

We conclude that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} -\frac{\dot{z}}{x} & 0 \\ 0 & \frac{\ddot{x}\dot{z} - \dot{x}\ddot{z}}{\dot{z}^2 + \dot{x}^2} \end{pmatrix} = \frac{1}{\dot{S}} \begin{pmatrix} \frac{\dot{S} \sin \psi}{x} & 0 \\ 0 & \frac{\dot{S}^2 \dot{\psi}}{\dot{S}^2} \end{pmatrix} = \begin{pmatrix} \frac{\sin \psi}{x} & 0 \\ 0 & \frac{\dot{\psi}}{\dot{S}} \end{pmatrix},$$

leading to

$$H = \frac{\sin \psi}{x} + \frac{\dot{\psi}}{\dot{S}} \text{ and } K = \frac{\dot{\psi} \sin \psi}{x \dot{S}}.$$

Finally we have that

$$dA = \|\underline{r}_s \wedge \underline{r}_\phi\| d\phi dS = x \sqrt{\dot{z}^2 + \dot{x}^2} d\phi dS = x \dot{S} d\phi dS. \quad \square$$

From the lemma we can now state  $F_b$ ,  $F_G$  and  $F_l$  for vesicles as sketched in Figure 11:

$$\begin{aligned} F_b &= 2\pi \frac{\kappa^{(\alpha)}}{2} \int_0^{t_1} x \dot{S} \left( \frac{\sin \psi}{x} + \frac{\dot{\psi}}{\dot{S}} - C_0^{(\alpha)} \right)^2 dt + 2\pi \frac{\kappa^{(\beta)}}{2} \int_{t_1}^{t_2} x \dot{S} \left( \frac{\sin \psi}{x} + \frac{\dot{\psi}}{\dot{S}} - C_0^{(\beta)} \right)^2 dt, \\ F_G &= 2\pi \kappa_G^{(\alpha)} \int_0^{t_1} \frac{\dot{\psi} \sin \psi}{x \dot{S}} x \dot{S} dt + 2\pi \kappa_G^{(\beta)} \int_{t_1}^{t_2} \frac{\dot{\psi} \sin \psi}{x \dot{S}} x \dot{S} dt, \end{aligned}$$

and

$$F_l = \int_{\partial\beta} \sigma dl = \int_{\partial\alpha} \sigma dl = 2\pi \sigma x(t_1).$$

Since the functions  $x$ ,  $S$  and  $\psi$  are sufficiently smooth except possibly at the point  $t = t_1$ , the integrands are regulated and hence the integrals make sense. We can simplify the expression for the Gauss curvature by noting that  $d/dt(\cos \psi) = -\dot{\psi} \sin \psi$ . Thus we get the following lemma.

**Lemma 3.2** (Gauss Bonnet for surface of revolutions which are not smooth at a point).

$$F_G = 2\pi \left[ \kappa_G^{(\alpha)} + \kappa_G^{(\beta)} + \lim_{\varepsilon \downarrow 0} \left( \kappa_G^{(\beta)} \cos \psi(t_1 + \varepsilon) - \kappa_G^{(\alpha)} \cos \psi(t_1 - \varepsilon) \right) \right].$$

*Proof.* Using the fact that  $d/dt(\cos \psi) = -\dot{\psi} \sin \psi$  and the dominated convergence theorem we get

$$\int_0^{t_1} \dot{\psi} \sin \psi dt = \lim_{\varepsilon \downarrow 0} \int_0^{t_1 - \varepsilon} \dot{\psi} \sin \psi dt = \lim_{\varepsilon \downarrow 0} \left[ -\cos \psi \right]_0^{t_1 - \varepsilon} = 1 - \lim_{\varepsilon \downarrow 0} \cos \psi(t_1 - \varepsilon)$$

and

$$\int_{t_1}^{t_2} \dot{\psi} \sin \psi dt = \lim_{\varepsilon \downarrow 0} \int_{t_1 + \varepsilon}^{t_2} \dot{\psi} \sin \psi dt = \lim_{\varepsilon \downarrow 0} \left[ -\cos \psi \right]_{t_1 + \varepsilon}^{t_2} = 1 + \lim_{\varepsilon \downarrow 0} \cos \psi(t_1 + \varepsilon),$$

giving

$$F_G = 2\pi \left[ \kappa_G^{(\alpha)} + \kappa_G^{(\beta)} + \lim_{\varepsilon \downarrow 0} \left( \kappa_G^{(\beta)} \cos \psi(t_1 + \varepsilon) - \kappa_G^{(\alpha)} \cos \psi(t_1 - \varepsilon) \right) \right]. \quad \square$$

Since the first two terms of this are constant then we only need to worry about the third term, which reduces to  $2\pi(\kappa_G^{(\beta)} - \kappa_G^{(\alpha)}) \cos \psi(t)$  if  $\psi$  is continuous at  $t_1$ .

### 3.2 Deriving the Shape Equations

We now introduce new notation so  $\underline{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) := (\psi, x, z, S, \gamma, \eta)$  and hence  $\dot{\underline{\zeta}} = (\dot{\zeta}_1, \dot{\zeta}_2, \dot{\zeta}_3, \dot{\zeta}_4, \dot{\zeta}_5, \dot{\zeta}_6) := (\dot{\psi}, \dot{x}, \dot{z}, \dot{S}, \dot{\gamma}, \dot{\eta})$ . Since we want to minimise the energy  $F$  of the vesicle subject to the constraints that  $A^{(\alpha)}$ ,  $A^{(\beta)}$  and  $V$  are fixed, we introduce Lagrangian multipliers  $\Sigma^{(\alpha)}$ ,  $\Sigma^{(\beta)}$  and  $P$  respectively. We then want to minimise the functional

$$\hat{F}(\underline{\zeta}) := F_b(\underline{\zeta}) + F_G(\underline{\zeta}) + F_l(\underline{\zeta}) + \Sigma^{(\alpha)} A^{(\alpha)} + \Sigma^{(\beta)} A^{(\beta)} + PV, \quad (3.1)$$

subject to varying the Lagrangian multipliers in order to achieve the prescribed surface areas and volume. Since we have introduced the Lagrangian multipliers to account for the surface area and volume being fixed, we want to express  $A^{(\alpha)}$ ,  $A^{(\beta)}$  and  $V$  in terms of integrals of  $t$ :

$$A^{(\alpha)} = 2\pi \int_0^{t_1} x \dot{S} dt, \quad A^{(\beta)} = 2\pi \int_{t_1}^{t_2} x \dot{S} dt \text{ and } V = \pi \int_0^{t_1} x^2 \dot{S} \sin \psi dt + \pi \int_{t_1}^{t_2} x^2 \dot{S} \sin \psi dt.$$

We can now write the functional  $\hat{F}$  in terms of the parametrisation using previous results, although we need to introduce Lagrangian functions  $\gamma : ((0, t_1) \cup (t_1, t_2)) \mapsto \mathbb{R}$  and  $\eta : ((0, t_1) \cup (t_1, t_2)) \mapsto \mathbb{R}$  in order to account for the geometric relations  $\dot{x} = \dot{S} \cos \psi$  and  $\dot{z} = -\dot{S} \sin \psi$ , which must hold everywhere. Thus we want to minimise  $\hat{F}(\underline{\zeta}) = 2\pi(\hat{F}^{(\alpha)}(\underline{\zeta}) + \hat{F}^{(\beta)}(\underline{\zeta}) + f(\underline{\zeta}))$ , where

$$\begin{aligned} \hat{F}^{(\alpha)}(\underline{\zeta}) &= \int_0^{t_1} L^{(\alpha)}(t, \underline{\zeta}, \dot{\underline{\zeta}}) dt, \quad \hat{F}^{(\beta)}(\underline{\zeta}) = \int_{t_1}^{t_2} L^{(\beta)}(t, \underline{\zeta}, \dot{\underline{\zeta}}) dt, \\ L^{(i)} &= \frac{\kappa^{(i)}}{2} x \dot{S} \left( \frac{\sin \psi}{x} + \frac{\dot{\psi}}{\dot{S}} - C_0^{(i)} \right)^2 + \Sigma^{(i)} x \dot{S} + \frac{1}{2} P x^2 \dot{S} \sin \psi + \gamma(\dot{x} - \dot{S} \cos \psi) + \eta(\dot{z} + \dot{S} \sin \psi), \\ f(\underline{\zeta}) &= \sigma x(t_1) + \lim_{\varepsilon \downarrow 0} \left( \kappa_G^{(\beta)} \cos \psi(t_1 + \varepsilon) - \kappa_G^{(\alpha)} \cos \psi(t_1 - \varepsilon) \right). \end{aligned}$$

**Definition.** We define  $\Omega_1 := (0, t_1) \cup (t_1, t_2)$  and  $\Omega_2 := [0, t_2]$ .

Since we have not placed any assumption on the continuity of  $\psi$  at  $t_1$ , we should consider exactly what smoothness conditions we have on  $\underline{\zeta}$ . We have  $x, z, S \in C^0(\Omega_2) \cap C^2(\Omega_1)$ ,  $\psi \in C^1(\Omega_1)$  and we also assume that  $\eta, \gamma \in C^1(\Omega_1)$  so that we can perform variational calculus.

**Theorem 3.3.** If  $\underline{\xi}$  minimises  $\hat{F}$  then  $\underline{\xi}$  satisfies the following equation for  $\underline{v}$  differentiable on  $\Omega_1$ :

$$\begin{aligned} \sum_{n=1}^6 \left\{ \int_0^{t_1} \left( \frac{\partial L^{(\alpha)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) - \frac{d}{dt} \frac{\partial L^{(\alpha)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) \right) v_n dt + \int_{t_1}^{t_2} \left( \frac{\partial L^{(\beta)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) - \frac{d}{dt} \frac{\partial L^{(\beta)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) \right) v_n dt \right\} \\ + \sum_{n=1}^6 \left\{ v_n \frac{\partial L^{(\alpha)}}{\partial \dot{\zeta}_n} \Big|_0^{t_1} + v_n \frac{\partial L^{(\beta)}}{\partial \dot{\zeta}_n} \Big|_{t_1}^{t_2} \right\} + \frac{d}{dh} f(\underline{\xi} + h\underline{v}) \Big|_{h=0} = 0. \end{aligned} \quad (3.2)$$

*Proof.* We follow a similar approach to the proof of the Euler Lagrange equations in [DA09]. If we have a minimum  $\underline{\xi}$  of  $\hat{F}$  then define

$$\phi(h) := \frac{1}{2\pi} \hat{F}(\underline{\xi} + h\underline{v})$$

where  $\underline{v}$  is differentiable on  $\Omega_1$ , and we choose  $h$  in a sufficiently small interval such that  $\hat{F}(\underline{\xi} + h\underline{v})$  is differentiable on this interval. For  $\phi$  to be a minimum, we require that  $\phi(0) \leq \phi(h)$ , and hence it is sufficient that:

$$\phi'(0) = \frac{d}{dh} \hat{F}^{(\alpha)}(\underline{\xi} + h\underline{v}) \Big|_{h=0} + \frac{d}{dh} \hat{F}^{(\beta)}(\underline{\xi} + h\underline{v}) \Big|_{h=0} + \frac{d}{dh} f(\underline{\xi} + h\underline{v}) \Big|_{h=0} = \langle \delta \hat{F}, \underline{v} \rangle = 0$$

This is then equivalent to (justified by results measure theory)

$$\int_0^{t_1} \frac{d}{dh} L^{(\alpha)}(t, \underline{\xi} + h\underline{v}, \dot{\underline{\xi}} + h\dot{\underline{v}}) dt + \int_{t_1}^{t_2} \frac{d}{dh} L^{(\beta)}(t, \underline{\xi} + h\underline{v}, \dot{\underline{\xi}} + h\dot{\underline{v}}) dt + \frac{d}{dh} f(\underline{\xi} + h\underline{v}) \Big|_{h=0} = 0,$$

and using the chain rule we get that (ignoring the last term for now)

$$\sum_{n=1}^6 \int_0^{t_1} \dot{v}_n \frac{\partial L^{(\alpha)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) + v_n \frac{\partial L^{(\alpha)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) dt + \int_{t_1}^{t_2} \dot{v}_n \frac{\partial L^{(\beta)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) + v_n \frac{\partial L^{(\beta)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) dt.$$

Finally, using integration by parts, gives us:

$$\begin{aligned} \sum_{n=1}^6 \left\{ \int_0^{t_1} \left( \frac{\partial L^{(\alpha)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) - \frac{d}{dt} \frac{\partial L^{(\alpha)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) \right) v_n dt + v_n \frac{\partial L^{(\alpha)}}{\partial \dot{\zeta}_n} \Big|_0^{t_1} \right\} + \\ \sum_{n=1}^6 \left\{ \int_{t_1}^{t_2} \left( \frac{\partial L^{(\beta)}}{\partial \zeta_n}(t, \underline{\xi}, \dot{\underline{\xi}}) - \frac{d}{dt} \frac{\partial L^{(\beta)}}{\partial \dot{\zeta}_n}(t, \underline{\xi}, \dot{\underline{\xi}}) \right) v_n dt + v_n \frac{\partial L^{(\beta)}}{\partial \dot{\zeta}_n} \Big|_{t_1}^{t_2} \right\} + \frac{d}{dh} f(\underline{\xi} + h\underline{v}) \Big|_{h=0} = 0. \quad \square \end{aligned}$$

### 3.2.1 Bulk Terms

To understand what equations  $\underline{\xi}$  must satisfy to minimise  $\hat{F}$ , we consider the terms inside the integral and outside the integral separately. We firstly consider the bulk terms inside the integral.

**Definition.** The Hamiltonian functions are defined as

$$\mathcal{H}^{(i)} := -\frac{\partial L^{(i)}}{\partial \dot{S}} \equiv \frac{1}{\dot{S}} \left( -L^{(i)} + \dot{\psi} \frac{\partial L^{(i)}}{\partial \dot{\psi}} + \dot{x} \frac{\partial L^{(i)}}{\partial \dot{x}} + \dot{z} \frac{\partial L^{(i)}}{\partial \dot{z}} \right).$$

**Lemma 3.4.** If  $\underline{\xi}$  minimises  $\hat{F}$  then it must satisfy the following systems of equations for

$t_1^i < t < t_2^i$ :

$$\kappa^{(i)} \left( \frac{\ddot{\psi}}{\dot{S}} - \frac{\ddot{S}\dot{\psi}}{\dot{S}^2} \right) = \frac{1}{2} Px\dot{S} \cos \psi + \frac{\eta \dot{S} \cos \psi}{x} + \frac{\gamma \dot{S} \sin \psi}{x} - \frac{\kappa^{(i)} \dot{\psi} \cos \psi}{x} + \frac{\kappa^{(i)} \dot{S} \cos \psi \sin \psi}{x^2} \quad (3.3)$$

$$\dot{\gamma} = \frac{\kappa^{(i)} \dot{S}}{2} \left( \frac{\dot{\psi}}{\dot{S}} - C_0^{(i)} \right)^2 - \frac{\kappa^{(i)} \dot{S} \sin^2 \psi}{2x^2} + \Sigma^{(i)} \dot{S} + Px\dot{S} \sin \psi \quad (3.4)$$

$$\dot{\eta} = 0 \quad (3.5)$$

$$\dot{\mathcal{H}}^{(i)} = 0 \quad (3.6)$$

$$\dot{x} = \dot{S} \cos \psi \quad (3.7)$$

$$\dot{z} = -\dot{S} \sin \psi \quad (3.8)$$

*Proof.* We note that for each  $n$  we can choose  $v_m = 0$  for  $m \neq n$ , and thus each term in the sum of (3.2) must be zero. Since we can independently vary  $v$  in the regions  $(0, t_1)$  and  $(t_1, t_2)$  we must also have both integrals also equal to zero. Using the fundamental lemma of calculus of variations [DA09] and the fact that  $\partial L^{(i)} / \partial z = \partial L^{(i)} / \partial S = \partial L^{(i)} / \partial \dot{\eta} = \partial L^{(i)} / \partial \dot{\gamma} = 0$  are trivially satisfied, we get that the following equations need to hold for  $t_1^i < t < t_2^i$ :

$$\frac{\partial L^{(i)}}{\partial \psi} - \frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{\psi}} = 0, \frac{\partial L^{(i)}}{\partial x} - \frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{x}} = 0, -\frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{z}} = 0, -\frac{d}{dt} \frac{\partial L^{(i)}}{\partial \dot{S}} = 0, \frac{\partial L^{(i)}}{\partial \gamma} = 0, \frac{\partial L^{(i)}}{\partial \eta} = 0.$$

Inserting the definition of  $L^i$  gives the desired result, similarly to the one phase membranes.  $\square$

We now remove the variable  $S$  from equations (3.3)-(3.8) by reparametrising by arclength. From now on we denote derivatives with respect to  $S$  by primes, and those with respect to  $t$  by over dots. Using the relations that  $\zeta'_i = \dot{\zeta}_i / \dot{S}$  and  $\psi'' \dot{S} = \ddot{\psi} / \dot{S} - \ddot{S} \dot{\psi} / \dot{S}^2$ , equations (3.3),(3.4),(3.7) and (3.8) become

$$\kappa^{(i)} \psi'' = \frac{Px \cos \psi}{2} + \frac{\eta \cos \psi}{x} + \frac{\gamma \sin \psi}{x} - \frac{\kappa^{(i)} \psi' \cos \psi}{x} + \frac{\kappa^{(i)} \cos \psi \sin \psi}{x^2} \quad (3.9)$$

$$\gamma' = \frac{\kappa^{(i)}}{2} \left( \psi' - C_0^{(i)} \right)^2 - \frac{\kappa^{(i)} \sin^2 \psi}{2x^2} + \Sigma^{(i)} + Px \sin \psi \quad (3.10)$$

$$x' = \cos \psi \quad (3.11)$$

$$z' = -\sin \psi. \quad (3.12)$$

We can also calculate that

$$\mathcal{H}^{(i)} = \frac{x \kappa^{(i)}}{2} \left[ (\psi')^2 - \left( \frac{\sin \psi}{x} - C_0^{(i)} \right)^2 \right] - \Sigma^{(i)} x - \frac{1}{2} Px^2 \sin \psi + \gamma \cos \psi - \eta \sin \psi.$$

### 3.2.2 Boundary Terms

We now denote  $v_n = v_{\zeta_n}$ , and we note that  $v_n$  must satisfy the same smoothness conditions as  $\zeta_n$ , so  $v_x, v_z, v_S \in C^0(\Omega_2) \cap C^2(\Omega_1)$  and  $v_\psi \in C^1(\Omega_1)$ . By considering the boundary terms, we get that the following equations need to hold (recalling that  $f(\underline{\zeta})$  is only a function of the variables  $x$  and  $\psi$  and  $\partial L^{(i)} / \partial \dot{\eta} = \partial L^{(i)} / \partial \dot{\gamma} = 0$ ):

$$v_z \frac{\partial L^{(\alpha)}}{\partial \dot{z}} \Big|_0^{t_1} + v_z \frac{\partial L^{(\beta)}}{\partial \dot{z}} \Big|_{t_1}^{t_2} = 0 \quad (3.13)$$

$$v_S \frac{\partial L^{(\alpha)}}{\partial \dot{S}} \Big|_0^{t_1} + v_S \frac{\partial L^{(\beta)}}{\partial \dot{S}} \Big|_{t_1}^{t_2} = 0 \quad (3.14)$$

$$v_x \frac{\partial L^{(\alpha)}}{\partial \dot{x}} \Big|_0^{t_1} + v_x \frac{\partial L^{(\beta)}}{\partial \dot{x}} \Big|_{t_1}^{t_2} + \frac{d}{dh} f(\underline{\xi} + hv_x) \Big|_{h=0} = 0 \quad (3.15)$$

$$v_\psi \frac{\partial L^{(\alpha)}}{\partial \dot{\psi}} \Big|_0^{t_1} + v_\psi \frac{\partial L^{(\beta)}}{\partial \dot{\psi}} \Big|_{t_1}^{t_2} + \frac{d}{dh} f(\underline{\xi} + hv_\psi) \Big|_{h=0} = 0. \quad (3.16)$$

**Proposition 3.5.** *For a vesicle of spherical topology minimising the total energy we have  $\eta(t) = 0$  and  $\mathcal{H}^{(i)}(t) = 0$  when  $t_1^{(i)} < t < t_2^{(i)}$ .*

*Proof.* Firstly, we have for  $t_1^{(i)} < t < t_2^{(i)}$

$$\frac{\partial L^{(i)}}{\partial \dot{z}}(t) = \eta \text{ and } \frac{\partial L^{(i)}}{\partial \dot{S}}(t) = \mathcal{H}^{(i)}.$$

Using the fact that  $\dot{\eta} = \dot{\mathcal{H}}^{(i)} = 0$ , we get that  $\eta = \eta^{(i)}$  on  $\mathcal{H}^{(i)} = h^{(i)}$  on  $t_1^{(i)} < t < t_2^{(i)}$ . Substituting this into equations (3.13) and (3.14) and using the continuity of  $v_z$  and  $v_S$  we get that the following two equations must hold:

$$\begin{aligned} \eta^{(\beta)} v_z(t_2) + (\eta^{(\alpha)} - \eta^{(\beta)}) v_z(t_1) - \eta^{(\alpha)} v_z(0) &= 0 \\ h^{(\beta)} v_S(t_2) + (h^{(\alpha)} - h^{(\beta)}) v_S(t_1) - h^{(\alpha)} v_S(0) &= 0. \end{aligned}$$

Noting that we can vary  $v_z(t_2)$  and  $v_z(t_1)$  independently since we are working with a vesicle with spherical topology gives us that  $\eta^{(\beta)} = 0$ ,  $\eta^{(\alpha)} - \eta^{(\beta)} = 0$ , hence  $\eta \equiv 0$ . A similar argument also works for the other equation, since we can vary  $v_S(t_2)$  and  $v_S(t_1)$  independently, giving us  $\mathcal{H}^{(\beta)} = 0$ ,  $\mathcal{H}^{(\alpha)} - \mathcal{H}^{(\beta)} = 0$ , hence  $\mathcal{H}^{(i)} \equiv 0$ .  $\square$

We see that the condition  $\mathcal{H}^{(i)}(t) = 0$  actually gives us an obvious matching condition, which we state in the following proposition.

**Proposition 3.6** (Matching Condition 1). *A vesicle of spherical topology minimising the total energy must satisfy the condition*

$$0 = \lim_{\delta \downarrow 0} \left( \mathcal{H}^{(\beta)}(t_1 + \delta) - \mathcal{H}^{(\alpha)}(t_1 - \delta) \right) \Leftrightarrow 0 = \lim_{\delta \downarrow 0} \left( \mathcal{H}^{(\beta)}(S_1 + \delta) - \mathcal{H}^{(\alpha)}(S_1 - \delta) \right).$$

We now turn our attention to the other two equations, which involve the function

$$f(\xi) = \sigma x(t_1) + \lim_{\varepsilon \downarrow 0} \left( \kappa_G^{(\beta)} \cos \psi(t_1 + \varepsilon) - \kappa_G^{(\alpha)} \cos \psi(t_1 - \varepsilon) \right).$$

These equations provide additional matching conditions that are essentially used to provide a difference between the end conditions for the  $\alpha$  phase membrane and the initial conditions for the  $\beta$  phase membrane for  $\psi'$ ,  $\psi$  and  $\gamma$ .

**Proposition 3.7** (Matching Condition 2). *A vesicle of spherical topology minimising the total energy must satisfy the condition*

$$\sigma = \lim_{\delta \downarrow 0} (\gamma(S_1 + \delta) - \gamma(S_1 - \delta)).$$

*Proof.* Since

$$\frac{d}{dh} f(\xi + hv_x) = \sigma v_x(t_1),$$

equation (3.16) becomes (using the continuity of  $v_x$ ):

$$\lim_{\delta \rightarrow 0} \left( v_x(t_2) \frac{\partial L^{(\beta)}}{\partial \dot{x}} \Big|_{t_2-\delta} - v_x(t_1) \frac{\partial L^{(\beta)}}{\partial \dot{x}} \Big|_{t_1+\delta} + v_x(t_1) \frac{\partial L^{(\alpha)}}{\partial \dot{x}} \Big|_{t_1-\delta} - v_x(0) \frac{\partial L^{(\alpha)}}{\partial \dot{x}} \Big|_\delta \right) + \sigma v_x(t_1) = 0.$$

Using the conditions  $v_x(0) = v_x(t_2) = 0$  (since  $x(0) = x(t_2) = 0$  are fixed) results in the first and fourth terms vanishing, and we are left with

$$\sigma v_x(t_1) = v_x(t_1) \lim_{\delta \rightarrow 0} \frac{\partial L^{(\beta)}}{\partial \dot{x}} \Big|_{t_1+\delta} - v_x(t_1) \lim_{\delta \rightarrow 0} \frac{\partial L^{(\alpha)}}{\partial \dot{x}} \Big|_{t_1-\delta}.$$

Assuming that  $v_x(t_1) \neq 0$ , and using the fact that  $\partial L^{(i)}/\partial \dot{x} = \gamma$ , we require that

$$\sigma = \lim_{\delta \downarrow 0} (\gamma(t_1 + \delta) - \gamma(t_1 - \delta)) \Leftrightarrow \sigma = \lim_{\delta \downarrow 0} (\gamma(S_1 + \delta) - \gamma(S_1 - \delta)). \quad \square$$

We note that Proposition 3.7 becomes a continuity condition for  $\gamma$  at  $S_1$  if  $\sigma = 0$ , as in the case for a one phase membrane.

**Proposition 3.8** (Matching Conditions 3a & 3b). *A vesicle of spherical topology minimising the total energy must satisfy the condition*

$$\lim_{\delta \downarrow 0} \kappa^{(\alpha)} \psi'(S_1 - \delta) = \lim_{\delta \downarrow 0} \left[ \kappa_G^{(\alpha)} C_0^{(\alpha)} - (\kappa_G^{(\alpha)} + \kappa^{(\alpha)}) \frac{\sin \psi(S_1 - \delta)}{x(S_1 - \delta)} \right],$$

$$\lim_{\delta \downarrow 0} \kappa^{(\beta)} \psi'(S_1 + \delta) = \lim_{\delta \downarrow 0} \left[ \kappa_G^{(\beta)} C_0^{(\beta)} - (\kappa_G^{(\beta)} + \kappa^{(\beta)}) \frac{\sin \psi(S_1 + \delta)}{x(S_1 + \delta)} \right].$$

*Proof.* We first note that (we justify switching limits and differentiation by results in analysis)

$$\frac{d}{dh} f(\underline{\xi} + hv_\psi) = \lim_{\delta \downarrow 0} \left[ \kappa_G^{(\alpha)} \sin(\xi_\psi + hv_\psi)(t_1 - \delta) v_\psi(t_1 - \delta) - \kappa_G^{(\beta)} \sin(\xi_\psi + hv_\psi)(t_1 + \delta) v_\psi(t_1 + \delta) \right].$$

Hence we get that (using the continuity of  $v_\psi$  at 0 and  $t_2$ )

$$\begin{aligned} \lim_{\delta \downarrow 0} & \left[ v_\psi(t_2) \frac{\partial L^{(\beta)}}{\partial \dot{\psi}} \Big|_{t_2-\delta} - v_\psi(t_1 + \delta) \frac{\partial L^{(\beta)}}{\partial \dot{\psi}} \Big|_{t_1+\delta} + v_\psi(t_1 - \delta) \frac{\partial L^{(\alpha)}}{\partial \dot{\psi}} \Big|_{t_1-\delta} - v_\psi(0) \frac{\partial L^{(\alpha)}}{\partial \dot{\psi}} \Big|_\delta \right] \\ &= \lim_{\delta \downarrow 0} \left[ \kappa_G^{(\beta)} \sin \psi(t_1 + \delta) v_\psi(t_1 + \delta) - \kappa_G^{(\alpha)} \sin \psi(t_1 - \delta) v_\psi(t_1 - \delta) \right]. \end{aligned}$$

The first and fourth terms of the equation disappear since  $v_\psi(0) = v_\psi(t_2) = 0$  (as  $\psi(0) = 0$  and  $\psi(t_2) = 0$  are fixed), and hence we require that

$$\begin{aligned} \lim_{\delta \downarrow 0} & \left[ -v_\psi(t_1 + \delta) \frac{\partial L^{(\beta)}}{\partial \dot{\psi}} \Big|_{t_1+\delta} + v_\psi(t_1 - \delta) \frac{\partial L^{(\alpha)}}{\partial \dot{\psi}} \Big|_{t_1-\delta} \right] \\ &= \lim_{\delta \downarrow 0} \left[ \kappa_G^{(\beta)} \sin \psi(t_1 + \delta) v_\psi(t_1 + \delta) - \kappa_G^{(\alpha)} \sin \psi(t_1 - \delta) v_\psi(t_1 - \delta) \right]. \quad (3.17) \end{aligned}$$

This then gives us the independent matching conditions which need to be satisfied, which are (noting the mistake in sign in [JL96]):

$$\begin{aligned} \lim_{\delta \downarrow 0} \frac{\partial L^{(\alpha)}}{\partial \dot{\psi}} \Big|_{t_1-\delta} &= -\lim_{\delta \downarrow 0} \kappa_G^{(\alpha)} \sin \psi(t_1 - \delta) \\ \lim_{\delta \downarrow 0} \frac{\partial L^{(\beta)}}{\partial \dot{\psi}} \Big|_{t_1+\delta} &= -\lim_{\delta \downarrow 0} \kappa_G^{(\beta)} \sin \psi(t_1 + \delta). \end{aligned}$$

Using the definition of  $\partial L^{(i)}/\partial \dot{\psi}$ , and reparametrising by arclength, we get the following two equations have to be satisfied provided that  $x(S_1) \neq 0$  (where  $\delta^{(\alpha)} = -\delta$  and  $\delta^{(\beta)} = \delta$ ):

$$\lim_{\delta \downarrow 0} \kappa^{(i)} \psi'(S_1 + \delta^{(i)}) = \lim_{\delta \downarrow 0} \left[ \kappa^{(i)} C_0^{(i)} - (\kappa_G^{(i)} + \kappa^{(i)}) \frac{\sin \psi(S_1 + \delta^{(i)})}{x(S_1 + \delta^{(i)})} \right]. \quad \square$$

If we allowed  $\psi$  to be continuous at  $t_1$  then we would only get one matching condition from (3.16), which is the difference between the two matching conditions. This follows from considering (3.17) and hence we get Matching Condition 3 [JL96]:

$$\lim_{\delta \downarrow 0} \left( \kappa^{(\beta)} \psi'(S_1 + \delta) - \kappa^{(\alpha)} \psi'(S_1 - \delta) \right) = \kappa^{(\beta)} C_0^{(\beta)} - \kappa^{(\alpha)} C_0^{(\alpha)} + (\kappa_G^{(\alpha)} + \kappa^{(\alpha)} - (\kappa_G^{(\beta)} + \kappa^{(\beta)})) \frac{\sin \psi(S_1)}{x(S_1)}.$$

### 3.3 General Shape Equations

We are now in a position to state the complete set of Shape Equations and additional constraints than govern a vesicle of spherical topology that minimises the total energy. Noting that we have the boundary conditions  $x(0) = x(S_2) = 0$ ,  $\psi(0) = 0$ ,  $\psi(S_2) = \pi$  and  $\gamma(0) = \gamma(S_2) = 0$  (by substituting the boundary conditions for  $x$  and  $\psi$  into the expression for the Hamiltonian) and making the substitution  $U = \psi'$ , we get the following theorem.

h

**Theorem 3.9.** *A vesicle of spherical topology that minimises the total energy satisfies the following systems of differential equations on  $S_1^i < S < S_2^i$ :*

$$\psi' = U, \quad \psi(0) = 0, \quad \psi(S_2) = \pi, \quad (3.18)$$

$$\kappa^{(i)} U' = \frac{Px \cos \psi}{2} + \frac{\gamma \sin \psi}{x} - \frac{\kappa^{(i)} U \cos \psi}{x} + \frac{\kappa^{(i)} \cos \psi \sin \psi}{x^2}, \quad (3.19)$$

$$\gamma' = \frac{\kappa^{(i)}}{2} \left( U - C_0^{(i)} \right)^2 - \frac{\kappa^{(i)} \sin^2 \psi}{2x^2} + \Sigma^{(i)} + Px \sin \psi, \quad \gamma(0) = 0, \quad \gamma(S_2) = 0, \quad (3.20)$$

$$x' = \cos \psi, \quad x(0) = 0, \quad x(S_2) = 0 \quad (3.21)$$

$$z' = -\sin \psi, \quad (3.22)$$

and matching conditions;

$$0 = \lim_{\delta \downarrow 0} (\mathcal{H}^{(\beta)}(S_1 + \delta) - \mathcal{H}^{(\alpha)}(S_1 - \delta)). \quad (3.23)$$

$$\sigma = \lim_{\delta \downarrow 0} (\gamma(S_1 + \delta) - \gamma(S_1 - \delta)), \quad (3.24)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa^{(\alpha)} C_0^{(\alpha)} - \kappa^{(\alpha)} U(S_1 - \delta) - (\kappa_G^{(\alpha)} + \kappa^{(\alpha)}) \frac{\sin \psi(S_1 - \delta)}{x(S_1 - \delta)} \right], \quad (3.25)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa^{(\beta)} C_0^{(\beta)} - \kappa^{(\beta)} U(S_1 + \delta) - (\kappa_G^{(\beta)} + \kappa^{(\beta)}) \frac{\sin \psi(S_1 + \delta)}{x(S_1 + \delta)} \right], \quad (3.26)$$

### 3.4 Membranes with zero Neck

In section 3.2, we assumed that  $x(S_1) \neq 0$ , since clearly this is not biologically feasible. We now seek to find out under what conditions we have  $x(S_1) = 0$ , i.e. zero neck. Initially, we assume  $\kappa^{(i)} = \kappa_G^{(i)} = 0$ .

**Proposition 3.10.** *A necessary and sufficient condition for the existence of a two phase membrane with zero neck is that there exist  $V^{(\alpha)}$  and  $V^{(\beta)}$  with  $V^{(\alpha)} + V^{(\beta)} = V$  and*

$$v_\alpha := 6\sqrt{\pi} \frac{V^{(\alpha)}}{(A^{(\alpha)})^{\frac{3}{2}}} \leq 1 \text{ and } v_\beta := 6\sqrt{\pi} \frac{V^{(\beta)}}{(A^{(\beta)})^{\frac{3}{2}}} \leq 1.$$

*Proof.* Recalling that a sphere maximises volume for a fixed surface area, we have that  $V^{(i)} = \frac{4\pi R^3}{3}$  and  $A^{(i)} = 4\pi R^2$ , and eliminating  $R$  we get that a sphere satisfies

$$v_i = 6\sqrt{\pi} \frac{V^{(i)}}{(A^{(i)})^{\frac{3}{2}}} = 1.$$

Thus, if we have the reduced volumes for each phase satisfying  $v_\alpha \leq 1$  and  $v_\beta \leq 1$  then we can find a shape which has zero neck (non unique if  $v_\alpha < 1$  and  $v_\beta < 1$ ). This is a necessary and sufficient condition since if either  $v_\alpha > 1$  or  $v_\beta > 1$  then we have to open up the sphere by moving  $x(S_1)$  outwards to accommodate fitting in more volume for a fixed surface area, resulting in a non-zero neck.  $\square$

We now consider the case when  $\kappa^{(i)} \neq 0$  and  $\kappa_G^{(i)} = 0$ .

**Proposition 3.11.** *A necessary and sufficient condition for the existence of a two phase membrane with zero neck is that there exist  $V^{(\alpha)}$  and  $V^{(\beta)}$  with  $V^{(\alpha)} + V^{(\beta)} = V$  such that we can find a solution to the one phase Shape Equations for associated Lagrange multipliers  $(\bar{P}_\alpha, \bar{\Sigma}_\alpha)$  and  $(\bar{P}_\beta, \bar{\Sigma}_\beta)$  by our earlier conjecture.*

*Proof.* We first check that if  $x(S_1) = 0$  that each phase must satisfy the Shape Equations. Since we have  $x(S_1) = 0$  then by continuity of  $x$  we have that  $\lim_{\delta \rightarrow 0} x(S_1 - \delta) = \lim_{\delta \rightarrow 0} x(S_1 + \delta) = 0$ . Thus Matching Conditions 3a and 3b become

$$\lim_{\delta \rightarrow 0} \kappa^{(\alpha)} \sin \psi(S_1 - \delta) = 0 \text{ and } \lim_{\delta \rightarrow 0} \kappa^{(\alpha)} \sin \psi(S_1 + \delta) = 0.$$

By considering the geometry of the vesicle, this then gives that

$$\lim_{\delta \rightarrow 0} \psi(S_1 - \delta) = \pi \text{ and } \lim_{\delta \rightarrow 0} \psi(S_1 + \delta) = 0.$$

By considering the other two matching conditions, we get that

$$\lim_{\delta \rightarrow 0} \gamma(S_1 + \delta) + \lim_{\delta \rightarrow 0} \gamma(S_1 - \delta) = 0 \text{ and } \lim_{\delta \rightarrow 0} \gamma(S_1 + \delta) - \lim_{\delta \rightarrow 0} \gamma(S_1 - \delta) = 0,$$

hence

$$\lim_{\delta \rightarrow 0} \gamma(S_1 + \delta) = \lim_{\delta \rightarrow 0} \gamma(S_1 - \delta) = 0.$$

If we now rescale all Lagrangian multipliers, we get that the  $\alpha$  phase has to satisfy the (one phase) Shape Equations for  $C_0 = C_0^\alpha$ ,  $\bar{P}_\alpha = P/\kappa^{(\alpha)}$  and  $\bar{\Sigma}_\beta = \Sigma^{(\alpha)}/\kappa^{(\alpha)}$  and similarly for the  $\beta$  phase. Thus, if we are given area constraints  $A^{(\alpha)}$  and  $A^{(\beta)}$  and volume constraint  $V^{(\alpha)}$  and  $V^{(\beta)}$ , we look for Lagrange Multipliers  $\bar{P}_i$  and  $\bar{\Sigma}_i$  which give us the required area and volume constraints. It is clear that if each phase satisfies the Shape Equations then we must have  $x(S_1) = 0$ .  $\square$

### 3.5 Numerical Method

We now discuss how we would go about solving the Shape Equations in Theorem 3.9 numerically, and we deal with the initial condition  $x(0) = 0$  by choosing  $\psi(0) = \varepsilon$  and  $x(0) = \sin \varepsilon / U(0)$ . We also choose  $z(0) = 0$ . We now ensure we have enough equations and initial/boundary conditions to calculate a solution if it exists. We use the condition  $x(S_2) = 0$  to determine  $S_2$ , while we use (3.25) to determine  $S_1$ , so  $S_1$  solves

$$C_0^{(\alpha)} - U(S) - \left(1 + \frac{\kappa_G^{(\alpha)}}{\kappa^{(\alpha)}}\right) \frac{\sin \psi(S)}{x(S)} = 0.$$

We now define

$$[x_1, \psi_1, U_1, \gamma_1, z_1] := \lim_{\delta \downarrow 0} [x, \psi, U, \gamma, z] \Big|_{S_1 - \delta} \text{ and } [x_2, \psi_2, U_2, \gamma_2, z_2] := \lim_{\delta \downarrow 0} [x, \psi, U, \gamma, z] \Big|_{S_1 + \delta},$$

then the remaining three matching conditions (3.23), (3.24) and (3.26) are used to determine the jump that  $\psi$ ,  $U$  and  $\gamma$  make at  $S_1$ . Using the fact that  $x$  and  $z$  are continuous, and equation (3.24) we conclude that

$$x_2 = x_1, \quad z_2 = z_1 \text{ and } \gamma_2 = \sigma + \gamma_1. \tag{3.27}$$

By using equation (3.26) we get

$$U_2 = C_0^{(\beta)} - \left(1 + \frac{\kappa_G^{(\beta)}}{\kappa^{(\beta)}}\right) \frac{\sin \psi_2}{x_1}. \tag{3.28}$$

Finally, combining equations (3.23), (3.27), (3.28) gives that  $\psi_2$  needs to satisfy

$$0 = \left( \Sigma^{(\alpha)} - \Sigma^{(\beta)} \right) x_1 + (\sigma + \gamma_1) \cos \psi_2 - \gamma_1 \cos \psi_1 + \frac{\left( \kappa_G^{(\beta)} \right)^2 \sin^2 \psi_2}{2\kappa^{(\beta)} x_1} - \frac{\left( \kappa_G^{(\alpha)} \right)^2 \sin^2 \psi_1}{2\kappa^{(\alpha)} x_1} \\ + \frac{Px_1^2}{2} (\sin \psi_1 - \sin \psi_2) + \kappa_G^{(\beta)} \sin \psi_2 \left( \frac{\sin \psi_2}{x_1} - C_0^{(\beta)} \right) - \kappa_G^{(\alpha)} \sin \psi_1 \left( \frac{\sin \psi_1}{x_1} - C_0^{(\alpha)} \right). \quad (3.29)$$

We also have five initial/boundary conditions ( $z(0) = 0$ ,  $\psi(0) = 0$ ,  $\psi(S_1) = \pi$ ,  $\gamma(0) = 0$ ,  $x(0) = 0$ ) for the three first order and one second order differential equations and hence can specify a solution to the Shape Equations. To actually solve the Shape Equations, we perform a shooting method to find the solution, choosing  $U(0) = u$  such that the vesicle also satisfies  $\psi(S_2) = \pi$  and  $\gamma(S_2) = 0$ , where  $S_2$  is defined as the solution of  $x(S_2) = 0$ . We also define  $S_1$  as above and solve these differential equations for  $0 < S < S_1$ ;

$$\begin{aligned} \psi' &= U & \psi(0) &= \varepsilon \\ \kappa^{(\alpha)} U' &= \frac{Px \cos \psi}{2} + \frac{\gamma \sin \psi}{x} - \frac{\kappa^{(\alpha)} U \cos \psi}{x} + \frac{\kappa^{(\alpha)} \cos \psi \sin \psi}{x^2}, & U(0) &= u, \\ \gamma' &= \frac{\kappa^{(\alpha)}}{2} \left( U - C_0^{(\alpha)} \right)^2 - \frac{\kappa^{(\alpha)} \sin^2 \psi}{2x^2} + \Sigma^{(\alpha)} + Px \sin \psi, & \gamma(0) &= 0, \\ x' &= \cos \psi & x(0) &= \sin \varepsilon / u, \\ z' &= -\sin \psi, & z(0) &= 0, \end{aligned}$$

and these on  $S_1 < S < S_2$  where  $[x_2, \psi_2, U_2, \gamma_2, z_2]$  are as above;

$$\begin{aligned} \psi' &= U, & \psi(S_1) &= \psi_2, \\ \kappa^{(\beta)} U' &= \frac{Px \cos \psi}{2} + \frac{\gamma \sin \psi}{x} - \frac{\kappa^{(\beta)} U \cos \psi}{x} + \frac{\kappa^{(\beta)} \cos \psi \sin \psi}{x^2}, & U(S_1) &= U_2, \\ \gamma' &= \frac{\kappa^{(\beta)}}{2} \left( U - C_0^{(\beta)} \right)^2 - \frac{\kappa^{(\beta)} \sin^2 \psi}{2x^2} + \Sigma^{(\beta)} + Px \sin \psi, & \gamma(S_1) &= \gamma_2, \\ x' &= \cos \psi, & x(S_1) &= x_2, \\ z' &= -\sin \psi, & z(S_1) &= z_2. \end{aligned}$$

### 3.6 Numerical Simulation

Using the above calculations, we are able to use Matlab to calculate numerical simulations of the differential equations and plot them, for more detail of this see Appendix A2. Despite careful checking of the equations, I was unable to match the results in Stinner's and Elliotts paper [ES10] when  $\kappa_G^{(i)} = 0$ . However, if we consider when  $-2\kappa^{(i)} < \kappa_G^{(i)} < 0$  as suggested in [HE11b] and [HE12] then we are able to produce the results similar to those seen in [BHW03]. We fix  $\kappa^{(i)} = 1$  and  $\kappa_G^{(i)} = -1$ , since these constants depend on the the cell type which is fixed. We additionally fix  $P = 5$  and also choose to fix  $C_0^{(i)} = -0.1$  for the purpose of the section. Thus we have four variables that we have not fixed:  $u$ ,  $\sigma$ ,  $\Sigma^{(\alpha)}$  and  $\Sigma^{(\beta)}$ . Since we have so many free variables, we choose not to make a similar catalogue to the one phase membrane case, and we only explore solutions for  $\sigma = 10$  and a few possible values of  $\Sigma^{(\alpha)}$  and  $\Sigma^{(\beta)}$  in Appendix C2. Instead, we showcase the effect of  $\sigma$  on the numerical solutions, as illustrated in Figure 12.

Clearly the assumption that both phases have the same bending rigidity and spontaneous curvature will be in general false, and modifying this will lead to a rich variety of solutions to the Shape Equations. Also, these equations would be better modelled using finite element analysis to produce numerical solutions as in [ES10], which would also allow us to consider surfaces which are no longer axisymmetric.

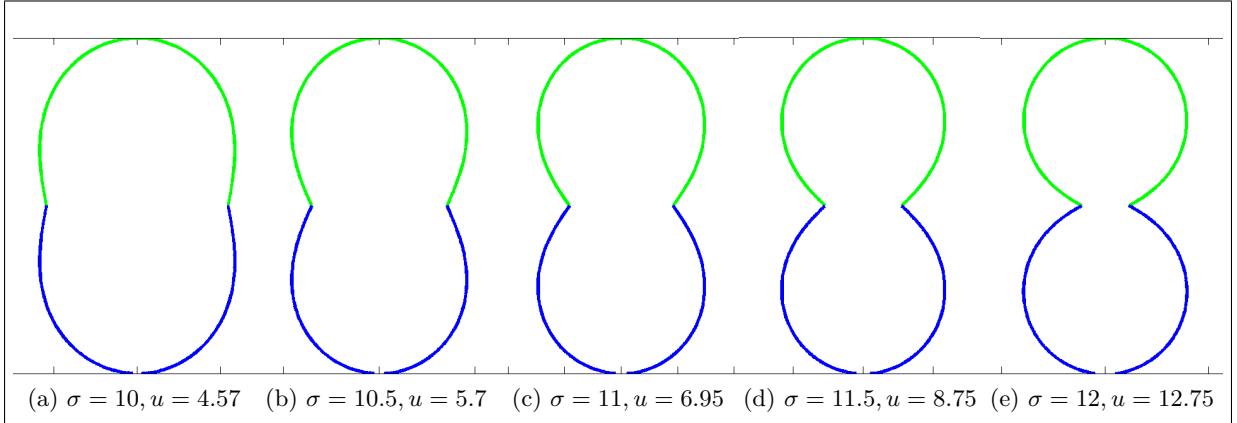


Figure 12: The effect of  $\sigma$  of the numerical solutions when  $P = 5$  and  $\Sigma^{(\alpha)} = \Sigma^{(\beta)} = -7$ .

## 4 Multiple Phase Membranes

We now wish to generalise the Shape Equations we have derived, for example we will now consider three phase membranes, although no more since the assumption of axisymmetry is otherwise no longer justified [JL96]. We will also consider two phase membranes with a low flip flop, which results in the two phases not fully separating, with the image on the right of Figure 2 as motivation for this. Finally, we also allow the possibility of ghost interfaces, which is where the vesicle has kinks not at phase boundaries. All of these cases can be treated in a similar way, as we will show below. To account for the phase changing, we introduce a phase field  $v : [0, S_\infty] \mapsto \{-1, 0, 1\}$ , where  $S_\infty$  represents the contour length between the north and south pole, which has been  $S_1$  and  $S_1 + S_2$  previously. In Figure 13 we show the phase field  $v$  we assumed for a vesicle with two phases and a high flip flop, and we also show a possible function for three phases with low flip flop. The phase boundaries are marked by the points of discontinuity of  $v$ , which are  $A := \{S_1, S_2, \dots, S_n\}$ , and for obvious reasons we assume we have finitely many phase boundaries.

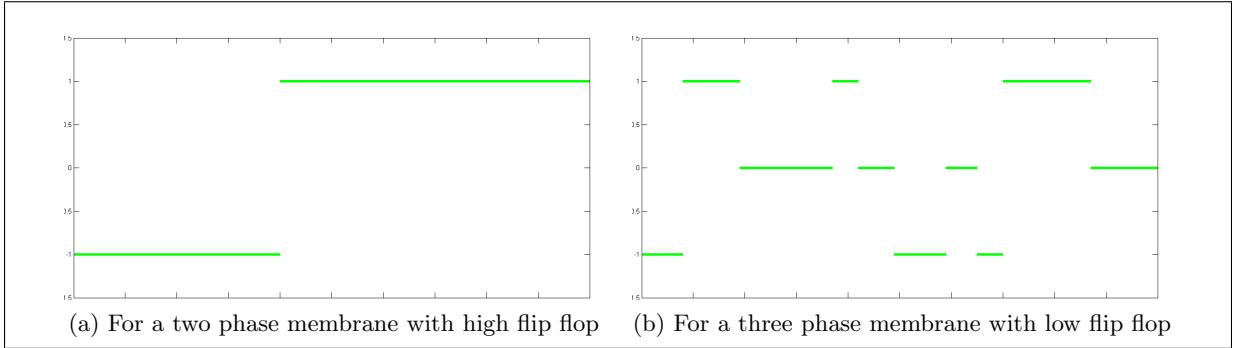


Figure 13: Examples of the phase field  $v$ .

We can define  $\kappa, C_0, \kappa_G : \{-1, 0, 1\} \mapsto \mathbb{R}$ , setting  $\kappa(-1) = \kappa^{(\alpha)}, \kappa(0) = \kappa^{(\beta)}, \kappa(1) = \kappa^{(\gamma)}$  and similarly for  $C_0$  and  $\kappa_G$ , where  $\alpha, \beta$  and  $\gamma$  are the three phases. We also introduce the set  $B := \{W_1, W_2, \dots, W_k\}$ , which is the points where we have ghost interfaces, so  $\psi$  is discontinuous at these points. Using exactly the same analysis as before, we introduce Lagrangian multipliers  $\Sigma(v)$ ,  $P$  and Lagrangian functions  $\eta, \gamma$ , and then we seek to minimise  $\hat{F}(\underline{\zeta}) = 2\pi[F(\underline{\zeta}) + f(\underline{\zeta}) + g(\underline{\zeta})]$ , where

$$F(\underline{\zeta}) = \int_0^{S_\infty} \frac{\kappa(v)}{2} x \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0(v) \right)^2 + \Sigma(v)x + \frac{1}{2} Px^2 \sin \psi + \gamma(\dot{x} - \cos \psi) + \eta(\dot{z} + \sin \psi) dS,$$

$$f(\underline{\zeta}) = \sum_{i=1}^n \left( \sigma x(S_i) + \lim_{\varepsilon \downarrow 0} [\kappa_G(v(S_i + \varepsilon)) \cos \psi(S_i + \varepsilon) - \kappa_G(v(S_i - \varepsilon)) \cos \psi(S_i - \varepsilon)] \right),$$

$$g(\underline{\zeta}) = \sum_{i=1}^k \left( \sigma x(W_i) + \kappa_G(v(W_i)) \lim_{\varepsilon \downarrow 0} [\cos \psi(W_i + \varepsilon) - \cos \psi(W_i - \varepsilon)] \right).$$

Using a similar approach to before, we consider the bulk and boundary terms separately and thus we can then derive the Shape Equations that need to hold.

**Theorem 4.1.** *For a vesicle of spherical topology to be in equilibrium, it must satisfy the following system of differential equations for  $S \in [0, S_\infty] \setminus (A \cup B)$ ;*

$$\psi' = U \quad \psi(0) = 0, \quad \psi(S_\infty) = \pi, \quad (4.1)$$

$$\kappa(v)U' = \frac{Px \cos \psi}{2} + \frac{\gamma \sin \psi}{x} - \frac{\kappa(v)U \cos \psi}{x} + \frac{\kappa(v) \cos \psi \sin \psi}{x^2}, \quad (4.2)$$

$$\gamma' = \frac{\kappa(v)}{2}(U - C_0(v))^2 - \frac{\kappa^{(i)} \sin^2 \psi}{2x^2} + \Sigma(v) + Px \sin \psi, \quad \gamma(0) = 0, \quad \gamma(S_\infty) = 0, \quad (4.3)$$

$$x' = \cos \psi \quad x(0) = 0, \quad x(S_\infty) = 0, \quad (4.4)$$

$$z' = -\sin \psi, \quad (4.5)$$

and matching conditions for  $i = 1, \dots, n$  and  $j = 1, \dots, k$ ;

$$\sigma = \lim_{\delta \downarrow 0} (\gamma(S_i + \delta) - \gamma(S_i - \delta)), \quad (4.6)$$

$$\sigma = \lim_{\delta \downarrow 0} (\gamma(W_j + \delta) - \gamma(W_j - \delta)), \quad (4.7)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa(v)C_0(v) - \kappa(v)U - (\kappa_G(v) + \kappa(v)) \frac{\sin \psi}{x} \right]_{S_i - \delta}, \quad (4.8)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa(v)C_0(v) - \kappa(v)U - (\kappa_G(v) + \kappa(v)) \frac{\sin \psi}{x} \right]_{S_i + \delta}, \quad (4.9)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa(v)C_0(v) - \kappa(v)U - (\kappa_G(v) + \kappa(v)) \frac{\sin \psi}{x} \right]_{W_j - \delta}, \quad (4.10)$$

$$0 = \lim_{\delta \downarrow 0} \left[ \kappa(v)C_0(v) - \kappa(v)U - (\kappa_G(v) + \kappa(v)) \frac{\sin \psi}{x} \right]_{W_j + \delta}, \quad (4.11)$$

$$0 = \lim_{\delta \downarrow 0} [\mathcal{H}(S_i + \delta) - \mathcal{H}(S_i - \delta)], \quad (4.12)$$

$$0 = \lim_{\delta \downarrow 0} [\mathcal{H}(W_j + \delta) - \mathcal{H}(W_j - \delta)]. \quad (4.13)$$

Whilst we note that in principle we can solve this in Matlab similarly to how we did for the two phase membrane, we encounter difficulties since at each phase boundary or ghost interface we actually have two possible values of  $\psi$  which satisfy the matching condition, so we would have to check  $2^{n+k}$  possibilities each time we find a solution. Instead, we end this section by showing that we indeed have enough initial, boundary and matching conditions to specify a solution of the Shape Equations if it exists. We can arbitrarily choose an initial condition  $z(0) = 0$ , and then we use the condition  $x(S_\infty) = 0$  to determine  $S_\infty$ . We use equations (4.8) and (4.10) to determine  $S_i$  and  $W_j$ . The three matching conditions (4.6), (4.9) and (4.12) are used to determine the jump that  $\gamma$ ,  $\psi$  and  $U$  make at  $S_i$ , while the conditions (4.7), (4.11) and (4.13) are used to determine the jump that  $\gamma$ ,  $\psi$  and  $U$  make at  $W_j$ . Then we have five initial/boundary conditions ( $z(0) = 0$ ,  $\psi(0) = 0$ ,  $\psi(S_\infty) = \pi$ ,  $\gamma(0) = 0$ ,  $x(0) = 0$ ) for the three first order and one second order differential equations and hence can specify a solution to the Shape Equations.

## 5 Existence of Solutions for Two Phase Membranes

The functional we wish to study is the energy of a vesicle, which is given by

$$E \equiv \sum_{J=\pm} \int_{M_j} \kappa^j (H - C^j)^2 + \kappa_G^j K dA + \sigma |\partial M^+|,$$

for all closed surfaces of revolution  $M$  with  $M = M^+ + M^-$  and  $M^+ \cap M^- = \emptyset$ , satisfying area constraints on  $M^\pm$  and a volume constraint. We want to show that  $E$  has a minimum, and we firstly turn to the Direct Method of Calculus of Variations (Appendix A.3), but checking that  $E$  is both coercive and lower semi-continuous is very difficult. Thus we approximate  $E$  by a sequence of equi-coercive functionals that  $\Gamma$ -converge to  $E$ , and then we find the minimum of  $E$  is given by the limit of the minimums of the approximating energies. For the approximating functionals we use sufficiently regular surfaces homeomorphic to the sphere, although the limiting surface has no such conditions.

### 5.1 $\Gamma$ -convergence

In this section we introduce  $\Gamma$ -convergence, state some properties of it and state the theorem we require.  $\Gamma$ -convergence is not the only notion of variational convergence, but it is the most widely used. This is because it has several advantages such as the  $\Gamma$ -limit being stable under continuous perturbations and having good compactness properties. We will state the necessary results from both [AB02] and [DM93].

**Definition** ( $\Gamma$ -convergence). We say a sequence of functions  $F_n : X \mapsto \overline{\mathbb{R}}$   $\Gamma$ -converges in  $X$  to  $F_\infty : X \mapsto \overline{\mathbb{R}}$  if

1. for each  $x \in X$ , for each  $(x_n) \rightarrow x$ ,  $F_\infty(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n)$ ,
2. and there is a sequence  $(y_n) \rightarrow x$  such that  $F_\infty(x) \geq \limsup_{n \rightarrow \infty} F_n(y_n)$ .

We denote  $F_\infty(x) = \Gamma\text{-}\lim_{n \rightarrow \infty} F_n$ . Although we have stated the definition for  $\Gamma$ -convergence in terms of sequences, we can also state it in terms of supremum and infimum of sets similarly to lower semi-continuity (Appendix A.3). We immediately get an obvious corollary from the definition.

**Corollary 5.1.** *There is a sequence  $y_n \rightarrow x$  which satisfies  $F_\infty(x) = \lim_{n \rightarrow \infty} F_n(y_n)$ .*

It is clear that  $\Gamma$ -convergence is a very weak notion of convergence, and can occur even when no point-wise limit exists, as we show in the following example of  $F_n(x) = \sin(nx)$ .

**Example.** We will show that the sequence of functionals  $\sin(nx)$   $\Gamma$ -converges to the constant function  $-1$  but has no point-wise limit. Firstly, it is clear that  $\lim_{n \rightarrow \infty} \sin(nx)$  only exists for  $x = 0$ , since otherwise we can choose sequences  $n_k = 2\pi k/x$  and  $n_m = (2\pi m + \frac{3\pi}{2})/x$ . We can see that  $f_\infty = -1$  clearly satisfies the first part of the definition for  $\Gamma$ -convergence since  $f_n(x_n) \geq -1$  for every sequence  $(x_n)$ . We can write any rational multiple of  $\pi$  as  $x = 2\pi q$  for  $q \in \mathbb{Q}$ , and we can note that if  $m \in \mathbb{N}$  then  $n = mq \in \mathbb{N}$  for sufficiently large  $m$ . Then  $y_m = 3\pi/2m + 2\pi n/m$  converges to  $x$ , and  $\sin(my_m) = \sin(3\pi/2 + 2\pi n) = -1$  for large enough  $m$ . Since we can approximate any real number by a sequence of these rational multiples of  $\pi$ , and  $F_n$  is continuous then we have shown the second part of the definition holds for all  $x \in \mathbb{R}$ .

**Proposition 5.2.** *Assume that  $F_n$   $\Gamma$ -converges to  $F$  in  $X$  and  $G_n$   $\Gamma$ -converges to  $G$  in  $X$ . Then some properties of  $\Gamma$  convergence are as follows:*

1. *If the limit exists, then it is unique.*
2.  *$F$  is lower semi-continuous.*
3. *If  $H_n = H$  for each  $n$  then  $H_n$   $\Gamma$ -converges to  $H$  iff  $H$  is lower semi-continuous.*
4.  *$\Gamma$ -convergence is stable under continuous perturbations (so if  $g : X \mapsto \overline{\mathbb{R}}$  is continuous then  $F_n + g$   $\Gamma$ -converges to  $F_\infty + g$ ).*
5. *Any subsequence  $F_{n_k}$   $\Gamma$ -converges to  $F$  in  $X$ .*
6. *If  $F_n \leq G_n$  for all  $n$  then  $F \leq G$ .*

7. If  $F_n + G_n$   $\Gamma$ -converges to  $H$  in  $X$  then  $F + G \leq H$ .

**Definition** (Equi-coercive). The sequence of functions  $F_n : X \mapsto \overline{\mathbb{R}}$  is (sequentially) equi-coercive if for each  $t \in \mathbb{R}$  there exists a closed sequentially compact set  $K_t$  such that  $\{F_n \leq t\} \subseteq K_t$  for each  $n \in \mathbb{N}$ .

We are now in a position to state the theorem which connects the minimum points of the functionals  $F_n$  and the functional  $F$ , which is proved in [DM93].

**Theorem 5.3.** *If  $(F_n)$  is a sequence of equi-coercive functionals which  $\Gamma$ -converge to  $F$ , then  $F$  is coercive and*

$$\min_{x \in X} F(x) = \lim_{n \rightarrow \infty} \inf_{x \in X} F_n(x).$$

## 5.2 Results about surfaces of revolutions

We follow a similar approach to [HE11b], who states rigorously some results we have already proved for surfaces of revolution which are not smooth at one point. We also differ slightly by considering rotating the x-axis around the z-axis like we did earlier, as opposed to Helmers' approach to rotate the y-axis around the x-axis. We assume that  $\gamma : I \mapsto \mathbb{R}^2$  is a Lipschitz parametrised curve, and thus it is differentiable almost everywhere. We assume that  $x(t) \geq 0$  for all  $t \in \bar{I}$ , and let  $M_\gamma$  be the surface of revolution, which is the image of  $\bar{I} \times [0, 2\pi]$  under the map

$$\Phi(t, \theta) = (x(t) \cos \theta, x(t) \sin \theta, z(t))$$

We can reparametrise so that the curve has constant speed  $|\gamma'| \equiv \mathcal{L}_\gamma/I := q_\gamma$ , where  $\mathcal{L}_\gamma$  is the length of the curve. In what follows, we get the same results as in section 3.1 with  $\dot{S}$  replaced by the now constant speed  $q_\gamma$ .

**Lemma 5.4.** *We have the following results if  $\gamma$  is as above:*

1. *If  $J$  is a measurable subset of  $I$  then  $\mathcal{A}_\gamma(J) = 2\pi \int_J |\gamma'| x dt$  (We denote  $\mathcal{A}_\gamma(I) = \mathcal{A}_\gamma$ ).*
2. *If  $J$  is a measurable subset of  $I$  then  $\mathcal{V}_\gamma(J) = \pi \int_J y' x^2 dt$  (We denote  $\mathcal{V}_\gamma(I) = \mathcal{V}_\gamma$ ).*
3. *If we consider rotationally symmetric  $f : M_\gamma \mapsto \mathbb{R}^k$  to be a function  $F : \bar{I} \times [0, 2\pi] \mapsto \mathbb{R}^k$  then  $|\nabla_{M_\gamma} f(t, \theta)| = |F'(t)|/|\gamma'(t)|$ .*
4. *If additionally  $\gamma \in W_{loc}^{2,1}(I, \mathbb{R}^2)$  then we have that (almost everywhere)*

$$H = \frac{\sin \psi}{x} + \frac{\dot{\psi}}{|\gamma'|} \text{ and } K = \frac{\dot{\psi} \sin \psi}{x |\gamma'|}.$$

*Proof.* We have already proved parts 1,2 and 4 for curves which are not differentiable at one point. The proof for rotating the y-axis around the x-axis is given in [HE11b]. The proof is not particularly challenging, but it requires delving further into geometry than we wish to at this point.  $\square$

If we additionally add a couple more geometrical constraints (which only slightly restrict the curve  $\gamma$  we can choose) then we can state some extra properties of these surfaces of revolution which will be useful in considering the limit surface. Throughout the rest of this section we denote  $\{x > 0\} := \{t \in I | x(t) > 0\}$ . Recall also that  $H^2 - 2K = k_1^2 + k_2^2$ , where  $k_i$  are the curvatures in orthogonal directions.

**Theorem 5.5.** *Let*

$$\mathcal{D} := \left\{ \gamma \in C^{0,1}(I, \mathbb{R}^2) \cap W_{loc}^{2,1}(\{x > 0\}, \mathbb{R}^2) \mid |\gamma'| = c > 0, x \geq 0, \int_{M(\{x > 0\})} H^2 - 2K dA < \infty \right\}$$

*Then if  $\gamma \in \mathcal{D}$  the following properties hold [HE11b]:*

1.  $\gamma \in W_{loc}^{2,2}(\{x > 0\}, \mathbb{R}^2)$  and  $x \in W^{2,1}(\{x > 0\}, \mathbb{R}^2)$ .
2. The number of components of  $\{x > 0\}$  is finite, hence  $M_\gamma$  has finitely many components which are connected at the axis of revolution.
3. Each component is a  $C^1$  surface and a  $W^{2,2}$  surface away from the axis of revolution.

### 5.3 Approximate energy of the vesicle

We now seek to define the approximate energy for a surface of revolution  $M_\gamma$ , as in [HE11b]. To do this we also need to define a phase field  $u : M_\gamma \mapsto \mathbb{R}$  which defines what phase we are in at each point of the surface, which is clearly a radial function. It is natural to want to think of  $u$  as just a step function like we considered in section 4, but by defining it on the real line we allow each point to contribute to both phases  $\pm$ . That is, for each point we consider the amount in phase  $+$  to be  $u_+$  and the amount in phase  $-$  to be  $u_-$ , with  $u_+ + u_- = 1$  and  $u = u_+ - u_-$ . So if  $u = 0.7$  then we would have  $u_+ = 0.85$  and  $u_- = 0.15$ . The approximate energy is given by

$$\mathcal{E}_\varepsilon(\gamma, u) \equiv \int_{M_\gamma} \kappa(u) (H - C(u))^2 + \kappa_G(u) K \, dA + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) \, dA,$$

where  $W : \mathbb{R} \mapsto [0, \infty)$  is a positive, continuous double well function such as  $(1 - u^2)^2$ , that is zero only at  $\pm 1$ . We also assume for simplicity it is symmetric, and that it satisfies  $W(u) \rightarrow \infty$  as  $u \rightarrow \pm\infty$ , and also for technical reasons that it is  $C^2$  in the neighbourhood of  $\pm 1$ . The first integral represents the usual Helfrich energy which we have considered so far, while the second integral term represents the diffuse interface energy which comes from Cahn-Hilliard theory [CH58]. We also define  $C, \kappa, \kappa_G : \mathbb{R} \mapsto \mathbb{R}$  to be continuous bounded extensions of  $C^\pm, \kappa^\pm, \kappa_G^\pm$  respectively which satisfy  $C(\pm 1) = C^\pm$ ,  $\kappa(\pm 1) = \kappa^\pm$  and  $\kappa_G(\pm 1) = \kappa_G^\pm$ . We also place a couple of technical conditions on the bending energies, which are

$$\inf_{u \in \mathbb{R}} \kappa(u) > 0, \sup_{u \in \mathbb{R}} \kappa_G(u) < 0 \text{ and } \kappa(u) > -\kappa_G(u)/2.$$

Finally, we deal with the area constraints of  $M_\gamma$ . If the areas of the phases are  $\mathcal{A}^+$  and  $\mathcal{A}^-$  then we get the constraints

$$\mathcal{A}_\gamma = \mathcal{A}_0 := \mathcal{A}^+ + \mathcal{A}^- \text{ and } \int_{M_\gamma} u \, dA = \mathcal{A}^+ - \mathcal{A}^-.$$

We now wish to consider the conditions on closed surfaces and phase fields we require, ensuring that the energy is well defined, satisfies area and volume constraints and also has the technical conditions from Theorem 5.5 to ensure the limit surface is well behaved. We consider  $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ , where [HE11b]

$$\mathcal{C}_\varepsilon := \left\{ \gamma \in C^{0,1}(I, \mathbb{R}^2) \cap W_{loc}^{2,1}(I, \mathbb{R}^2) \mid \begin{aligned} |\gamma'| &= c, x(\partial I) = 0, x(I) > 0, z' \geq 0, \int_{M_\gamma} H^2 - 2K \, dA < \infty, \mathcal{A}_\gamma = \mathcal{A}_0, \mathcal{V}_\gamma = \mathcal{V}_0 \end{aligned} \right\}$$

and

$$\mathcal{P}_\varepsilon := \left\{ u \in W_{loc}^{1,1}(I) \mid \int_{M_\gamma} |\nabla_{M_\gamma} u|^2 \, dA < \infty, \|u\|_\infty \leq C, \int_{M_\gamma} u \, dA = \mathcal{A}^+ - \mathcal{A}^- \right\}$$

Recalling that  $\gamma \in C^{0,1}$  means that  $\gamma$  is Lipschitz, the first three terms of  $\mathcal{C}_\varepsilon$  ensure that  $M_\gamma$  is a closed surface and can be parametrised with constant speed. Since we can also reparametrise the curve without changing the energy, restricting to constant speed curves is no geometric

restriction, but makes calculations easier. The fourth term is a technical condition that orientates the surface  $M_\gamma$  and also ensures there will be no overlapping in the limit surfaces. The fifth term ensures we have the properties of the surface of revolution we require by Theorem 5.5, and also ensures that the energy  $\mathcal{E}_\varepsilon$  is well defined, along with the first two conditions on  $\mathcal{P}_\varepsilon$ . The sixth condition for  $\mathcal{C}_\varepsilon$  and third condition for  $\mathcal{P}_\varepsilon$  are the area constraints for  $M_\gamma$ , while the last condition for  $\mathcal{C}_\varepsilon$  is the volume constraint.

#### 5.4 $\Gamma$ -Convergence of the approximate energy

We now wish to understand the limit energy, and what spaces  $\gamma$  and  $u$  are in the limit. It turns out that  $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ , which are defined as [HE11b]

$$\mathcal{C} := \left\{ \gamma \in C^{0,1}(I, \mathbb{R}^2) \cap W_{loc}^{2,1}(\{x > 0\}, \mathbb{R}^2) \mid |\gamma'| = c, x(\partial I) = 0, x \geq 0, z' \geq 0, \right. \\ \left. \mathcal{H}^0(\{x = 0\}) < \infty, \int_{M_\gamma(\{x > 0\})} H^2 - 2K \, dA < \infty, \mathcal{A}_\gamma = \mathcal{A}_0, \mathcal{V}_\gamma = \mathcal{V}_0 \right\}$$

and

$$\mathcal{P} := \left\{ u : I \mapsto \{-1, 1\} \text{ piecewise constant} \mid \int_{M_\gamma} u \, dA = \mathcal{A}^+ - \mathcal{A}^-, 2\pi \sum_{s \in S_u} x(s) < \infty \right\}.$$

We have defined  $S_u$  to be the countable jump set of  $u$  in  $x > 0$ , i.e all the points where  $u$  is discontinuous when not on the axis of revolution.  $\mathcal{H}^0$  is the zero dimensional Hausdorff measure and in this case just counts the number of points  $t$  where  $x(t) = 0$ . The difference between  $\mathcal{C}$  and  $\mathcal{C}_\varepsilon$  is that  $\gamma \in \mathcal{C}$  may touch the axis of revolution in the interior of  $I$ , but only finitely many times, and also now allows different components to touch. The limit energy is then

$$\mathcal{E}(\gamma, u) \equiv \int_{M_\gamma} \kappa(u) (H - C(u))^2 \, dA + 2\pi\sigma \sum_{s \in S_u} x(s),$$

where

$$\sigma = 2 \int_{-1}^1 \sqrt{W(u)} \, du.$$

The term with the Gauss bending energy disappears in the limit since we are approximating the surface with closed, smooth surfaces which satisfy the Gauss Bonnet Theorem. To be able to use Theorem 5.5 we wish to fix the domain, so we extend the functionals  $\mathcal{E}$  and  $\mathcal{E}_\varepsilon$  to the larger space  $W^{1,1}(I, \mathbb{R}^2) \times L^1(I)$  by defining  $\mathcal{E}(\gamma, u) = \mathcal{E}_\varepsilon(\gamma, u) = \infty$  if  $(\gamma, u)$  is not in either  $\mathcal{C} \times \mathcal{P}$  or  $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ . We can now state Helmers' theorem from [HE11b].

**Theorem 5.6.** *The functionals  $\mathcal{E}_\varepsilon : W^{1,1}(I, \mathbb{R}^2) \times L^1(I) \mapsto \mathbb{R}$  are equi-coercive, and  $\mathcal{E}_\varepsilon$   $\Gamma$ -converges to  $\mathcal{E}$  as  $\varepsilon \rightarrow 0$ .*

We firstly note that by restricting  $\mathcal{P}_\varepsilon$  to step functions  $u : I \mapsto \{-1, 1\}$  the theorem is still valid, since equi-coerciveness still holds and  $\Gamma$ -convergence is unaffected by setting some of the energy integrals to infinity. We also note this does not just apply to surfaces which are axisymmetric, since in [HE11a] Helmers proves a similar result for one dimensional curves which do not have any symmetry properties. We omit the proof of Theorem 5.6 since it is beyond the scope of this project, but we comment on the difficulty of it. The first issue is he uses a different definition of equi-coerciveness to the standard definition we have provided, and the main difficulty in proving the equi-coerciveness is understanding Sobolev spaces and the notions of convergence required such as weak-\* convergence. The lower bound inequality follows relatively easily from the properties of compactness and convergence proved for equi-coerciveness. The upper bound inequality is by far the hardest part of the proof, and Helmers firstly shows we

have finitely many interfaces in  $(\gamma, u)$ . He then goes on to construct very carefully the sequence required on different intervals separately, based on Modica-Mortola theory.

Instead of restricting the domain of  $\gamma$  with the volume and area constraints, we could instead introduce this constraint via penalty terms in the energy integral. We would just add the following term to the energy:

$$\frac{1}{\varepsilon}(\mathcal{V}_\gamma - \mathcal{V}_0) + \frac{1}{\varepsilon}(\mathcal{A}_\gamma - \mathcal{A}_0).$$

## 5.5 Extensions

In [HE11b] Helmers only looks at surfaces which have no kinks in them, so we now introduce the possibilities of ghost interfaces, as in Helmers' recent paper [HE12]. The approximate energy we now consider is :

$$\begin{aligned} \mathcal{F}_\varepsilon(\gamma, u) \equiv & \int_{M_\gamma} u^2 \kappa(u) (H - C(u))^2 + u^2 \kappa_G(u) K \, dA \\ & + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) \, dA + \varepsilon \int_{M_\gamma} H^2 - 2K \, dA. \end{aligned}$$

The  $u^2$  term is forced to 1 in the limit, and hence the first part of the energy is the Helfrich energy, while the second part is the interface energy, with the third integral term penalising regions of large curvature and accounting for a kink's bending energy in the limit. The limit of this energy in the new notion of convergence (which is slightly weaker than  $\Gamma$ -convergence, but still sufficient to connect minimum points of  $\mathcal{F}$  and  $\mathcal{F}_\varepsilon$ ) is given by [HE12]

$$\begin{aligned} \mathcal{F}(\gamma, u) \equiv & \int_{M_\gamma(\{x>0\}\setminus S)} \kappa(u) (H - C(u))^2 + \kappa_G(u) K \, dA \\ & + 2\pi \sum_{s \in S} (\sigma + \hat{\sigma}|[\gamma'](s)|x(s)) + 2\pi\hat{\sigma}\mathcal{L}_\gamma(\{x=0\}). \end{aligned}$$

$S$  is the set of ghost interfaces and phase boundary's with  $|[\gamma'](s)|$  measuring the angle made by the two one-sided tangent vectors at  $s$  modulo  $2\pi$  and  $\mathcal{L}_\gamma(\{x=0\})$  denoting the length of the segment  $\gamma(\{x=0\})$ .  $\hat{\sigma}$  depends on the double well potential  $W(u)$  and is given by  $2\sqrt{W(0)}$ . Whilst we still choose the same sets  $\mathcal{C}_\varepsilon$  and  $\mathcal{P}_\varepsilon$  for the approximating energies, we choose a slightly wider set  $\mathcal{C}^*$  for the limit surface to be in, although we keep the same set  $\mathcal{P}$  for the phase field. Extending these energies to  $C^0(I, \mathbb{R}^2) \times L^1(I)$ , we get that  $\mathcal{F}_\varepsilon$  is equi-coercive, and converges to  $\mathcal{F}$  in this new sense of convergence, see [HE12] for more details.

We finally consider extensions of Theorem 5.6 for when we have three phases, and also the simpler case for when we only have one phase. We adjust the phase field function  $u$ , which results in having to adjust  $W$  and also the area constraints. We summarise these in the table below.

Table 2: Properties of the phase field  $u$  depending on the number of phases.

Property	One Phase	Two Phase	Three Phase
Limit of $u$	$u \equiv 1$	$u : \mathbb{R} \mapsto \{-1, 1\}$	$u : \mathbb{R} \mapsto \{-1, 0, 1\}$
Example of $W(u)$	$(1-u)^2$	$(1-u)^2(1+u)^2$	$(1-u)^2(1+u)^2u^2$
$W(u)$ zero at	1	$\pm 1$	$0, \pm 1$
Area constraint in $\mathcal{C}$	$\mathcal{A}_\gamma = \mathcal{A}^+$	$\mathcal{A}_\gamma = \mathcal{A}_0 := \mathcal{A}^+ + \mathcal{A}^-$	$\mathcal{A}_\gamma = \mathcal{A}_0 := \mathcal{A}^+ + \mathcal{A}^0 + \mathcal{A}^-$
Area constraint in $\mathcal{P}$	None	$\int_{M_\gamma} u \, dA = \mathcal{A}^+ - \mathcal{A}^-$	$\int_{M_\gamma} u \, dA = \mathcal{A}^+ - \mathcal{A}^-$ , $\int_{M_\gamma} u + 1 \, dA = 2\mathcal{A}^+ + \mathcal{A}^0$

## References

- [YL08] Y Li, *Phase Separation in Giant Vesicles*, Potsdam University 2008.
- [W12a] Wikipedia entry on Cell Membrane [http://en.wikipedia.org/wiki/Cell\\_membrane](http://en.wikipedia.org/wiki/Cell_membrane), last modified 4 March 2012 at 18:14.
- [W12b] Wikipedia entry on Vesicle [http://en.wikipedia.org/wiki/Vesicle\\_\(biology\)](http://en.wikipedia.org/wiki/Vesicle_(biology)), last modified 22 February 2012 at 21:07.
- [W12c] Wikipedia entry on Membrane Curvature [http://en.wikipedia.org/wiki/Membrane\\_curvature](http://en.wikipedia.org/wiki/Membrane_curvature), last modified on 3 March 2012 at 19:08.
- [BHW03] T Baumgart, S Hess, W Webb, *Imaging coexisting fluid domains in biomembrane models coupling curvature and line tension*, Letters to Nature 425 (2003), 821–824.
- [SBL91] U Seifert, K Berndl, R Lipowsky, *Shape transformation of vesicles: Phase diagram for spontaneous-curvature and bilayer-coupling models*, Phys. Rev. A 44 (1991), 1182–1202.
- [JS94] F Jülicher, U Seifert, *Shape Equations for axisymmetrical vesicles: A clarification*, Phys. Rev. E 49 (1994), 4728–4731.
- [JL96] F Jülicher, R Lipowsky, *Shape transformations of vesicles with intermembrane domains*, Phys. Rev. E 53 (1996), 2670–2683.
- [DJB09] S Das, J Jenkins, T Baumgart, *Neck geometry and shape transitions in vesicles with co-existing fluid phases: Role of Gaussian curvature stiffness vs. spontaneous curvature*, EPL Journal 86 (2009), 48003.
- [DA09] B Dacorogna, *Introduction to the Calculus of Variations*, Imperial College Press 2009, *Second Edition*.
- [ES10] C Elliott, B Stinner, *Modeling and computation of two phase geometric biomembranes using surface finite elements*, J. Comput. Phys. (2010), doi:10.1016/j.jcp.2010.05.014 .
- [AB02] A Braides,  $\Gamma$ -*convergence for Beginners*, Oxford Lecture Series in Mathematics and its Applications volume 22 (Oxford University Press 2002), *First Edition*.
- [DM93] G Dal Maso, *An introduction to  $\Gamma$ -convergence*, Birkhäuser Boston 1993, *First Edition*.
- [HE11a] M Helmers, *Snapping elastic curves as a one dimensional analogue of two-component lipid bilayer*, Mathematical Models and Methods in Applied Sciences (M3AS) Volume: 21, Issue: 5(2011), 1027–1042.
- [HE11b] M Helmers, *Convergence of an approximation for rotationally symmetric two-phase*, preprint [http://www.iam.uni-bonn.de/fileadmin/user\\_upload/helmers/preprints/membranes\\_regular.pdf](http://www.iam.uni-bonn.de/fileadmin/user_upload/helmers/preprints/membranes_regular.pdf).
- [HE12] M Helmers, *Kinks in two-phase lipid bilayer membranes*, preprint [http://www.iam.uni-bonn.de/fileadmin/user\\_upload/helmers/preprints/membranes\\_singualar.pdf](http://www.iam.uni-bonn.de/fileadmin/user_upload/helmers/preprints/membranes_singualar.pdf).
- [CH58] J Cahn, J Hilliard *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys. (1958), 28(2), 258–267.
- [WAGC] B Bowditch, *Warwick University Geometry of Curves and Surfaces lecture course* <http://www.warwick.ac.uk/~masgak/cas/notes/casnotes.pdf>.
- [UCPD] J Hunter, *University of California, Davis Partial Differential Equations lecture course* <http://www.math.ucdavis.edu/~hunter/pdes/ch6A.pdf>.

## A Mathematical Background

### A.1 Surface Curvature

We consider Gauss and mean curvature [WAGC], so let  $\underline{r} : U \mapsto \mathbb{R}$  be a smooth, parametrised surface  $S$ .

**Definition** (Unit Normal). The unit normal is defined as  $\underline{n} := \frac{\underline{r}_u \wedge \underline{r}_v}{\|\underline{r}_u \wedge \underline{r}_v\|}$ .

For simplicity in the next two definitions we use the following standard notation:  $E := \underline{r}_u \cdot \underline{r}_u$ ,  $F := \underline{r}_u \cdot \underline{r}_v$ ,  $G := \underline{r}_v \cdot \underline{r}_v$ ,  $L := \underline{r}_{uu} \cdot \underline{n}$ ,  $M := \underline{r}_{uv} \cdot \underline{n}$ ,  $N := \underline{r}_{vv} \cdot \underline{n}$ .

**Definition** (Mean Curvature). The mean curvature is defined as  $H : U \mapsto \mathbb{R}$

$$H := \text{Trace} \left[ \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \right].$$

**Definition** (Gauss Curvature). The Gauss curvature is defined as  $K : U \mapsto \mathbb{R}$

$$K := \left| \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \right| = \frac{LN - M^2}{EG - F^2}.$$

### A.2 Gauss Bonnet Theorem

**Definition.** A closed surface in  $\mathbb{R}^3$  is a surface  $S \subset \mathbb{R}^3$  that is closed (in the topological sense), bounded and path connected. [WAGC]

**Theorem A.1.** *Let  $S$  be a closed, smooth surface, then*

$$\int_S K \, dA = 2\pi\chi(S).$$

We give a proof of this for surfaces of revolution in section 2, and for a full proof of the theorem for all smooth surfaces see [WAGC]. This result only holds for smooth surfaces, and we prove the analogue of the theorem for surfaces of revolution when we relax the smoothness condition at a point in section 3.1, Lemma 3.2.

### A.3 Direct Method of Calculus of Variations

The material for this section comes from [DM93].

**Definition** (Lower semi-continuity). The function  $f : X \mapsto \overline{\mathbb{R}}$  is (sequentially) lower semi-continuous at  $x \in X$  if for all  $x_n \rightarrow x$ ,

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

We say  $f$  is (sequentially) lower semi-continuous if it is (sequentially) lower semi-continuous for each  $x \in X$ . Equivalently,  $f$  is lower semi-continuous at  $x \in X$  if

$$f(x) \leq \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} f(y) \Leftrightarrow f(x) = \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} f(y),$$

where  $\mathcal{N}(x)$  is the set of all open neighbourhoods containing  $x$  that are in  $X$ .

**Definition** (Coercive). The function  $f : X \mapsto \overline{\mathbb{R}}$  is (sequentially) coercive if  $\overline{\{f \leq t\}} := \{x \in X | f(x) \leq t\}$  is sequentially compact in  $X$  (so every sequence has a convergent subsequence) for every  $t \in \mathbb{R}$ .

Clearly, if we have a continuous functions on a compact set then it achieves its minimum, but this is not a necessary condition. We can relax both continuity (to lower semi-continuity) and compactness (to coercive) and still have that the function achieves its minimum, as we see below.

**Theorem A.2** (Direct Method of Calculus of Variations). *If  $f : X \mapsto \overline{\mathbb{R}}$  is (sequentially) coercive and (sequentially) lower semi-continuous then  $f$  has a minimum point in  $X$ .*

*Proof.* If  $f \equiv \infty$  then we are done, so assume not. Then we can find a minimising sequence  $(x_n)$  which satisfies  $\inf_{y \in X} f(y) = \lim_{n \rightarrow \infty} f(x_n) < \infty$ , and this clearly exists by properties of the infimum. By taking  $t = \lim_{n \rightarrow \infty} f(x_n) + 1$  we have  $x_n \in \overline{\{f \leq t\}}$  since  $f$  is (sequentially) coercive and thus  $x_n \rightarrow x$  for some  $x \in X$ . Then  $f$  is lower semi-continuous, hence we have the following inequalities:

$$\inf_{y \in X} f(y) \leq f(x) \leq \liminf_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(x_n) = \inf_{y \in X} f(y),$$

and thus  $f(x) = \inf_{y \in X} f(y)$ . □

#### A.4 Generalisations of Sobolev Spaces to $\mathbb{R}^k$

Most books only define  $L^p$  spaces and Sobolev spaces for functions  $f : \Omega \mapsto \mathbb{R}$ , but we want to work with vector valued functions on surfaces of revolutions. From [UCPD] we get the generalised result, which we will use in section 5 for  $\Omega = I$  and  $k = 2$ . Recall also that all norms on  $\mathbb{R}^k$  are equivalent .

**Definition** ( $L^p$  space). We define  $L^p(\Omega, \mathbb{R}^k)$  as the space of all measurable functions  $f : \Omega \mapsto \mathbb{R}^k$  such that

$$\|f\|_{L^p(\Omega, \mathbb{R}^k)} := \left( \int_{\Omega} \|f\|_{\mathbb{R}^k}^p dx \right)^{\frac{1}{p}}.$$

**Definition** (Weakly differentiable). We define  $f \in L^1_{loc}(\Omega, \mathbb{R}^k)$  to be weakly differentiable with weak derivative  $D^\alpha f = g$  if

$$\int_{\Omega} D^\alpha \phi f_i dx = (-1)^{|\alpha|} \int_{\Omega} g_i \phi dx,$$

for each  $\phi \in C_c^\infty(\Omega)$  and each  $i$  where  $f = (f_1, f_2, \dots, f_k)$  and  $g = (g_1, g_2, \dots, g_k)$ .

**Definition** (Sobolev space). We define the Sobolov space  $W^{n,p}(\Omega, \mathbb{R}^k)$  to consist of measurable functions  $f : \Omega \mapsto \mathbb{R}^k$  whose weak derivative for  $|\alpha| \leq n$  is in  $L^p(\Omega, \mathbb{R}^k)$ , that is

$$W^{n,p}(\Omega, \mathbb{R}^k) := \{f \in L^p(\Omega, \mathbb{R}^k) : D^\alpha f \in L^p(\Omega, \mathbb{R}^k) \ \forall |\alpha| \leq n\}.$$

## B Intermediate Steps for Shape Equations

### B.1 One Phase Membranes

We wish to prove the following theorem which we stated in section 2.2.

**Theorem B.1** (Shape Equations for a general vesicle). *A general axisymmetric vesicle that minimises the bending energy  $E$  subject to volume and area constraints satisfies the following equations for  $S_0 < S < S_1$ :*

$$\ddot{\psi} = \frac{\cos \psi \sin \psi}{x^2} - \frac{\dot{\psi}}{x} \cos \psi + \frac{\bar{P}}{2} x \cos \psi + \frac{\gamma}{x} \sin \psi + \frac{\eta}{x} \cos \psi \quad (\text{B.1})$$

$$\dot{\gamma} = \frac{1}{2}(\dot{\psi} - C_0)^2 - \frac{\sin^2 \psi}{2x^2} + \bar{\Sigma} + \bar{P}x \sin \psi \quad (\text{B.2})$$

$$\dot{\eta} = 0 \quad (\text{B.3})$$

$$\dot{x} = \cos \psi \quad (\text{B.4})$$

$$\dot{z} = -\sin \psi. \quad (\text{B.5})$$

To prove the theorem we need to solve the following equations:

$$\frac{\partial L}{\partial \psi} - \frac{d}{dS} \frac{\partial L}{\partial \dot{\psi}} = 0, \quad \frac{\partial L}{\partial x} - \frac{d}{dS} \frac{\partial L}{\partial \dot{x}} = 0, \quad \frac{\partial L}{\partial z} - \frac{d}{dS} \frac{\partial L}{\partial \dot{z}} = 0, \quad \frac{\partial L}{\partial \gamma} = 0, \quad \frac{\partial L}{\partial \eta} = 0,$$

where

$$L(\psi, x, z, \gamma, \eta) = \frac{x}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 + \bar{\Sigma} x + \frac{\bar{P}}{2} x^2 \sin \psi + \gamma(\dot{x} - \cos \psi) + \eta(\dot{z} + \sin \psi).$$

We can calculate that

$$\frac{\partial L}{\partial \psi} = x \left( \dot{\psi} + \frac{\sin \psi}{x} - C_0 \right) \frac{\cos \psi}{x} + \frac{\bar{P}x^2}{2} \cos \psi + \gamma \sin \psi + \eta \cos \psi \text{ and } \frac{\partial L}{\partial \dot{\psi}} = x \left( \dot{\psi} + \frac{\sin \psi}{x} - C_0 \right)$$

and thus,

$$\frac{d}{dS} \frac{\partial L}{\partial \dot{\psi}} = \dot{x} \left( \dot{\psi} + \frac{\sin \psi}{x} - C_0 \right) + x \left( \ddot{\psi} + \frac{\dot{\psi} \cos \psi}{x} - \frac{\dot{x} \sin \psi}{x^2} \right).$$

Equating these two expressions, we conclude that

$$\begin{aligned} \frac{\partial L}{\partial \psi} - \frac{d}{dS} \frac{\partial L}{\partial \dot{\psi}} &= \dot{\psi} \cos \psi + \frac{\sin \psi \cos \psi}{x} - C_0 \cos \psi + \frac{\bar{P}x^2}{2} \cos \psi + \gamma \sin \psi + \eta \cos \psi - \dot{x} \dot{\psi} \\ &\quad - \frac{\dot{x} \sin \psi}{x} + \dot{x} C_0 - x \ddot{\psi} - \dot{\psi} \cos \psi + \frac{\dot{x} \sin \psi}{x} = 0. \end{aligned}$$

Finally, recalling that  $\dot{x} = \cos \psi$  gives us that

$$x \ddot{\psi} = \frac{\sin \psi \cos \psi}{x} + \frac{\bar{P}x^2}{2} \cos \psi + \gamma \sin \psi + \eta \cos \psi - \dot{\psi} \cos \psi,$$

and hence B.1 follows.

We now deal with the second equation, and we can calculate that

$$\frac{\partial L}{\partial x} = \frac{1}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right)^2 - x \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right) \frac{\sin \psi}{x^2} + \bar{\Sigma} + \bar{P}x \sin \psi \text{ and } \frac{\partial L}{\partial \dot{z}} = \gamma,$$

thus

$$\frac{\partial L}{\partial x} - \frac{d}{dS} \frac{\partial L}{\partial \dot{x}} = \frac{1}{2} \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 \right) \left( \frac{\sin \psi}{x} + \dot{\psi} - C_0 - \frac{2 \sin \psi}{x} \right) + \bar{\Sigma} + \bar{P}x \sin \psi - \dot{\gamma} = 0.$$

Hence we have that

$$\dot{\gamma} = \frac{1}{2} \left( \dot{\psi} - C_0 + \frac{\sin \psi}{x} \right) \left( \dot{\psi} - C_0 - \frac{\sin \psi}{x} \right) + \bar{\Sigma} + \bar{P}x \sin \psi = \frac{1}{2} (\dot{\psi} - C_0)^2 - \frac{\sin^2 \psi}{2x^2} + \bar{\Sigma} + \bar{P}x \sin \psi.$$

The third equation is easy to solve, since we have

$$\frac{\partial L}{\partial z} = 0 \text{ and } \frac{\partial L}{\partial \dot{z}} = \eta,$$

thus

$$\frac{\partial L}{\partial z} - \frac{d}{dS} \frac{\partial L}{\partial \dot{z}} = 0 - \dot{\eta} = 0$$

and hence  $\dot{\eta} = 0$  follows. The fourth and fifth equations are also easy to solve, since

$$\frac{\partial L}{\partial \gamma} = \dot{x} - \cos \psi = 0 \text{ and } \frac{\partial L}{\partial \eta} = \dot{z} + \sin \psi = 0,$$

which gives B.4 and B.5.

## C Matlab Code

### C.1 One Phase Membranes

To set up the Shape Equations in Matlab, I used one m-file ‘mem’ (Figure 14) to define the system of first order differential equations and another m-file ‘mevents’ (Figure 15) to stop the ode solver when the  $x$  coordinate got close to the origin. (Due to the fact  $x(S_1) = 0$ , and the difficulties this generates we instead tell the solver to stop when  $x = 0.01$ , but only if the derivative is decreasing).

```
function dxds = mem(s,x)
P=5; %Volume Lagrange Multiplier
sigma=-2; %Area Lagrange Multiplier
c=-0.1; %Spontaneous Curvature
% Define new variables: x=x(1); phi=x(2); u=x(3); gamma=x(4) z=x(5);

dxds = zeros(5,1);
dxds(1) = cos(x(2));
dxds(2) = x(3);
dxds(3) = ((-x(3)*cos(x(2)))/x(1))+((cos(x(2))*sin(x(2)))/((x(1))^2))+...
           ((x(4)*sin(x(2)))/(x(1)))+(0.5*P*x(1)*cos(x(2)));
dxds(4) = (0.5*((x(3)-c)^2))-(0.5*((sin(x(2)))^2)/((x(1))^2))+...
           (P*x(1)*sin(x(2)))+sigma;
dxds(5) = -sin(x(2));

end
```

Figure 14: Matlab m-file ‘mem’.

```
function [value,isterminal,direction] = mevents(s,x)
% Find when the x-coordinate next gets within 0.01 of the origin after the
% intial condition.
value = x(1)-0.01;
isterminal= 1;
direction = -1;
end
```

Figure 15: Matlab m-file ‘mevents’.

We can then use any of the ode solvers (ode23, ode45, ode113 etc) to solve the system of equations, since experimentation found that they agree on solutions. All of the solutions in Appendix D.1 were done generated with ode113, since Matlab recommends it for ‘problems with stringent error tolerances or for solving computationally intensive problems’. In the Matlab workspace (Figure 16), we first need to define  $l$  (the maximum value that  $S_1$  can take) and  $u$  (which is the value of the initial curvature,  $U(0)$ ). We also define  $e = 10^{-25}$ , which is small enough for our needs, and is used to define the initial conditions for  $\psi$  and  $x$ . We then instruct Matlab to use the ode solver and also plot a graph of  $x$  against  $z$  and  $-x$  against  $z$  to get a full phase portrait of the solution, which is the figure output (Figure 17).

We also get a matrix (Figure 18) which represents the vector  $\underline{x}$  (which is made up from the five variables we are interested in,  $\underline{x} = [x, \psi, u, \gamma, z]$ ) at arclength intervals 0.001 apart. ‘xe’ represents the vector where we choose to stop the solver, and ‘ie’ represents how many times this occurs, we chose this to be just once. The ‘events’ mfile stops the solver at  $x = 0.01$ , and we can read off that  $S_1 = 4.30$  in this case, and also that  $\psi(S_1) \approx \pi$  and  $\gamma(S_1) \approx 0$ . Whenever ‘mevents’ stops the solver, we need to check that that  $\psi(S_1) \approx \pi$  and  $\gamma(S_1) \approx 0$ , since otherwise it is not a solution to the Shape Equations.

```

>> l=20
l =
20
>> e=10^-25
e =
1.0000e-025
>> u=-0.9
u =
-0.9000
>> a=[sin(e)/u,e,u,0,0]
a =
-0.0000    0.0000   -0.9000         0         0
>> [s1,x,S_1,xe,ie]=ode113(@mem,[0:0.001:1],a,odeset('Events',@mevents))
>> plot(x(:,1),x(:,5),'g',-x(:,1),x(:,5),'g','linewidth',4), axis equal

```

Figure 16: Matlab input

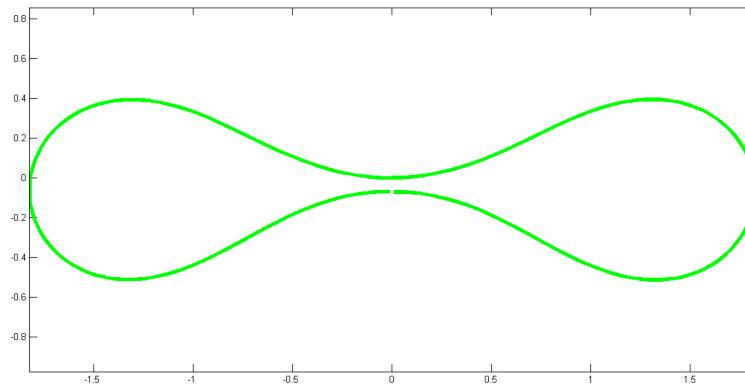


Figure 17: Matlab figure output when  $\bar{P} = 5$ ,  $\bar{\Sigma} = -2$ ,  $u = -0.9$  and  $c_0 = -0.1$

We also need to discard any solutions whose phase portrait intersects, since this is clearly not biological feasible. An example to show that this is possible despite  $\psi(S_1) \approx \pi$  and  $\gamma(S_1) \approx 0$  is given in Figure 19.

```

0.0223    3.1316   -2.4313    0.1053   -0.0695
0.0213    3.1290   -2.5801    0.1062   -0.0695
0.0203    3.1263   -2.7508    0.1074   -0.0695
0.0193    3.1234   -2.9466    0.1089   -0.0695
0.0183    3.1203   -3.1707    0.1107   -0.0695
0.0173    3.1173   -3.4121    0.1121   -0.0696
0.0163    3.1137   -3.7242    0.1150   -0.0696
0.0153    3.1097   -4.1006    0.1185   -0.0696
0.0143    3.1054   -4.5552    0.1228   -0.0697
0.0133    3.1010   -5.0597    0.1248   -0.0697
0.0123    3.0956   -5.7684    0.1313   -0.0697
0.0113    3.0893   -6.6778    0.1394   -0.0698
0.0103    3.0820   -7.8506    0.1494   -0.0698
0.0100    3.0825   -7.7496    0.1124   -0.0699

S_1 =
4.3023

xe =
0.0100    3.0825   -7.7496    0.1124   -0.0699

ie =
1

```

Figure 18: Matlab numerical output.

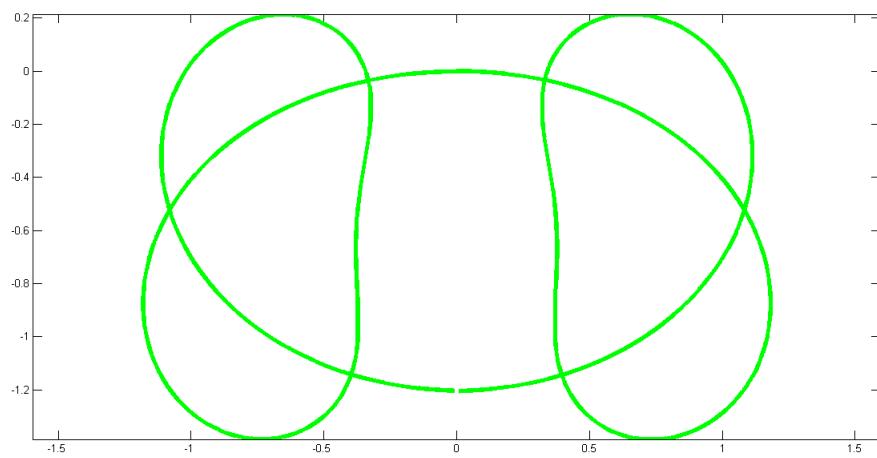


Figure 19: Example of a solution whose phase portrait intersects, which occurs when  $\bar{P} = 5$ ,  $\bar{\Sigma} = -2.1$ ,  $u = 0.6$  and  $c_0 = -0.1$

## C.2 Two Phase Membranes

For the implementation of the two phase Shape Equations in Matlab, I used two m-files for defining the system of first order differential equations on each phase; ‘cell1’ for the alpha phase (Figure 20) and ‘cell2’ for the beta phase (Figure 21). Additionally, I also used two m-files ‘cevents’ (Figure 22) and ‘devents’ (Figure 23) to stop the numerical solver where desired. The ‘cevents’ file stops the numerical solver at the distance  $S_1$ , which is the first solution of the equation

$$C_0^{(\alpha)} - U(S) - \left(1 + \frac{\kappa_G^{(\alpha)}}{\kappa^{(\alpha)}}\right) \frac{\sin \psi(S)}{x(S)} = 0,$$

and the other file ‘devents’ stops the numerical solver when the  $x$ -coordinate gets close to the origin. The command “global” allows us to define all the constants in the first m-file ‘cell1’ and then the other m-files and workspace can use these constants.

```

function dxds = cell1(s,x)
global P;
P=5; %Volume Lagrange Multiplier
global sigma;
sigma=11.5; %Line tension
global sigma_alpha;
sigma_alpha=-7; %Area Lagrange Multiplier alpha
global sigma_beta;
sigma_beta=-7; %Area Lagrange Multiplier beta
global c_alpha;
c_alpha=-0.1; %Spontaneous Curvature of alpha
global c_beta;
c_beta=-0.1; %Spontaneous Curvature of beta
global k_alpha;
k_alpha=1; %Bending rigidity of alpha
global k_beta;
k_beta=1; %Bending rigidity of beta
global k_g_alpha;
k_g_alpha=-1; %Bending rigidity of alpha
global k_g_beta;
k_g_beta=-1; %Bending rigidity of beta
% Define new variables: x=x(1); phi=x(2); u=x(3); gamma=x(4) z=x(5);

dxds = zeros(5,1);
dxds(1) = cos(x(2));
dxds(2) = x(3);
dxds(3) = ((-x(3)*cos(x(2)))/x(1))+((cos(x(2))*sin(x(2)))/((x(1))^2))+...
           ((x(4)*sin(x(2)))/(k_alpha*x(1)))+((0.5*P*x(1)*cos(x(2)))/k_alpha);
dxds(4) = (0.5*k_alpha*((x(3)-c_alpha)^2)+(P*x(1)*sin(x(2)))*...
           -(0.5*k_alpha*((sin(x(2)))^2)/((x(1))^2))+sigma_alpha;
dxds(5) = -sin(x(2));
end

```

Figure 20: Matlab m-file ‘cell1’.

We use ode113 solver for the same reasons as above, and again in the Matlab workspace we first need to define  $l$ ,  $u$  and  $e = 10^{-25}$ . Instead of typing the solver into the Matlab workspace, we now use a script file ‘run1’ (Figure 24) to carry out solving the differential equations for the two phases and the matching conditions connecting them. We use the ‘fzero’ numerical solver in Matlab to solve equation (3.29)‘matchfun’ , and then ‘run1’ starts ‘fzero’ at 0 and ‘run2’ starts ‘fzero’ at  $\pi$ , which in most cases gives us both solutions of the equation. Matlab plots a graph of  $x$  against  $z$  and  $-x$  against  $z$  for each phase to get a full phase portrait of the solution, the figure output (Figure 25), with the alpha phase green and the beta phase blue. We also get two matrices (Figure 26) which represent the vector  $\underline{x}$  at arclength intervals 0.001

```

function dxds = cell2(s,x)
global P; %Volume Lagrange Multiplier
global sigma_beta; %Area Lagrange Multiplier beta
global c_beta; %Spontaneous Curvature of beta
global k_beta;%Bending rigidity of beta

% Define new variables: x=x(1); phi=x(2); u=x(3); gamma=x(4) z=x(5);

dxds = zeros(5,1);
dxds(1) = cos(x(2));
dxds(2) = x(3);
dxds(3) = ((-x(3)*cos(x(2)))/x(1))+((cos(x(2))*sin(x(2)))/((x(1))^2))+...
           ((x(4)*sin(x(2)))/(k_beta*x(1)))+((0.5*P*x(1)*cos(x(2)))/k_beta);
dxds(4) = (0.5*k_beta*((x(3)-c_beta)^2)+(P*x(1)*sin(x(2)))...
           -(0.5*k_beta*((sin(x(2)))^2)/((x(1))^2))+sigma_beta;
dxds(5) = -sin(x(2));

end

```

Figure 21: Matlab m-file ‘cell2’.

```

function [value,isterminal,direction] = cevents(s,x)
% Find the distance S_1 by stopping S when the following condition reaches
% zero
global c_alpha;
global k_alpha;
global k_g_alpha
value = c_alpha*x(1)-x(3)*x(1)-(1+(k_g_alpha/k_alpha))*(sin(x(2)));
isterminal= 1;
direction = 0;
end

```

Figure 22: Matlab m-file ‘cevents’.

```

function [value,isterminal,direction] = devents(s,x)
% Find when the x-coordinate gets within 0.01 of the origin
value = x(1)-0.01;
isterminal= 1;
direction = -1;
end

```

Figure 23: Matlab m-file ‘devents’.

apart. The ‘cevents’ m-file stops the solver at  $S_1 = 0.3233$  in this case, and ‘devents’ gives us that  $S_2 = 0.3139$ , making a total arclength of  $S = 0.6372$ . We can see that  $\psi(S_1) \approx \pi$  and  $\gamma(S_1) \approx 0$ . Whenever ‘devents’ stops the numerical solver, we need to check that that  $\psi(S_1) \approx \pi$  and  $\gamma(S_1) \approx 0$ , since otherwise it is not a possible solution to the Shape Equations.

```

global P; global sigma; global sigma_alpha; global sigma_beta; global c_alpha;
global c_beta; global k_alpha; global k_beta; global k_g_beta; global k_g_alpha;
a=[sin(e)/u,e,u,0,0];
[s,x,S_1,xe,ie]=ode113(@cell1, [0:0.001:1],a,odeset('Events',@cevents));
plot(x(:,1),x(:,5),'g',-x(:,1),x(:,5),'g','linewidth',4), axis equal; hold on;
theta=fzero('matchfun',0);
b=[xe(1),theta,c_beta-(1+(k_g_beta/k_beta))*(sin(theta)/xe(1)),xe(4)+sigma,xe(5)];
[s,x,S_2,xe1,ie1]=ode113(@cell2, [0:0.001:1],b,odeset('Events',@devents));
plot(x(:,1),x(:,5),'b',-x(:,1),x(:,5),'b','linewidth',4), hold off, axis equal

function r = matchfun(t)
global P; global sigma; global sigma_alpha; global sigma_beta; global c_alpha;
global c_beta; global k_alpha; global k_beta; global k_g_beta;
global k_g_alpha; global xe;
r=((sigma_alpha-sigma_beta)*xe(1))+(sigma*cos(t))+(xe(4)*(cos(t)-cos(xe(2))))+...
(0.5*P*((xe(1))^2)*(sin(xe(2))-sin(t)))+((sin(t)*sin(t))/(2*xe(1)))-...
(((k_g_alpha)^2)*sin(xe(2))*sin(xe(2)))/(2*k_alpha*xe(1))...
+(k_g_beta*sin(t)*((sin(t)/xe(1))-c_beta))...
-(k_g_alpha*sin(xe(2))*((sin(xe(2))/xe(1))-c_alpha));
end

```

Figure 24: Matlab script ‘run1’ and ‘matchfun’

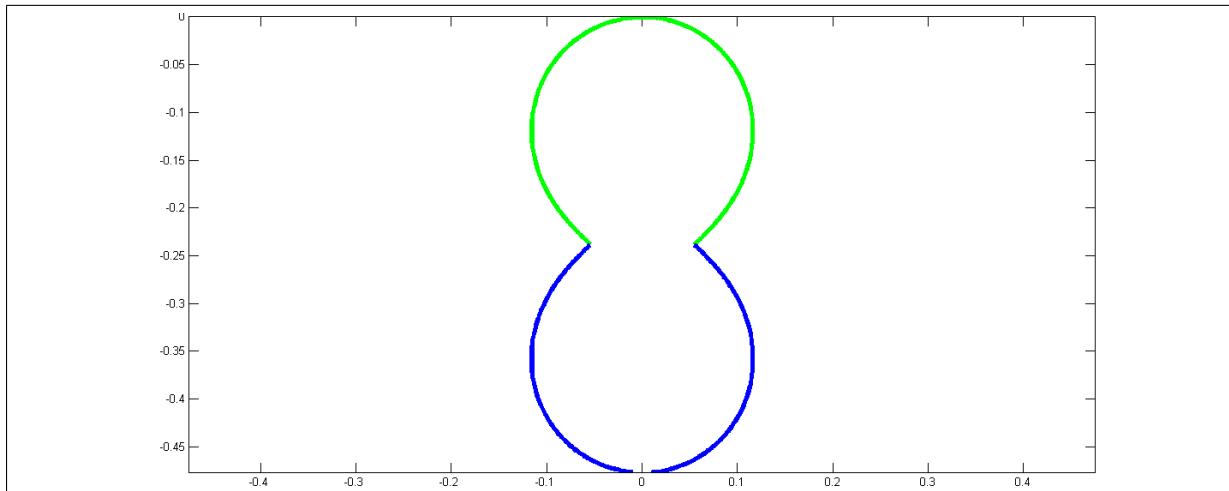


Figure 25: Matlab figure output.

```

0.0218    2.9479    8.6069    0.0880   -0.4752
0.0208    2.9565    8.5960    0.0798   -0.4754
0.0198    2.9651    8.5833    0.0713   -0.4756
0.0188    2.9737    8.5685    0.0626   -0.4758
0.0179    2.9822    8.5513    0.0537   -0.4759
0.0169    2.9908    8.5322    0.0446   -0.4761
0.0159    2.9993    8.5081    0.0349   -0.4762
0.0149    3.0078    8.4787    0.0248   -0.4764
0.0139    3.0163    8.4450    0.0144   -0.4765
0.0129    3.0247    8.4003    0.0030   -0.4766
0.0119    3.0331    8.3436   -0.0093  -0.4767
0.0109    3.0414    8.2708   -0.0226  -0.4768
0.0100    3.0489    8.1853   -0.0361  -0.4769

s_2 =
0.3139

xe1 =
0.0100    3.0489    8.1853   -0.0361  -0.4769

ie1 =
1

```

Figure 26: Matlab numerical output.

## D Catalogue of Solutions

### D.1 One Phase Membranes

The following is a non-exhaustive catalogue of solutions for one phase membranes when  $\bar{P} = 5$  and  $C_0 = -0.1$

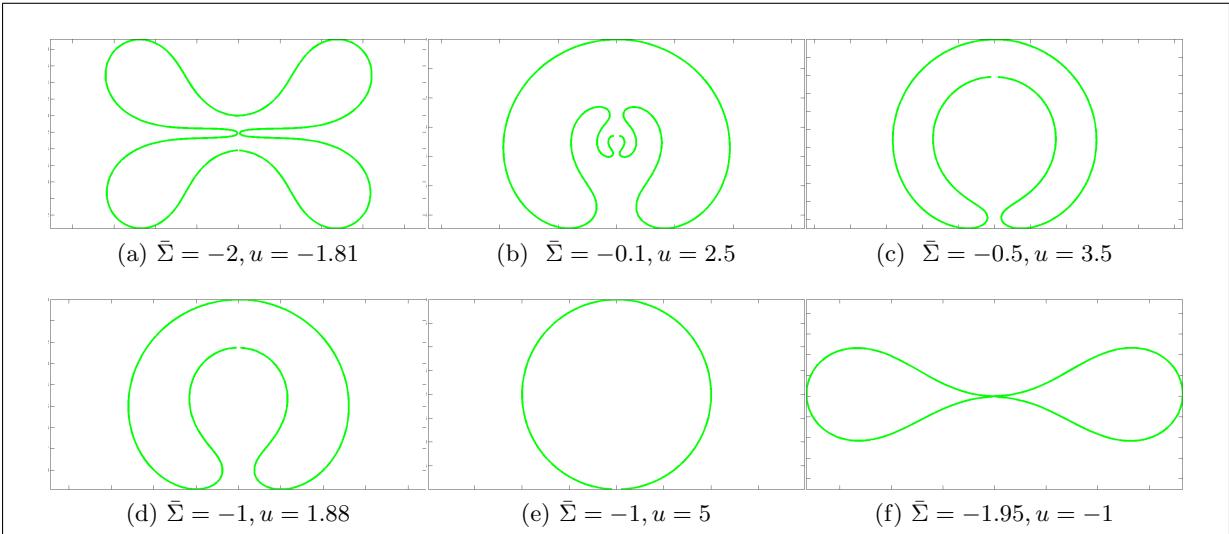


Figure 27: Solutions for miscellaneous  $\bar{\Sigma}$

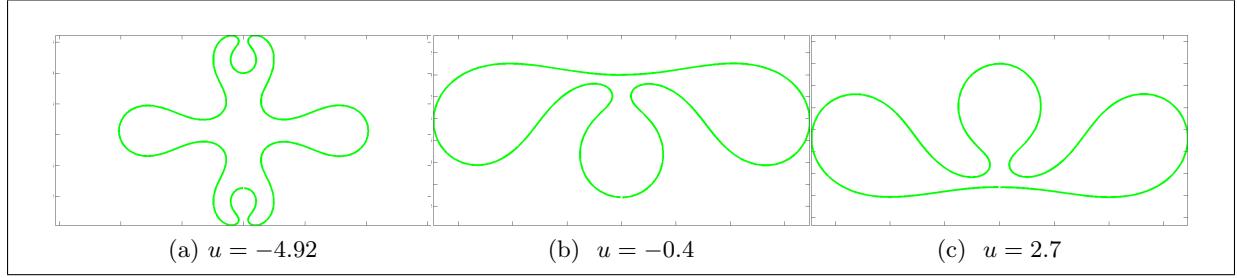


Figure 28: Solutions when  $\bar{\Sigma} = -1.8$

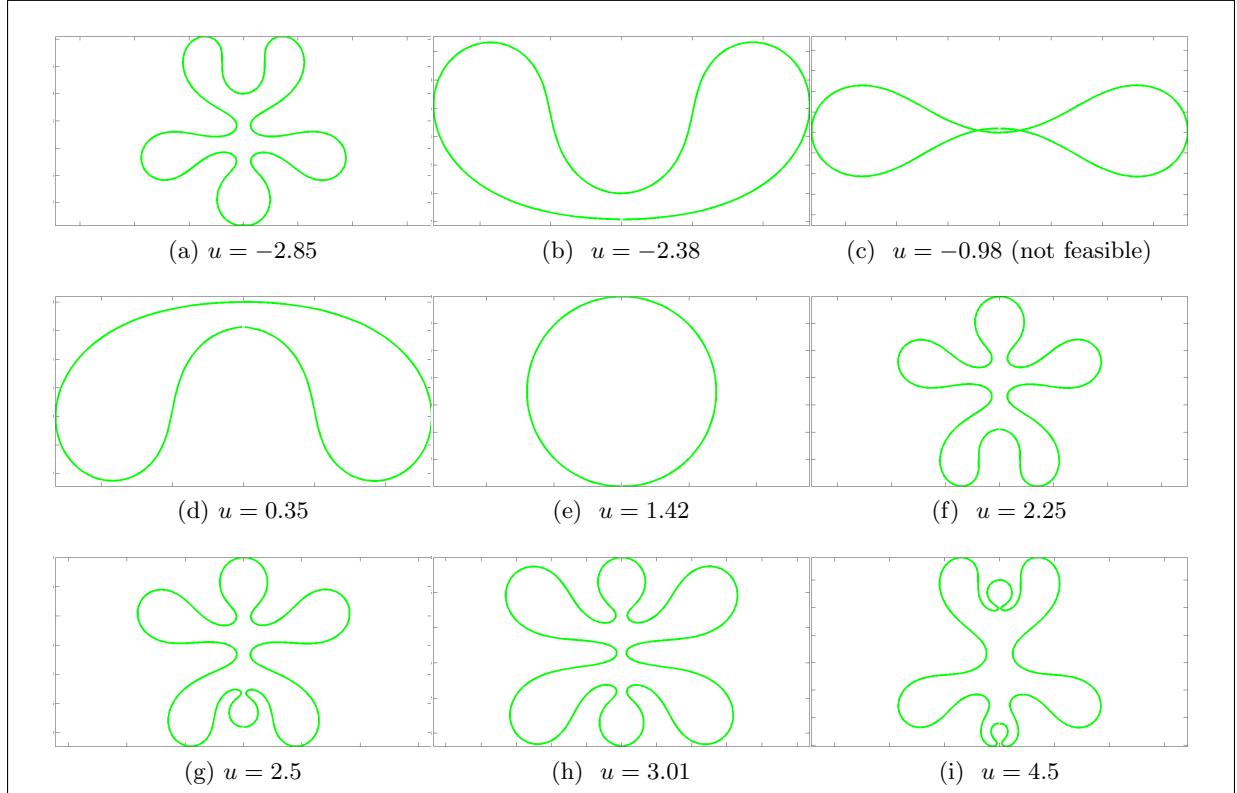


Figure 29: Solutions when  $\bar{\Sigma} = -1.9$

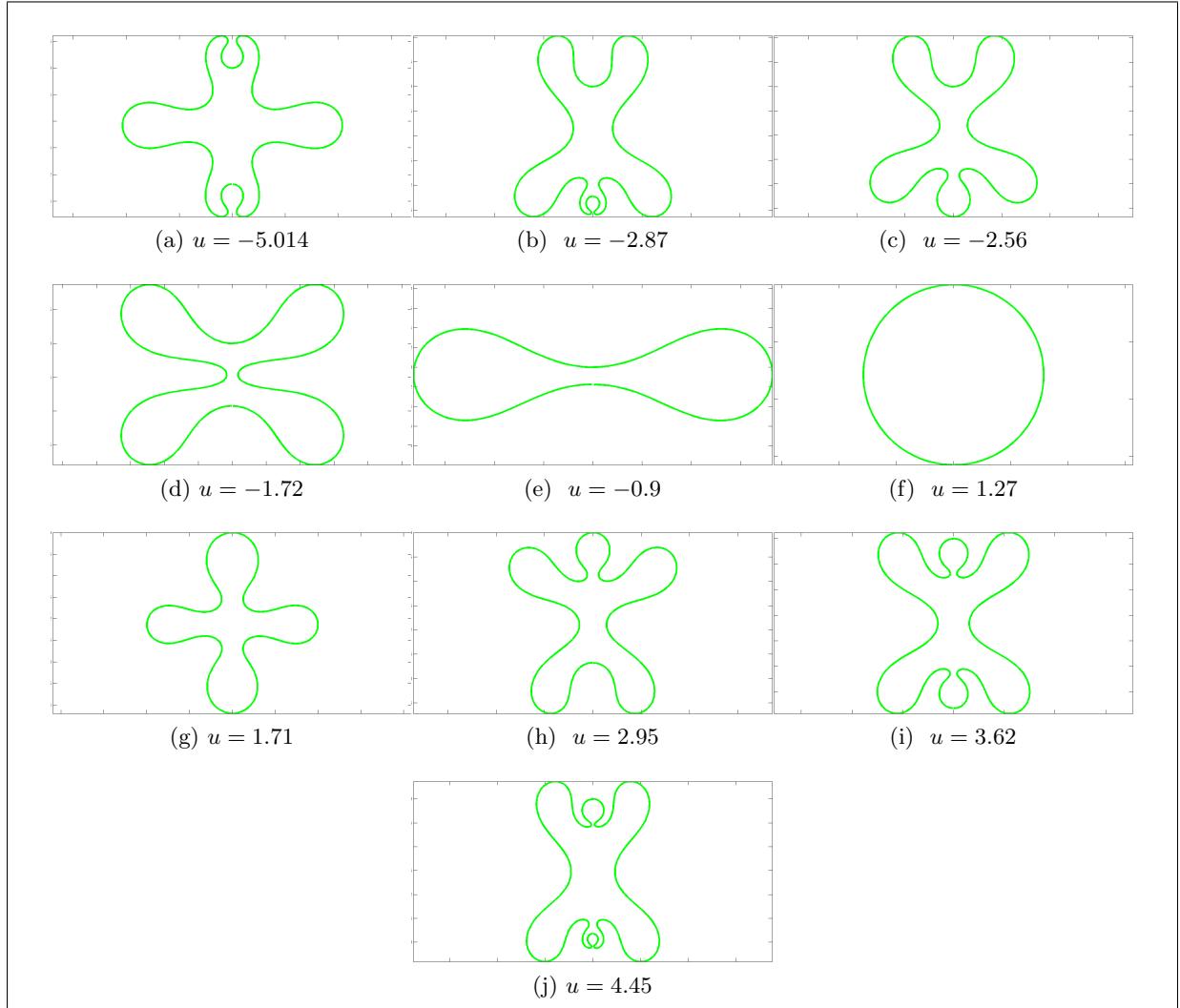


Figure 30: Solutions when  $\bar{\Sigma} = -2.1$

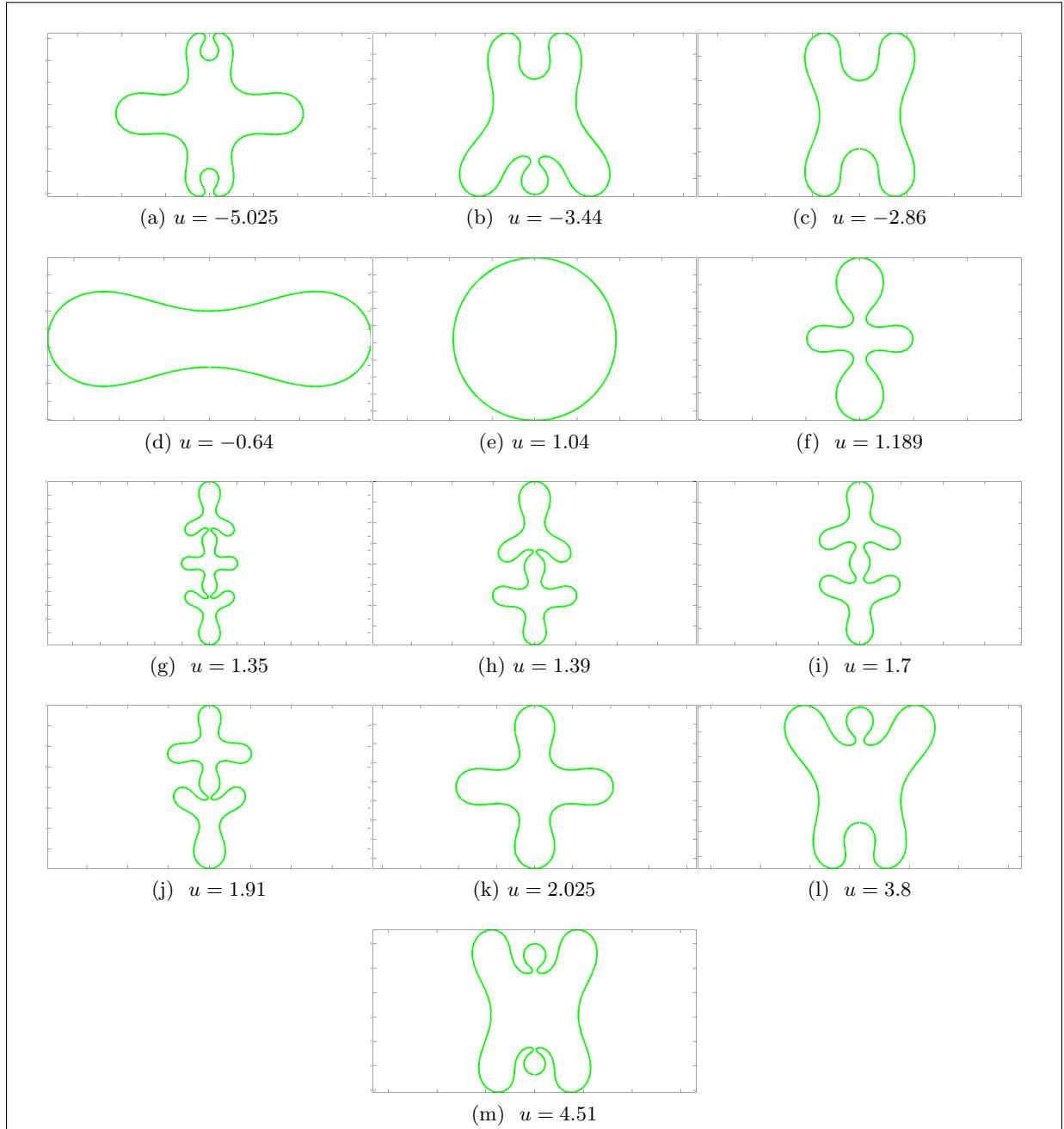


Figure 31: Solutions when  $\bar{\Sigma} = -2.5$

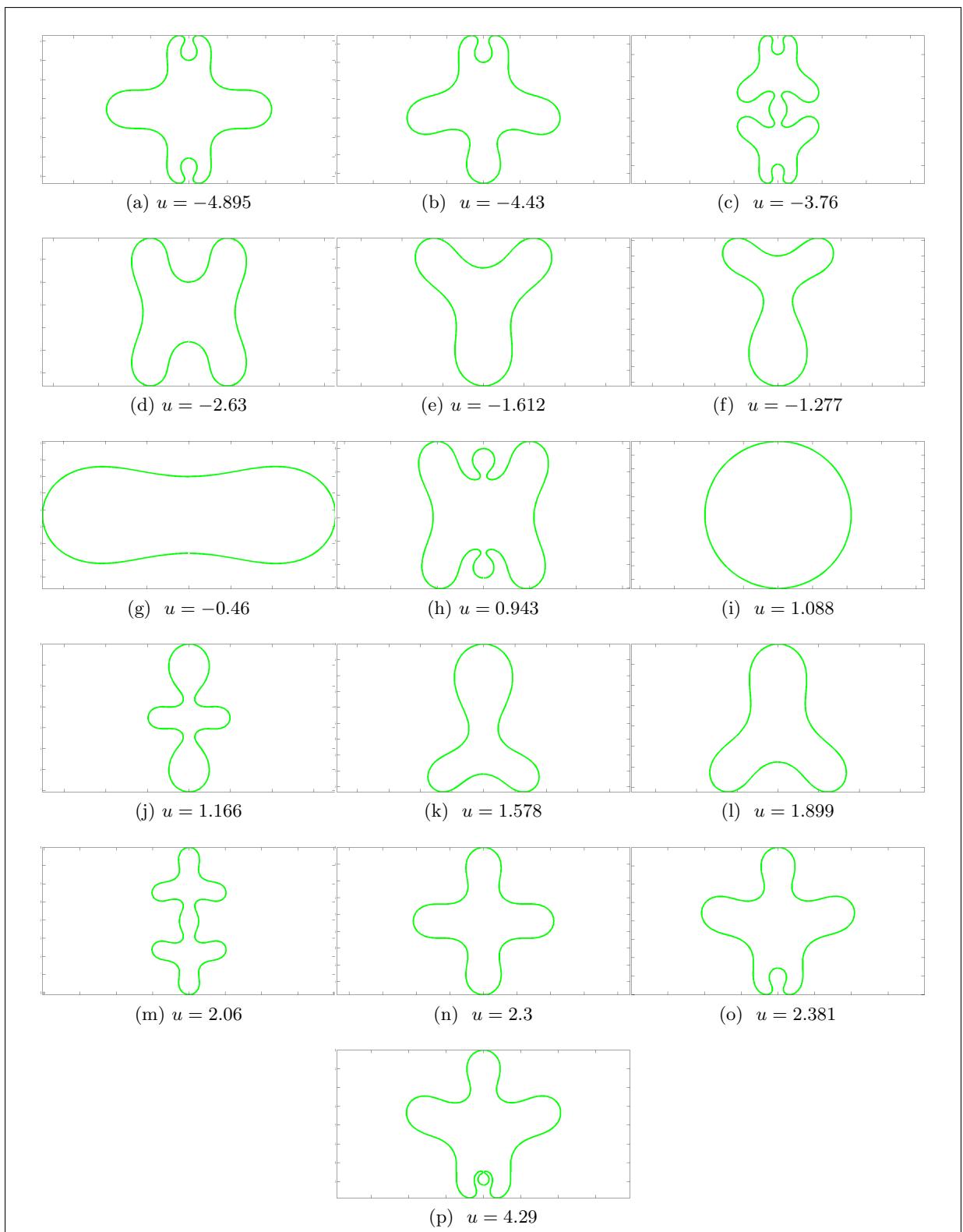


Figure 32: Solutions when  $\bar{\Sigma} = -2.75$

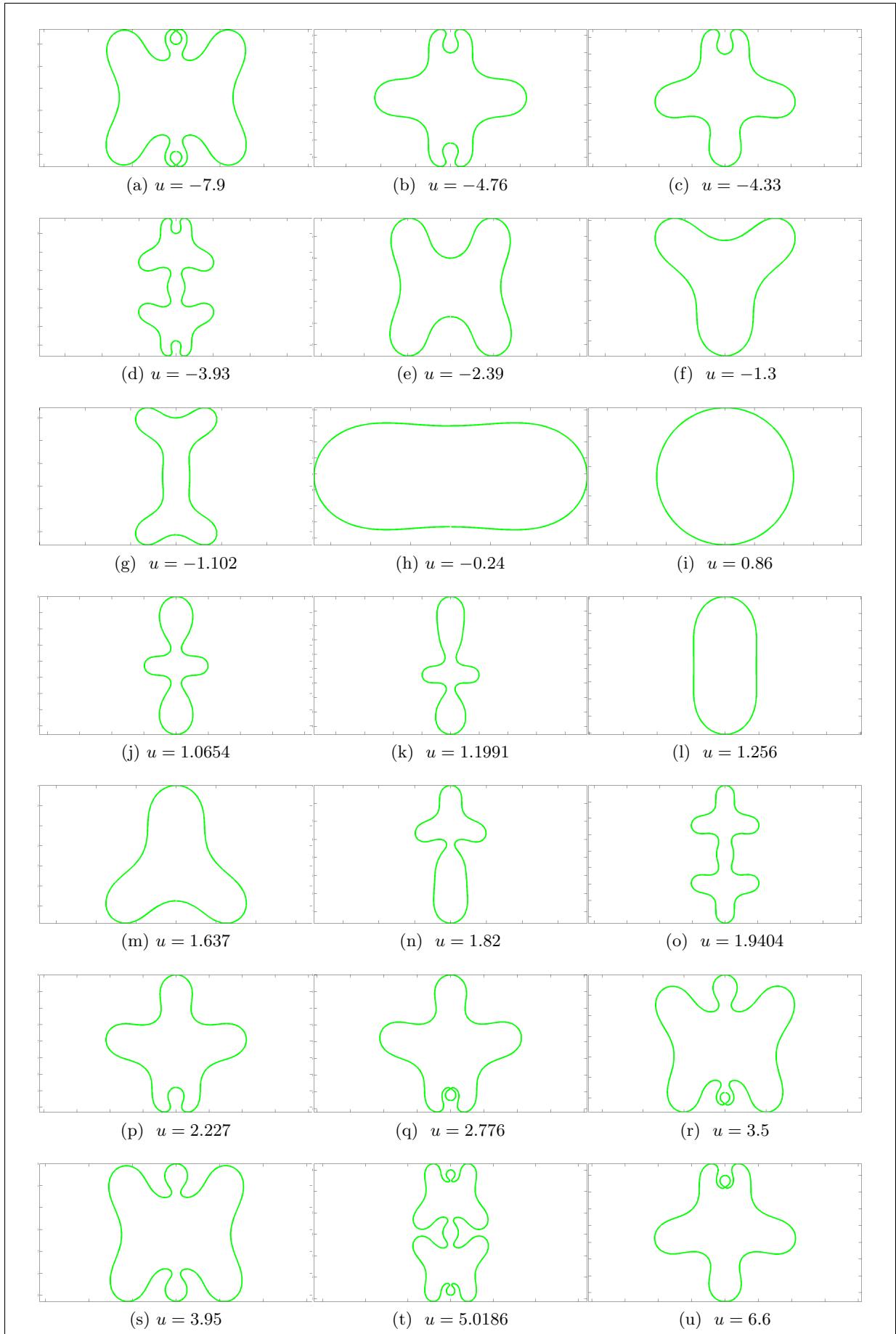


Figure 33: Solutions when  $\bar{\Sigma} = -3$

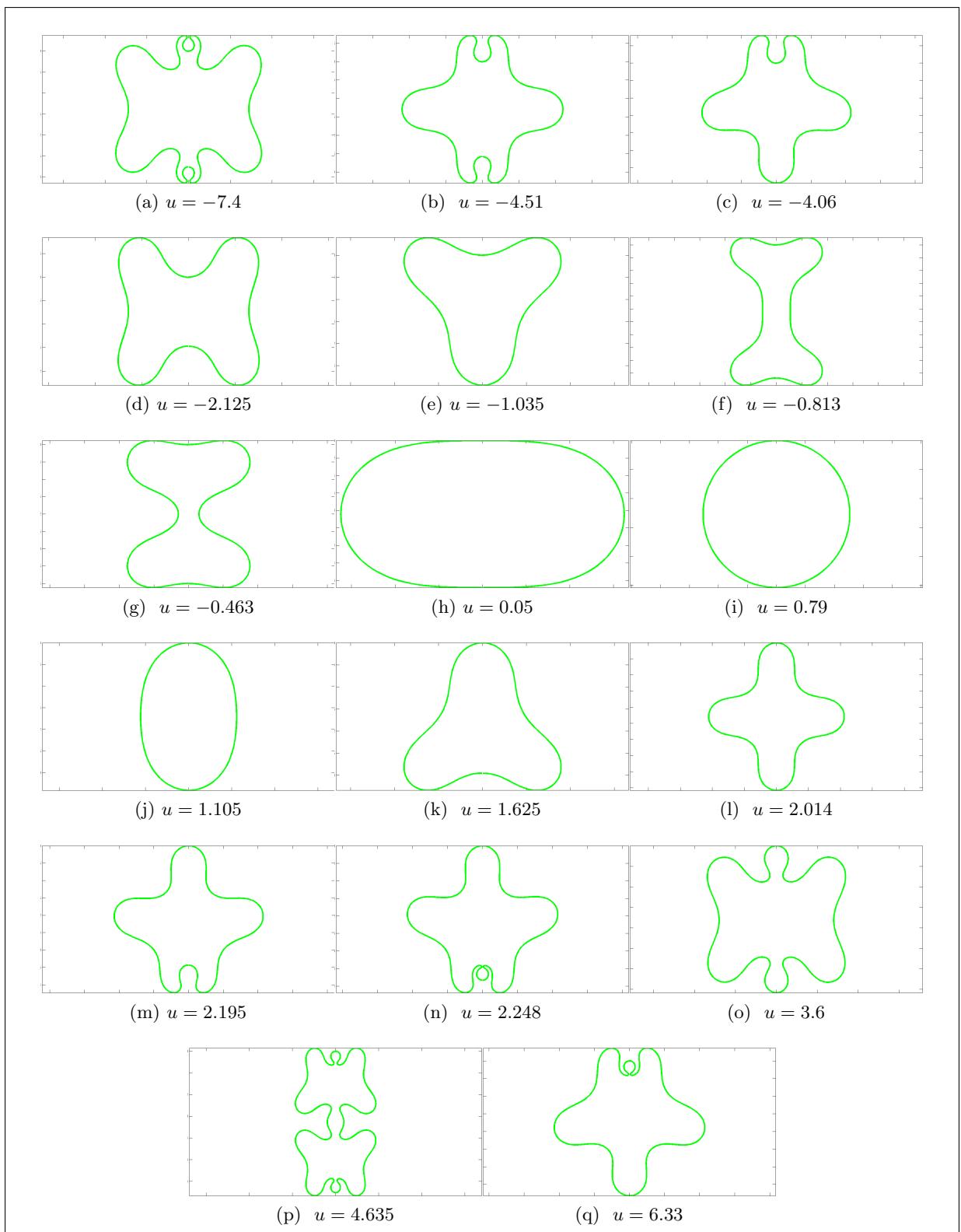


Figure 34: Solutions when  $\bar{\Sigma} = -3.25$

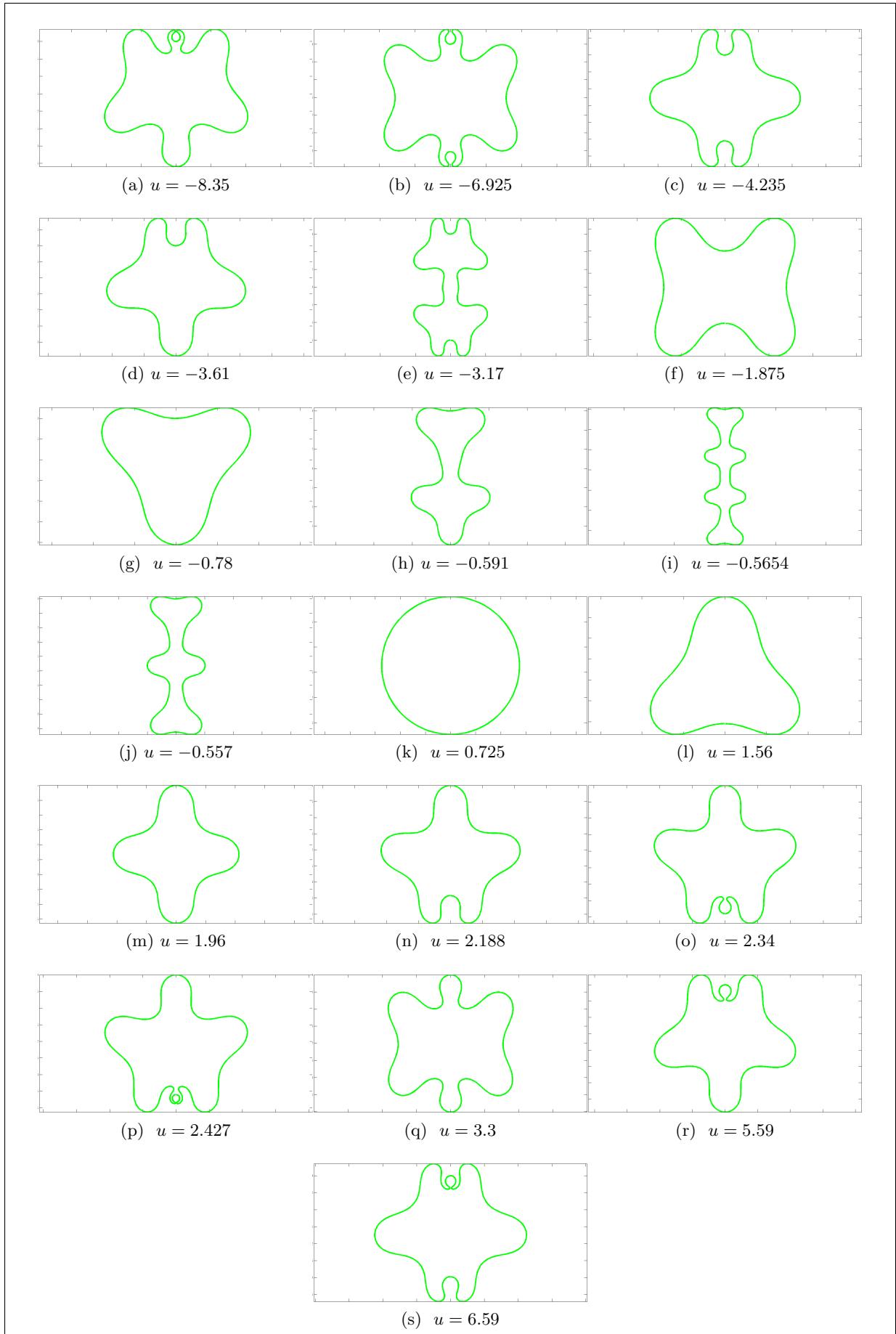


Figure 35: Solutions when  $\bar{\Sigma} = -3.5$

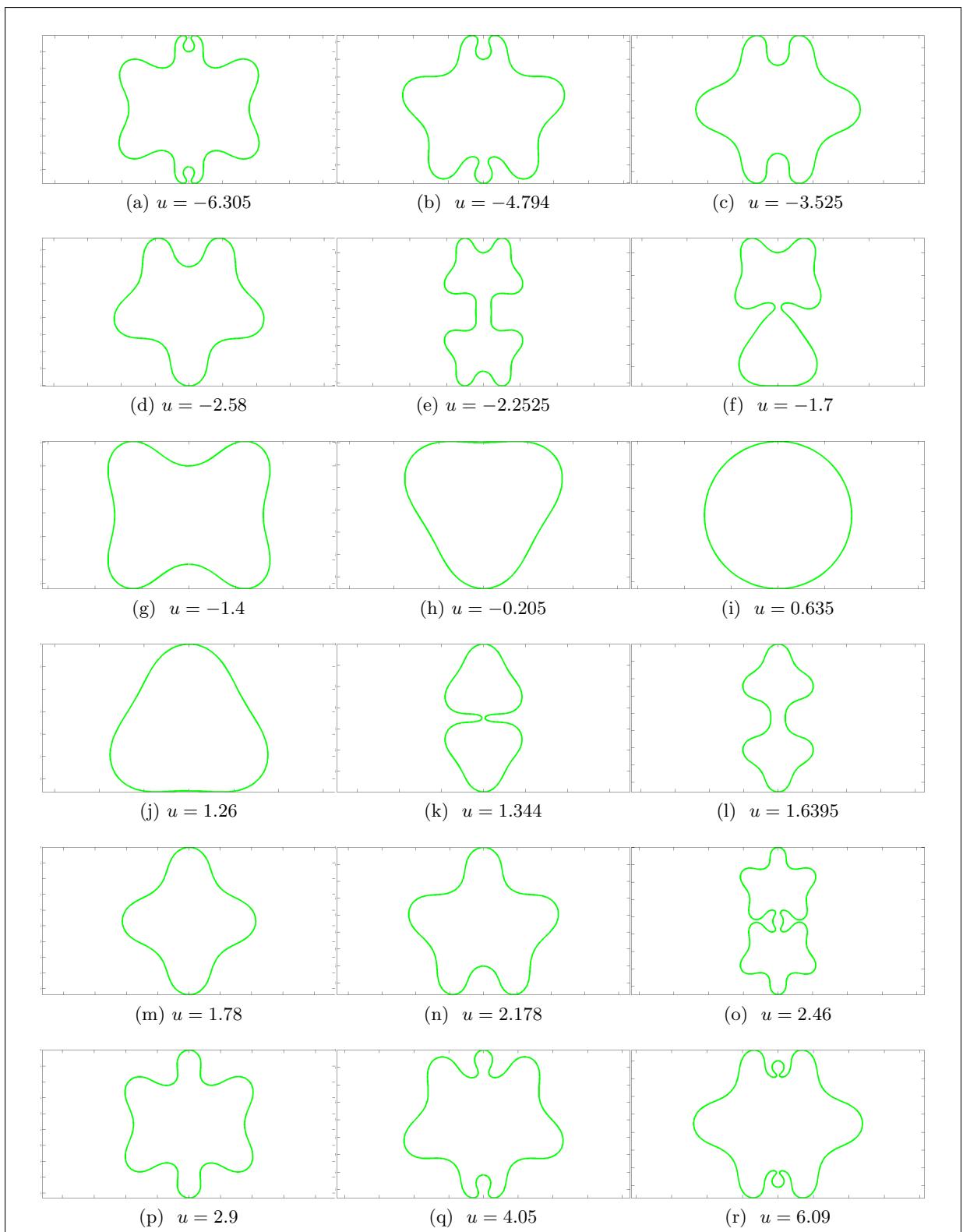


Figure 36: Solutions when  $\bar{\Sigma} = -4$

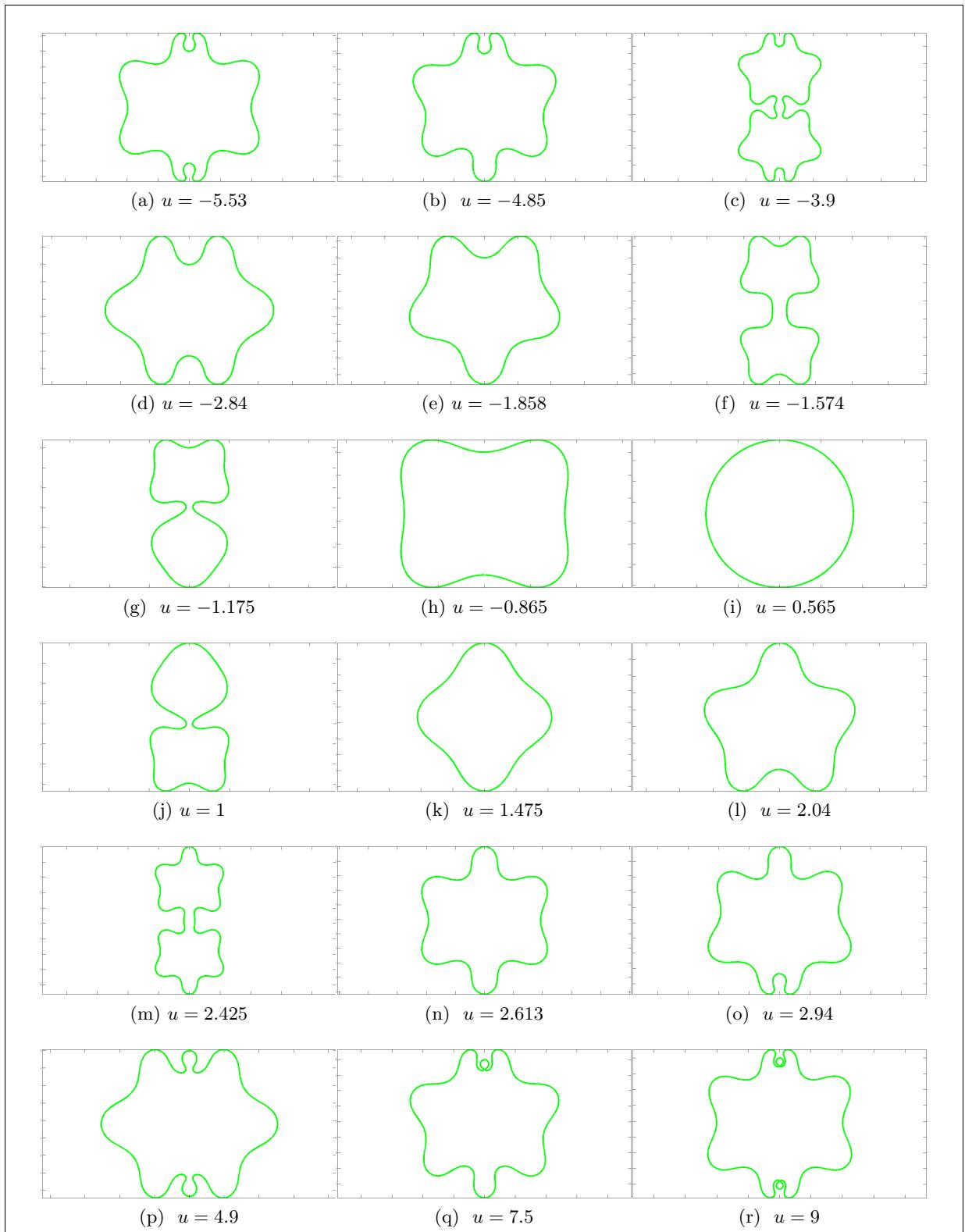


Figure 37: Solutions when  $\bar{\Sigma} = -4.5$

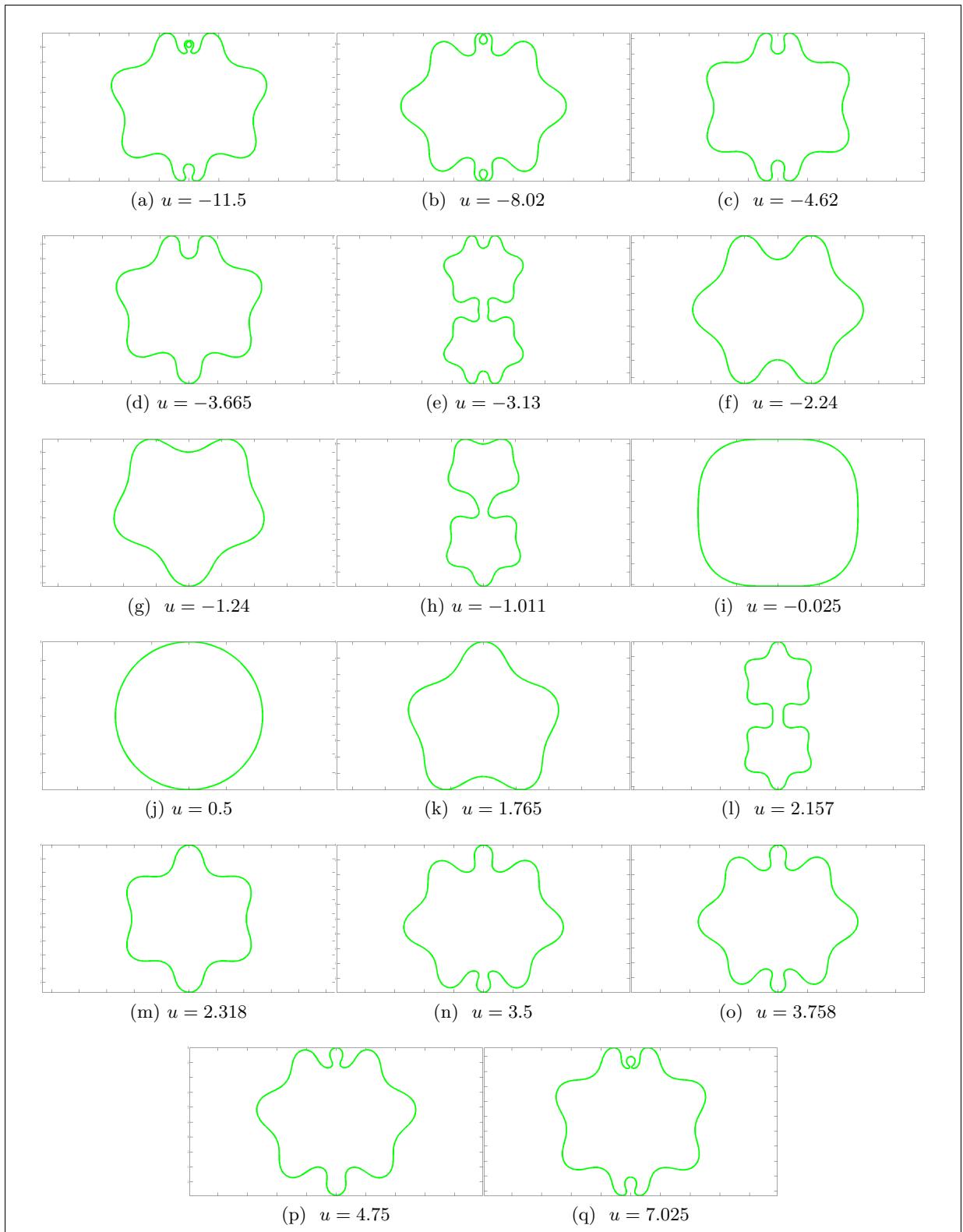


Figure 38: Solutions when  $\bar{\Sigma} = -5$

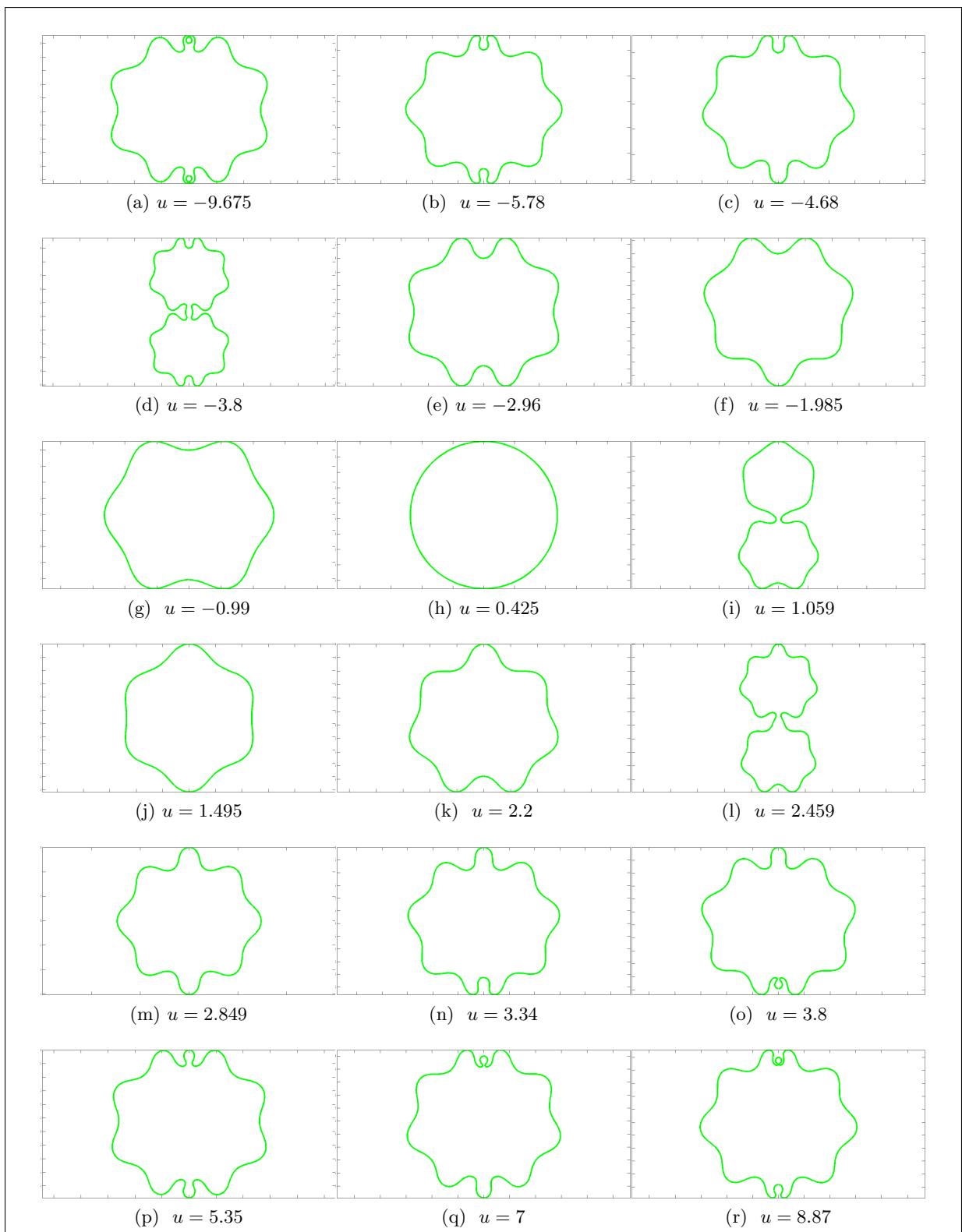


Figure 39: Solutions when  $\bar{\Sigma} = -6$

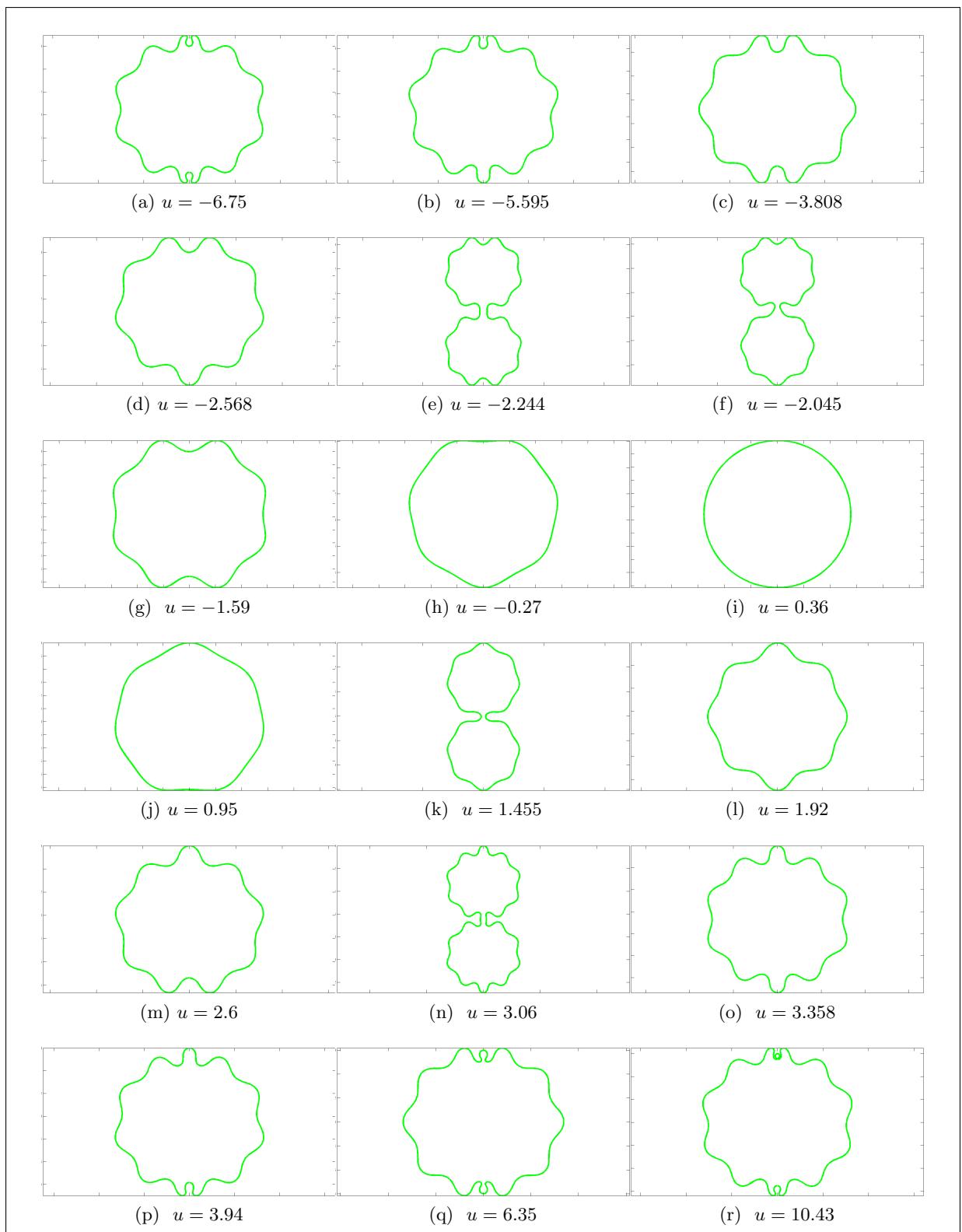


Figure 40: Solutions when  $\bar{\Sigma} = -7$

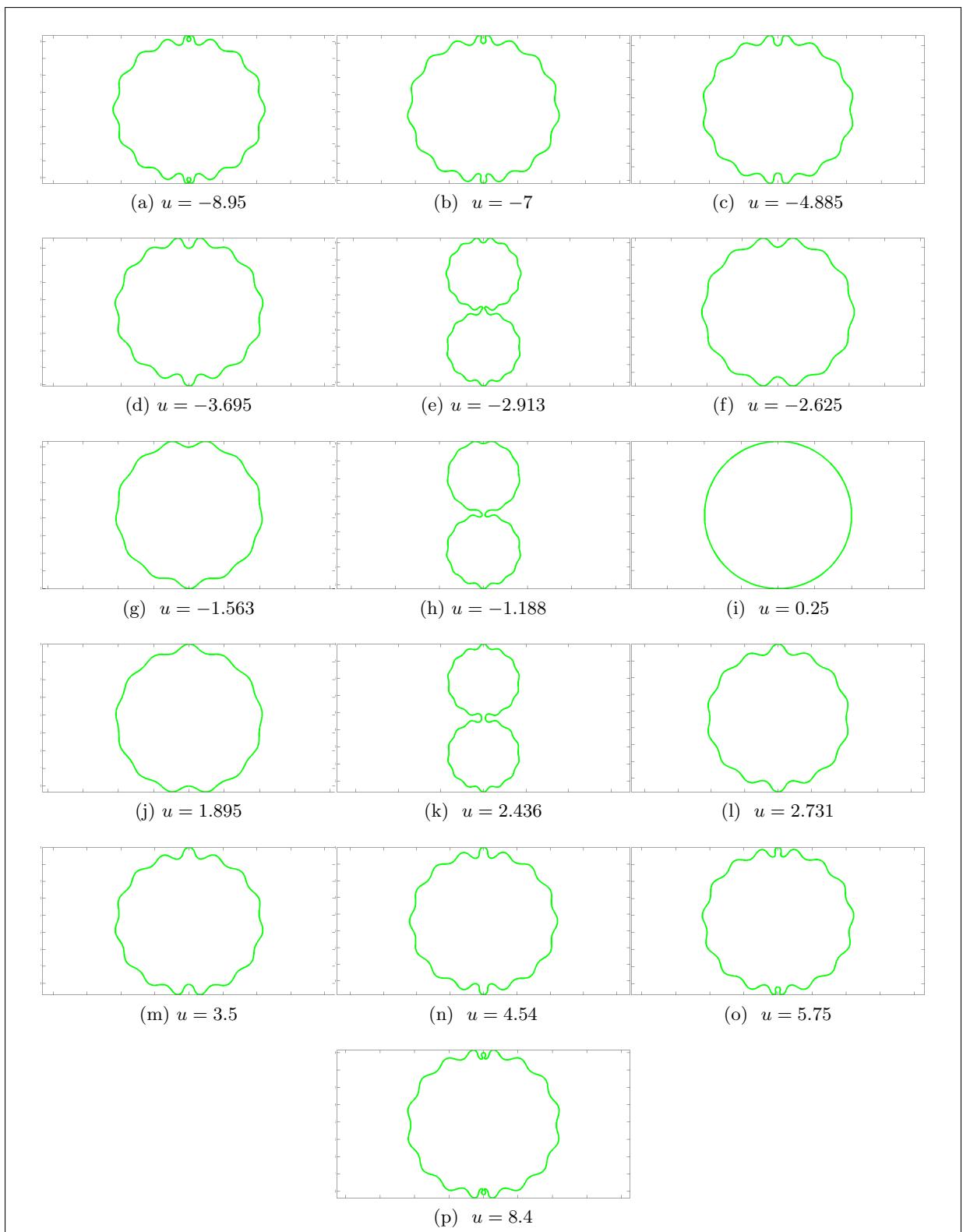


Figure 41: Solutions when  $\bar{\Sigma} = -10$

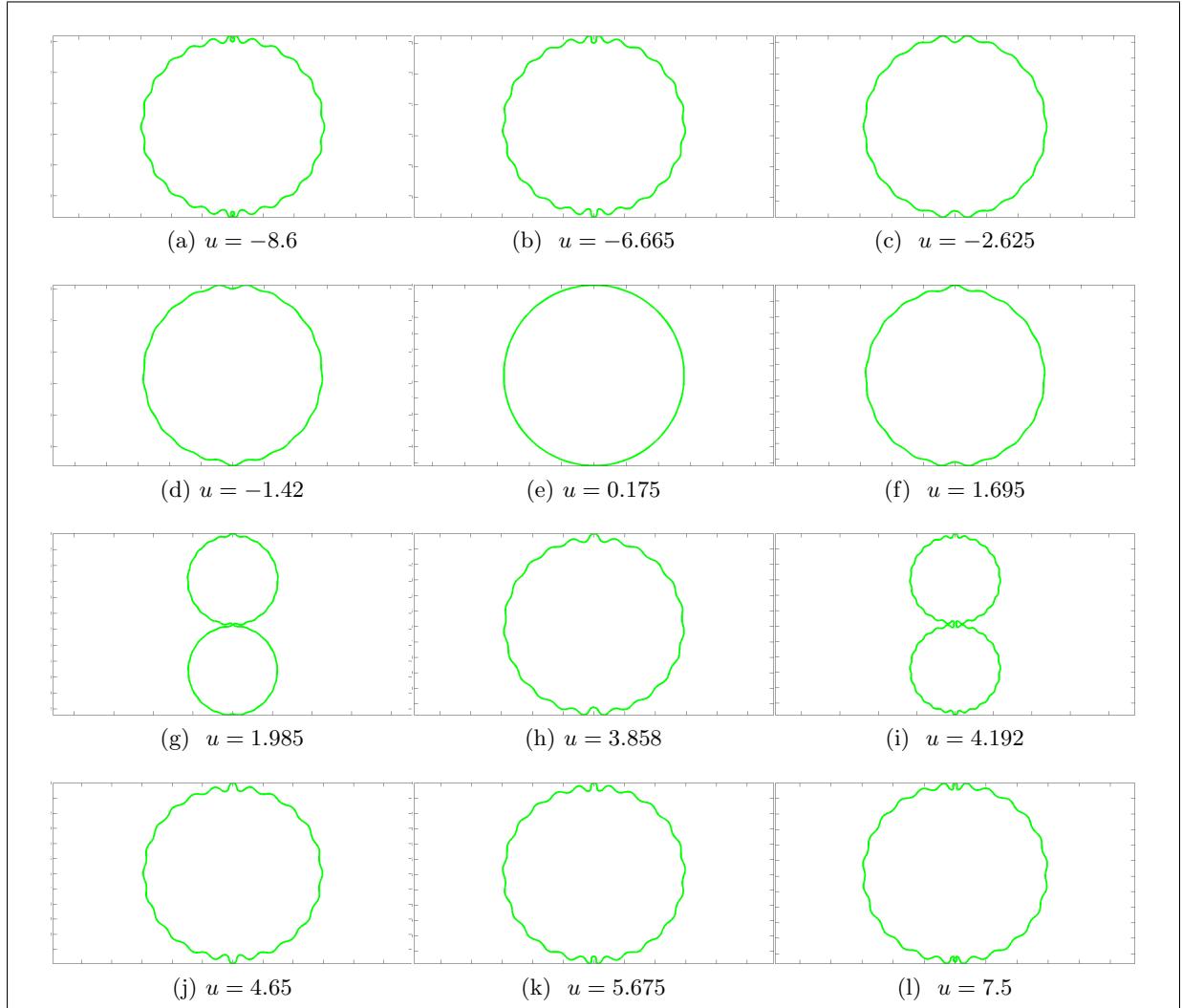


Figure 42: Solutions when  $\bar{\Sigma} = -14$

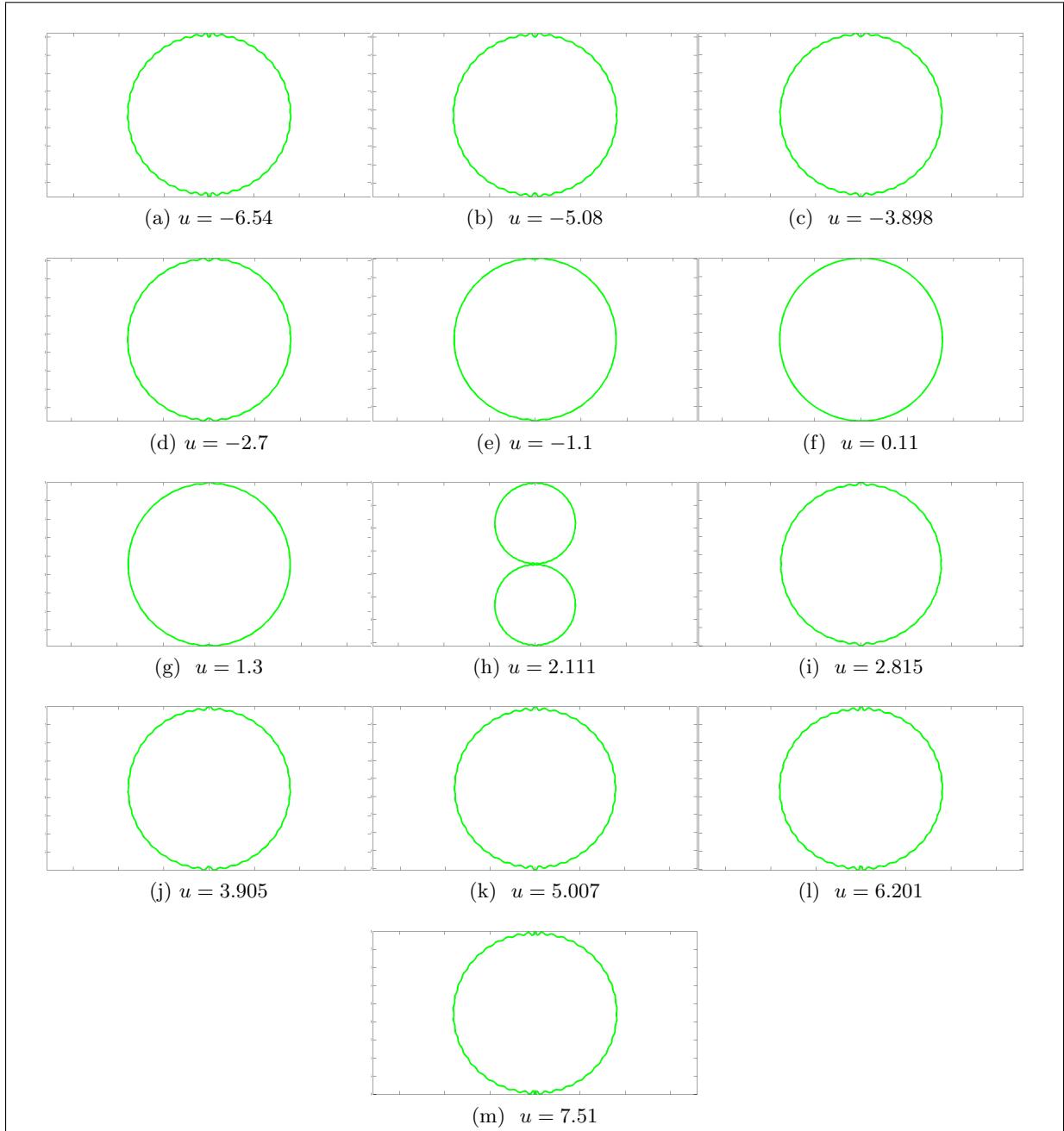


Figure 43: Solutions when  $\bar{\Sigma} = -22$

## D.2 Two Phase Membranes

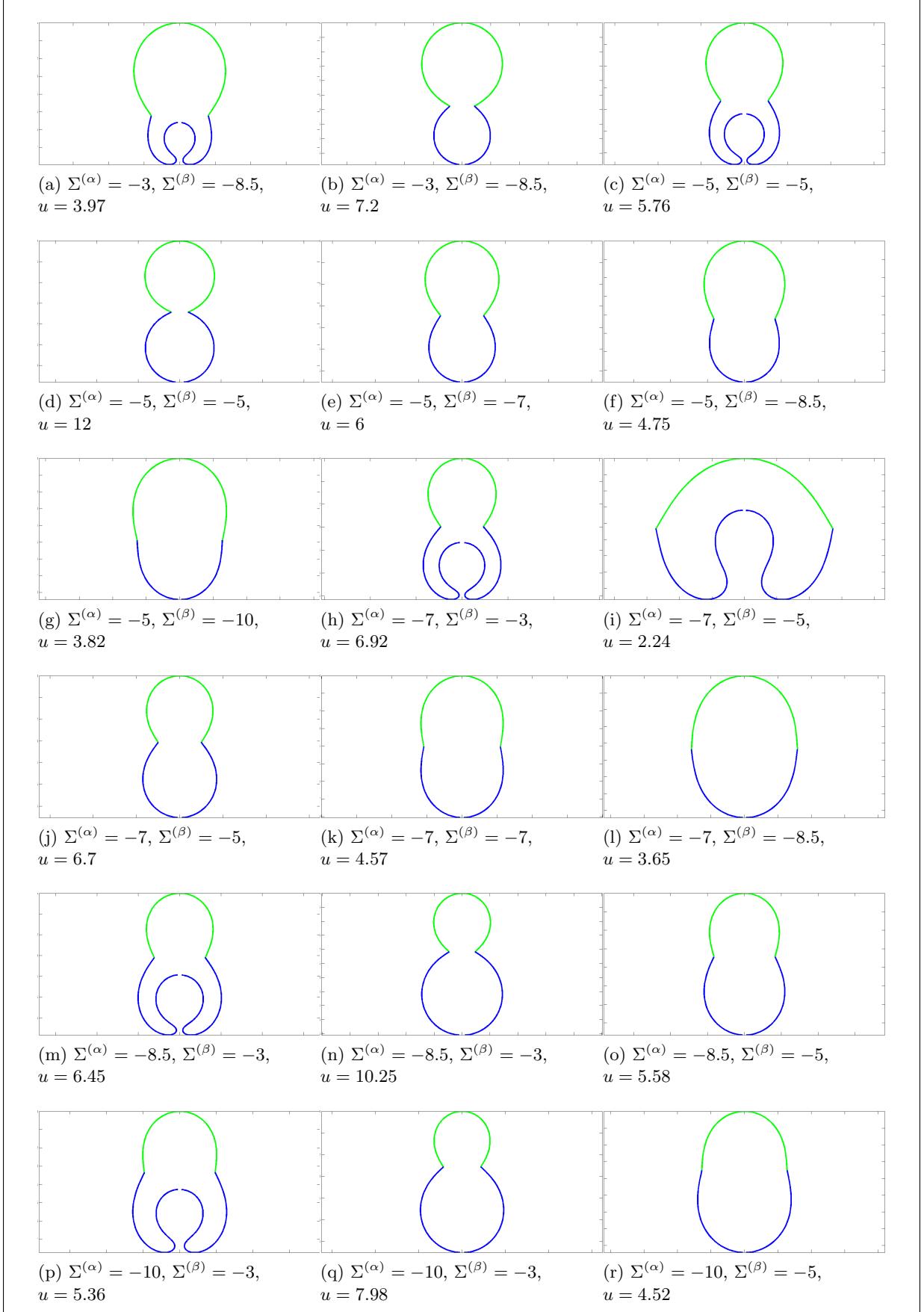


Figure 44: Solutions when  $C_0^{(\alpha)} = C_0^{(\beta)} = -0.1, \kappa^{(\alpha)} = \kappa^{(\beta)} = 1, \kappa_G^{(\alpha)} = \kappa_G^{(\beta)} = -1, P = 5$  and  $\sigma = 10$ .

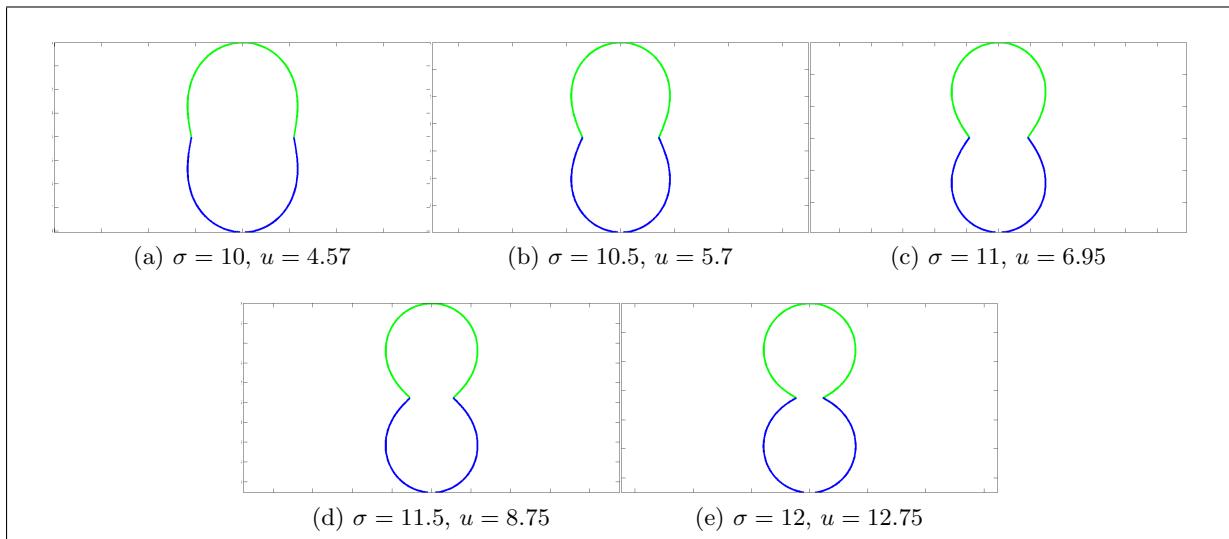


Figure 45: Solutions when  $C_0^{(\alpha)} = C_0^{(\beta)} = -0.1$ ,  $\kappa^{(\alpha)} = \kappa^{(\beta)} = 1$ ,  $\kappa_G^{(\alpha)} = \kappa_G^{(\beta)} = -1$ ,  $P = 5$ ,  $\Sigma^{(\alpha)} = -7$  and  $\Sigma^{(\beta)} = -7$ .

## E Classifying Shapes of Cells

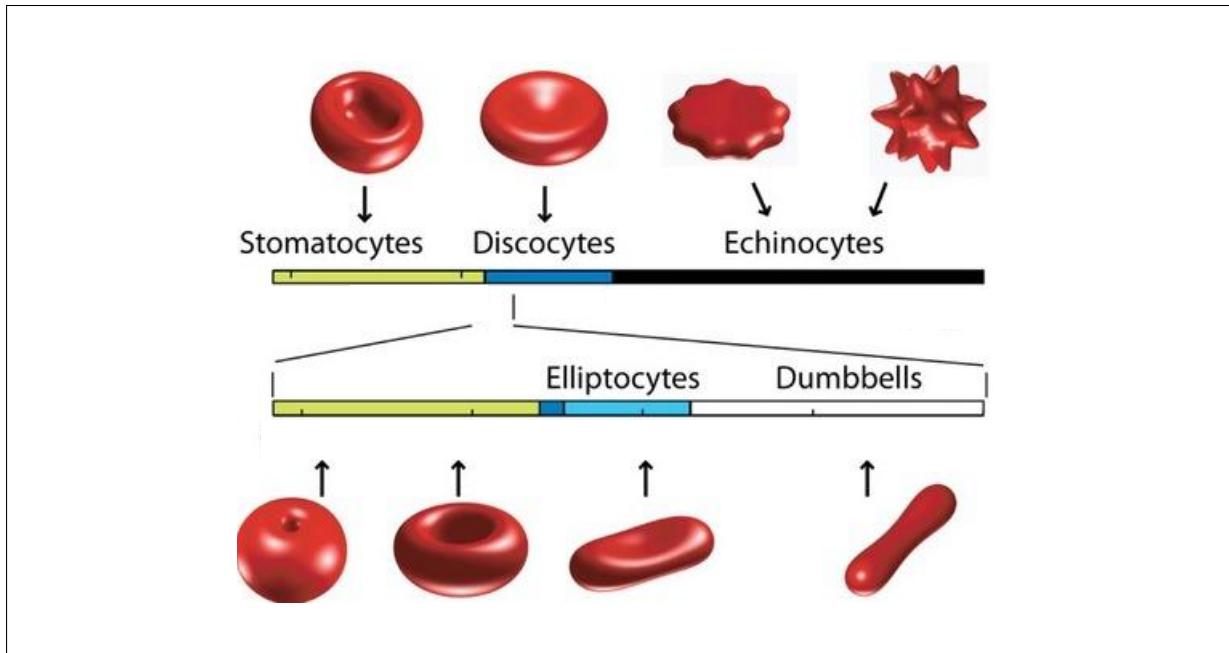


Figure 46: We use the above as a reference to classify the shapes of cells [http://www.springerimages.com/Images/Engineering/1-10.1007\\_s12195-008-0019-5-0](http://www.springerimages.com/Images/Engineering/1-10.1007_s12195-008-0019-5-0).