1 The Integrator

We are trying to solve a system of differential equations of the form

$$\dot{p} = f_1(p, q) \tag{1}$$

$$\dot{q} = f_2(p, q) \tag{2}$$

in order to find $q(t)$ and $p(t)$. Any differential equation $\dot{f}$ can be approximated by the truncated series

$$\dot{f}[t] \approx \dot{f}_0 + a_1 t + a_2 t^2 + ... + a_7 t^7 \tag{3}$$

where the constant term is determined by the initial condition. By introducing h $\equiv t/dt$, and $b_k \equiv a_k dt^k$ we can write

$$\dot{f}[h] \approx \dot{f}_0 + b_1 h + b_2 h^2 + ... + b_7 h^7 \tag{4}$$

We now rewrite the expansion in terms of the coefficients g_k, which can be converted into b_k by comparing equations (4) and (5)

$$\dot{f}[h] \approx \dot{f}_0 + g_1 h + g_2 h(h - h_1) + ... + g_7 h(h - h_1)...(h - h_7) \tag{5}$$

Here $h_1, ..., h_7$ are simply coefficients in the range (0,1). Plugging each respective h_k value into our expansion allows us to solve for the g_k coefficients, because all higher order terms become zero:

$$h = h_1 \qquad \text{gives} \qquad g_1 = \frac{\dot{f}(h_1) - \dot{f}_0}{h_1} \tag{6}$$

$$h = h_2 \qquad \text{gives} \qquad g_2 = \frac{\dot{f}(h_2) - \dot{f}_0 - g_1 h_2}{h_2(h_2 - h_1)} \tag{7}$$

$$...$$

After using these g_k values to solve for new values of the b_k coefficients, we now have a more accurate approximation of $\dot{f}(t)$. By integrating equation (4), we can write an approximation of $f(t)$:

$$f[h] \approx p_0 + hdt(\dot{p}_0 + \frac{h}{2}(b_1 + \frac{2h}{3}(b_2 + ...))) \tag{8}$$

We are able to solve for $q(t)$ and $p(t)$ using this scheme. In the very first iteration we set all b_k values to 0. Using approximations for $p(t)$ and $q(t)$ provided by eq (8), we can calculate each value of g_k for the expansion of $\dot{p}(t)$. After converting the g_k coefficients into b_k coefficients we have now slightly improved the accuracy of our expansions for $\dot{p}(t)$ and $p(t)$. We can now perform the same method on our expansion of $\dot{q}(t)$. This process is repeated, with the accuracy of our expansions increasing after every iteration.

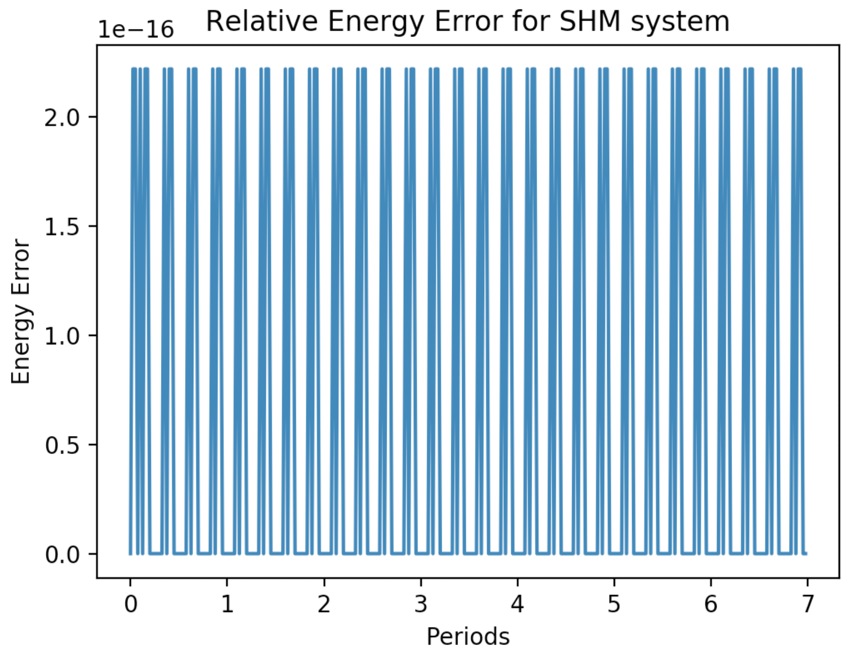

Figure 1: Energy Error using $\Delta t = T/40$

2 Accuracy of the Scheme

In order to test the accuracy of this scheme, I integrated a system of linked differential equations corresponding to a 1-d harmonic oscillator:

$$\begin{aligned} \dot{p} &= -q \\ \dot{q} &= p \end{aligned} \tag{9}$$

with initial conditions:

$$\begin{aligned} q(0) &= 0 \\ p(0) &= 1 \end{aligned} \tag{10}$$

For my first test I converged the scheme over 3 iterations, using a time-step $\Delta t = 1/40$th of a period. The relative change in energy was kept on the order of 10^{-16} (figure 1)

Using a larger time-step of $\Delta t = 1/10$th a period I was able to achieve the same order of energy error in 5 iterations (figure 2).

Although this has shown that the integrator has the potential to be extremely accurate, using 12 iterations makes it nowhere near as computationally efficient as the similar IAS15 algorithm. So far, I have been unable to find a way to

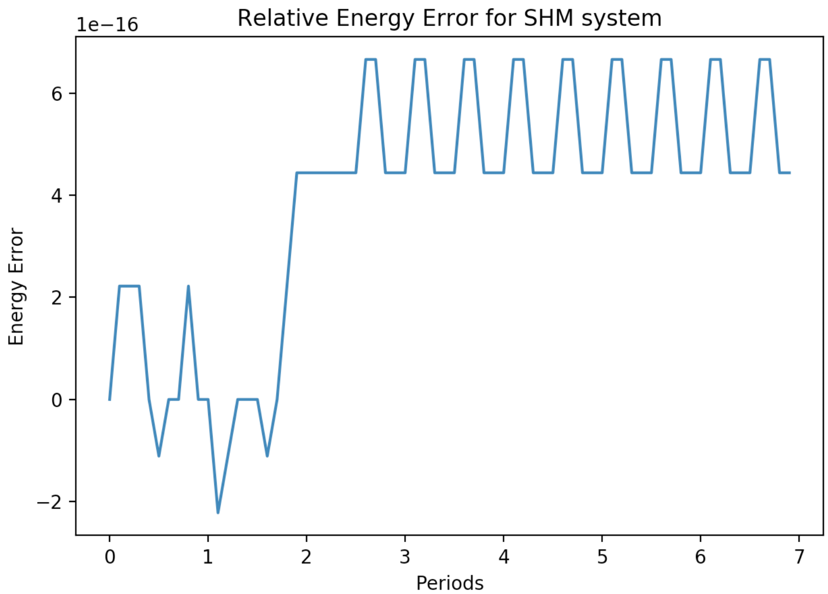

Figure 2: Energy Error using $\Delta t = T/10$

dynamically terminate the predictor-corrector loop that can keep energy error on the order of 10^{-15} or less.