1 Equations of a Particle in a Constant Electric and Magnetic Field

We want to solve the differential equation

$$\frac{d\vec{p}}{dt} = \vec{g} + \vec{p} \times \vec{\Omega} \tag{1}$$

where

$$\vec{g} = g_x\hat{x} + g_y\hat{y} + g_z\hat{z}$$
$$\vec{\Omega} = \Omega\hat{z}$$

We will solve this in the case that the magnetic field points in the $\hat{z}$ direction, however these solutions can be rotated in order to fit any arbitrary fields. The parametric solution to this system is given by

$$p_x(t) = [p_x(0) - \frac{g_y}{\Omega}]cos(\Omega t) + [\frac{g_x}{\Omega} + p_y(0)]sin(\Omega t) + \frac{g_y}{\Omega}$$
$$p_y(t) = [p_y(0) + \frac{g_x}{\Omega}]cos(\Omega t) + [\frac{g_y}{\Omega} - p_x(0)]sin(\Omega t) + \frac{g_x}{\Omega}$$
$$p_z(t) = g_z t + p_z(0)$$

Because we want to find the coefficients for the expansion of $\dot{p}(t)$ we will take the derivative of each component:

$$\dot{p}_x(t) = [g_y - \Omega p_x(0)]sin(\Omega t) + [g_x + \Omega p_y(0)]cos(\Omega t) \tag{2}$$

$$\dot{p}_y(t) = -[g_x + \Omega p_y(0)]sin(\Omega t) + [g_y - \Omega p_x(0)]cos(\Omega t) \tag{3}$$

$$p_z(t) = g_z \tag{4}$$

The coefficients we are trying to find are the b_k values in the equation:

$$\dot{p}[h] \approx \dot{p_0} + b_1 h + b_2 h^2 + ... + b_7 h^7 \tag{5}$$

By taylor expanding equations (2) and (3) we can obtain the exact values for these coefficients.

For the coefficients in the expansion of $\dot{p}_x(t)$:

$$b_0 = \Omega dt(g_y - \Omega p_x(0))$$
$$b_1 = -\frac{\Omega^2 dt^2}{2!}(g_x + \Omega p_y(0)$$
$$b_2 = -\frac{\Omega^3 dt^3}{3!}(g_y - \Omega p_x(0))$$
$$...$$

And for the coefficients in the expansion of $\dot{p}_y(t)$:

$$
\begin{aligned}
b_0 &= -\Omega dt(g_x + \Omega p_y(0)) \\
b_1 &= -\frac{\Omega^2 dt^2}{2!}(g_y - \Omega p_x(0) \\
b_2 &= \frac{\Omega^3 dt^3}{3!}(g_x + \Omega p_y(0)) \\
&\ldots
\end{aligned}
$$

By plugging these coefficients into equation (5), along with the initial condition, you can obtain an approximation for $\dot{p}(t)$ and $p(t)$.

2 Rotating the solution for arbitrary fields

This expansion assumed a magnetic field pointing in the $+\hat{z}$ direction. This is general enough to describe any arbitrary constant magnetic and electric field, as long as we can rotate our reference frame to allow the magnetic field to point in another direction. We can define an orthogonal coordinate system using the vectors $\vec{B}$, $\vec{B}\times(\vec{E}\times\vec{B})$, and $\vec{E}\times\vec{B}$. However, we must normalize these vectors in order to create an orthonormal coordinate system. Calling these vectors $\vec{U}, \vec{V}$, and $\vec{W}$ respectively we can now rotate our coordinate frame so that $\vec{B}$ is facing in the $+\hat{z}$ direction using the rotation matrix:

$$
R = \begin{bmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{bmatrix} \tag{6}
$$

In order to get $\vec{E}$ in this coordinate system we only need to multiply this matrix with

$$
\vec{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{7}
$$

on the right hand side. This will give us $\vec{E'}$ which can be used along with $\vec{B'}$ in order to calculate the expansion coefficients (as described in the previous section). In order get our expansion back into the original reference frame, we only need to multiply them by the inverse rotation matrix:

$$
R^{-1} \begin{bmatrix} \dot{p'}_x(h) \\ \dot{p'}_y(h) \\ \dot{p'}_z(h) \end{bmatrix} => \begin{bmatrix} \dot{p}_x(h) \\ \dot{p}_y(h) \\ \dot{p}_z(h) \end{bmatrix} \tag{8}
$$

Using these new equations we can thus work out good guesses for our coefficients. Hopefully, this will allow our predictor-corrector scheme to converge to the correct answer far more quickly.