First Year Electromagnetism Notes

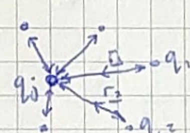
Structure of the Course

1. Electrostatics: Charges create 'electric fields' which represent the resulting force experienced by a small test charge : Gauss Law
2. Magnetostatics: Electrical currents create 'magnetic fields' which create forces on moving test charges. There are no magnetic monopoles. : Ampere's current Law
3. Induction: A time-varying magnetic flux through an area creates an electromotive force along the area's rim : Faraday / LENS LAW
4. Electromagnetic Waves: A time-varying electric flux through an area creates a magnetic field along the area's rim : EM wave propagation

Electrostatics

$$\varepsilon_0 = 8.854 \times 10^{-12} \frac{As}{Vm}$$

Properties of charge:
- Both positive and electric charge exists, charge is quantised (Millikan 1913)
- Coulomb's Law: the force between two point charges varies with the inverse square of their distance $\underline{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{\underline{r}}$
- Superposition: The force between two point charges varies linearly with the amount of each charge, hence the forces resulting from individual charges superimpose in an assembly of charges $F = \sum F_i$

↳  $\rightarrow \underline{F}_j = \frac{q_j}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i}{r_{ij}^2} \hat{\underline{r}}_{ij}$

vector points towards the one we are finding the force on

↳ Superposition also works for electric fields

The electric field:
- The electric field at point $\underline{r}$, generated by a distribution of charges q_i is defined as the force per unit charge that a test charge would experience if placed at $\underline{r}$
- A point test charge q experiences a force $\underline{F}$ due to a field $\underline{E}$
$$\underline{F} = q \cdot \underline{E} = q \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\underline{e}}_r$$
- Electric field due to a point charge always radially points away from + charge: $\underline{E} = \underline{F}/q = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\underline{r}}$
- As above, the principle of superposition also holds for the electric field: the electric field generated by a distribution of charges is equal to the sum of the electric fields generated by the individual charges

The electrostatic potential:

- The electrostatic potential difference between A and B is defined as the work done to move a unit charge between A and B

This is the work done to move a charge q, but electrostatic p.d is for a unit charge

$$V_{AB} = \frac{W_{AB}}{q} = -\int_A^B \underline{E}\cdot d\underline{L}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|r - r_i|} + \text{const}$$

r_i is from point to charge in question

here the second term is often set to zero by taking $V(r \to \infty) = 0$

$$\hookrightarrow W_{AB} = -\int_{r_A}^{r_B} \underline{F}\cdot d\underline{L} = -q\int_{r_A}^{r_B} \underline{E}\cdot d\underline{L}$$

- Since the work done is related linearly to the electric field, the principle of superposition also holds for the work done.

- Note that any field configuration can be made up of the sum of infinitesimal point charges. So we can now check that the work done by moving a test charge through the electric field of a point charge doesn't depend on the path taken.

Proof that the E field is conservative

For a point charge Q $\quad \underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\underline{r}}$

Work done to move a charge Q from A to B

$$W_{AB} = -q\int_A^B \underline{E}\cdot d\underline{L} \to \text{in spherical coordinates } d\underline{L} = dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}} + r\sin\theta\,d\phi\,\hat{\underline{\phi}}$$

$$\therefore \underline{E}\cdot d\underline{L} = \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$W_{AB} = -q\int_A^B \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,dr$$

$$= \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \to \text{only depends on initial and final radial separation - independent of path}$$

$\hookrightarrow$ Electric field is conservative

The relationship between electric field and potential and just a quick summary:

- The electric field at a point $\underline{r}$, generated by a distribution of charges q_i, is equal to the force F per unit charge q that a small test charge would experience at $\underline{r}$

$$\underline{E}(\underline{r}) = \frac{\underline{F}(\underline{r})}{q}$$

- The electric potential V at a point $\underline{r}$ is the energy W required per unit charge q to move a small test charge q from a reference point to $\underline{r}$.

$$V(\underline{r}) = \frac{W(\underline{r})}{q}$$

- The electric field and potential are related through:

$$V(\underline{r}) = -\int_{\underline{r}_0}^{\underline{r}} \underline{E}(\underline{r}')\cdot d\underline{r}' \qquad \underline{E}(\underline{r}) = -\underline{\nabla} V(\underline{r}) \qquad \underline{\nabla} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Energy of a system of charges:

If we calculate the energy to bring the charges up from infinity

$U =$ the first charge : none

$+$ the second charge q_2: $q_2 \left(\frac{q_1}{4\pi\varepsilon_0 r_{12}} \right)$

$+$ the third charge q_3: $q_3 \left(\frac{q_1}{4\pi\varepsilon_0 r_{13}} + \frac{q_2}{4\pi\varepsilon_0 r_{12}} \right)$

$+ \dots$

Compare to W, the sum of the potential energy of each charge from the field of the others $W = \sum_i \sum_{j \neq i} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$ $\Rightarrow$ we can see that $U = \frac{1}{2} W$

Hence $U = \frac{1}{2} W = \frac{1}{2} \sum_i q_i V_i$ where $V_i = \sum_{j(\neq i)} \frac{q_j}{4\pi\varepsilon_0 r_{ij}}$

Summary of assembly of discrete charge systems

The electric field $\underline{E}$ and potential V of a distribution of point charges q_i placed at positions $\underline{r}_i$ are:

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{(\underline{r} - \underline{r}_i)^2} \frac{\underline{r} - \underline{r}_i}{|\underline{r} - \underline{r}_i|}$$

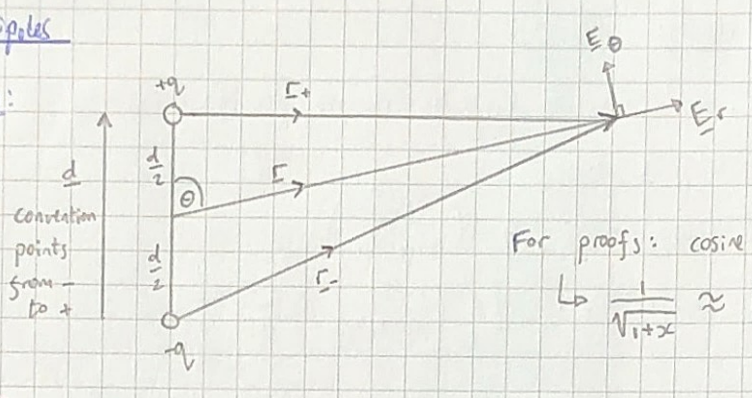
$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\underline{r} - \underline{r}_i|}$$

The energy U required to assemble a system of point charges q_i by bringing them to positions $\underline{r}_i$ from infinity is given by:

$$U = \frac{1}{2} \sum_i q_i V_i = \frac{1}{8\pi\varepsilon_0} \sum_i q_i \sum_{j \neq i} \frac{q_j}{r_{ji}}$$

Electric multipoles

Electric dipole:



dipole moment: $\underline{p} = q\underline{d}$

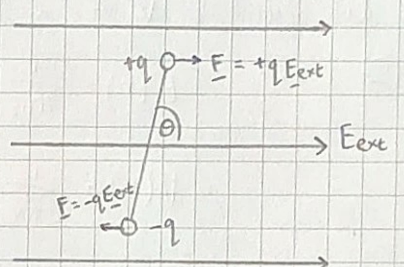
For proofs: cosine rule $r_{+/-} = \sqrt{r^2 + (\frac{d}{2})^2 \mp dr\cos\theta}$

$\hookrightarrow \frac{1}{\sqrt{1+x}} \approx \left(1 - \frac{1}{2}x\right)$ retained up to $\frac{d}{r}$ (and including)

$$\underbrace{V = \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\underline{p}\cdot\underline{r}}{4\pi\varepsilon_0 r^3}}_{\text{Far-field } (r>>d) \text{ potential}} \; ; \quad \underbrace{E_r = \frac{2\underline{p}\cdot\underline{r}}{4\pi\varepsilon_0 r^4} \quad E_\theta = \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3} \quad E_\phi = 0}_{\text{Far-field } (r>>d) \text{ electric field}}$$

Torque: $\underline{T} = \underline{p} \times \underline{E}_{ext}$

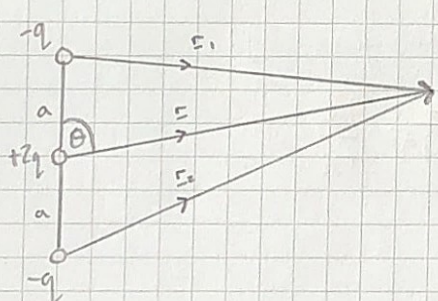
Energy of a dipole in an external E-field:
(taking $\theta = \frac{\pi}{2}$ as the zero of potential energy)



$U = -p\,E_{ext}\cos\theta$

$U = -\underline{p}\cdot\underline{E}_{ext}$

Electric quadrupole:

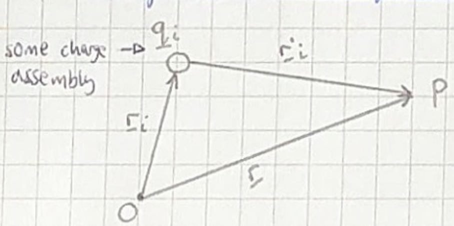


$$V = \frac{qa^2}{4\pi\varepsilon_0 r^3}\left(1 - 3\cos^2\theta\right)$$

For proof: cosine rule $r_{1/2} = \sqrt{r^2 + a^2 \mp 2ar\cos\theta}$

$\hookrightarrow \frac{1}{\sqrt{1+x}} \approx \left(1 - \frac{1}{2}x + \frac{3}{8}x^2\right)$ retain only up to and including $\left(\frac{a}{r}\right)^2$

The general multipole expansion



Potential at P: $V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\underline{r}_i'|}$

where $\underline{r}_i' = \underline{r} - \underline{r}_i$

$\rightarrow r_i' = \sqrt{r^2 + r_i^2 - 2rr_i\cos\theta_i} = r\sqrt{1 - 2\frac{r_i\cos\theta_i}{r} + \frac{r_i^2}{r^2}}$

$\therefore \frac{1}{|\underline{r}_i'|} \approx \frac{1}{r}\left[1 + \frac{r_i\cos\theta_i}{r} - \frac{r_i^2}{2r^2} + \frac{3}{2}\frac{r_i^2}{r^2}\cos^2\theta_i + \dots\right]$

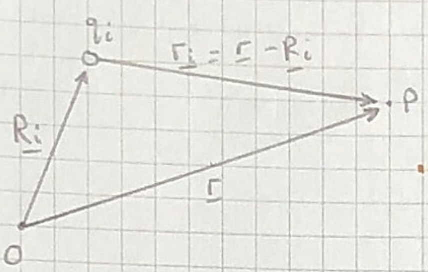
$$\therefore V(\underline{r}) \approx \underbrace{\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r}}_{\text{monopole term}} + \underbrace{\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i r_i\cos\theta_i}{r^2}}_{\text{dipole term}}$$

$$+ \underbrace{\frac{1}{4\pi\varepsilon_0}\sum_i \frac{\frac{1}{2}q_i r_i^2(3\cos^2\theta_i - 1)}{r^3}}_{\text{quadrupole term}} + \dots$$

Continuous Charge Distributions

For a continuous charge distribution $\sum_i V_i \rightarrow \int dV$, Hence $\boxed{V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{|\underline{r} - \underline{R}|}}$

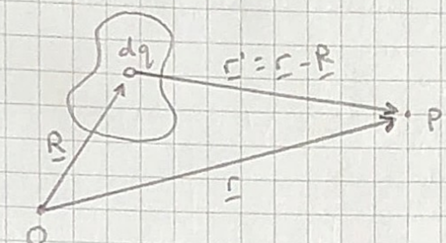
Discrete



Alternatively: $V(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{R})\, dV}{|\underline{r} - \underline{R}|}$

$$E(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{R})\,(\underline{r} - \underline{R})}{|\underline{r} - \underline{R}|^3} dV$$

Continuous



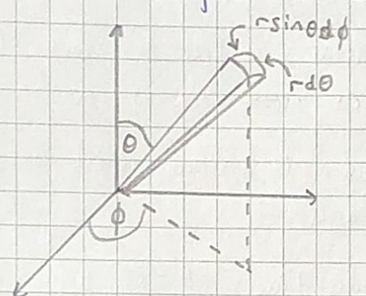
line charges: $dq = \lambda\, dL$

surface charges: $dq = \sigma\, dA$

volume charges: $dq = \rho\, dV$

Gauss Law

Solid angles: Consider an element of area on a sphere. Define a vector of surface element $\underline{da}$ normal to the surface



$$\underline{da} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\underline{r}}$$

define: $d\Omega = \sin\theta\, d\theta\, d\phi$ an element of solid angle

[Note that $d\Omega$ is independent of r]

$$\int_{surface} d\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 4\pi$$ ← holds for any closed surface

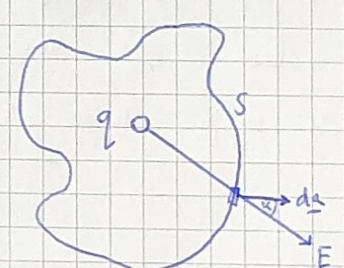
Gauss derivation: calculate the flux $d\phi = \underline{E} \cdot \underline{da}$ through an infinitesimal area $\underline{da}$ of surface S at a distance r away from point charge q

$$d\phi_E = \underline{E} \cdot \underline{da} = E\, da \cos\alpha$$

electric flux

$$\therefore d\phi_E = \left(\frac{q}{4\pi\varepsilon_0} \frac{\hat{\underline{r}}}{r^2}\right) r^2 \sin\theta\, d\theta\, d\phi\, \hat{\underline{r}}$$

$$d\phi_E = \frac{q}{4\pi\varepsilon_0} \sin\theta\, d\theta\, d\phi = \frac{q}{4\pi\varepsilon_0} d\Omega$$



Therefore for any closed surface

$$\oint_S \underline{E} \cdot \underline{da} = \frac{q}{4\pi\varepsilon_0} \int d\Omega = \frac{q}{4\pi\varepsilon_0} 4\pi$$ → Gauss Law: $\oint_{\text{closed surface}} \underline{E} \cdot \underline{da} = \frac{q}{\varepsilon_0}$ [q = enclosed charge]

The principle of superposition applies to Gauss Law so $\oint_S \underline{E}\cdot d\underline{a} = \frac{\overbrace{\sum q_i}^{\text{total enclosed charge}}}{\varepsilon_0} = \frac{\int_V \rho(\underline{r})\, dV}{\varepsilon_0}$

Gauss Law allows:
- Finding the total charge enclosed inside a closed surface if the field in known on the surface, and vice versa
- Allows straightforward calculation of field for symmetrical charge distributions

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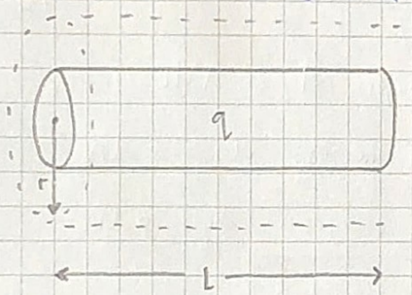
Gauss Law example: Long uniformly charged rod



Cylindrical gaussian surface, symmetry $\underline{E}$ is // to $d\underline{a}$

$$\oint_S \underline{E}\cdot d\underline{a} = \frac{q}{\varepsilon_0}$$

$$E \oint_S da = \frac{q}{\varepsilon_0}$$

$$2\pi r L E = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{L}\frac{1}{2\pi\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Electric field inside a conductor:

Inside a conductor, one or more electrons per atom are free to move throughout the material. We ~~can~~ are considering electroSTATICS (static charge) As a result:

i) $E = 0$ inside a conductor (free charge moves to surface until the internal electric field is cancelled)

ii) $\rho = 0$ inside a conductor (from Gauss Law: $\underline{E} = 0$ hence $\rho = 0$)

iii) Therefore any net charge resides on the surface

iv) A conductor is an equipotential (since $\underline{E} = 0$, $V(\underline{r}_1) = V(\underline{r}_2)$)
inside a conductor we usually set $V = 0$

v) At the surface of a conductor, $\underline{E}$ is perpendicular to the surface (otherwise charges will flow until the tangential component becomes zero when equilibrium is reached).

The surface charge will oppose any external applied field

Poisson and Laplace Equations

The expression derived previously is the 'integral form' of Gauss Law

$$\oint_S \underline{E} \cdot d\underline{a} = \frac{1}{\varepsilon_0} \int_V \rho \, dv \quad \text{over volume } V$$

We can express Gauss Law in differential form using the divergence theorem

$$\boxed{\int_V \underline{\nabla} \cdot \underline{F} \, dv = \oint_S \underline{F} \cdot d\underline{a} \; : \text{divergence theorem}}$$

$$\oint_S \underline{E} \cdot d\underline{a} = \int_V \underline{\nabla} \cdot \underline{E} \, dv = \frac{1}{\varepsilon_0} \int_V \rho \, dv$$

$$\hookrightarrow \boxed{\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \; : \text{Gauss Law in differential form}}$$

Then, if we use $\underline{E} = -\underline{\nabla} V \rightarrow$ $\boxed{\underline{\nabla}^2 V = -\frac{\rho}{\varepsilon_0} \; : \text{Poisson's Equation}}$

In regions where $\rho = 0$, we get $\rightarrow$ $\boxed{\underline{\nabla}^2 V = 0 \; : \text{Laplace's Equation}}$

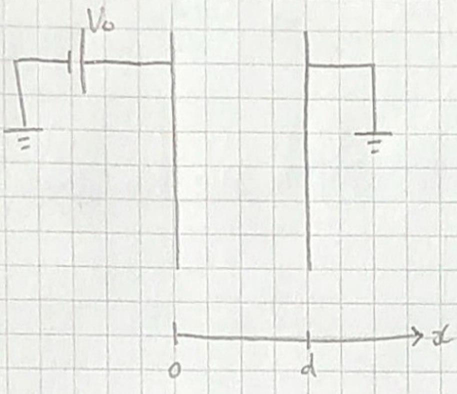
Uniqueness theorem: The potential V inside a volume is uniquely determined, if the following are specified:
- The charge density througout the region
- The value of V on all boundaries

$\hookrightarrow$ Proof: - Suppose there are two solutions V_1 and V_2 to Laplace's equation for potential inside the volume (taking $\rho = 0$)

- $\underline{\nabla}^2 V_1 = 0$, $\underline{\nabla}^2 V_2 = 0$ and $V_1 = V_2$ on the boundary surface S
- Define the difference $V_3 = V_1 - V_2$
- Then $\underline{\nabla}^2 V_3 = \underline{\nabla}^2 V_1 - \underline{\nabla}^2 V_2 = 0$
- V_3 also obeys Laplace's equation
- But on the boundary $V_3 = V_1 - V_2 = 0$! V_1 and V_2 are the same !

This is a contradiction $\rightarrow$ V must be unique

Example: Laplace's equation in cartesian coordinates - parallel plate capacitor



$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(by symmetry)}$$

$\therefore$ need to solve $\frac{\partial^2 V}{\partial x^2} = 0$

$\frac{\partial V}{\partial x} = C_1$ constant

$V = C_1 x + C_2$

values on capacitor plate define boundary condition

$$V(x=0) = V_0$$
$$V(x=d) = 0$$
$$\therefore \; C_2 = V_0$$
$$C_1 d + V_0 = 0$$
$$C_1 = \frac{-V_0}{d}$$

$$V(x) = V_0\left(1 - \frac{x}{d}\right)$$

$$\underline{E} = -\underline{\nabla} V$$

$$\underline{E} = -\frac{\partial}{\partial x} V \hat{\underline{x}}$$

$$\underline{E} = \frac{V_0}{d}\hat{\underline{x}}$$

You can also use Laplace's equation in spherical coordinates ~~xxxx~~ for charge distributions with azimuthal symmetry and obtain the multipole expansion terms we saw before

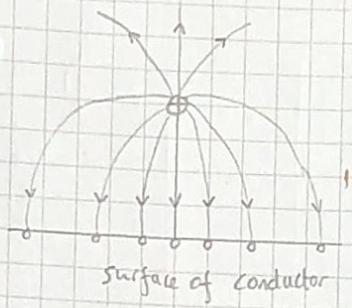
The method of images:

- The method of images is useful for calculating potentials created by charges placed in the vicinity of metal conductors
- Replace the conducting elements with imaginary charges ("image charges") which replicate the boundary conditions of the problem ~~xx~~ on a surface (the surface of the conducting material)
- The Uniqueness Theorem guarantees that within the region bounded by the surface, the potential calculated for the "imagined" charge distribution is identical to that of the "real" situation
- If a suitable replacement "image charge distribution" is chosen, the calculation of the potential becomes mathematically much simpler

This is why it works

remember the properties of charge and electric field inside a conductor

Method of images example: point charge a distance d above a grounded metal plate



Boundary conditions:

1. At the metal ($z = 0$), the parallel component of $\underline{E} = 0$, field lines are
2. Surface is an equipotential $\rightarrow V = 0$ V is a constant, I think we are defining it as 0
3. Far from charge and metal plate: $V \rightarrow 0$

- The two assemblies share the same charge distribution and boundary conditions for the upper volume half
- The Uniqueness Theorem states that the potential in those regions must therefore be identical
  ↳ in the upper half, the fields in both scenarios are identical

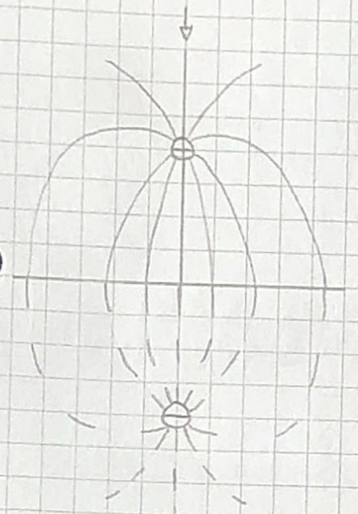


Image charges due to a pair of plates at 90°

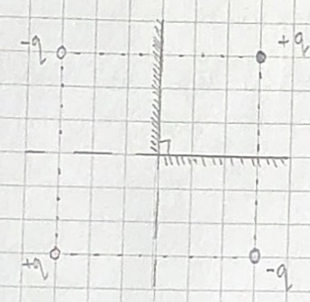
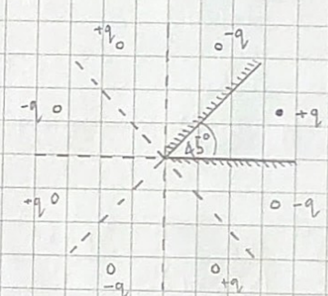
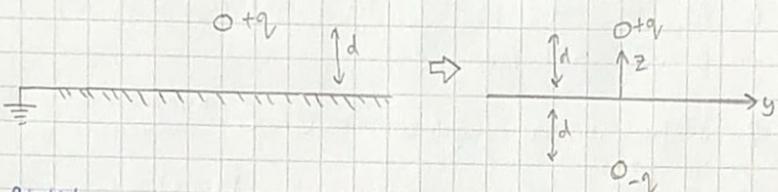


Image charges due to a pair of plates at 45°



Fully worked method of images example with induced surface charge, total plate charge, force, and energy calculated:

Point charge above a metal plate



Potential

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{+q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \rightarrow$$ This gives $V=0$ when $z=0$, $V=0$ when far away ✓

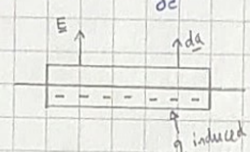
$\therefore$ all boundary conditions satisfied

↳ if we wanted to, $\underline{E}$ at any point can be calculated with $\underline{E} = -\nabla V$

Induced surface charge

At metal surface $E_{\parallel} = 0$, $E_{\perp} = -\frac{\partial V}{\partial z}$

Gauss Law near the conductor surface:



↳ $\oint_s \underline{E} \cdot d\underline{a} = E_{\perp}\, da = \frac{q_{induced}}{\varepsilon_0}$

use $\sigma_{induced} = \frac{q_{induced}}{da}$ ← charge / unit area

$\therefore\ \sigma_{induced} = \varepsilon_0 E_{\perp}$

$\sigma_{induced} = -\varepsilon_0 \frac{\partial V}{\partial z}$

$$\frac{\partial V}{\partial z} = \left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right] \times \frac{1}{4\pi\varepsilon_0}$$

$$\sigma_{induced}(x,y) = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$\sigma_{induced}(x,y) = -\frac{1}{2\pi}\frac{q d}{(x^2+y^2+d^2)^{3/2}}$$

This is the surface charge on the plate in the (x,y) plane

$\sigma_{induced}$ is $-ve$ and largest at $x=y=0$

Total charge induced

$$\sigma_{induced} = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}} \xrightarrow{\text{polars}} = \frac{-1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}}$$

$$q_{induced} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma\, dx\, dy = \int_0^{\infty}\int_0^{2\pi} \sigma\, r\, dr\, d\phi = \int_0^{\infty} \sigma (2\pi r)\, dr = \int_0^{\infty} -\frac{1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}}\, 2\pi r\, dr$$

$$q_{induced} = \left[\frac{qd}{\sqrt{r^2+d^2}}\right]_0^{\infty} = -q$$

→ Total charge on the plate is $-q$, as might be expected logically or from Gauss Law

Force between charge and plate: $\underline{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{\underline{z}}$ [reduces to the case between two point charges]

Energy stored in the electric field:

$$W = \int_{\infty}^{d} \underline{F}\cdot d\underline{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{(2z)^2}\, dz = -\frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{4z}\right]_{\infty}^{d}$$

$$W = -\frac{1}{16\pi\varepsilon_0}\frac{q^2}{d}$$

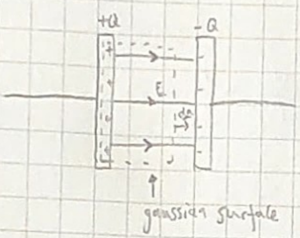
(This is half that of bringing two point charges to the same distance)

<u>Capacitance</u>: - capacitors store electrostatic energy, by keeping two different charge accumulations on different metallic surfaces

- **Capacitance** is defined as the charge that is stored per unit voltage applied between the two surfaces

$$C = \frac{\text{Stored charge } Q}{\text{Voltage applied}}$$

- The charge is equal and opposite on both surfaces



<u>Parallel plate capacitor</u>

$$\oint_s \underline{E} \cdot d\underline{a} = \frac{Q}{\varepsilon_0}$$

$$EA = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{\varepsilon_0 A}$$

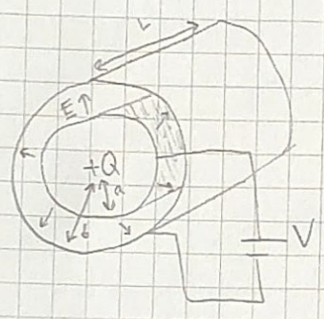
Voltage between plates $= V_{-+} = -\int_d^0 E \cdot dx$

$$V_{-+} = \frac{Qd}{\varepsilon_0 A}$$

$$C = \frac{\varepsilon_0 A}{d}$$

$$C = \frac{Q}{V} \Rightarrow \underline{C = \frac{\varepsilon_0 A}{d}}$$

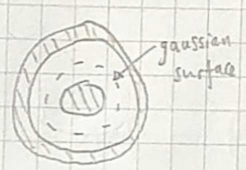
<u>Cylindrical capacitor</u>: Battery supplies +Q on inner surface, -Q is induced on the outer



$$\oint_s \underline{E} \cdot d\underline{a} = E \cdot 2\pi r L = \frac{Q}{\varepsilon_0}$$

$$\underline{E} = \frac{Q/L}{2\pi\varepsilon_0 r} \hat{\underline{r}} \quad \text{for } a \leq r \leq b$$

$$\underline{E} = 0 \quad \text{for } 0 < r < a \text{ and } r > b$$



$$V_{-+} = -\int_b^a E\,dr$$

$$V_{-+} = -\int_b^a \frac{Q/L}{2\pi\varepsilon_0 r}\,dr$$

$$V_{-+} = \frac{Q/L}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

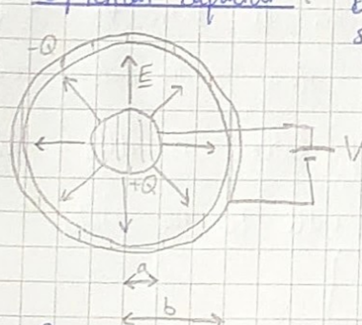
$$\rightarrow C = \frac{Q}{V}$$

$$C = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)} \times L$$

$$C' = \frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}$$

capacitance per unit length

<u>Spherical capacitor</u>: Battery supplies +Q on the inner sphere, -Q is induced on the outer

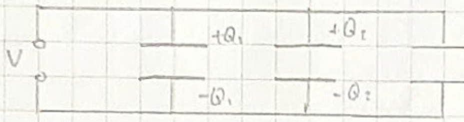


$$V_{-+} = \int_b^a E\,dr = \int_b^a \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

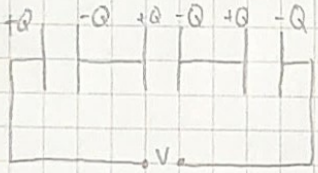
$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\underline{r}} \quad \text{(gauss)}$$

Capacitance networks: capacitors in parallel



- voltage is the same across each capacitor
- Total charge: $Q = Q_1 + Q_2 + Q_3 \cdots$
$= C_1 V + C_2 V + C_3 V \cdots$
- Total capacitance $C = C_1 + C_2 + C_3 + \cdots$

Capacitors in series
- Charge is the same ~~from~~ on each capacitor plate (inner plates are isolated from the outside) with $Q_{tot} = 0$



- Total voltage: $V = V_1 + V_2 + V_3$

$$\frac{1}{C} = \frac{V}{Q} = \frac{V_1}{Q_1} + \frac{V_2}{Q_2} + \frac{V_3}{Q_3} + \cdots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \cdots$$

Energy stored in a capacitor

- Capacitor is initially uncharged: add a small amount of charge
- Further charge will have to be brought up against the potential created by the existing charge
  - $\hookrightarrow dW = V(q)\, dq$

$VC = q \rightarrow dq = C dV$

Then $W = \int_0^{Q_0} V(q)\, dq$

$W = \int_0^{V_0} CV\, dV$

$W = \frac{1}{2} C V_0^2 = U_0$ (energy stored)

Hence energy stored in a capacitor charge Q_0 voltage V_0:

$$U_C = \frac{1}{2} C V_0^2 = \frac{1}{2} Q_0 V_0 = \frac{1}{2} Q_0^2 / C$$

Changing C at constant V

- Battery maintains capacitor at constant V, what happens if C changes?
  - Energy stored $= U_C = \frac{1}{2} CV^2 \rightarrow \boxed{dU_C = \frac{1}{2} V^2 dC}$
  - Hence if C increases, U_C increases
- If C increases, battery has to supply charge to maintain the same V. Since $Q = CV$, charge on capacitor increases, and energy stored in the battery decreases
- Battery supplies dQ at constant $V \rightarrow$ energy change of battery is $dU_B = -V dQ$
  - $\hookrightarrow$ -ve because battery loses energy in providing $+dQ$ to the plates
- $Q = CV$, $dQ = V dC \rightarrow$ hence $\boxed{dU_B = -V^2 dC}$
- This is a general result, if U_C increases at constant V, this is matched by a $\times 2$ decrease in battery energy $\rightarrow$ difference in energy is the work done to change C (push/pull plates together)

Force between capacitor plates

- Capacitor plates are oppositely charged → an attractive force F exists between them
- By pulling the plates apart we peform work on the capacitor/battery system

Work done in pulling apart the plates = $W = -\int \underline{F} \cdot d\underline{x}$

Energy stored in capacitor = $U_C = \frac{1}{2}\frac{Q^2}{C} \; (= \frac{1}{2}CV^2)$

Energy stored in battery = $U_B = VQ \; (= CV^2)$

Cons. energy: $dU_B = dU_C + dW = 0$

Background

Case 1: Pull capacitor plates apart at constant charge Q (battery disconnected, $dU_B = 0$)

$$\rightarrow F = -\frac{\partial U_C}{\partial x} = -\frac{1}{2}Q^2 \frac{d}{dx}\left(\frac{1}{C}\right) = -\frac{1}{2}Q^2 \frac{d}{dx}\left(\frac{x}{\varepsilon_0 A}\right)$$

$$\rightarrow \boxed{F = -\frac{1}{2}\frac{Q^2}{\varepsilon_0 A}}$$

→ Mechanical work required to separate the plates from position d_1 to d_2

$$W = -\int_{d_1}^{d_2} \underline{F} \cdot d\underline{x} \rightarrow \boxed{W = \frac{1}{2}\frac{Q^2}{\varepsilon_0 A}(d_2 - d_1)}$$

Case 2: Plates pulled apart at constant voltage (which is supplied by the battery)

$$\rightarrow F = -\frac{\partial U_{total}}{\partial x} = -\frac{\partial}{\partial x}\left(\underbrace{\frac{1}{2}CV^2}_{\text{capacitor}} - \underbrace{CV^2}_{\text{battery}}\right) = \frac{1}{2}V^2\frac{dC}{dx}$$

$$F = \frac{1}{2}V^2\frac{d}{dx}\left(\frac{\varepsilon_0 A}{x}\right) \rightarrow \boxed{F = -\frac{1}{2}V^2\frac{\varepsilon_0 A}{x^2}}$$

$$W = -\int_{d_1}^{d_2} \underline{F} \cdot d\underline{x} = \frac{1}{2}V^2\varepsilon_0 A\left(\frac{1}{d_1} - \frac{1}{d_2}\right) \rightarrow \boxed{W = \frac{1}{2}V^2(C_1 - C_2)}$$

Energy density of the electric field

Consider the parallel plate capacitor: $U_C = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon_0 E^2 \underbrace{Ad}_{\text{volume}}$ $[E = V/d, \; C = \frac{\varepsilon_0 A}{d}]$

Energy density between plates = $u_p = U_C/[\text{unit volume}]$

Note; principle of superposition does not apply to energy density → $u_p = \frac{1}{2}\varepsilon_0 E^2$ → This is a general result for any E-field in a region of space [the volume can be made arbitrarily small]

Example: energy of a hollow spherical shell carrying charge q

$$\begin{cases} E = 0 & \text{for } 0 \le r \le a \text{ (inside sphere)} \\ E = \frac{q}{4\pi\varepsilon_0 r^2} & \text{for } a \le r \text{ (radial)} \end{cases}$$

$U_{tot} = \frac{1}{2}\varepsilon_0 \int_V E^2 dV$

Same result if you do energy required to bring up charge dq

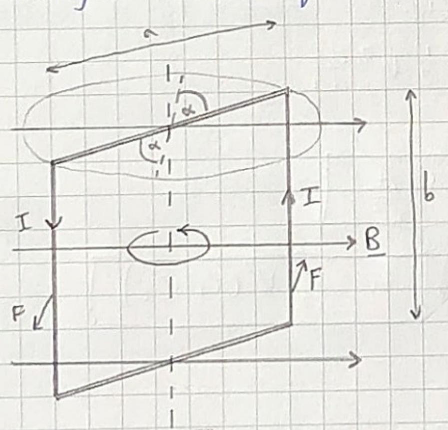
$$U_{tot} = \frac{1}{2}\varepsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} \frac{q^2}{16\pi^2\varepsilon_0^2 r^4}(r^2 dr \sin\theta \, d\theta \, d\phi) = \frac{1}{2}\varepsilon_0 \frac{4\pi q^2}{16\pi^2\varepsilon_0^2}\int_a^{\infty}\frac{1}{r^2}dr \rightarrow U = \frac{q^2}{8\pi\varepsilon_0 a}$$

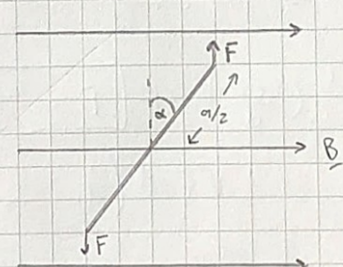
Magnetostatics

Magnetism overview:
- Minerals found in ancient Greek city Magnesia ("magnetite", Fe_3O_4) attract small metal objects
- Materials containing certain atoms such as iron (Fe), cobalt (Co), Nickel (Ni) can exhibit 'permanent' magnetic dipoles
- Forces exist between pairs of current-carrying wires (attractive for currents in the same direction and vice versa)
- An electric current through a wire creates a magnetic field whose field lines loop around the wire
- Magnetic field lines form closed loops. They do not originate from "magnetic monopoles"
- The fundamental generators of magnetic fields are dipoles that may result from electrical current loops or inherent magnetic properties such as aligned angular momentum of charged particles
- Magnetic flux density denoted by $\underline{B}$. $[B] = T$
- Magnetic field strength denoted by $\underline{H} = \frac{1}{\mu_0}\underline{B}$. (in non-magnetic materials) $[H] = Am^{-1}$

$\mu_0 = 4\pi \times 10^{-7}\ NA^{-2}$ (magnetic permeability)

Measuring B: From torque on a wire loop carrying current I in field $\underline{B}$:





$$\underline{T} = \underline{r} \times \underline{F}$$

$$|T| = 2 \times \frac{a}{2}\sin(\alpha)\ \overbrace{I\,b\,B}^{F}$$

$$|T| = IBA\sin\alpha \quad (A = \text{loop area})$$

The Lorenz Force: - Force on a current carrying wire in a B-field:

$$d\underline{F} = I\,d\underline{L} \times \underline{B} \quad \text{or} \quad d\underline{L}\,\underline{I} \times \underline{B}$$

- If we zoom into a wire segment and assume it's the + charge moving

$$d\underline{F} = dL\,\underline{I} \times \underline{B}$$

$$I = \frac{dq}{dt} \quad \text{and} \quad V = \frac{dL}{dt}$$

$$I = \frac{dq}{dL}\cdot\frac{dL}{dt} = V\frac{dq}{dL}$$

$$\hookrightarrow \underline{I}\,dL = \underline{v}\,dq$$

$$d\underline{F} = \underline{I}\,dL \times \underline{B}$$

$$d\underline{F} = dq\,\underline{V} \times \underline{B} \qquad \underline{F} = q\,\underline{v} \times \underline{B} \text{ : Lorentz force}$$

- Any charge moving with velocity $\underline{v}$ in a magnetic ~~field~~ flux density $\underline{B}$ experiences a Lorentz force $\underline{F} = q\,\underline{v} \times \underline{B}$ $\perp$ to both
- Work done in moving charge

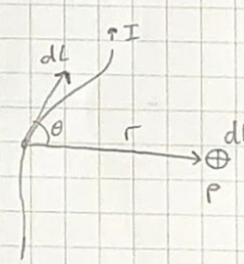
$$\hookrightarrow dW = \underline{F}\cdot d\underline{L} = q(\underline{v} \times \underline{B})\cdot\underline{v}\,dt = \underline{0}$$

MAGNETIC FIELDS DO NO WORK

The Biot-Savart Law: for calculating magnetic field

The Biot-Savart is taken here as an empirical starting point for calculation of magnetic fields, but can be derived from the Lorentz Force and Coulomb's Law (or Maxwells equations - see later)

- The Biot-Savart Law states that the field at point P:



$$d\underline{B} = \frac{\mu_0 I}{4\pi r^2}\,d\underline{L} \times \hat{\underline{r}} \quad \text{or} \quad \underline{B} = \frac{\mu_0}{4\pi}\int I\,\frac{d\underline{L} \times \hat{\underline{r}}}{r^2}$$

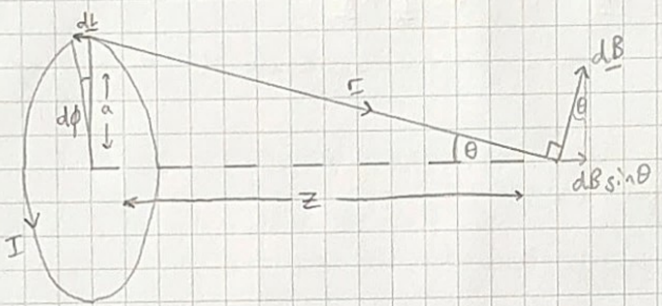
The Biot-Savart Law in terms of current density

Define current density $\underline{J}$: $\quad dI = \underline{J}\cdot d\underline{a} \quad$ or $\quad \underline{J} = \frac{dI}{da_\perp}$

$$\hookrightarrow \text{Also } I\,d\underline{L} = (\underline{J}\cdot d\underline{a})\,dL$$
$$I\,d\underline{L} = \underline{J}\,dV$$

$$\hookrightarrow \text{Hence: } \underline{B} = \int_V \mu_0\,\frac{\underline{J} \times \hat{\underline{r}}}{4\pi r^2}\,dV$$

Example: B-field of circular current loop



Field due to element $d\underline{L}$: $\quad d\underline{B} = \mu_0 I \dfrac{d\underline{L} \times \hat{\underline{r}}}{4\pi r^2}$

→ Components of field perpendicular to z axis cancel due to symmetry

→ Field is along the z-axis

$$|d\underline{B}| = \frac{\mu_0 I \, dL}{4\pi r^2} \qquad [dL \perp \text{ to } \hat{\underline{r}}]$$

component of $d\underline{B}$ // to z-axis

$$B = \int \sin\theta \, dB$$

$$= \int \frac{a}{r} dB \qquad\qquad r = (a^2+z^2)^{1/2}$$

$$= \int \frac{\mu_0 I}{4\pi r^2} \frac{a}{r} dL$$

$$= \frac{\mu_0 I a}{4\pi r^3} \underbrace{\int dL}_{=2\pi a}$$

$$= \frac{\mu_0 I a}{4\pi (a^2+z^2)^{3/2}} (2\pi a)$$

$$B = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \quad \text{or} \quad = \frac{\mu_0 I}{2a} \sin^3\theta \qquad \left[\sin\theta = \frac{a}{\sqrt{z^2+a^2}}\right]$$

The magnetic dipole: A small current loop defines a magnetic dipole

magnetic dipole momement: $\underline{m} = I\,\underline{A}$

$\underline{m}$ = [current] × [Area of loop]

Electric dipole moment	magnetic dipole (current loop) moment
$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$	$B_r = \frac{2\mu_0 m\cos\theta}{4\pi r^3}$
$E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$	$B_\theta = \frac{\mu_0 m\sin\theta}{4\pi r^3}$
$E_\phi = 0$	$B_\phi = 0$

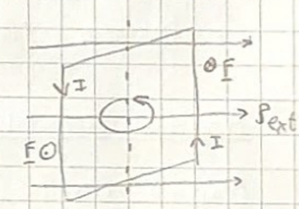
(derivation of B-fields components are beyond course scope)

↳ the magnetic and electric dipole ~~moments~~ komponents have exactly the same form

Torque on a magnetic dipole in a B-field

net $T = \int_{loop} I\,\underline{dl} \times \underline{B_{ext}} = 0$ (loop doesn't move in any direction, it only rotates)

$|T| = IBA\sin\theta$ ← steps before this but basically ~~[illegible]~~ obvious



$\underline{T} = I\,\underline{A} \times \underline{B_{ext}}$

$\underline{T} = \underline{m} \times \underline{B_{ext}}$ → $\underline{T} = \underline{m} \times \underline{B_{ext}}$

↳ this is general for any shape not just rectangular

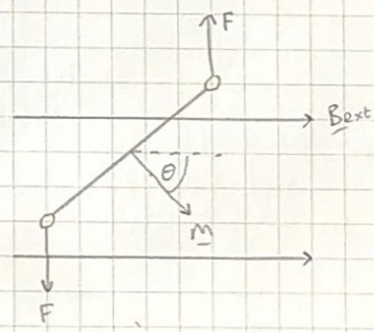
Energy of a magnetic dipole in a B-field

The energy of a magnetic dipole placed in a magnetic field $\underline{B_{ext}}$ is equal to the work done in rotating dipole into its position:

$dW = T\,d\theta$

zero energy chosen at $\theta = \frac{\pi}{2}$

$W = \int_{\pi/2}^{\theta} \underbrace{m B_{ext}\sin\theta'}_{torque}\, d\theta' = -mB_{ext}\left[\cos\theta\right]_{\pi/2}^{\theta}$

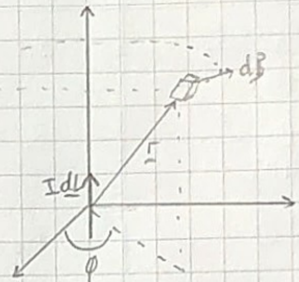


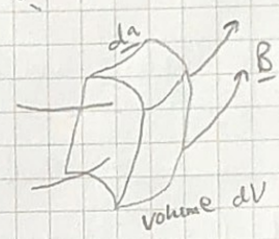
$W = -m B_{ext}\cos\theta$

~~[illegible]~~ $W = -\underline{m} \cdot \underline{B_{ext}}$

Divergence of $\underline{B}$

- Place a current element $I\underline{dl}$ at the origin pointing along the z-axis
- The Biot-Savart Law gives the field at point $P \rightarrow d\underline{B} = \mu_0 I \frac{\underline{dl} \times \hat{\underline{r}}}{4\pi r^2}$
- $d\underline{B}$ is $\perp$ to $\underline{r}$ and $\hat{\underline{z}}$
- Rotate $\underline{r}$ around ϕ and it can be seen the lines of $\underline{B}$ are circles in planes perpendicular to $\underline{dl}$ and centred on it $\rightarrow$ The net outward flux of $\underline{B}$ due to $\underline{dl}$ through the surface of the volume element dV is zero





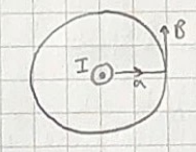
- Any volume can be made up of volume elements as dV
- Hence, $\oint_S \underline{B} \cdot d\underline{a} = 0 \rightarrow$ no magnetic monopoles
- Divergence Theorem: $\oint_S \underline{B} \cdot d\underline{a} = \int_V (\underline{\nabla} \cdot \underline{B})\, dV \rightarrow \underline{\nabla} \cdot \underline{B} = 0$

Ampere's Circuital Law

Ampere's circuital law can be derived formally from the Biot-Savart Law and vector calculus but is beyond the scope of this lecture

- But to derive the law for a special case: B-field from an infinite straight wire

$B = \frac{\mu_0 I}{2\pi a} \rightarrow$ separately:



closed loop integral: $\oint_c \underline{B} \cdot d\underline{l} = \frac{\mu_0 I}{2\pi a} \cdot 2\pi a = \mu_0 I$

$\hookrightarrow$ This is Ampere's Circuital Law: $\oint_{\text{closed loop}} \underline{B} \cdot d\underline{l} = \mu_0 I = \mu_0 \int \underline{J} \cdot d\underline{a}$

current enclosed

This allows for straightforwards calculations of B-fields along loops where B is constant

(Note: this needs to be amended in the case of a time-varying magnetic field)

Stokes: $\oint_c \underline{B} \cdot d\underline{l} = \mu_0 I$

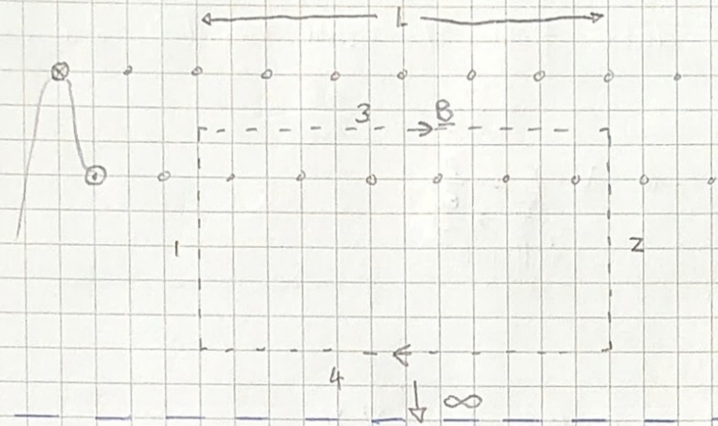
$$\int_S (\underline{\nabla} \times \underline{B})\, d\underline{a} = \mu_0 \oint_S \underline{J} \cdot d\underline{a}$$

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$ in differential form

can use this to find B-fields: inside & outside wires, in infinite solenoids, in toroidal coils

Example: B-field of a long solenoid

- solenoid carrying current I
- Amperean path is a rectangle inside and outside the solenoid



$$\oint \underline{B} \cdot d\underline{l} = BL = \mu_0 N I$$

$$B = \frac{\mu_0 N}{L} I \quad \text{inside solenoid}$$

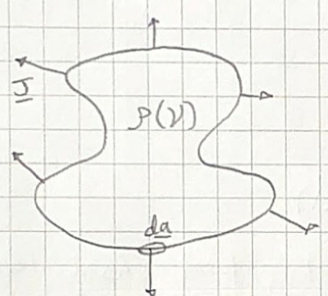
$\downarrow$

Same as for Biot-Savart but easier

(sides 2 & 3 cancel and take 3 to ∞)

Conservation of charge

Consider a volume $\mathcal{V}$ bounded by a surface S



The integral of current density flowing out (or into) the surface $\underline{J} \cdot d\underline{a}$ is equal to the charge lost by the volume per unit time

$$\int_S \underline{J} \cdot d\underline{a} = I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho(V)\, dV$$

Divergence theorem: $\int_V \underline{\nabla} \cdot \underline{J}\, dV = \frac{-d}{dt} \int_V \rho(V)\, dV$

This gives the continuity equation $\rightarrow$ $\underline{\nabla} \cdot \underline{J} = \frac{-\partial \rho}{\partial t}$

constant B-fields: $\frac{dJ}{dt} = 0$ for steady current

constant E-fields: $\frac{d\rho}{dt} = 0$ for stationary charges

For both cases $\leftarrow$

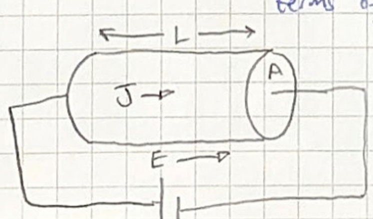
$\underline{\nabla} \cdot \underline{J} = 0$

current density and Ohm's Law $\rightarrow$ $V = IR$

$$EL = JAR$$

Ohm's Law in terms of J: $J = \frac{L}{RA} E$ $\rightarrow$ conductivity $\sigma = L/RA$

resistivity $\rho = \frac{1}{\sigma} = \frac{RA}{L}$



Summary of electrostatics

1. Coulomb's Law:

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{R})}{(\underline{r}-\underline{R})^3} (\underline{r}-\underline{R})\, dV$$

- An electric charge generates an electric field. Electric field lines begin and end on charge or at ∞

2. Gauss Law: $\underbrace{\oint_S \underline{E} \cdot d\underline{a} = \frac{Q_{encl}}{\epsilon_0}}_{\text{integral form}} \rightarrow \underbrace{\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0}}_{\text{differential form}}$

3. The electric field is conservative:
   - A well-defined potential V such that $\underline{E} = -\underline{\nabla} V$
     $\rightarrow \oint \underline{E} \cdot d\underline{l} = 0$ (work is independent of path)
   - Using the vector identity: $\underline{\nabla} \times \underline{E} = -\underline{\nabla} \times \underline{\nabla} V = 0$
   - Hence $\underline{\nabla} \times \underline{E} = 0$

Summary of magnetostatics

1. Biot-Savart Law: $\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\underline{J}(\underline{R})}{(\underline{r}-\underline{R})^3} \times (\underline{r}-\underline{R})\, dV$
   - There are no magnetic monopoles. Magnetic field lines form closed loops

2. Gauss Law of magnetostatics: $\underbrace{\oint_S \underline{B} \cdot d\underline{a} = 0}_{\text{integral form}} \rightarrow \underbrace{\underline{\nabla} \cdot \underline{B} = 0}_{\text{differential form}}$
   (no magnetic monopoles)

3. Ampere's Law: magnetic fields are generated by electric currents
   $\rightarrow \oint \underline{B} \cdot d\underline{l} = \mu_0 I_{encl} \rightarrow \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

4. Continuity equation: $\int_S \underline{J} \cdot d\underline{a} = -\frac{d}{dt} \int_V \rho(r)\, dV \rightarrow \underline{\nabla} \cdot \underline{J} = -\frac{d}{dt}(\rho)$
   (Charge Conserved)

Electromagnetic Induction

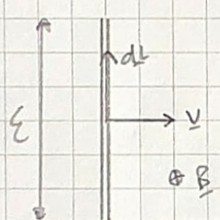
A change with time in the magnetic flux density through a circuit causes an "electromotive force" that moves charge along the circuit

Lorentz Force on a point charge moving in a B-field:

→ Any point charge q moving with velocity v in a magnetic flux density B experiences a Lorentz force $\perp$ to $\underline{v}$ and $\underline{B}$, with:

$$\underline{F} = q\,\underline{v} \times \underline{B}$$

Electromotance resulting from forces on charges in a conductor moving with respect to a B-field: A Lorentz force on a charge in a circuit element $d\underline{l}$ moving with velocity $\underline{v}$ with respect to a magnetic flux density $\underline{B}$ causes an electromotance $\mathcal{E}$ in the circuit, with:



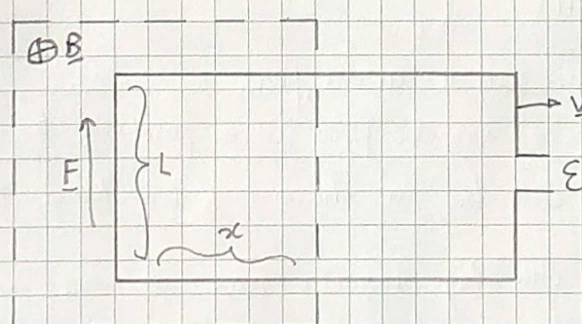
$$\mathcal{E} = \int_L (\underline{v} \times \underline{B}) \cdot d\underline{l}$$

$$\text{EMF} = \mathcal{E} = \int \frac{dw}{q} \qquad V = \text{work done per unit charge}$$

$$\mathcal{E} = \int \frac{\underline{F} \cdot d\underline{L}}{q}$$

$$\text{Hence } \mathcal{E} = \int (\underline{v} \times \underline{B}) \cdot dL$$

Magnetic flux: consider a wire loop being pulled with velocity v out of a region with a $\underline{B}$-field



EMF along vertical side:

$$\mathcal{E} = \int \underline{v} \times \underline{B} \cdot dL = VBL$$

define the magnetic flux: $\phi = \int_S \underline{B} \cdot d\underline{a}$

unit = Weber = Wb = Tm²

$$\frac{d\phi}{dt} = \frac{d}{dt}\int_S B\,da = \frac{d}{dt}(BA) = \frac{d}{dt}(BLx)$$

$$\frac{d\phi}{dt} = BL\frac{dx}{dt} = -BLV$$

In general: $\mathcal{E} = -\frac{d}{dt}\int_S \underline{B} \cdot d\underline{a} = \frac{-d\phi}{dt}$ $\quad \therefore \frac{d\phi}{dt} = -BvL = -\mathcal{E}$

Faraday's Law: The induced electromotance (EMF) $\mathcal{E}$ in any closed circuit is equal to (the negative of) the time rate of change of the magnetic flux ϕ through the circuit

Lenz's Law: The induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux

Faraday's Law in differential form:

Net potential around a circuit loop due to an induced Emf: $\mathcal{E} = \int \underline{E} \cdot d\underline{l}$

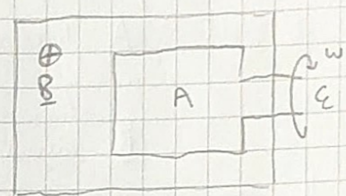
Faraday's: $\mathcal{E} = -\frac{d}{dt}\int_s \underline{B} \cdot d\underline{a}$

$$\int_L \underline{E} \cdot d\underline{l} = -\frac{d}{dt}\int_s \underline{B} \cdot d\underline{a}$$

$$\int_s \underline{\nabla} \times \underline{E} \cdot d\underline{s} = -\frac{d}{dt}\int_s \underline{B} \cdot d\underline{a} \rightarrow \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla} \times \underline{E} = -\frac{d\underline{B}}{dt}$$

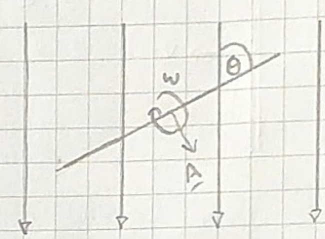
Example: coil rotating in a $\underline{B}$-field



flux $\phi = \int_s \underline{B} \cdot d\underline{a}$

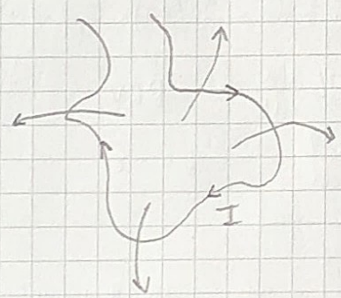
$\phi = BAN\sin\theta = BAN\sin\omega t$

$\mathcal{E} = -\frac{d\phi}{dt} = -NAB\omega\cos\omega t$



Self Inductance: Units = Henry = [H]

- Take a closed-loop circuit through which current flows
- The current I has an associated magnetic field which penetrates the circuit $B \propto I$
- If the current changes there will be a changing $\underline{B}$-field through the loop



Faraday: the changing magnetic flux ϕ induces an emf in the loop itself

Lenz: The Emf will act in a direction so as to oppose the change in flux that caused it

Define self inductance: $L = \frac{\phi}{I}$ since $\phi \propto I \rightarrow L = \frac{d\phi}{dI}$

$$L = \frac{d\phi}{dt} \Big/ \frac{dI}{dt} = -\frac{\mathcal{E}}{\frac{dI}{dt}} \rightarrow \mathcal{E} = -L\frac{dI}{dt} \rightarrow V = L\frac{dI}{dt} \text{ as we know}$$

L depends solely on the geometry of the circuit

Example: self inductance of a long coil area A, N turns, length L

$\rightarrow$ Amperes: $\oint \underline{B} \cdot d\underline{L} = \mu_0 I$

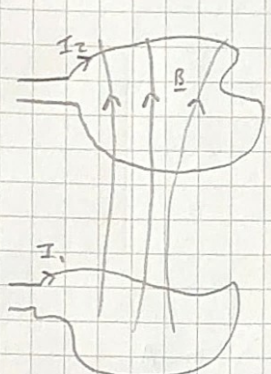
$$B = \frac{\mu_0 N I}{L}$$

$$\phi = \int \underline{B} \cdot d\underline{a}$$
$$= NAB$$
$$= \frac{\mu_0 N^2 A I}{L}$$

$\rightarrow$ $L = \phi / I$

$$L = \frac{\mu_0 N^2 A}{L}$$

Mutual inductance



- current I_1 through circuit loop 1 generates magnetic field density B_1 which penetrates circuit loop 2

- A change in current I_1 will induce an Emf in circuit loop 2

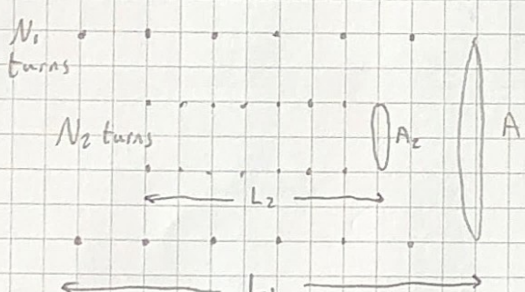
define mutual inductance M:

$$M_{21} = \frac{\phi_2}{I_1} = \frac{d\phi_2}{dI_1} = M_{12} = \frac{\phi_1}{I_2} = \frac{d\phi_1}{dI_2}$$

$\rightarrow$ In general: ~~[illegible]~~ $M = K\sqrt{L_1 L_2}$ K = coefficient of coupling $(K \leq 1)$

Example: Mutual induction of two coaxial solenoids

$\hookrightarrow$ current through coil 1 creates $\underline{B}$ field through coil 2



$$B_1 = \frac{\mu_0 N_1 I_1}{L_1}$$

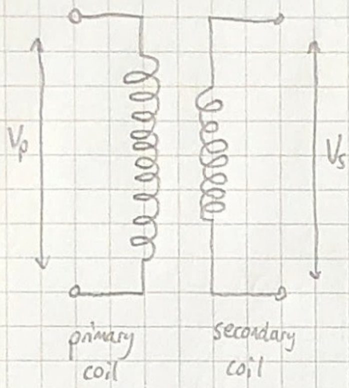
$$\phi_2 = N_2 A_2 B_1 = \frac{\mu_0 N_1 I_1}{L_1} A_2 N_2$$

$$M_{21} = \frac{\phi_2}{I_1} = \frac{\mu_0 N_1 N_2}{L_1} A_2$$

$$\mathcal{E} = -\frac{d\phi_2}{dt} = -\frac{\mu_0 A_2 N_1 N_2}{L_1} \frac{dI_1}{dt} = -M_{21} \frac{dI_1}{dt}$$

$\hookrightarrow$ $\boxed{\mathcal{E} = -M_{21} \frac{dI_1}{dt}}$ $\rightarrow$ compare to $\mathcal{E} = -L\frac{dI}{dt}$ for self inductance $\rightarrow$ Should really account for M but we neglect it

Transformers



The primary coil creates a flux $\phi_p = A_p B_p$ per winding

→ results in emf in secondary coil $\mathcal{E} = -\frac{d\phi_s}{dt}$ per winding

The coils are coupled $\phi_s = K\phi_p$ where $\boxed{K = 1 \text{ for an ideal transformer}}$

→ K depends on geometry, coupling etc

Ratio of EMFs: $\mathcal{E}_p = -N_p \frac{d\phi_p}{dt}$, $\mathcal{E}_s = -N_s \frac{d\phi_s}{dt}$

$$\frac{V_s}{V_p} = \frac{d\phi_s}{d\phi_p} \times \frac{N_s}{N_p} = K \frac{N_s}{N_p}$$

Power $V_s I_s = V_p I_p$ → $\frac{I_s}{I_p} = \frac{V_p}{V_s} = \frac{1}{K}\frac{N_p}{N_s}$

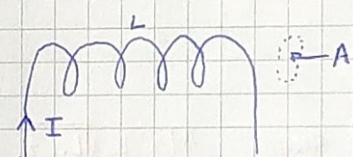
- Transformer will step up or step down applied voltage V_p by the winding ratio
- Ideally there is no power consumed in the transformer if coils have zero resistance

$$\frac{I_s}{I_p} = \frac{V_p}{V_s} = \frac{1}{K}\frac{N_p}{N_s}$$ ($K = 1$ for an ideal transformer)

$$K = \frac{d\phi_s}{d\phi_p}$$

Energy of the magnetic field

Consider the energy stored in an inductor L:



Change in current results in a back EMF

We need to put in energy to change the current

Power = work/unit time

$$P = \frac{dw}{dt} = \frac{V\,dQ}{dt} = VI$$

Stored energy $U = \int VI\,dt$

$$U = \int \left(L\frac{dI}{dt}\right) I\,dt$$

$$U = \int LI\,dI$$

$$U = \frac{1}{2}LI^2 = \frac{1}{2}\phi I \quad \left(\text{since } L = \frac{\phi}{I}\right)$$

for a coil: $L = \frac{\mu_0 N^2}{L} A$ and $B = \frac{\mu_0 N}{L} I$

$$U = \frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_0 N^2}{L} A \left(\frac{B^2 L^2}{\mu_0^2 N^2}\right) = \frac{1}{2}\frac{B^2}{\mu_0} AL = \frac{1}{2}\frac{B^2}{\mu_0} \mathcal{V}$$

$\frac{1}{2}\frac{B^2}{\mu_0}$ = energy density

In the general case: $U = \frac{1}{2\mu_0}\int B^2\,d\mathcal{V}$ over all space

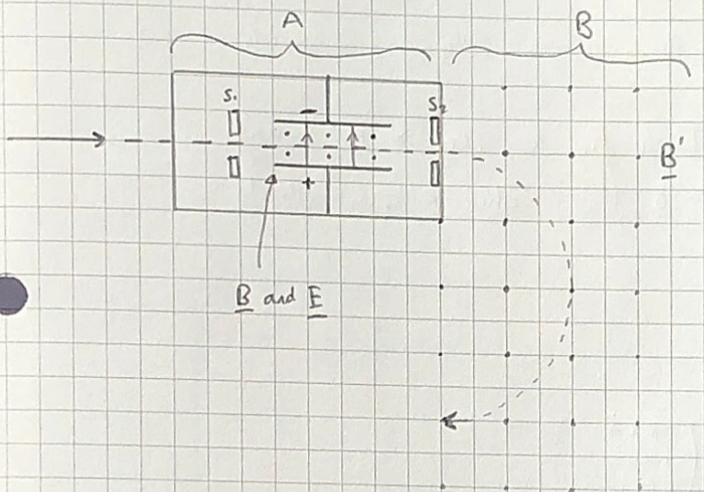
Charged Particles in E and B Fields

Force on a charged particle in an $\underline{E}$ and $\underline{B}$ field: $\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$

along $\underline{E}$ — $\perp$ to both $\underline{v}$ and $\underline{B}$

N II: $\underline{F} = m\underline{a} = m\ddot{\underline{r}} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$

<u>The mass spectrometer</u>: used for detecting small charged particles (molecules, ions) by their mass m.



<u>Stage A</u>: The velocity filter
- The particle will pass through both slits if it experiences no net force inside the filter
- The region has both $\underline{E}$ and $\underline{B}$ fields
- Will filter particles with $v = \frac{|\underline{E}|}{|\underline{B}|}$ and the spread $\pm \Delta v$ is given by the slit width

<u>Stage B</u>: The mass filter
- This region has only a $\underline{B}$-fields

$$m\ddot{\underline{r}} = q\,\dot{\underline{r}} \times \underline{B}$$

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = q\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \dot{x} & \dot{y} & \dot{z}\\ 0 & 0 & B\end{vmatrix}$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \frac{q}{m}\begin{pmatrix}\dot{y}B\\ -\dot{x}B\\ 0\end{pmatrix}$$

$$\ddot{r}^2 = \ddot{x}^2 + \ddot{y}^2 = \frac{q^2}{m^2}\left(\dot{x}^2 + \dot{y}^2\right)B^2 = \frac{q^2}{m^2}\dot{r}^2 B^2$$

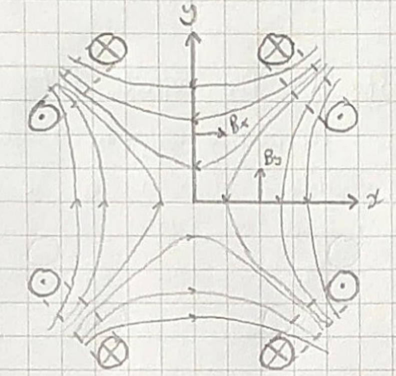
circular motion: $\ddot{r} = \frac{q}{m} vB = \frac{v^2}{R} \rightarrow$ radius of curvature $= R = \frac{mv}{qB} \rightarrow R \propto m$

v, q, and B are constant

Magnetic Lenses

- Magnetic lenses are used for focusing and collimating charged particle beams, used in electron microscopes, particle accelerators etc.
- Quadrupole lens: four identical coils aligned in z-direction
- Sum of 4 dipole fields: for small values of x, y close to the axis of symmetry

$$B_x \propto y, \quad B_y \propto x$$



- Along x-axis: Only B_y component
- Along y-axis: Only B_x component

Inside the lens close to the z-axis

$$\underline{B} = \begin{pmatrix} Kx \\ Ky \\ 0 \end{pmatrix} \quad \text{where } K \text{ is a constant}$$

Eq^n of motion: $\underline{F} = q\, \underline{v} \times \underline{B}$

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dot{x} & \dot{y} & \dot{z} \\ Kx & Ky & 0 \end{vmatrix} = q \begin{pmatrix} -Kx\dot{z} \\ +Ky\dot{z} \\ K(\dot{x}x - \dot{y}y) \end{pmatrix}$$

Assume particle travels at small angles wrt z axis $\rightarrow \dot{x}, \dot{y} \approx 0$

$$\hookrightarrow m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} -Kx\dot{z} \\ +Ky\dot{z} \\ 0 \end{pmatrix}$$

$\hookrightarrow \ddot{z} = 0 \rightarrow \dot{z}$ is constant $= v$

$\hookrightarrow \ddot{x} = -\frac{q}{m} Kvx, \quad \ddot{y} = \frac{q}{m} Kvy$: Equations of motion

Solutions:

$$x(t) = A \sin \sqrt{\frac{qKv}{m}}\, t + B \cos \sqrt{\frac{qKv}{m}}\, t$$

$$y(t) = C \sinh \sqrt{\frac{qKv}{m}}\, t + D \cosh \sqrt{\frac{qKv}{m}}\, t$$

Boundary conditions

At $t=0$, $z=0$, $x = x_0$ and $\dot{x} = 0$, $y = y_0$ and $\dot{y} = 0$

Solutions become $x(z) = x_0 \cos \sqrt{\frac{qK}{mv}}\, z$: focussing

$y(z) = y_0 \cosh \sqrt{\frac{qK}{mv}}\, z$: divergent/de-focussing

Bit odd, but in practise you put one on top of the other rotated 90° so you focuss first in x, then in y

- Focal points in x-direction at $f_n = \frac{\pi}{2}\sqrt{\frac{mv}{qK}} + n\pi\sqrt{\frac{mv}{qK}}$
- Use lens pair with 90° angle for collimating a charged beam

Electromagnetic Waves

Electrodynamics "before Maxwell"

1) Gauss Law: $\oint_S \underline{E} \cdot d\underline{a} = \frac{Q_{encl}}{\epsilon_0} \rightarrow \underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0}$

2) No magnetic monopoles: $\oint_S \underline{B} \cdot d\underline{a} = 0 \rightarrow \underline{\nabla} \cdot \underline{B} = 0$

3) Faraday's Law: $\int_L \underline{E} \cdot d\underline{L} = -\frac{\partial}{\partial t} \oint_S \underline{B} \cdot d\underline{a} \rightarrow \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t}$

4) Ampere's Law: $\oint_L \underline{B} \cdot d\underline{L} = \mu_0 I_{encl} \rightarrow \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

Time varying B-fields generate E-fields. However, time-varying E-fields do not seem to create B-fields in this version. Is something wrong

Revisiting Ampere's Law

Ampere's Law: $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

~~Applying divergence theorem~~ $\underbrace{\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B})}_{\text{Always zero}} = \underbrace{\mu_0 \underline{\nabla} \cdot \underline{J}}_{\boxed{\text{NOT Always zero}}}$

Recall continuity equation: $\int_S \underline{J} \cdot d\underline{a} = -\frac{\partial}{\partial t} \int_V \rho(V)\, dV$

divergence theorem $\rightarrow$ $\underline{\nabla} \cdot \underline{J} = -\frac{\partial}{\partial t} \rho \rightarrow \mu_0 \underline{\nabla} \cdot \underline{J}$ is not zero

Therefore Ampere's Law in its current form violates the continuity equation! But this is not surprising since we derived Ampere's Law assuming that $\frac{\partial}{\partial t} \rho = 0$

We need to add a term to Ampere's Law to make it compatible with the continuity equation: $\underline{\nabla} \cdot \underline{J} = -\frac{\partial}{\partial t}(\rho)$

Gauss law: $\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \rightarrow \rho = \epsilon_0 \underline{\nabla} \cdot \underline{E}$

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial}{\partial t} \epsilon_0 (\underline{\nabla} \cdot \underline{E})$$

$$\underline{\nabla} \cdot \underline{J} = -\underline{\nabla} \cdot \left(\epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\underline{\nabla} \cdot \left(\underline{J} + \epsilon_0 \frac{\partial E}{\partial t}\right) = 0$$ $\rightarrow$ This one is always zero!

stoke's

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \underbrace{\epsilon_0 \frac{\partial E}{\partial t}}\right)$$ $\leftarrow$ Implies we have to add $\epsilon_0 \frac{\partial E}{\partial t}$ to $\underline{J}$ in Ampere's Law

$$\oint_C \underline{B} \cdot d\underline{L} = \underbrace{\mu_0 \int_S \underline{J} \cdot d\underline{a}}_{\mu_0 I_{encl}} + \mu_0 \epsilon_0 \int_S \frac{\partial E}{\partial t} \cdot d\underline{a}$$

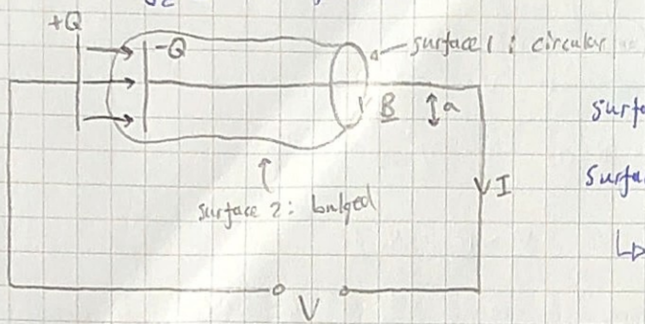
$\epsilon_0 \frac{\partial E}{\partial t}$ is called the "displacement current" J_D but it's actually a time-varying electric field

Time-varying E field now generate B fields and vice versa. Also satisfies charge conservation

Example: Ampere's law and a charging capacitor

↳ An example of Ampere's Law failing without the displacement current

$$\oint_c \underline{B} \cdot d\underline{l} = \mu_0 I_{enclo}$$



surface 1: circular → $I_{encl} = I$

Surface 2: bulged → $I_{encl} = 0$

↳ original Ampere's would give 2 different values of B

but $E = \frac{Q}{\varepsilon_0 A}$ → $\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A} \frac{\partial Q}{\partial t} = \frac{I}{\varepsilon_0 A}$

↳ works if we use the new ampere's law

$$\oint_c \underline{B} \cdot d\underline{l} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int_s \frac{\partial \underline{E}}{\partial t} \cdot d\underline{a}$$

For surface 1

$$\oint_c \underline{B} \cdot d\underline{l} = \mu_0 I + 0 \rightarrow B = \frac{\mu_0 I}{2\pi a}$$

The same!

For surface 2

$$\oint_c \underline{B} \cdot d\underline{l} = 0 + \mu_0 \varepsilon_0 \int_s \frac{\partial \underline{E}}{\partial t} \cdot d\underline{a} = \mu_0 \varepsilon_0 \left(\frac{I \times A}{\varepsilon_0 A} \right) = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi a}$$

For another good example see slide 6 of lecture 19

Quick Summary: Maxwell's Equations

Gauss Law: Charge generates an electric field. Electric field lines begin ~~and reach~~ on charge and end on opposite charge/infinity → $\int_s \underline{E} \cdot d\underline{a} = \frac{Q}{\varepsilon_0}$ ⟷ $\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$

Faraday's Law: Time varying magnetic fields create electric fields (induction)

↳ $\oint \underline{E} \cdot d\underline{l} = -\frac{d}{dt} \int_s \underline{B} \cdot d\underline{a}$ ⟷ $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

No magnetic monopoles: There are no magnetic monopoles. Magnetic field lines form closed loops

↳ $\oint_s \underline{B} \cdot d\underline{a} = 0$ ⟷ $\underline{\nabla} \cdot \underline{B} = 0$

Ampere's Law: electric currents and time-varying electric fields generate B-fields

↳ $\oint \underline{B} \cdot d\underline{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt} \int_s \underline{E} \cdot d\underline{a}$ ⟷ $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

Electromagnetic waves in a vacuum: In the absence of electric charge or current;

$$\rho = 0, \quad \underline{J} = 0$$

Maxwell's equations become:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 0 \\ \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \underline{\nabla} \times \underline{B} &= \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \end{aligned}}$$

Apply curl to Faraday's Law:

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t} \underline{\nabla} \times \underline{B}$$

USE the vector identity:

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} = \underbrace{\underline{\nabla}(\underline{\nabla} \cdot \underline{E})}_{=0} - \underline{\nabla}^2 \underline{E}$$

$$\downarrow$$

use ampere's law

$$-\underline{\nabla}^2 \underline{E} = -\frac{\partial}{\partial t} \overbrace{\underline{\nabla} \times \underline{B}}$$

$$-\underline{\nabla}^2 \underline{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

This gives us a wave equation in $\underline{E}$: $\underline{\nabla}^2 \underline{E} - \varepsilon_0 \mu_0 \ddot{\underline{E}} = 0$

We can instead apply the curl to Ampere's Law and follow very similar steps to get a wave equation in $\underline{B}$ in addition to our original one for $\underline{E}$

↳ $\underline{\nabla}^2 \underline{E} - \varepsilon_0 \mu_0 \ddot{\underline{E}} = 0$ and $\underline{\nabla}^2 \underline{B} - \varepsilon_0 \mu_0 \ddot{\underline{B}} = 0$

These equations have a general solution:

$$E(x,t) = F(x-ct) + G(x+ct) \quad \text{and} \quad B(x,t) = F'(x-ct) + G'(x+ct)$$

where F, G, F', and G' are any functions of $(x-ct)$ and $(x+ct)$

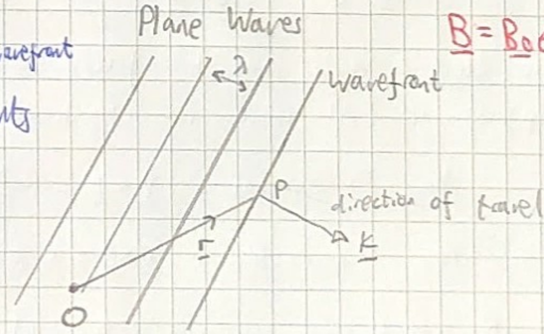
↳ from here on we will only consider the simplest solution, the plane wave solution

↳ $\boxed{\underline{E} = \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}, \quad \underline{B} = \underline{B}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}}$

just same in red

$\underline{E} = \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$, $\underline{B} = \underline{B}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$

Plane Waves



- All points P with the same phase form a wavefront
- $\underline{k}$ is in the direction normal to the wavefronts
- Maxima are separated by the wavelength λ where $\lambda = \frac{2\pi}{k}$
- Phase velocity (or propagation velocity) of wavefronts is given by $c = \frac{\omega}{k}$

Divergence, time derivative, and curl of $\underline{E}$ and $\underline{B}$

↳ We are just doing this to make our manipulations of maxwell's equations easier

The divergence of $\underline{E}$

$$\nabla \cdot \underline{E} = \nabla \cdot \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \underline{E}_0 e^{i(\omega t - k_x x - k_y y - k_z z)}$$
$$= \left[(-i) k_x E_x + (-i) k_y E_y + (-i) k_z E_z\right] e^{i(\omega t - \underline{k} \cdot \underline{r})}$$
$$= (-i)\, \underline{k} \cdot \underline{E}$$

$$\boxed{\text{Hence } \nabla \cdot \equiv -i\underline{k}}$$

Time derivative of $\underline{E}$

$$\frac{\partial}{\partial t} \underline{E} = \frac{\partial}{\partial t} \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$$
$$= i\omega \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$$
$$= i\omega \underline{E}$$

$$\boxed{\text{Hence } \frac{\partial}{\partial t} \equiv i\omega}$$

The curl of $\underline{E}$

$$\nabla \times \underline{E} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} = (-i) \begin{pmatrix} k_y E_z - k_z E_y \\ k_z E_x - k_x E_z \\ k_x E_y - k_y E_x \end{pmatrix} = (-i)\, \underline{k} \times \underline{E}$$

Note that these are the same, just showing it works for div and curl

[obviously these all work for $\underline{B}$ too]

$$\boxed{\text{Hence } \nabla = -i\underline{k}}$$

Electromagnetic waves speed of propagation

Sub $\underline{E} = \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$ into wave equation $\nabla^2 \underline{E} = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \underline{E}$

use $\nabla = -i\underline{k}$ and $\frac{\partial}{\partial t} = i\omega$ → $-k^2 \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})} = -\omega^2 \varepsilon_0 \mu_0 \underline{E}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$

↳ $\boxed{k^2 = \omega^2 \varepsilon_0 \mu_0}$

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \, ms^{-1}$$

Relationship between $\underline{E}$ and $\underline{B}$

↳ using: $\nabla \cdot \underline{E} = 0$ and $\nabla \cdot \underline{B} = 0$

↳ $\nabla \cdot \underline{E} = -i\underline{k} \cdot \underline{E}$, $\nabla \cdot \underline{B} = -i\underline{k} \cdot \underline{B}$ → Hence $\underline{k} \cdot \underline{E} = 0$, $\underline{k} \cdot \underline{B} = 0$ → Hence both $\underline{E}$ and $\underline{B}$ $\perp$ to direction of propagation

Hence electric and magnetic fields in vacuum are perpendicular to direction of propagation → EM waves are transverse

$\underline{B} = \frac{1}{\omega} \underline{k} \times \underline{E}$, $\underline{E} = -\frac{c^2}{\omega} \underline{k} \times \underline{B}$

Now Faraday's Law → $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$, $-i\underline{k} \times \underline{E} = -i\omega \underline{B}$, $\boxed{\underline{B} = \frac{1}{\omega} \underline{k} \times \underline{E}}$

Also Ampere's Law → $\nabla \times \underline{B} = \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$, $-i\underline{k} \times \underline{B} = \mu_0 \varepsilon_0 (-i\omega) \underline{E}$, $\boxed{\underline{E} = -\frac{c^2}{\omega} \underline{k} \times \underline{B}}$

∴ $\underline{E}$, $\underline{B}$, and $\underline{k}$ are mutually orthogonal, $\underline{E}$ and $\underline{B}$ are in phase and lie in the plane of the wavefront

Field magnitude ratio: $\frac{|\underline{E}|}{|\underline{B}|} = \frac{c^2}{\omega} k = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Characteristic impedance of free space

Take the ratio $Z = \frac{|E|}{|H|}$ $\quad \left[|H| = \frac{1}{\mu_0} |B| \right]$

Z has units of $\frac{[Vm^{-1}]}{[Am^{-1}]}$ = OHMS

Z is called the characteristic impedance of free space

$Z = \mu_0 c = 376.7\Omega$

$$Z = \mu_0 \frac{|E|}{|B|} = \mu_0 c = \frac{\mu_0}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\Omega$$

Polarisation

- Linearly (or plane) polarised wave: E has one specific direction
- Circularly polarised wave: Two linear components of E superimposed at a right angle and phase shifted by $\pi/2$
- Elliptically polarised wave: As above but with unequal amplitudes
- Unpolarised: E superimposed with all orientations (with no fixed phase relationships between components)

Energy flow and the Poynting Vector

Recall: Energy of the electric field $U_e = \int_V \frac{1}{2} \epsilon_0 \underline{E}^2 dV$

Energy of the magnetic field $U_m = \int_V \frac{1}{2\mu_0} \underline{B}^2 dV$

Total EM energy in volume V:

$$U = \int_V \frac{1}{2} \left(\epsilon \underline{E} \cdot \underline{E} + \frac{1}{\mu_0} \underline{B} \cdot \underline{B} \right) dV$$

In free space $(J = 0, \rho = 0)$

$$\underbrace{\frac{\partial E}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \underline{\nabla} \times \underline{B}}_{\text{Ampere's Law}} \quad ; \quad \underbrace{\frac{\partial B}{\partial t} = - \underline{\nabla} \times \underline{E}}_{\text{Faraday's Law}}$$

Calculate the rate of change of energy in V:

$$\frac{du}{dt} = \int_V \left(\epsilon_0 \underline{E} \cdot \underline{\dot{E}} + \frac{1}{\mu_0} \underline{B} \cdot \underline{\dot{B}} \right) dV$$

$$\frac{du}{dt} = \int_V \left(\frac{\epsilon_0}{\mu_0 \epsilon_0} (\underline{E} \cdot \underline{\nabla} \times \underline{B}) - \frac{1}{\mu_0} \underline{B} \cdot (\underline{\nabla} \times \underline{E}) \right) dV$$

$$\boxed{\frac{du}{dt} = - \frac{1}{\mu_0} \int_V \underline{\nabla} \cdot (\underline{E} \times \underline{B}) \, dV}$$

Energy flow and the Poynting Vector continued

Energy flow out of volume V for unit time

$$\frac{du}{dt} = -\frac{1}{\mu_0}\int_V \nabla \cdot (\underline{E} \times \underline{B})\, dV$$

Apply divergence theorem

$$\frac{du}{dt} = -\oint_S \frac{1}{\mu_0}(\underline{E} \times \underline{B})\, d\underline{a}$$

$$\downarrow$$

$$\frac{du}{dt} = -\oint_S \frac{1}{\mu_0}(\underline{E} \times \underline{B}) \cdot d\underline{a} = -\oint_S \underline{N} \cdot d\underline{a}$$

where $\underline{N} = \frac{1}{\mu_0}\underline{E} \times \underline{B}$

Poynting vector $\underline{N}$ is the power per unit area flowing through the surface bounded by volume V (it also gives the direction of flow)

↳ units = Wm^{-2}

For EM waves the intensity is the time-average of $|\underline{N}|$

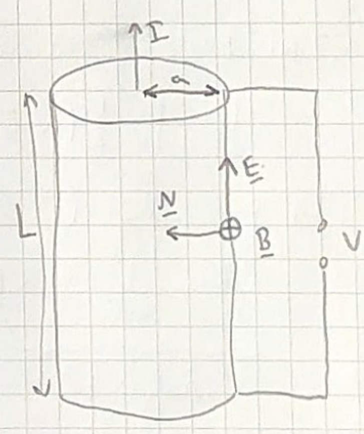
$$I = \langle |\underline{N}| \rangle = \frac{1}{\mu_0} E_0 B_0 \underbrace{\langle \cos^2(\omega t - \underline{k} \cdot \underline{r}) \rangle}_{1/2} = \frac{1}{2\mu_0 c} E_0^2$$

$$I = \frac{1}{2c\mu_0} E_0^2$$

Example: Poynting Vector for a long resistive cylinder

- Calculate Poynting Vector at the surface of the wire with p.d V and current I

$$\underline{N} = \frac{1}{\mu_0}\underline{E} \times \underline{B}$$



- Electric field along the wire axis: $\underline{E} = V/L$

Magnetic flux density at wire surface:

$$\oint \underline{B} \cdot d\underline{L} = B \cdot 2\pi a = \mu_0 I$$

(Note this is tangential)

- $$N = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a}$$

(in radial direction pointing inwards)

↳ wire heats up!

- Hence $N = \frac{VI}{2\pi L a}$

- Total power dissipated in wire: $P = \int_S \underline{N} \cdot d\underline{a} = VI$ as expected from circuit theory

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