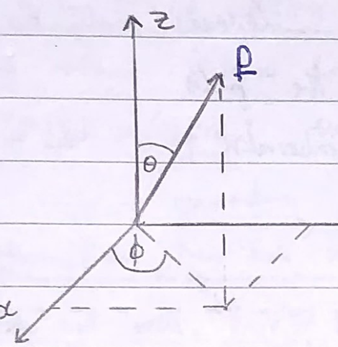
Classical Mechanics Notes First Year (MT)

Introduction

Classical mechanics applies in scales that are not too fast ($<0.1c$), not too small (Quantum mechanics), and not too large (curvature of space is ignored).

vector components



x component $= |F|\sin\theta\cos\phi$

y component $= |F|\sin\theta\sin\phi$

z component $= |F|\cos\theta$

scalar product: $\underline{a}\cdot\underline{b} = |a||b|\cos\theta$

↳ commutative, distributive over addition, not associative. ↳ if ⊥ then $=0$

$|\underline{a}+\underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2\underline{a}\cdot\underline{b}$

$= |\underline{a}|^2 + |\underline{b}|^2 + 2|a||b|\cos\theta$

vector product: $\underline{a}\times\underline{b} = |a||b|\sin\theta\,\hat{\underline{n}}$ where $\hat{\underline{n}} \perp$ to $\underline{a}$ and $\underline{b}$ by RHR

↳ not associative, "anti" commutative, distributive over addition

↳ if // then vector product $=0$

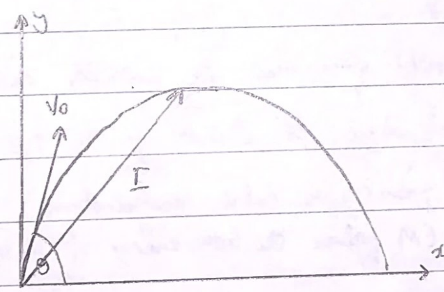
$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \varepsilon_{ijk}\, a_j b_k$ (i^{th} component)

$\underline{a}\cdot(\underline{b}\times\underline{c}) = \varepsilon_{ijk}\, a_i b_j c_k$

Example: Scalar product → work done by force from position 1 to 2 $= W_{12} = \int_1^2 \underline{F}\cdot\underline{dr}$

vector product → torque about point is perp. to both $\underline{r}$ and $\underline{F}$ $=$ $\underline{T} = \underline{r}\times\underline{F}$

Differentiation: $\dot{y} = \frac{dy}{dt}$, $y' = \frac{dy}{dx}$, $\dot{\underline{a}} = \dot{a}_x\underline{i} + \dot{a}_y\underline{j} + \dot{a}_z\underline{k}$



$\underline{v} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V_0\cos\theta \\ V_0\sin\theta - gt \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} V_0\cos\theta\, t \\ V_0\sin\theta\, t - \frac{1}{2}gt^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

eliminate t

$x = V_0\cos\theta\, t \Rightarrow t = \frac{x}{V_0\cos\theta}$

$y = V_0\sin\theta\, t - \frac{1}{2}gt^2 = \frac{V_0\sin\theta\, x}{V_0\cos\theta} - \frac{1}{2}g\left(\frac{x}{V_0\cos\theta}\right)^2$

$y = \tan\theta\, x - \frac{g}{2V_0^2}\sec^2\theta\, x^2$

Kepler's Third Law: $P^2 = \frac{4\pi^2}{GM_0} r^3$

(1st = ellipses, 2nd = equal areas in equal time)

NI: Every body continues in a state of rest or in uniform motion unless acted upon by an external force

NII: The rate of change of momentum is equal to the applied force; where the momentum is defined as product of mass and velocity

NIII: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body

cons. momentum: In an isolated system, the total momentum is conserved

Frames of Reference

Inertial reference frame: Newton's first Law is satisfied

Kinetic and potential Energy

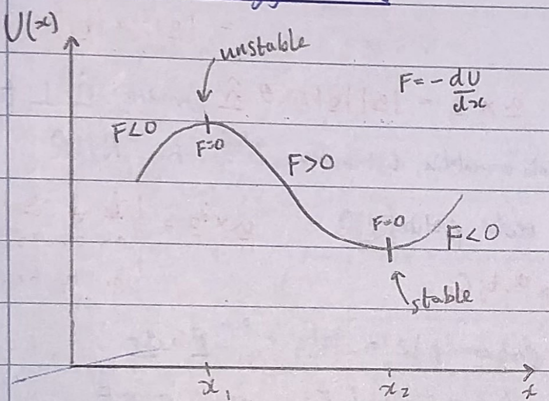
Conservative forces: $W_{ab} = \int_a^b \underline{F} \cdot d\underline{r} = U(a) - U(b) = -U_{ab} = T_b - T_a$

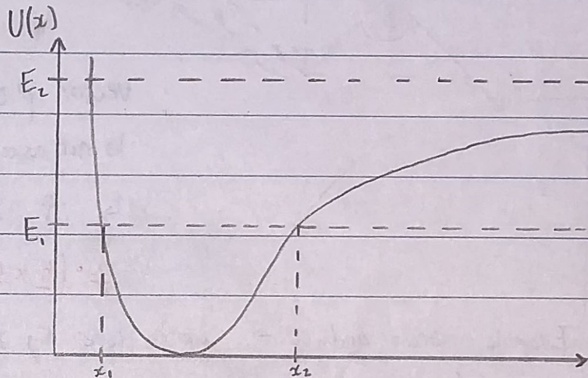
$U(x) = -\int_{x_0}^{x} \underline{F} \cdot d\underline{r}$ $\qquad T_2 + \left(-\int_{x_0}^{x_2} F\,dx\right) = T_1 + \left(-\int_{x_0}^{x_1} F\,dx\right)$

For a conservative field of force, the work done depends only on the initial and final positions of the particle <u>independent of the path</u>

For any force: $W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$, Only for conservative force: $W_{ab} = U(a) - U(b)$

Potential energy curve





bounded motion: $E_{Total} = E_1 \rightarrow x$ contrained $x_1 \le x \le x_2$

unbounded motion: $E_{Total} = E_2 \rightarrow x$ uncontrained at high x

Centre of mass

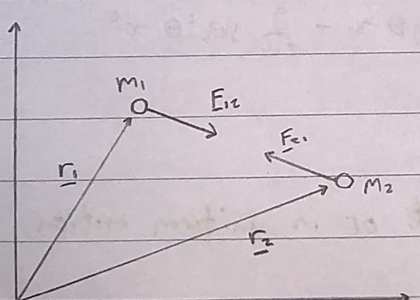
The Labratory Frame: this is the frame where measurements are actually made

The Centre of mass Frame: this is the frame where the C.O.M of the system is at rest

↳ also called the zero momentum frame as total momentum is zero

↳ in CM frame the total energy of the system is a min. compared to all other inertial frames

Internal forces and reduced mass



$\underline{F}_{12} = m_1\ddot{\underline{r}}_1$, $\underline{F}_{21} = m_2\ddot{\underline{r}}_2$

$\ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = \frac{\underline{F}_{21}}{m_2} - \frac{\underline{F}_{12}}{m_1}$

$\underline{F}_{21} = -\underline{F}_{12} = \underline{F}_{int}$

$\underline{r} = \underline{r}_2 - \underline{r}_1$

$\ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1$

$\rightarrow \ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1$

$\underline{F}_{int} \times \left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \ddot{\underline{r}}$

Let $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$\underline{F}_{int} = \mu\ddot{\underline{r}}$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

centre of mass: the point where the mass weighted position vectors relative to that point sum to zero. The mean location of a distribution of mass in space

Lab → CM frame: $\underline{V_i'} = \underline{V_i} - \underline{V_{cm}}$

$\underline{r_{cm}} = \frac{1}{M}\sum_i m_i \underline{r_i}$

$\underline{V_{cm}} = \frac{1}{M}\sum_i m_i \underline{V_i}$

$[M = \sum_i m_i]$

CM of a continuous Volume

$0 = \sum_i m_i(\underline{r_i} - \underline{r_{cm}})$

$0 = \int \rho(\underline{r})(\underline{r} - \underline{r_{cm}})\,dV$

$\underline{r_{cm}} = \frac{1}{M}\int \rho(\underline{r})\,\underline{r}\,dV$

Kinetic Energy and the CM

Lab total KE = $T_{lab} = \frac{1}{2}\sum_i m_i v_i^2$

CM total KE $\Rightarrow$ $T_{lab} = \frac{1}{2}\sum_i m_i (v_i' + V_{cm})^2 = \frac{1}{2}\sum_i m_i v_i'^2 + \underbrace{\sum_i m_i v_i' \cdot V_{cm}}_{=0} + \frac{1}{2}\sum m_i V_{cm}^2$

$T_{lab} = \frac{1}{2}\sum_i m_i v_i'^2 + \frac{1}{2}\sum_i m_i V_{cm}^2$

$T_{lab} = T_{cm} + \frac{1}{2} M V_{cm}^2$

Two body Elastic collisions

During collision: internal force causes change of momentum $F = \frac{dp}{dt}$

Perfectly elastic: relative speed before collision = relative speed after collision

Use cons. of momentum and energy

Transformations From Lab to CM Frame

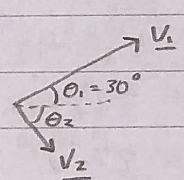
1. Draw out well. 2. Find V_{cm} relative to lab 3. Transform initial velocities to CM frame 4. use the cons. momentum and energy 5. Convert back to lab frame

If elastic: $|u_1'| = |v_1'|$ & $|u_2'| = |v_2'|$

Elastic Collision Example

Lab before: $u_1 = u_0$, M; $u_2 = 0$, $2m$

Lab After:



Find θ_2

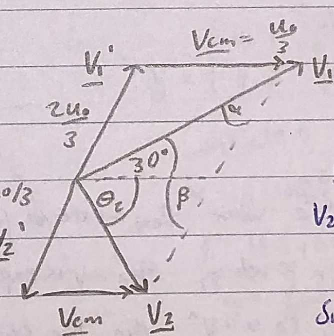
$V_{cm} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{3}$

$u_1' = u_0 - V_{cm} = \frac{2u_0}{3}$

$u_2' = -V_{cm} = -\frac{u_0}{3}$

$|v_1'| = |u_1'| = \frac{2u_0}{3}$

$|v_2'| = |u_2'| = \frac{u_0}{3}$



$\frac{\sin 30°}{\frac{2}{3}u_0} = \frac{\sin\alpha}{\frac{u_0}{3}} \Rightarrow \sin\alpha = 1/4$

$\alpha = 14.5°$

$\beta = 180° - (180° - 30° - \alpha) = 30° + \alpha = 44.5°$

$v_2^2 = (\frac{u_0}{3})^2 + (\frac{u_0}{3})^2 - 2(\frac{u_0}{3})(\frac{u_0}{3})\cos(44.5)°$

$v_2 = \frac{1}{4}u_0$

$\frac{\sin 44.5°}{v_2} = \frac{\sin\theta_2}{\frac{u_0}{3}}$, $\theta_2 = 68.0°$

Inelastic Collisions

For a perfectly inelastic collision, objects are at rest in CM frame after the collision

$\Delta T_{cm} = \Delta T_{lab}$ $\hookrightarrow$ max energy that can be lost = $T_{cm} = T_{lab} - \frac{1}{2}MV_{cm}^2$

coefficient of restitution: $e = \left|\frac{v_2 - v_1}{u_1 - u_2}\right|$ $\quad 0 \le e \le 1$ $\quad$ e is same in Cm and Lab

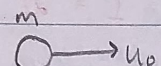
$T_{cm} = \frac{1}{2}\frac{m_1 m_2}{M}\dot{x}^2 = \frac{1}{2}\mu\dot{x}^2 = \frac{1}{2}\mu(u_1' - u_2')^2 = \frac{1}{2}\mu(u_1 - u_2)^2$ where x = distance between the 2 bodies

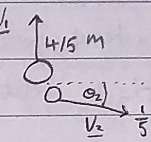
$e = \sqrt{1 - \frac{\Delta E}{T_{cm\,initial}}}$

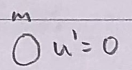
-ly

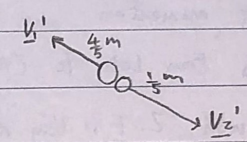
Inelastic Example

A calcium nucleus ($A=20$), mass m, travels with velocity u_0 ($u_0 << c$) in the Lab. It decays into a sulphur nucleus ($A=16$), mass $\frac{4}{5}m$, and an α-particle ($A=4$) mass $\frac{1}{5}m$. Energy ΔT is released as KE in the calcium rest frame (CM). A counter in the lab detects the sulphur at $90°$ to the line of travel. What is the speed and angle of the α-particle in the lab?

Lab before: 

Lab after: 

CM before: 

CM after: 

$V_{cm} = u_0$

momentum → $0 = -\frac{4}{5}mV_1' + \frac{1}{5}mV_2'$

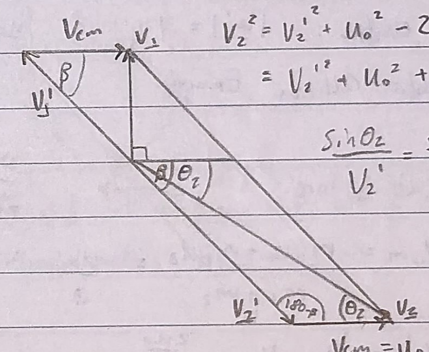
↳ $V_2' = 4V_1'$

energy → $\Delta T = \frac{1}{2}\left(\frac{4}{5}m\right)V_1'^2 + \frac{1}{2}\left(\frac{1}{5}m\right)V_2'^2$

$= \frac{1}{2}\left(\frac{4}{5}m\right)V_1'^2 + \frac{1}{2}\left(\frac{1}{5}m\right)(16V_1'^2)$

$= 2mV_1'^2$

$V_1' = \sqrt{\frac{\Delta T}{2m}}, \quad V_2' = \sqrt{\frac{8\Delta T}{m}}$



$\cos\beta = \frac{u_0}{V_1'} = \sqrt{\frac{2mu_0^2}{\Delta T}}$

$V_2^2 = V_2'^2 + u_0^2 - 2V_2'u_0\cos(180-\beta)$

$= V_2'^2 + u_0^2 + 2V_2'u_0\cos\beta$

$\frac{\sin\theta_2}{V_2'} = \frac{\sin(180-\beta)}{V_2} = \frac{\sin\beta}{V_2}$

... ~~just~~ finish trig

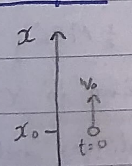
Resisted motion

Newton II: $m\frac{dv}{dt} = F_{ext} + F_R$ where F_{ext} = external force and F_R = resistive force

↳ if $F_{ext} = 0$ and $F_R \propto$ velocity then $v \propto \exp(-\alpha t)$

↳ if $F_{ext} \neq 0$ and eg. $F_R \propto -v^n$ then there exists a limiting speed corresponding to $\frac{dv}{dt} = 0$ that satisfies $F_R = -F_{ext}$

Example 1: resistive force $F_R \propto v$ → body fired vertically upwards under gravity



$m\frac{dv}{dt} = -mg - \beta v$

$\int_{V_0}^{V} \frac{1}{g + \beta\frac{v'}{m}}\,dv' = -\int_0^t dt'$

let $\alpha = \beta/m$

$\left[\frac{1}{\alpha}\ln(g+\alpha v')\right]_{V_0}^{V} = \left[-t'\right]_0^t$

$\ln\left(\frac{g+\alpha V}{g+\alpha V_0}\right) = -\alpha t$

$\frac{g+\alpha v}{g+\alpha V_0} = e^{-\alpha t}$

$1 + \frac{\alpha v}{g} = \left(1 + \frac{\alpha V_0}{g}\right)e^{-\alpha t}$

$v(t) = \frac{g}{\alpha}\left(\left(1+\frac{\alpha V_0}{g}\right)e^{-\alpha t} - 1\right)$

→ max height: $v=0$, $t=t_{max}$

$e^{-\alpha t_{max}} = \frac{1}{1+\frac{\alpha V_0}{g}} \Rightarrow t_{max} = \frac{1}{\alpha}\ln\left(1+\frac{\alpha V_0}{g}\right)$

distance travelled

$x = \int_0^t \frac{g}{\alpha}\left(\left(1+\frac{\alpha V_0}{g}\right)e^{-\alpha t'} - 1\right)dt'$

work done

$W = \int F\,dx$ ...

Classical mechanics Notes (HT)

Structure: - Rocket Motion & non-inertial reference frames
- Central forces (orbits)
- Rotational Dynamics
- Lagrangian Dynamics

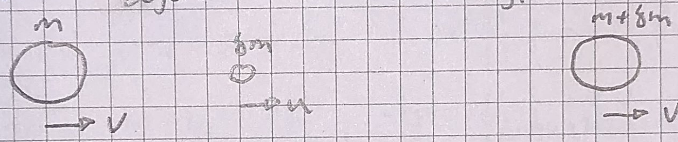
Rocket Motion

Variable mass: a body acquiring mass

A body of mass m has velocity v. In time δt it aquires mass δm, which is moving along v direction with velocity u

~~then change~~

Before After



case 1: no external force

change of momentum = $\delta p = (m+\delta m)(v+\delta v) - (mv + \delta m\, u)$

no external force → $0 = \cancel{mv} + m\delta v + \delta m\, v + \cancel{\delta m \delta v} - \cancel{mv} - \delta m\, u$

vanish in limit

$$\frac{\delta p}{\delta t} = 0 = m\frac{\delta v}{\delta t} + (v-u)\frac{\delta m}{\delta t}$$

write $w = v - u$ [relative velocity] and take $\delta t \to 0$

$$m\frac{dv}{dt} + w\frac{dm}{dt} = 0$$

$$\frac{dv}{dt} = -\frac{w}{m}\frac{dm}{dt}$$

in case $w > 0$, $\frac{dv}{dt}$ is -ve as expected

case 2: application of an external force F

NII: change of momentum = $\delta p = F\delta t = m\delta v + w\delta m$

$$\hookrightarrow \quad m\frac{dv}{dt} + w\frac{dm}{dt} = F$$

Example - the raindrop

An idealised raindrop has an initial mass m_0, is at height h above ground and has zero initial velocity. As it falls it aquires water (added from rest) such that its increase in mass at speed v is given by $dm/dt = bmv$ where b is a constant. The air resistance is $F = Kmv^2$

Find: - Terminal velocity
- mass vs. distance
- speed at ground level

$\boxed{\text{Terminal velocity}}$

$$m\frac{dv}{dt} + w\frac{dm}{dt} = F$$

Now, $F = mg - Kmv^2$

$w = v \quad \text{since} \quad u = 0$ (drops picked up from rest)

$$\frac{dm}{dt} = bmv$$

$$m\frac{dv}{dt} + v\,bmv = mg - Kmv^2$$

$$\frac{dv}{dt} + (b+K)v^2 = g$$

at V_T, $\frac{dv}{dt} = 0 \rightarrow$ $V_T = \sqrt{\frac{g}{b+K}}$

Now, $\frac{dm}{dt} = bmv \rightarrow \frac{dm}{dx} = \frac{dm}{dt} \times \frac{dt}{dx}$

$\boxed{\text{we want } m(x)}$

$$\frac{dm}{dx} = bmv \times \frac{1}{v}$$

$$\frac{dm}{dx} = bm$$

$$\int_{m_0}^{m} \frac{dm}{m} = b\int_0^x dx$$

$$\ln\left(\frac{m}{m_0}\right) = bx$$

$$m = m_0 e^{bx} \quad \text{(exp growth)}$$

$\boxed{\text{speed at ground}}$

we have $\frac{dv}{dt} + (b+K)v^2 = g$

$$\frac{dx}{dt}\frac{dv}{dx} + (b+K)v^2 = g$$

$$v\frac{dv}{dx} + (b+K)v^2 = g$$

$$\int_0^{V_h} \frac{v\,dv}{g-(b+K)v^2} = \int_0^h dx$$

$$h = \left[-\frac{1}{2(b+K)}\ln\left(g-(b+K)v^2\right)\right]_0^{V_h}$$

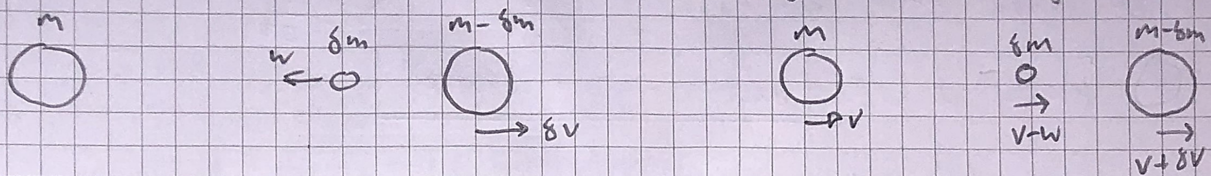
$$\rightarrow \frac{-1}{2(b+K)}\ln\left(\frac{g-(b+K)V_h^2}{g}\right) = h$$

$$V_h = \sqrt{\frac{g}{b+K}\left(1-e^{-2h(b+K)}\right)}$$

Ejecting mass: The rocket equation

A body of mass m has velocity v. In time δt it ejects mass δm, with relative velocity w to the body

Before in CM | After in CM | Boost R.F by v: Before | After



Change of momentum δp

$$\delta p = \delta m(v-w) + (m-\delta m)(v+\delta v) - mv$$

$$= \cancel{v\delta m} - w\delta m + \cancel{mv} + m\delta v - \cancel{v\delta m} - \cancel{\delta m \delta v} - \cancel{mv}$$

$$= m\delta v - w\delta m$$

with external force $F = \frac{\delta p}{\delta t} = m\frac{\delta v}{\delta t} - w\frac{\delta m}{\delta t}$

In limit $\delta t \to 0$, $\frac{\delta m}{\delta t} \to -\frac{dm}{dt}$ because rocket is losing mass

$\hookrightarrow$ $F = m\frac{dv}{dt} + w\frac{dm}{dt}$ again

The Vertical Rocket Launch

$F = -mg$, $\frac{dm}{dt} = -\alpha \Rightarrow m = m_0 - \alpha t$

Now:

$$F = m\frac{dv}{dt} + w\frac{dm}{dt}$$

$$-mg = m\frac{dv}{dt} - \alpha w$$

$$\int_{t_i}^{t_f} (-mg + \alpha w)\,dt = \int_{v_i}^{v_f} m\,dv$$

$$\int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} \left(-g + \frac{w\alpha}{m_0 - \alpha t}\right) dt$$

$$v_f - v_i = -g(t_f - t_i) - w\ln\left[\frac{m_0 - \alpha t_f}{m_0 - \alpha t_i}\right]$$

$$\boxed{v_f - v_i = -g(t_f - t_i) + w\ln\left[\frac{m_i}{m_f}\right]}$$

cont. The rocket starts from rest at $t=0$, half the mass is fuel. what is the velocity and height reached by the rocket at burn-out time $t=T$?

$$\rightarrow v = -gt - w\ln\frac{m_0-\alpha t}{m_0} = -gt - w\ln\left(1-\frac{\alpha}{m_0}t\right) = \frac{dx}{dt}$$

At burnout $t=T$, $m = m_0 - \alpha T = \frac{1}{2}m_0 \Rightarrow \alpha = \frac{m_0}{2T}$

Velocity at $t=T$, $\boxed{V_{max} = -gT + w\ln(2)}$

Height at burnout,

$$\frac{dx}{dt} = V = -gt - w\ln\left(1-\frac{\alpha t}{m_0}\right)$$

$$\int_0^{x_{max}} dx = \int_0^T \left(-gt - w\ln\left(1-\frac{\alpha t}{m_0}\right)\right)dt$$

Standard integral: $\int \ln z\, dz = z\ln z - z \rightarrow z = 1 - \frac{\alpha t}{m_0}$; $dz = -\frac{\alpha}{m_0}dt$

$$\int_0^{x_{max}} dx = \int_0^T (-gt)\,dt - w\int_1^{1/2}\left(-\frac{m_0}{\alpha}\right)\ln z\, dz$$

$$x_{max} = \frac{-gT^2}{2} + w(2T)\left[z\ln z - z\right]_1^{1/2}$$

$$\boxed{x_{max} = \frac{-gT^2}{2} + wT(1-\ln(2))}$$

Non-inertial Reference Frames

A **non-inertial** frame: Newton's first Law not satisfied - the frame is accelerating

Galilean transformation

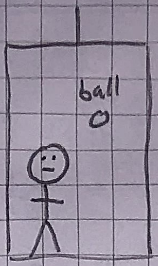
Recall Galilean transformation but now $\underline{u}$ varies with time

	Inertial Frame transformation	Non-inertial Frame transformation
Position	$\underline{r}' = \underline{r} - \underline{u}t$	$\underline{r}' = \underline{r} - \int \underline{u}(t)\,dt$
velocity	$\underline{v}' = \underline{v} - \underline{u}$	$\underline{v}' = \underline{v} - \underline{u}(t)$
Acceleration	$\frac{d\underline{v}'}{dt} = \frac{d\underline{v}}{dt}$	$\frac{d\underline{v}'}{dt} = \frac{d\underline{v}}{dt} - \frac{d\underline{u}}{dt}$
Force driving acceleration	$\underline{F}'(\underline{r}') = \frac{m\,d\underline{v}}{dt} = \underline{F}(\underline{r})$	$\underline{F}'(\underline{r}') = \underline{F}(\underline{r}) - m\frac{d\underline{u}}{dt}$

So even if $\underline{F}(\underline{r}) = 0$ there is an apparent/fictitious/inertial Force acting

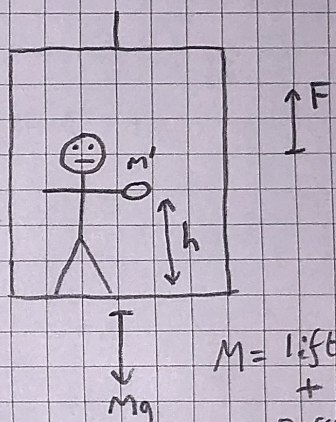
Example: accelerating lift

First consider the lift in free-fall



- The ball is "weightless" (stationary or moving with a constant velocity) according to an observer in the lift
  - → lift is now perfectly inertial frame in all directions (usually on earth they only exist for horizontal elements)
- To this observer the fictitious acceleration $\left(-\frac{du}{dt}\right)$ balances the gravitational acceleration

Now consider the lift accelerated upwards with force F. The passenger drops the ball mass m' from h



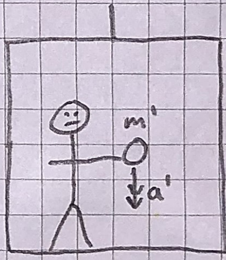
M = lift + passenger

Total force on lift $F_{tot} = F - Mg = Ma$

Acceleration of lift in ext. frame $a = \frac{F}{M} - g$

Switch to lift frame, what is the apparent acceleration of the ball?

↳ $= \underbrace{\left(\frac{F}{M} - g\right)}_{\text{acceleration of floor up}} + \underbrace{g}_{\text{acceleration of ball down}} = \frac{F}{M}$ downwards $= a'$



LIFT FRAME

Apparent weight of ball $= \frac{F}{M} \times m'$

<u>In lift frame how long to hit floor?</u>

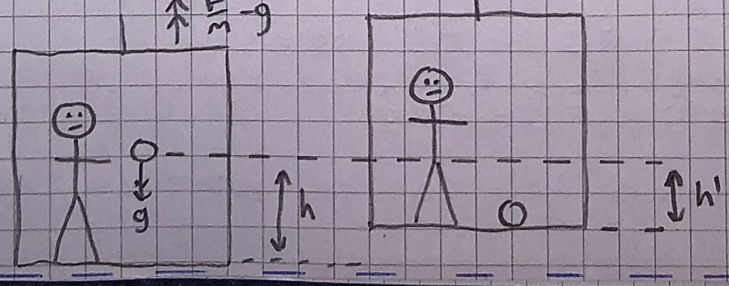
↳ ⇒ $h = \frac{1}{2}at^2$ → $t = \sqrt{\frac{2hM}{F}}$

the same, good!

<u>In inertial frame, how long to hit floor?</u>

↳ $t = \underbrace{\sqrt{\frac{2h'}{g}}}_{\text{ball falling}} = \underbrace{\sqrt{\frac{2(h-h')}{\frac{F}{M} - g}}}_{\text{Lift rising}}$ ⇒ $h' = \frac{Mgh}{F}$

$t = \sqrt{\frac{2hM}{F}}$



Central ~~Forces~~ Forces (orbits)

Introduction to Torque and Angular Momentum

vectors follow rules of differentiation:

- $\frac{d}{dt}\underline{a} = \frac{da_x}{dt}\underline{i} + \frac{da_y}{dt}\underline{j} + \frac{da_z}{dt}\underline{j} = \dot{a}_x\underline{i} + \dot{a}_y\underline{j} + \dot{a}_z\underline{k}$
- $\frac{d}{dt}(\underline{a}+\underline{b}) = \frac{d\underline{a}}{dt} + \frac{d\underline{b}}{dt} = \dot{\underline{a}} + \dot{\underline{b}}$
- $\frac{d}{dt}(c\underline{a}) = \frac{dc}{dt}\underline{a} + c\frac{d\underline{a}}{dt} = \dot{c}\underline{a} + c\dot{\underline{a}}$
- $\frac{d}{dt}(\underline{a}\cdot\underline{b}) = \frac{d\underline{a}}{dt}\cdot\underline{b} + \underline{a}\cdot\frac{d\underline{b}}{dt} = \dot{\underline{a}}\cdot\underline{b} + \underline{a}\cdot\dot{\underline{b}}$
- $\frac{d}{dt}(\underline{a}\times\underline{b}) = \frac{d\underline{a}}{dt}\times\underline{b} + \underline{a}\times\frac{d\underline{b}}{dt} = \dot{\underline{a}}\times\underline{b} + \underline{a}\times\dot{\underline{b}}$

$\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ can and do vary with time, unlike $\underline{i}, \underline{j}, \underline{k}$

$$\frac{d}{dt}(\hat{\underline{r}}\cdot\hat{\underline{r}}) = 2\hat{\underline{r}}\cdot\frac{d\hat{\underline{r}}}{dt}$$

but $\hat{\underline{r}}\cdot\hat{\underline{r}} = 1$ & so $\frac{d}{dt}(\hat{\underline{r}}\cdot\hat{\underline{r}}) = 0 = \hat{\underline{r}}\cdot\frac{d\hat{\underline{r}}}{dt}$

$\therefore \frac{d\hat{\underline{r}}}{dt} \perp$ to $\hat{\underline{r}}$

$\frac{d\hat{\underline{r}}}{dt}$ is in direction of $\hat{\underline{\theta}}$

↓ proof in more rigour

Derivatives of $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$

$\underline{r}$ in cartesians

$\underline{r} = r_0(\underline{i}\cos\theta + \underline{j}\sin\theta)$

unit vector $\hat{\underline{r}} = \underline{i}\cos\theta + \underline{j}\sin\theta$

and

$\hat{\underline{\theta}} = -\underline{i}\sin\theta + \underline{j}\cos\theta$

Now $\frac{d\hat{\underline{r}}}{dt} = -\underline{i}\sin\theta\,\dot{\theta} + \underline{j}\cos\theta\,\dot{\theta} \rightarrow \boxed{\dot{\hat{\underline{r}}} = \dot{\theta}\hat{\underline{\theta}}}$ $\therefore \frac{d\hat{\underline{r}}}{dt} \parallel \hat{\underline{\theta}}$

Similarly $\frac{d\hat{\underline{\theta}}}{dt} = -\underline{i}\cos\theta\,\dot{\theta} - \underline{j}\sin\theta\,\dot{\theta} \rightarrow \boxed{\dot{\hat{\underline{\theta}}} = -\dot{\theta}\hat{\underline{r}}}$ $\therefore \frac{d\hat{\underline{\theta}}}{dt} \parallel \hat{\underline{r}}$

Velocity vector in polar coordinates

$\underline{r} = r\hat{\underline{r}}$

$\underline{v} = \dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\hat{\underline{r}}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$\underline{v} = \dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

For circular motion: $\dot{r} = 0$ and so $v = r\dot{\theta}\hat{\underline{\theta}} \rightarrow \underline{v} = r\omega\hat{\underline{\theta}}$

Acceleration vector in polar coordinates

$$\underline{v} = \dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$

$$\frac{d}{dt}(\dot{r}\hat{\underline{r}}) = \ddot{r}\hat{\underline{r}} + \dot{r}\dot{\theta}\hat{\underline{\theta}} \qquad \frac{d}{dt}(r\dot{\theta}\hat{\underline{\theta}}) = \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\underline{r}})$$

$$= -r\dot{\theta}^2\hat{\underline{r}} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\underline{\theta}}$$

$$\therefore \underline{a} = \ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$$

circular motion

scalars $\dot{r} = \ddot{r} = \ddot{\theta} = 0 \rightarrow \underline{a} = -r\dot{\theta}^2\hat{\underline{r}} = -r\omega^2\hat{\underline{r}} = -\frac{v^2}{r}\hat{\underline{r}}$

Angular momentum and torque

The definition of angular momentum (or the moment of momentum) $\underline{J}$ for a single particle: $\underline{J} = \underline{r} \times \underline{p}$ where $\underline{r}$ is the displacement vector from the origin and $\underline{p}$ is the momentum

The direction of the angular momentum gives the direction $\perp$ to the plane of motion

Differentiate $\underline{J} \rightarrow \frac{d\underline{J}}{dt} = \underline{v} \times \underline{p} + \underline{r} \times \frac{d\underline{p}}{dt}$

$$\frac{d\underline{J}}{dt} = \underbrace{\underline{v} \times (m\underline{v})}_{=0} + \underline{r} \times \underline{F} \rightarrow \frac{d\underline{J}}{dt} = \underbrace{\underline{\tau}}_{\text{Torque}} = \underline{r} \times \underline{F}$$

Torque and angular momentum depend on the origin

Angular velocity $\underline{\omega}$ for rotation in a circle

Definition for angular velocity ω: $\dot{\underline{r}} = \underline{\omega} \times \underline{r}$

$\rightarrow \underline{\omega} = \omega\hat{\underline{n}}$, $\hat{\underline{n}}$ is $\perp$ to $\underline{r}$ and $\dot{\underline{r}}$

Relationship between $\underline{J}$ and $\underline{\omega}$

$$\underline{J} = \underline{r} \times \underline{p}$$

$$\underline{J} = m\underline{r} \times \dot{\underline{r}}$$

$$\underline{J} = m\underline{r} \times (\underline{\omega} \times \underline{r})$$

$$\underline{J} = mr^2\underline{\omega} - \underbrace{m(\underline{r}\cdot\underline{\omega})\underline{r}}_{=0\ (\underline{\omega} \perp \text{ to } \underline{r})} \qquad [\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}]$$

$\therefore \underline{J} = mr^2\underline{\omega} = I\underline{\omega}$ where I is the moment of inertia

generally $I = \sum_i m_i r_i^2$

Angular acceleration $\underline{\alpha}$ for rotation in a circle

$$\underline{\alpha} = \dot{\underline{\omega}}, \qquad \underline{\tau} = \frac{d}{dt}\underline{J} = I\underline{\alpha}$$

Angular motion: work and power

$$W = \int \underline{F} \cdot d\underline{s}$$

$$W = \int \underline{\tau} \cdot d\underline{\theta} = \int \underline{\tau} \cdot \underline{\omega}\, dt = \int P\, dt$$

$$dW = \underline{F} \cdot d\underline{s}$$

$$= \underline{F} \cdot (d\underline{\theta} \times \underline{r})$$

Scalar triple product $\rightarrow$ $dW = (\underline{r} \times \underline{F}) \cdot d\underline{\theta}$

$$dW = \underline{\tau} \cdot d\underline{\theta}$$

Correspondence between linear and angular quantities

linear quantities are re-formulated in a rotated frame:

Linear/translational quantities	Angular/rotational quantities
Displacement, position: $\underline{r}$ [m]	Angular displacement, angle: θ [rad]
Velocity: $\underline{v}$ [ms^{-1}]	Angular velocity: ω [rad s^{-1}]
Acceleration: $\underline{a}$ [ms^{-2}]	Angular acceleration: α [rad s^{-2}]
Mass: m [kg]	Moment of inertia: I [kg m^2 rad^{-1}]
Momentum: $\underline{p}$ [kg ms^{-1}]	Angular momentum: $\underline{J}$ [kg $m^2 s^{-1}$]
Force: $\underline{F}$ [N = kg ms^{-2}]	Torque: $\underline{\tau}$ [kg $m^2 s^{-2}$ rad^{-1}]
Work $dW = \underline{F} \cdot d\underline{x}$ [Nm]	Work $dW = \underline{\tau} \cdot d\underline{\theta}$ [Nm]

Reformulation of Newton's laws for angular motion

1. In the absence of a net applied torque, the angular velocity remains unchanged
2. Torque = Moment of inertia × angular acceleration → $\underline{\tau} = I\underline{\alpha}$
   ↳ applies to rotation around a single principal axis, usually the axis of symmetry
3. For every applied Torque, there is an equal and opposite reaction torque

Example: Derive the EOM of a simple pendulum using angular variables

$\underline{\tau} = \underline{r} \times \underline{F}$
$= -mgr\sin\theta\,\hat{\underline{k}}$

$\underline{J} = \underline{r} \times \underline{p}$
$= \underline{r} \times m\underline{v}$
$= mrv\,\hat{\underline{k}}$

$\underline{\tau} = \frac{d\underline{J}}{dt} = mr\dot{v}\hat{\underline{k}}$
$= mr(r\ddot{\theta})\hat{\underline{k}}$

$-mgr\sin\theta\,\hat{\underline{k}} = mr^2\ddot{\theta}\,\hat{\underline{k}}$

$mr\ddot{\theta} = -mg\,r\sin\theta$

$\ddot{\theta} + \frac{g}{r}\sin\theta = 0$

Central Forces

Central force: $\underline{F}$ act towards origin always

A central force is conservative

> A force $\underline{F}$ is conservative if it meets 3 equivalent conditions:
> 1. The curl of $\underline{F}$ is zero: $\nabla \times \underline{F} = 0$
> 2. Work over closed path $W \equiv \oint_C \underline{F}\cdot d\underline{r} = 0$ independent of path
> 3. $\underline{F}$ can be written in terms of scalar potential $\underline{F} = -\nabla U$

↳ Equivalence of 1 & 2 from Stokes' theorem

$\int_S (\nabla \times \underline{F})\cdot d\underline{a} = \int_C \underline{F}\cdot d\underline{r} = 0$

↳ Equivalence of 1 & 3 from vector calculus identity

$\nabla \times (\nabla U) = 0$

Central force: the equation of motion

$\underline{a} = \ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$

$F_r = m(\ddot{r} - r\dot{\theta}^2)$, $\quad F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$ $\quad$ [$F_\theta = 0$ for a central force]

consider $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$

$\therefore F_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\therefore r^2\dot{\theta}$ is a constant of motion

but $mr^2\dot{\theta} = |\underline{J}|$

$\boxed{\therefore \text{ for central forces } \underline{J} \text{ is a constant}}$

> we have $\underline{\tau} = \frac{d\underline{J}}{dt}$
> and $= \underline{r} \times \underline{F}$
> $= \underline{r} \times f(r)\hat{\underline{r}}$
> $= 0$
>
> $\therefore \frac{d\underline{J}}{dt} = 0 \rightarrow \underline{J}$ is a constant of motion

Motion in a plane

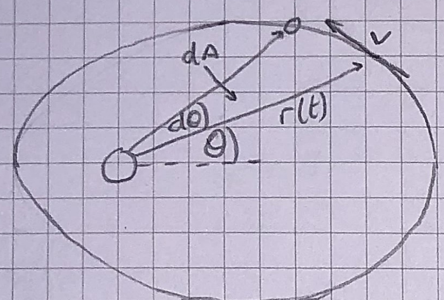
$\underline{J} = m\underline{r} \times \underline{v}$ is constant

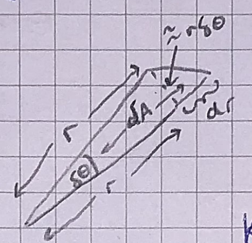
$\underline{J} \cdot \underline{r} = 0$ and $\underline{J} \cdot \underline{v} = 0$

So $\underline{J}$ is $\perp$ to $\underline{r}$ and $\underline{v}$ and remains so $\rightarrow$ motion under a central force lies in a plane

Sweeping out equal area in equal time

Take example: gravity $|F_r| = \frac{GMm}{r^2}$, ang. motion $|\underline{J}| = mr^2\dot{\theta}$ is constant





Kepler's II

$$dA = \frac{1}{2} r \cdot r\delta\theta + \frac{1}{2}\delta r \cdot r\delta\theta$$

limit $\rightarrow \delta r, \delta\theta = 0$

$$\delta A \rightarrow dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$$

Central Force: the total energy

$$E = T + U$$

$$= \frac{1}{2}mv^2 + U(r)$$

$$\underline{v} = \dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$

$$v^2 = \dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{r}^2 + r^2\dot{\theta}^2$$

(note cross-term vanishes as $\hat{\underline{r}} \cdot \hat{\underline{\theta}} = 0$)

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

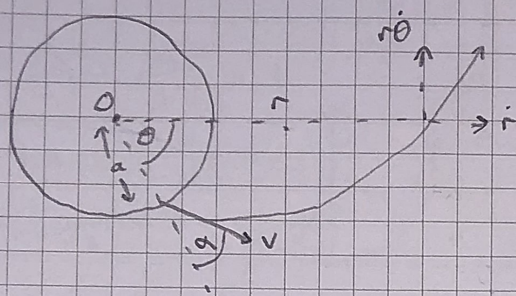
only depends on $r, \dot{r}$, no azimuthal dependence

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

$$U(r) = -\int_{r_{ref}}^{r} \underline{F} \cdot d\underline{r} = -\int_{r_{ref}}^{r} f(r)\,dr$$

Example: A projectile is fired from the Earth's surface with speed v at an angle α to the radius vector at the point of launch. Calculate the projectile's subsequent maximum distance from the earth's surface

$$U(r) = -\frac{GMm}{r}$$



$$|\underline{J}|_{\text{at launch}} = |\underline{r} \times m\underline{v}|$$
$$= mav\sin\alpha \text{ is constant}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{GMm}{r}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{ma^2v^2\sin^2\alpha}{2r^2} - \frac{GMm}{r}$$

↓ at max distance

$$E_{\text{at launch}} = \frac{1}{2}mv^2 - \frac{GMm}{a}$$

$$E_{max} = \frac{ma^2v^2\sin^2\alpha}{2r_{max}^2} - \frac{GMm}{r_{max}} = \frac{1}{2}mv^2 - \frac{GMm}{a}$$

$$\left(v^2 - \frac{2GM}{a}\right)r_{max}^2 = a^2v^2\sin^2\alpha - 2r_{max}GM$$

$$\left(v^2 - \frac{2GM}{a}\right)r_{max}^2 + 2GMr_{max} - a^2v^2\sin^2\alpha = 0$$

$$r_{max} = \frac{-2GM + \sqrt{(2GM)^2 + 4a^2v^2\sin^2\alpha\left(v^2 - \frac{2GM}{a}\right)}}{2\left(v^2 - \frac{2GM}{a}\right)}$$

Consider $r_{max} \to \infty$:

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = \frac{ma^2v^2\sin^2\alpha}{2r^2} - \frac{GMm}{r}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = 0$$

$$v_{esc} = \sqrt{\frac{2GM}{a}}$$

Effective Potential

Energy Equation (for central force)

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

Define an effective potential

$$U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)$$

Now problem becomes 1-D like: → $E = \frac{1}{2}m\dot{r}^2 + U_{eff}$

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

$$\frac{1}{2}m\dot{r}^2 = E - U_{eff}$$

↑ always > 0

Hence $E \geq U_{eff}$

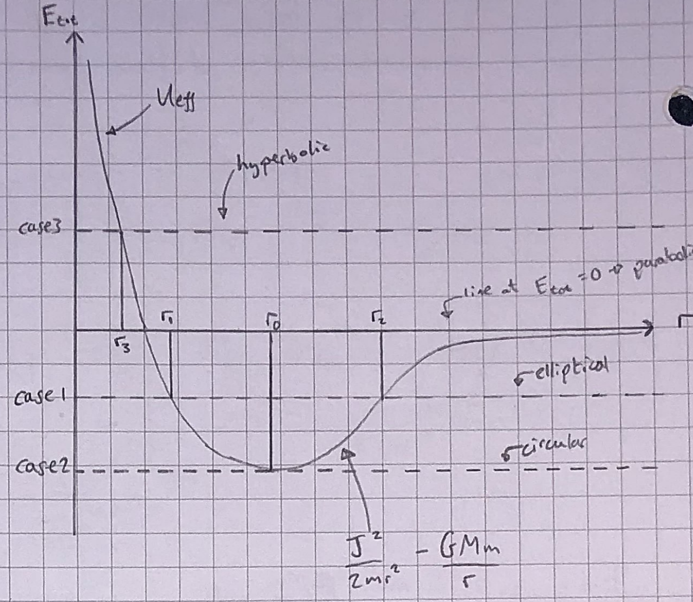
Allows us to predict features of motion without solving the radial eqn → The only locations where the particle is allowed to go are where $U_{eff} \leq E$

$U_{eff}(r)$ for inverse square law

U_{eff} for gravity: $U_{eff} = \frac{J^2}{2mr^2} + U$

$$U_{eff} = \frac{J^2}{2mr^2} - \frac{GMm}{r}$$

We require $U_{eff} \le E_{tot}$ for all r



Case 1: $E_{tot} < 0$

$\to$ bound orbit with $r_1 < r < r_2$ (elliptical)

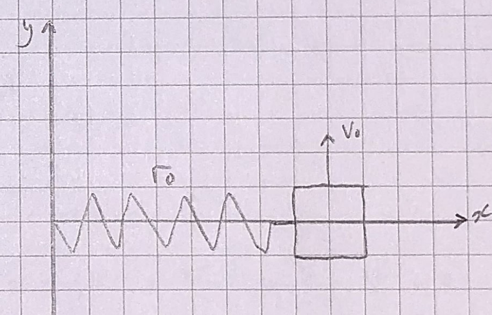
Case 2: E_{tot} at U_{eff} minimum

$\to$ $r = r_0$ (circular motion)

Case 3: $E_{tot} > 0$

$\to$ unbound orbit with $r > r_3$ (hyperbola)

Example: 2-D harmonic oscillator

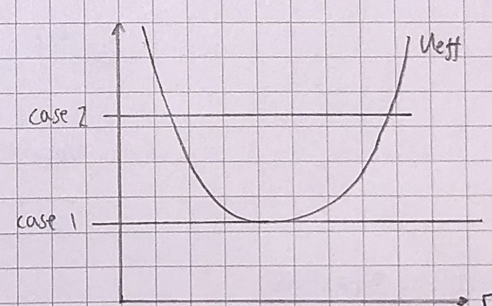


$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U$$

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

$$U_{eff} = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$$

$\underline{F} = -k\underline{r}$ (ignore natural length of spring)



Case 1: $E_{tot} = U_{eff}$ minimum

$\to$ fixed radius with circular motion

Case 2: varying radius

$\to$ bound 'orbit'

Consider circular motion defined at $\frac{dU_{eff}}{dr} = 0$

$$\frac{dU_{eff}}{dr} = -\frac{J^2}{mr^3} + kr = 0 \quad \text{at } r = r_0$$

$$J = mV_0 r_0 \to \frac{mV_0^2}{r_0} = kr_0 \quad (\checkmark \text{ centripetal force eqn})$$

$$r_0 = \sqrt{\frac{mV_0^2}{k}}$$

The orbit equation

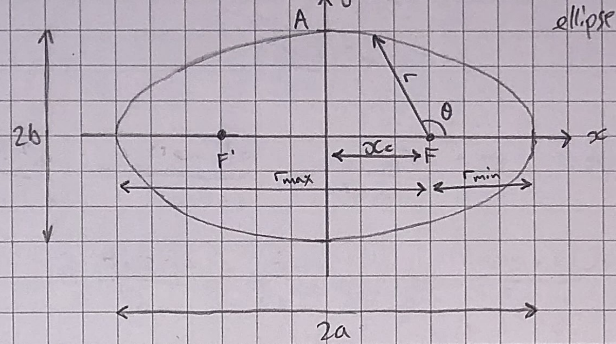
$$r(\theta) = \frac{r_0}{1+e\cos(\theta-\theta_0)} \quad \text{where } r_0 = \frac{J^2}{m\alpha} \quad [\alpha = GMm]$$

from focus

$$\equiv$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

ellipse area $= \pi ab$



closest approach = perigee

furthest approach = apogee

perigee: $\theta = 0$, $r_{min} = \frac{r_0}{1+e}$

apogee: $\theta = 180°$, $r_{max} = \frac{r_0}{1-e}$

$$r_{max} + r_{min} = 2a = \frac{2r_0}{1-e^2} \rightarrow r_0 = a(1-e^2), \quad r_{\substack{max\\min}} = a(1 \pm e)$$

$x_c + r_{min} = a$

$x_c = a - a(1-e)$

$x_c = ae$

$e = \frac{x_c}{a} \rightarrow \left[r_0 = \frac{b^2}{a}\right]$

$r_0 = a(1-e^2) = \frac{b^2}{a}$

$r_{\substack{max\\min}} = a(1 \pm e)$

~~[illegible]~~ $e = \frac{x_c}{a}$

$b = a\sqrt{1-e^2}$

At A: $r^2 = x_c^2 + b^2$, $\cos\theta = -\frac{x_c}{r}$

$$r = \frac{r_0}{1 - e\frac{x_c}{r}}$$

$$r = r_0 + ex_c$$

$$(r_0 + ex_c)^2 = x_c^2 + b^2$$

$$(a(1-e^2) + e^2a)^2 = e^2a^2 + b^2$$

$$a^2 = e^2a^2 + b^2$$

$$b = a\sqrt{1-e^2}$$

$e = 0$	for a circle
$e = 1$	for a parabola
$0 < e < 1$	for an ellipse
$e > 1$	for a hyperbola

Kepler's Law

KI: "The orbit of every planet is an ellipse with the Sun at one of the foci"
[shown in derivation of orbit equation - off syllabus]

KII: "A line joining a planet and the Sun sweeps out equal areas during equal intervals of time" [already shown]

KIII: "The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits"

Kepler's III Proof

Trivial for a circle: $\frac{mv^2}{r} = \frac{GMm}{r^2}$

$$m\omega^2 r_0 = \frac{GMm}{r_0^2}$$

$$m r_0 \left(\frac{2\pi}{T}\right)^2 = \frac{GMm}{r_0^2}$$

$$r_0^3 = T^2\left(\frac{GM}{4\pi^2}\right)$$

For an ellipse: $r = \frac{r_0}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{a^2(1-e^2)}{a(1+e\cos\theta)} = \frac{b^2}{a(1+e\cos\theta)}$

$$\frac{1}{r} = \frac{a(1+e\cos\theta)}{b^2} \xrightarrow{\frac{d}{dt}} \frac{\dot{r}}{r^2} = \frac{ae\sin\theta}{b^2}\dot{\theta}$$

$$\dot{r} = \frac{ae\sin\theta}{mb^2} J \qquad [J = mr^2\dot{\theta}]$$

and $\ddot{r} = \frac{ae\cos\theta J}{mb^2}\dot{\theta} = \frac{ae\cos\theta J^2}{m^2b^2r^2}$

$$\left[\text{also } \frac{a}{b^2} = \frac{1}{r} - \frac{ae\cos\theta}{b^2}\right]$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_r = \frac{ae\cos\theta J^2}{m^2b^2r^2} - \frac{J^2}{m^2r^3}$$

$$a_r = \frac{J^2}{m^2r^2}\left(\frac{ae\cos\theta}{b^2} - \frac{1}{r}\right)$$

$$a_r = -\frac{J^2}{m^2r^2}\left(\frac{a}{b^2}\right) = \frac{-J^2a}{m^2b^2r^2}$$

$$-\frac{mJ^2a}{m^2b^2r^2} = -\frac{GMm}{r^2}$$

$$GM = \frac{J^2a}{m^2b^2}$$

$$T = \frac{A}{\frac{dA}{dt}} \leftarrow = \frac{J}{2m} \quad [\text{lecture } 16]$$

$$T = \frac{\pi ab}{\frac{J}{2m}} \leftarrow \text{ellipse area} = \pi ab$$

$$J = \frac{2\pi abm}{T}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

Elliptical orbit via energy $E_{min} < E < 0$

$$E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$$

at r_{min}/r_{max} $\dot{r} = 0$

$$E = \frac{J^2}{2mr^2} - \frac{\alpha}{r}$$

$$r^2 + \frac{\alpha}{E} r - \frac{J^2}{2mE} = 0$$

$$r_{\substack{max \\ min}} = \frac{-\alpha}{2E}\left[1 \pm \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2}\right]$$

but $r_{\substack{max \\ min}} = a(1 \pm e)$ → same form if we associate $a = \frac{-\alpha}{2E}$ ← remember $E<0$

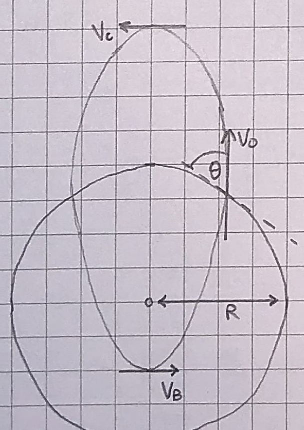
and $e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2}$

Total $E = \frac{m\alpha^2}{2J^2}(e^2 - 1)$

$$E = \frac{\alpha}{2r_0}(e^2 - 1)$$

Example: mistake in the direction of a satellite

↳ A satellite is in a circular orbit, and a mistake is made in trying to increase the speed. The magnitude of the final velocity after the boost is right but the direction is wrong. Therefore the energy is right but the angular momentum is wrong.



What is the perigee and apogee of the resulting orbit

$$|\underline{J}| = |\underline{r} \times \underline{p}| = R m V_0 \cos\theta \qquad m V_B r_B = m V_c r_c$$

$$E = E_{\text{intended for circular orbit}} = \frac{1}{2} m V_0^2 - \frac{\alpha}{R} = \frac{1}{2} m \dot{r} + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$$

centripetal force: $\frac{mV_0^2}{R} = \frac{\alpha}{R^2} \rightarrow V_0^2 = \frac{\alpha}{mR}$

$$\frac{1}{2} m V_0^2 - \frac{\alpha}{R} = \frac{m^2 V_0^2 R^2 \cos^2\theta}{2 m r_B^2} - \frac{\alpha}{r_B}$$

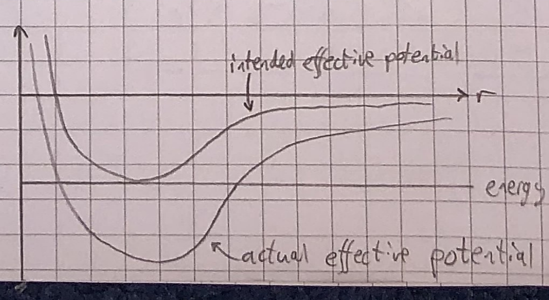
$$\frac{\alpha}{2R} - \frac{\alpha}{R} = \frac{\alpha R \cos^2\theta}{2 r_B^2} - \frac{\alpha}{r_B}$$

$$-\frac{1}{2R} = \frac{R\cos^2\theta}{2 r_B^2} - \frac{1}{r_B}$$

$$r_B^2 - 2R r_B + R^2 \cos^2\theta = 0$$

$$r_B = R - \sqrt{R^2 - R^2\cos^2\theta}, \quad r_c = R + \sqrt{R^2 - R^2\cos^2\theta}$$

$$r_B = R(1 - \sin\theta), \quad r_c = R(1 + \sin\theta)$$



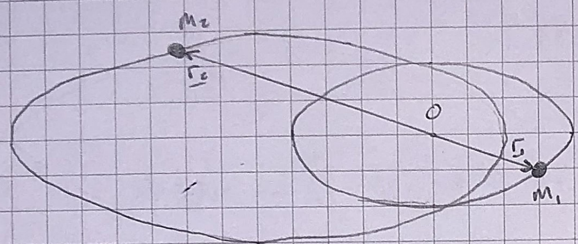
~~[illegible]~~

Mutual Orbits

Two bodies make ~~an~~ a mutual elliptical orbit on either side of the C of M in a straight line through the C of M

Relative position vector: $\underline{r} = \underline{r_2} - \underline{r_1}$

Definition of C of M about O: $m_1 \underline{r_1} + m_2 \underline{r_2} = 0$



$\underline{F_{12}} = m_1 \ddot{\underline{r_1}}$

$\underline{F_{21}} = m_2 \ddot{\underline{r_2}}$

$\underline{F_{12}} = -\underline{F_{21}}$

~~[illegible]~~

$\ddot{\underline{r}} = \ddot{\underline{r_2}} - \ddot{\underline{r_1}}$

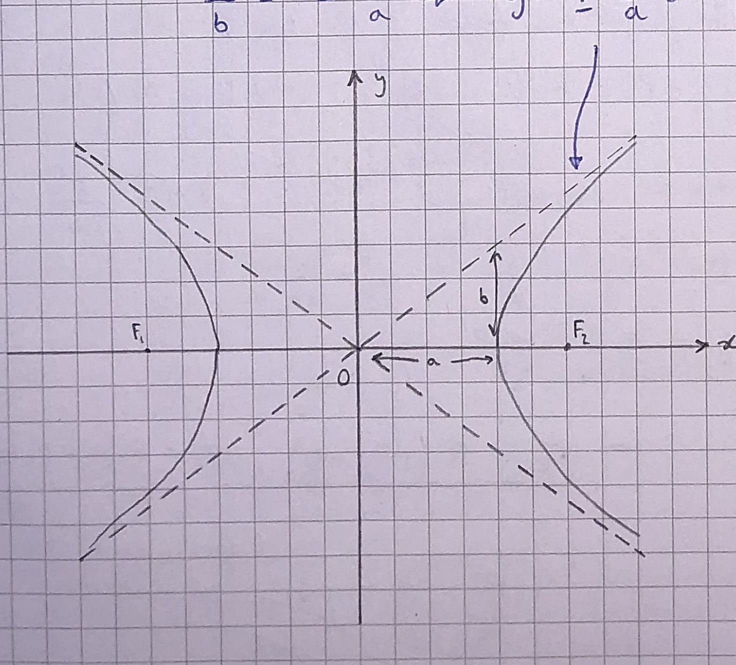
$\ddot{\underline{r}} = \dfrac{\underline{F_{21}}}{m_2} - \dfrac{\underline{F_{12}}}{m_1}$

$\ddot{\underline{r}} = \underline{F_{21}}\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right) \rightarrow \mu \ddot{\underline{r}} = \underline{F_{21}} = -\dfrac{G m_1 m_2}{|\underline{r_2} - \underline{r_1}|^2}\hat{\underline{r}} = \dfrac{-G\mu(m_1 + m_2)}{|\underline{r_2} - \underline{r_1}|^2}\hat{\underline{r}}$

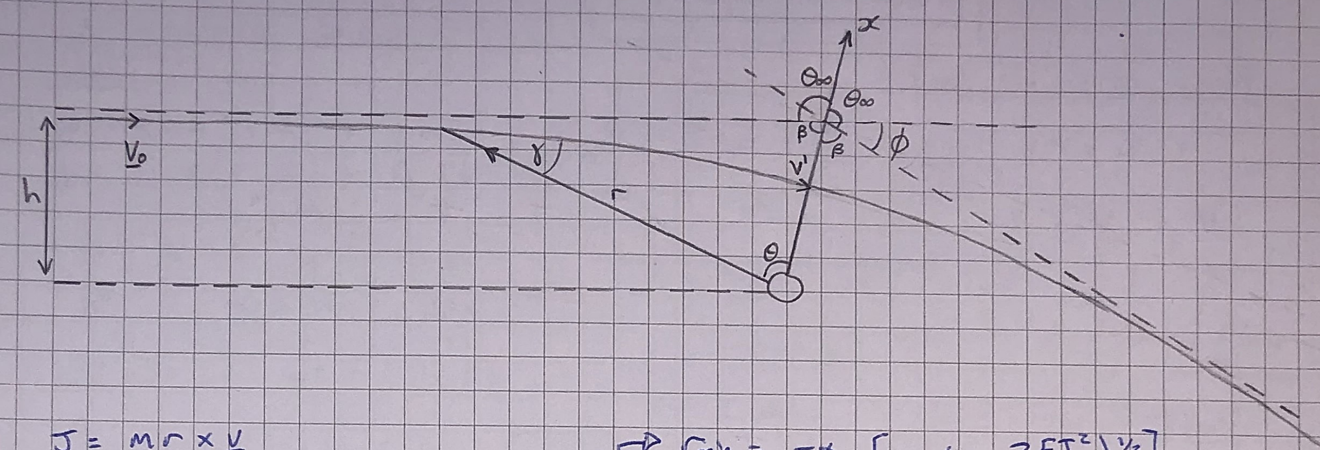
Hyperbolic Orbit

hyperbola: $\left(\dfrac{x}{a}\right)^2 - \left(\dfrac{y}{b}\right)^2 = 1 \Rightarrow$ in limit $x, y \rightarrow \infty$

$\dfrac{y}{b} = \pm \dfrac{x}{a} \rightarrow y = \pm \dfrac{b}{a} x$



Hyperbolic Orbit: distance of closest approach, velocity at closest, angle of deflection ϕ



$\underline{J} = m\underline{r} \times \underline{v}$

$|\underline{J}| = mrv\sin\gamma$

at $r \to \infty$, $r\sin\gamma \to h$, $v \to V_0$

$J = mV_0 h$

$E = \frac{1}{2}mV_0^2$

$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$

at closest approach: $\dot{r} = 0$, $r = r_{min}$

$E = \frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}}$

$r_{min}^2 + \frac{\alpha}{E}r_{min} - \frac{J^2}{2mE} = 0$

$\to r_{min} = \frac{-\alpha}{2E}\left[1 - \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2}\right]$

$\downarrow$

$r_{min} = -\frac{\alpha}{2E}[1 - e]$ same as ellipse

$J = mV'r_{min} = mV_0 h \to V' = \frac{V_0 h}{r_{min}}$

angle of deflection by Impulse

$\Delta P_x = -2p\sin\frac{\phi}{2}$ ①

$\Delta P_x = -2mV_0\sin\frac{\phi}{2}$

$\underline{F} = \frac{d\underline{P}}{dt}$ $\left[J = mr^2\frac{d\theta}{dt} \to \frac{dt}{d\theta} = \frac{mr^2}{J}\right]$

$\frac{dP_x}{dt} = F_x$

$\Delta P_x = \int_{-\infty}^{\infty} F_x\, dt$

$= \int_{-\infty}^{\infty} -\frac{\alpha}{r^2}\cos\theta\, dt$

$= \int_{-\theta_\infty}^{\theta_\infty} -\frac{\alpha}{r^2}\cos\theta\frac{dt}{d\theta}\,d\theta$

$= \int_{-\theta_\infty}^{\theta_\infty} -\frac{\alpha}{r^2}\cos\theta\frac{mr^2}{J}\,d\theta$

$= \int_{-\theta_\infty}^{\theta_\infty} -\frac{m\alpha\cos\theta}{J}\,d\theta$

$= -\frac{2m\alpha}{J}\int_0^{\theta_\infty}\cos\theta\, d\theta$

$= -\frac{2m\alpha}{J}\sin\theta_\infty$

$\theta_\infty = \frac{\pi}{2} + \frac{\phi}{2} \to \Delta P_x = \frac{-2m\alpha}{J}\cos\left(\frac{\phi}{2}\right)$ ②

$$\boxed{① \,\&\, ②: \cot\left(\frac{\phi}{2}\right) = \frac{JV_0}{\alpha} = \frac{mhV_0^2}{\alpha} = \frac{hV_0^2}{GM}}$$

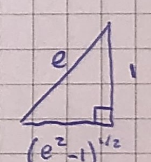
angle of deflection by orbit parameters

$r = \frac{r_0}{1 + e\cos\theta}$, as $r \to \infty$, $1 + e\cos\theta \to 0$

$1 + e\cos\theta_\infty = 0 \to \cos\theta_\infty = -\frac{1}{e}$

$\hookrightarrow \cos\left(\frac{\pi}{2} + \frac{\phi}{2}\right) = -\frac{1}{e}$

$\sin\left(\frac{\phi}{2}\right) = \frac{1}{e} \to$



$\cot\frac{\phi}{2} = (e^2 - 1)^{1/2}$

from r_{min} above: $e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2}$

$\cot\frac{\phi}{2} = \left(\frac{2EJ^2}{m\alpha^2}\right)^{1/2}$

$$\boxed{\cot\frac{\phi}{2} = \frac{mV_0^2 h}{\alpha} \text{ or } \frac{V_0^2 h}{GM}}$$

Rotational Dynamics

Kinetic energy and the CM

$$T = \sum_i^N \frac{1}{2} m_i \underline{v_i}^2$$

$$T = \frac{1}{2} \sum m_i \underline{v_i}^2 = \frac{1}{2}\left(\sum m_i \underline{v_i}'^2 + \sum m_i \underline{v_{cm}}^2 + 2 \underbrace{\sum m_i \underline{v_i}'}_{\text{Sum of P in CM}=0} \underline{v_{cm}} \right)$$

$$\underbrace{\underline{v_i}}_{\text{in lab}} = \underbrace{\underline{v_i}'}_{\text{in CM}} + \underbrace{\underline{v_{cm}}}_{\text{v of CM}}$$

$$T = \underbrace{T'}_{\text{KE in CM}} + \underbrace{\frac{1}{2} M \underline{v_{cm}}^2}_{\text{KE of CM}}$$

Translational motion

$$\sum_i^N m_i \frac{d^2}{dt^2}(\underline{r_i}) = \sum_i^N \underline{F_i}^{ext} + \underbrace{\sum_i^N \underline{F_i}^{int}}_{=0} = \sum_i^N \underline{F_i}^{ext}$$

$$\underline{r_{cm}} = \sum_i^N \frac{m_i \underline{r_i}}{M}$$

$$\underline{v_{cm}} = \dot{\underline{r}}_{cm} = \sum \frac{m_i \dot{\underline{r}}_i}{M}$$

$$\underline{P_{cm}} = \sum m_i \dot{\underline{r}}_i = \sum \frac{M m_i \dot{r}_i}{M} = M \underline{v_{cm}}$$

$$\underline{a_{cm}} = \ddot{\underline{r}}_{cm} = \sum \frac{m_i \ddot{\underline{r}}_i}{M} = \sum \frac{\underline{F_{ext}}^i}{M}$$

$$M \underline{a_{cm}} = \sum \underline{F_{ext}}^i$$

Laws of motion relate as expected

Angular momentum and the CM

$$\dot{\underline{J}} = \sum_i \underline{\tau_i}^{ext}$$ → For any system of particles, the rate of change of internal angular momentum about an origin is equal to the total torque of the external forces about the origin

$\underline{J}$ is in lab frame

lab to CM: $\underbrace{\underline{r_i}}_{\text{lab}} = \underbrace{\underline{r_i}'}_{\text{CM}} + \underline{r_{cm}}$

$$\underline{v_i} = \underline{v_i}' + \underline{v_{cm}}$$

$$\underline{J} = \sum_i^N \underline{J_i}$$

$$= \sum_i m_i (\underline{r_i}' + \underline{r_{cm}}) \times (\underline{v_i}' + \underline{v_{cm}})$$

$\sum m_i \underline{r_i}' = 0$ $\qquad$ $\sum m_i \underline{v_i}' = 0$

$$= \sum_i m_i (\underline{r_i}' \times \underline{v_i}') + \sum m_i (\underline{r_i}' \times \underline{v_{cm}}) + \sum m_i (\underline{r_{cm}} \times \underline{v_i}') + \sum m_i (\underline{r_{cm}} \times \underline{v_{cm}})$$

$$\Rightarrow \underline{J} = \underline{J}' + \underline{r_{cm}} \times M \underline{v_{cm}}$$

$$\underline{J} = \underbrace{\underline{J}'}_{\text{J wrt CM}} + \underbrace{\underline{r_{cm}} \times M \underline{v_{cm}}}_{\text{J of CM translation}}$$

Intro to moment of Inertia

Take 2 particles rotating in circular motion about a common axis of rotation with $\underline{\omega} = \omega \hat{\underline{z}}$

$$\underline{J} = \underline{r_1} \times (m_1 \underline{V_1}) + \underline{r_2} \times (m_2 \underline{V_2})$$

$$\underline{V_1} = \underline{\omega} \times \underline{r_1}$$

$$\underline{V_2} = \underline{\omega} \times \underline{r_2}$$

$$\underline{r} \times (\underline{\omega} \times \underline{r}) = \underline{\omega} r^2 - \underline{r}\underbrace{(\underline{\omega} \cdot \underline{r})}_{=0}$$

$$\underline{J} = \underbrace{(m_1 \underline{r_1}^2 + m_2 \underline{r_2}^2)}_{I}\underline{\omega}$$

I - moment of inertia

$$I_z = \int_V d^2 \rho \, dV = \int_V d^2 dm = \int dI$$

d = distance from z axis

$$I = \sum_i m_i r_i^2 \qquad \underline{J_z} = I_z \underline{\omega}$$

$$T_{rot} = \frac{1}{2} I_z \omega^2$$

$$I_{cylinder/disc} = \frac{1}{2} MR^2$$

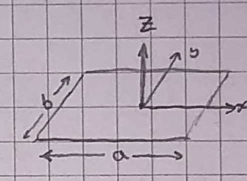
$$I_{hoop} = MR^2$$

$$I_{sphere} = \frac{2}{5} MR^2$$

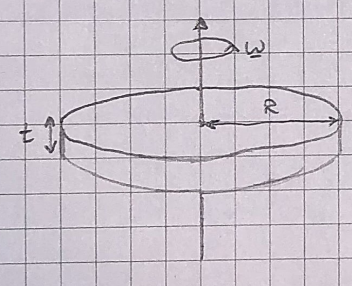
$$I_{rectangle-y} = \frac{Ma^2}{12}$$

$$I_{rectangle-z} = \frac{M(a^2+b^2)}{12}$$

note can be used for rods too



Example: MoI of a solid ~~sphere~~ cylinder



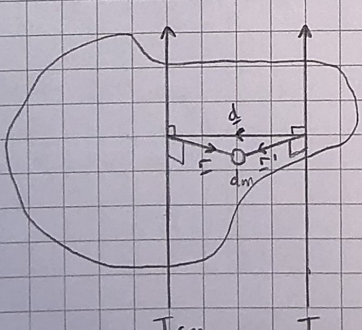
$$dm = \rho \, dV = \rho(2\pi r) dr dt$$

$$\rho = \frac{m}{\pi R^2 t}$$

$$I_z = \int r^2 dm = 2\pi \rho \int_0^R \int_0^t r^3 dt dr$$

$$I_z = \frac{1}{2} MR^2$$

Parallel axis theorem



$$I_{cm} = \int r^2 dm$$

$$\underline{r'} = \underline{d} + \underline{r}$$

$$I = \int r'^2 dm$$

$$r'^2 = d^2 + r^2 + 2\underline{d} \cdot \underline{r}$$

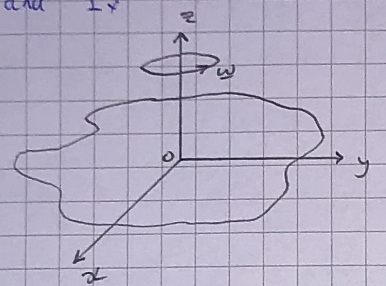
$$I = \int d^2 dM + \int r^2 dm + \int 2\underline{d} \cdot \underline{r} \, dm$$

$$I = \int d^2 dm + \int r^2 dm + 2\underline{d} \cdot \underbrace{\int \underline{r} \, dm}_{=0}$$

$$I = Md^2 + I_{cm}$$

Perpendicular axis theorem

Consider a rigid body that lies entirely within a plane. The perpendicular axis theorem links I_z (MoI about an axis perpendicular to the plane) with I_x and I_y



$$I_z = \int d^2\, dm$$

$$= \int x^2 + y^2\, dm$$

$$\Rightarrow \quad I_z = I_x + I_y$$

Example 1: solid ball rolling down slope – Find the speed

$$E = KE_{rot} + KE_{cm} + PE$$

$$E = \frac{1}{2} I \omega^2 + \frac{1}{2} M v^2 + Mgy$$

$$\omega = V/R$$

$$E_{start} = E_{end}$$

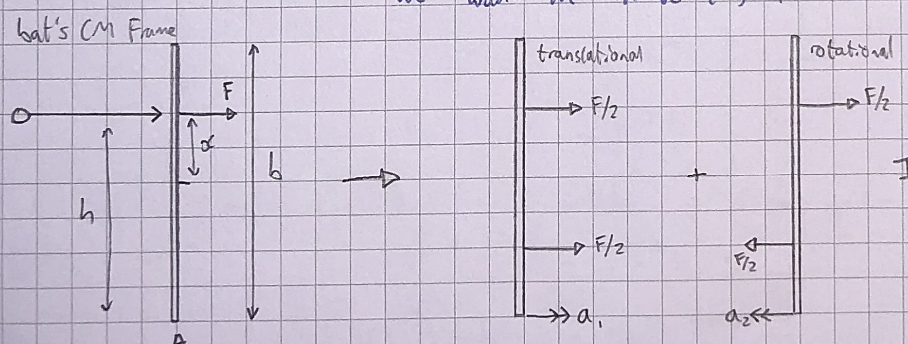
$$Mgh = \frac{1}{2} I \left(\frac{v}{R}\right)^2 + \frac{1}{2} M v^2$$

$$I_{sphere} = \frac{2}{5} MR^2$$

$$Mgh = \frac{1}{2} Mv^2 \left(\frac{2}{5} + 1\right)$$

$$V_{sphere} = \sqrt{\frac{10}{7} g h} \qquad \left(\text{compare to } V_{cylinder} = \sqrt{\frac{4}{3} g h}\right)$$

Example 2: where to hit a ball with a cricket bat.
we want the handle (A) to remain stationary after the hit



$$I_{cm} = \frac{1}{12} m b^2$$

For **A** to be stationary, $\underline{a}_1 = -\underline{a}_2$

$F = m a_1$ ①

$\tau = I(\ddot{\theta})$ ②

$a_2 = \ddot{\theta} \left(\frac{b}{2}\right)$

② $\rightarrow \quad 2 \times \frac{F}{2} \times x = \frac{1}{12} m b^2 \left(\frac{2}{b} a_2\right)$

$\hookrightarrow a_2 = \frac{6 F x}{m b}$

$|a_1| = |a_2|$

$\frac{F}{M} = \frac{6 F x}{M b} \rightarrow x = \frac{b}{6} \rightarrow h = \frac{2b}{3}$

Lagrangian Mechanics

- Lagrangian Mechanics: a very effective way to find the equations of motion for complicated dynamical systems using a scalar treatment.
  - Newton's Laws are vector relations. The Lagrangian is a single scalar function of the system variables
- Avoid the concept of force
  - For complicated situations, it may be hard to identify all the forces, especially if there are constraints
- The Lagrangian treatment provides a framework for relating conservation laws to symmetry
- The ideas may be extended to most areas of fundamental physics

Generalised coordinates: A set of parameters $q_k(t)$ that specifies the system configuration. (a geometrical parameter x, y, z, a set of angles etc.)

Degrees of Freedom: The number of degrees of freedom is the number of independent coordinates that is sufficient to describe the configuration of the system uniquely.

Constraints

A system has constraints if its components are not permitted to move freely in 3-D

The constraints are **Holonomic** if:
- The constraints are time independent
- The system can be described by relations between general coordinate variables and time
- The number of general coordinates is reduced to the number of degrees of freedom

The configuration space of a mechanical system is an n-dimensional space whose points determine the spatial position of the system in time. This space is parametrized by generalised coordinates $\underline{q} = (q_1, \dots q_n)$

The Lagrangian: $L = T - U$

Take 1-D case: $L = \frac{1}{2} m \dot{x}^2 - U(x)$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} = F \quad (N II)$$

Hence $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$: Euler - Lagrange equation

↳ Use of E-L: Find Lagrangian → apply E-L → obtain equation of motion

Generalise: The Lagrangian is a function of 2n variables: $L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n)$

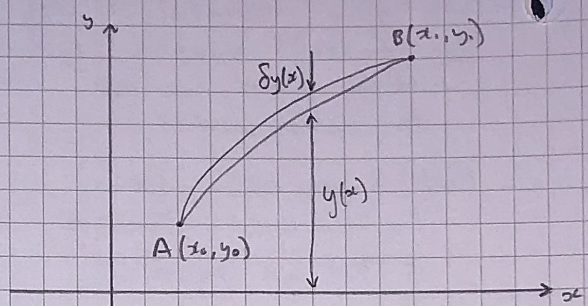
↳ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

Formal derivation of E-L equation : The calculus of variations

Take 2 points $A(x_0, y_0)$ and $B(x_1, y_1)$

The curve joining them is represented by the equation $y = y(x)$ such that y satisfies the boundary conditions

$y(x_0) = y_0$ and $y(x_1) = y_1$



want to find the curve that minimises

$I = \int_{x_0}^{x_1} f(x, y, y')\,dx \quad [\delta I = 0]$

~~We want to find the function $y=y(x)$ subject to the above conditions which minimises the path between the points (we are not minimising a set of variables here but a function)~~

We ~~want to~~ find a function that minimises:

$$\cancel{\int_{x_0, y_0}^{x_1, y_1} dL} = \int_{x_0}^{x_1} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

~~elemental path length dL~~

$$\cancel{dL^2 = dx^2 + dy^2}$$

$$\cancel{\frac{dL}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}}$$

Consider a particle mass m moving under F from A to B

Hamilton's Principle : the path that the particle will take from A to B is the one that makes the following functional stationary

$$I = \int_A^B L(q_1(t), \dots, q_n(t), \dot{q}_1(t), \dots, \dot{q}_n(t))\,dt$$

Lagrange showed that a particle will take a path where the integral of its Lagrangian is the lowest

We assume the unknown function f is a continuously differentiable scalar function, and the functional to be minimised depends on $y(x)$ and at most upon its first derivative $y'(x)$

We then wish to find the stationary values of the paths between points: an integral of the form $I = \int_{x_0}^{x_1} f(x, y, y')\,dx$ (want to minimise I)

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

← under a small change endpoints are unchanged

$\delta y(x_0) = 0$ and $\delta y(x_1) = 0$

$$\therefore \delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right] dx$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \underbrace{\frac{\partial f}{\partial y'}\frac{d}{dx}\delta y}_{\Delta}\right] dx$$

$$\Delta \quad \text{2nd term} = \underbrace{\left[\frac{\partial f}{\partial y'}\delta y\right]_{x_0}^{x_1}}_{=0 \text{ endpoints unchanged}} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\,dx$$

So far we have used x as the independent variable in a function of $(y(x), y'(x))$

Throughout we could have used other variables in particular time t and generalised coordinates $q_1, \dots, q_n$ and derivatives $\dot{q}_1, \dots, \dot{q}_n$

Instead we minimise $I = \int_{t_0}^{t_1} f(q_1(t), \dots, q_n(t), \dot{q}_1(t), \dots, \dot{q}_n(t))\,dt$

$$\to \frac{\partial f}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)$$

Hence $$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx$$

For I to be stationary, $\delta I = 0$ for any $\delta y(x)$

This is only possible if the integrand vanishes as well

$$\therefore \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

Conjugate Momentum and cyclic coordinates

E-L eqn: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$ with $L = T - U$

Define conjugate (generalised) momentum: $p_k = \frac{\partial L}{\partial \dot{q}_k}$

↳ note this is not necessarily linear momentum eg. could be angular

Following on, the E-L equation becomes: $\dot{p}_k = \frac{\partial L}{\partial q_k}$

↳ if L doesn't directly depend on q_k, the coordinate q_k is ~~ignorable~~ cyclic/ignorable

↳ with no q_k dependence: $\frac{\partial L}{\partial q_k} = 0$ and $p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant}$

$\boxed{\text{The momentum conjugate to a cyclic coordinate is a constant of motion}}$

Example: rotating bead

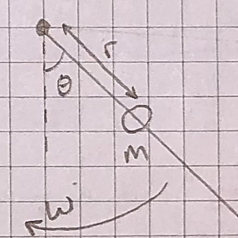
A bead slides on a wire rotating at constant angular speed ω in a horizontal plane

$\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

$L = T - U$ [here $U = 0$]

$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$

$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2$



E-L ⇒ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$

$\frac{d}{dt}\left(m\dot{r}\right) = m\omega^2 r$

$m\ddot{r} = m\omega^2 r$

$\ddot{r} = r\omega^2$

$r = Ae^{\omega t} + Be^{-\omega t}$

Lets say $r(0) = r_0$, $\dot{r}(0) = 0$ ⇨ $r_0 = A + B$, $\dot{r} = 0 = \omega(A - B)$ ⇒ $A = B = \frac{r_0}{2}$ ⇨ $r = \frac{r_0}{2}\left(e^{\omega t} + e^{-\omega t}\right)$

Example: rotating bead continued

What happens if the angular speed is now a free coordinate?

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$q_1 = r, \quad q_2 = \theta$$

E-L $\rightarrow$ $m\ddot{r} = mr\dot{\theta}^2$ & $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$

L indep of θ so θ is cyclic so conjugate momentum conserved

$$mr^2\ddot{\theta} = \frac{d}{dt}(\underbrace{mr^2\dot{\theta}}_{\text{conjugate momentum}}) = 0$$

$$J = mr^2\dot{\theta} \rightarrow \dot{\theta} = \frac{J}{mr^2}$$

$$\ddot{r} = \frac{J^2}{m^2r^3}$$

radial acceleration now drops with r, since J has to be conserved

The Lagrange Multiplier Method: Finding Forces with the Lagrangian

Add a constraint equation g into Lagrangian, multiplied by λ.

$$L(q_1, q_2 \ldots) = T(q_1, q_2 \ldots) - U(q_1, q_2, \ldots)$$

$$\downarrow$$

$$L'(q_1, q_2, \ldots, \lambda) = T(q_1, q_2 \ldots) - U(q_1, q_2, \ldots) + \lambda g(q_1, q_2, \ldots)$$

Treat λ as an additional coordinate, so must also evaluate E-L equation with respect to λ: $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}}\right) = \frac{\partial L'}{\partial \lambda}$

$Q_i = \lambda\left(\frac{\partial g}{\partial q_i}\right)$ is a force (or similar eg. torque) required to impose constraint

Example: Simple Pendulum

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + mgr\cos\theta$$

$$L' = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + mgr\cos\theta + \lambda(r-l)$$

E-L: for $\lambda \rightarrow$ $0 = r - l$

$r = l$
$\dot{r} = 0$ initial constraint
$\ddot{r} = 0$

for $\theta \rightarrow$ $\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = -mgr\sin\theta$$

for r: $\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 + mg\cos\theta + \lambda$

$\ddot{r} = 0 \rightarrow$ $0 = mr\dot{\theta}^2 + mg\cos\theta + \lambda$

$$\lambda = -\left(\frac{mv^2}{r} + mg\cos\theta\right)$$

constraint force $= \lambda\frac{\partial}{\partial r}(r-l)$

$T = \lambda$ (by coincidence in this example)

$$\therefore T = -\left(\frac{mv^2}{r} + mg\cos\theta\right)$$

The Lagrangian in various coordinate systems

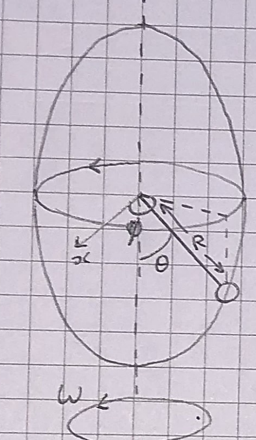
Cartesian: $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$

Cylindrical: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z)$

Spherical: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2) - U(r, \theta, \phi)$

Lagrange Example: bead on rotating hoop

A vertical circular hoop of radius R rotates about a vertical axis at a constant angular velocity ω. A bead of mass m can slide on the hoop without friction. Describe the motion of the bead.



$$T = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2 + (R\sin\theta)^2\dot{\phi}^2)$$

but $\dot{R} = 0$ and $\dot{\phi} = \omega$

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + (R^2\sin^2\theta)\omega^2)$$

$$U = -mgR\cos\theta$$

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + (R\sin\theta)^2\omega^2) + mgR\cos\theta$$

E-L for $\theta \rightarrow \frac{d}{dt}(mR^2\dot{\theta}) = mR^2\cos\theta\sin\theta\,\omega^2 - mgR\sin\theta$

$$\ddot{\theta} = \omega^2\cos\theta\sin\theta - \frac{g}{R}\sin\theta$$

$$\ddot{\theta} + (\omega_0^2 + \omega^2\cos\theta)\sin\theta = 0 \quad \text{where } \omega_0^2 = \frac{g}{R}$$

Now, if $\omega = 0$, $\ddot{\theta} + \omega_0^2\sin\theta = 0 \rightarrow$ SHM PENDULUM

if $\omega \neq 0$, look for equilibrium solutions ($\ddot{\theta} = 0$)

$\rightarrow \sin\theta(\omega_0^2 + \omega^2\cos\theta) = 0$

$\rightarrow \sin\theta = 0 \rightarrow \theta = 0$ = STABLE EQUILIBRIUM AT BOTTOM

$\theta = \pi$ = UNSTABLE EQUILIBRIUM AT TOP

$\rightarrow \cos\theta = \frac{\omega_0^2}{\omega^2} = \frac{g}{\omega^2 R}$ = STABLE EQUILIBRIUM ABOUT A CIRCLE

Hamiltonian Mechanics

Lagrangian mechanics: Allows us to find the equations of motion for a system in terms of an arbitrary set of generalised coordinates

Now extend the method as developed by Hamilton → use of the conjugate momenta $p_1, p_2, \ldots p_z$ replace the generalised velocities $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$

This has advantages when some of the conjugate momenta are constants of the motion and its well suited to finding conserved quantities

Remember $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$ and E-L eqn becomes: $\dot{p}_k = \dfrac{\partial L}{\partial q_k}$

Now: $L = L(t, q_k(t), \dot{q}_k(t))$

$$\hookrightarrow \frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \left(\frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial t} + \frac{\partial L}{\partial \dot{q}_k} \cdot \frac{\partial \dot{q}_k}{\partial t} \right)$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \left(\frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \right)$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \underbrace{\left(\dot{p}_k \dot{q}_k + p_k \ddot{q}_k \right)}_{\frac{d}{dt}(p_k \dot{q}_k)}$$

$$\frac{d}{dt} \Big[\underbrace{L - \sum_k p_k \dot{q}_k}_{-H} \Big] = \frac{\partial L}{\partial t}$$

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad \text{where } H \text{ is the Hamiltonian} \quad H = \sum_k p_k \dot{q}_k - L}$$

↳ If L does not depend explicitly on time, H is a constant of motion $\left(\frac{dH}{dt} = 0\right)$

Hamiltonian energy

General case in 3-D: $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$

$p_x = m\dot{x}, \ldots$

$H = (m\dot{x} \times \dot{x} + m\dot{y} \times \dot{y} + m\dot{z} \times \dot{z}) - (T - U)$

$H = 2T - T + U$

$H = T + U =$ total energy!

conclusion: in some systems $H = E_{total}$ (caution: often not the case eg. in driven systems)

Now $\dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t}$ so if L doesn't explicitly depend on t, then H is conserved so in (many) systems E is conserved

Furthermore if a coordinate does not appear in the Hamiltonian it is cyclic/ ignorable and the conjugate momentum is a constant of motion same as in Lagrangian.

Hamilton's Equations

$$H = \sum_i p_i \dot{q}_i(q,p) - L(q, \dot{q}(q,p))$$

$$\frac{\partial H}{\partial q_k} = \sum_i p_i \frac{\partial \dot{q}_i}{\partial q_k} - \frac{\partial L}{\partial q_k} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial q_k}$$

$$\frac{\partial H}{\partial q_k} = \cancel{\sum_i p_i \frac{\partial \dot{q}_i}{\partial q_k}} - \frac{\partial L}{\partial q_k} - \cancel{\sum_i p_i \frac{\partial \dot{q}_i}{\partial q_k}}$$

$$\frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} = -\dot{p}_k \rightarrow \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

$$\frac{\partial H}{\partial p_k} = \dot{q}_k + \sum_i p_i \frac{\partial \dot{q}_i}{\partial p_k} - \frac{\partial L}{\partial p_k}$$

$$= \dot{q}_k + \sum_i p_i \frac{\partial \dot{q}_i}{\partial p_k} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial p_k}$$

$$= \dot{q}_k + \cancel{\sum_i p_i \frac{\partial \dot{q}_i}{\partial p_k}} - \cancel{\sum_i p_i \frac{\partial \dot{q}_i}{\partial p_k}} \rightarrow \frac{\partial H}{\partial p_k} = \dot{q}_k$$

Lagrange eqns: set of n second-order differential equations

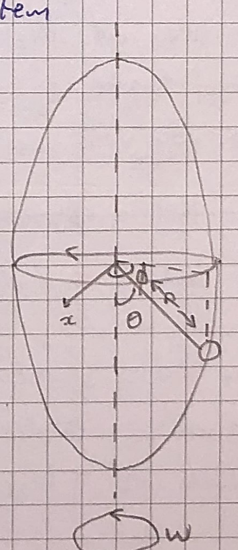
Hamilton's eqns: set of 2n first-order equations

Example: re-visit bead on rotating loop

First lets take the case of a free (undriven) system

$$L = \frac{1}{2} m(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2) + mgR\cos\theta$$

$$p_\theta = mR^2\dot{\theta}, \quad p_\phi = mR^2\sin^2\theta\,\dot{\phi}$$



$$H = mR^2\dot{\theta}\times\dot{\theta} + mR^2\sin^2\theta\dot{\phi}\times\dot{\phi} - L$$

$$H = m(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2) - L$$

$$H = \frac{1}{2} m(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2) - mgR\cos\theta$$

$$H = T + U = E_{total}$$

$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \rightarrow$ L doesn't explicitly depend on t $\rightarrow \frac{dH}{dt} = 0$

$\therefore$ H is a constant and so is E_{total}

Hamiltonian eqns: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow p_\phi = mR^2\sin^2\theta\,\dot{\phi} = J_z$ = constant of motion

Now consider a driven system, with the hoop rotating at a constant angular speed $\omega = \dot{\phi}$

$$L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) + mgR\cos\theta$$

$$H = \sum_k p_k \dot{q}_k - L \quad ; \quad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Now 1 generalised coordinate = θ

$$p_\theta = mR^2\dot{\theta}$$

$$\therefore H = mR^2\dot{\theta}^2 - L$$

$$H = mR^2\dot{\theta}^2 - \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) - mgR\cos\theta$$

$$H = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) - mgR\cos\theta - mR^2\omega^2\sin^2\theta$$

$$H = T + U - mR^2\omega^2\sin^2\theta$$

$$H = E_{tot} - mR^2\omega^2\sin^2\theta$$

Now $H \neq E_{tot}$

$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \rightarrow \frac{dH}{dt} = 0$ still so H is still conserved but E is not!

since external energy is being supplied to keep hoop at ω

Noether's theorem: not sure how examinable this is but woven in physics!

Noether's Theorem: whenever there is a continuous symmetry of the Lagrangian, there is an associated conservation law

- Symmetry means a transformation of the generalised coordinates q_k and $\dot{q}_k$ that leaves the value of the Lagrangian unchanged
- If a Lagrangian does not depend on a coordinate q_k (i.e. is cyclic) it is invariant (symmetric) under changes $q_k \rightarrow q_k + \delta q_k$; the corresponding generalised momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is conserved

~~1. For a Lagrangian that is symmetric under small rotations of~~

1. For a Lagrangian that is symmetric under changes $\underline{r} \rightarrow \underline{r} + \delta\underline{r}$, linear momentum $\underline{p}$ is conserved
2. For a Lagrangian that is symmetric under small rotations of angle $\theta \rightarrow \theta + \delta\theta$ about an axis $\hat{n}$, such a rotation transforms the Cartesian Coordinates by $\underline{r} \rightarrow \underline{r} + \delta\theta\,\hat{n} \times \underline{r}$. Then the component of $\underline{J}$ along $\hat{n}$ is conserved

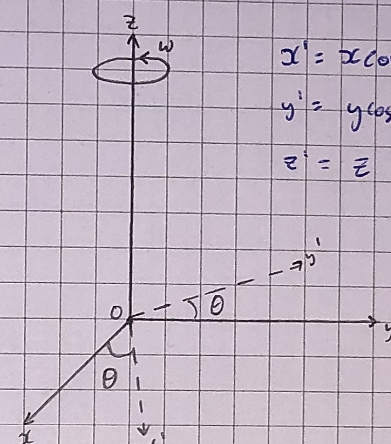
Similarly applies for a Lagrangian that is symmetric under changes $t \rightarrow t + \delta t$. Then $\frac{\partial L}{\partial t} = 0$ and the total energy H is conserved

Final E-L examples: a rotating coordinate system

Lagrangian of a free particle: $L = \frac{1}{2} m \dot{\underline{r}}^2$, $\underline{r} = (x, y, z)$ [with $U = 0$]

Measure the motion w.r.t a coordinate system rotating with angular velocity $\underline{\omega} = (0, 0, \omega)$

$\underline{r}' = (x', y', z')$ are coordinates in the rotating system



$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$
$$z' = z$$

$$\hookrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Take the inverse

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Substitute these expressions into L above

$$\dot{x}' = -(\omega\sin\omega t)x + (\cos\omega t)\dot{x} + (\omega\cos\omega t)y + (\sin\omega t)\dot{y}$$

Also $\omega y' = (-\omega\sin\omega t)x + (\omega\cos\omega t)y$

Hence $(\dot{x}' - \omega y') = \dot{x}\cos\omega t + \dot{y}\sin\omega t$

Similarly $(\dot{y}' - \omega x') = \dot{x}\sin\omega t + \dot{y}\cos\omega t$

Square and add $(\dot{x}' - y'\omega)^2 + (\dot{y}' + \omega x')^2 = \dot{x}^2\cos^2\omega t + \dot{y}^2\sin^2\omega t + 2\dot{x}\dot{y}\cos\omega t\sin\omega t + \dot{x}^2\sin^2\omega t + \dot{y}^2\cos^2\omega t$
$= \dot{x}^2 + \dot{y}^2$

So $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m\left[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2\right]$

$$\underline{\omega} \times \underline{r}' = \begin{vmatrix} \underline{i}' & \underline{j}' & \underline{k}' \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix} = -\omega y'\underline{i}' + \omega x'\underline{j}' \rightarrow L = \frac{1}{2} m(\dot{\underline{r}}' + \underline{\omega} \times \underline{r}')^2$$

So we see in the rotating frame: $\dot{\underline{r}} = \dot{\underline{r}}' + \underline{\omega} \times \underline{r}$

We now evaluate the E-L equations in these rotating coordinates i.e. three equation in $(x', \dot{x}')$, $(y', \dot{y}')$ and $(z', \dot{z}')$ which can be written together as: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{r}}'}\right) = \frac{\partial L}{\partial \underline{r}'}$

where $\frac{\partial}{\partial \underline{r}'} = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$

It can be shown (non-trivial) that:

$$\frac{\partial L}{\partial \underline{r}'} = m\left[\dot{\underline{r}}' \times \underline{\omega} - \underline{\omega} \times (\underline{\omega} \times \underline{r}')\right]$$

and we also have $\frac{\partial L}{\partial \dot{\underline{r}}'} = m(\dot{\underline{r}}' + \underline{\omega} \times \underline{r}')$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{r}}'}\right) - \frac{\partial L}{\partial \underline{r}'} = m\Big[\ddot{\underline{r}}' + \overbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r}')}^{\text{centrifugal force}} + \underbrace{2\underline{\omega} \times \dot{\underline{r}}'}_{\text{Coriolis force}}\Big] = 0$$