SDM5008 Fall 2025 | **Homework # 1**

Assigned September 12 | Due: Saturday, September 20, 2025

- This homework is mainly about the required math and Python coding background of this course. Depending on your previous training, you may need to read related materials online. Some background materials and tutorials are uploaded to Blackboard and course website.
- You can type your solution in Latex (encouraged but not required) or just hand write the solution on paper. Submit your homework (pdf) through Blackboard
- Report should be written in English. For codes, please turn in your Jupyter notebook file, including your codes and testing results.
- To receive credits, please write down all the necessary steps leading to the final answer.

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1. **Python Basics**: Please carefully study the Python tutorial posted on the class website, and complete the following questions:
   (a) Write a program to display the current date and time.
   (b) Write a program to print a specified list after removing the 0th, 4th and 5th elements.
      - **Sample List** : ['Red', 'Green', 'White', 'Black', 'Pink', 'Yellow']
      - **Expected Output** : ['Green', 'White', 'Black']

   (c) Define a class called *Student* that includes the student's name and age information. In addition, you should provide a method to display these information.

2. **Linear Algebra**: In this class, it is important to use Python to complete the linear algebra task. Let's get familiar with it now.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

   (a) Print the two matrices A and B.
   (b) Print the second row of A and the third column of B.
   (c) Print the results of $A + B$ and $A - B$.
   (d) Construct a new 4×6 matrix $[A, B]$ by appending B to the right of matrix A.
   (e) Compute $A^T B$

3. **Matplotlib**
   (a) Plot a unit circle
   (b) Plot 10 plus signs "+" uniformly distributed on the unit circle.

Linear Algebra: For this part, depending on your previous background, you may need to watch the linear algebra tutorials (`https://www.bilibili.com/video/BV1eA411F7RX/`) before working on the following problems.

4. *Column and Null Space*: Define

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & -1 & 1 & 0 \\ 1 & 5 & 4 & 3 \\ 1 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$

   (a) What are the dimensions of the null space and column space (i.e. range space) of A?
   (b) Find a set of basis vectors for null(A).
   (c) Find a set of basis vectors for col(A)
   (d) Is col(C) = col(A)? Justify your answer.
   (e) Find a matrix B of appropriate dimension such that $C = AB$. (You should be able to find B just by inspection).

Hint: Let a_1, a_2, a_3 be the three columns of A and $c_1, \ldots, c_4$ be the four columns of C. By inspection (simple calculation), the following relations hold

$$a_3 = 2a_1 - a_2, \quad c_1 = -a_1 + a_2, \quad c_2 = a_1 + 2a_2$$
$$c_3 = 2a_1 + a_2, \quad c_4 = a_1 + a_2$$

5. *Speak the Matrix Language*: Express the following statements in matrix language. You can assume that all matrices mentioned have appropriate dimensions. Here is an example: "Every column of C is a linear combination of the columns of B" can be expressed as "$C = BF$ for some matrix F". There can be several answers; one is good enough. You are expect to justify all of your answers.

   (a) For each i, row i of Z is a linear combination of rows $i, \ldots, n$ of Y.
   (b) W is obtained from V by permuting adjacent odd and even columns (i.e., 1 and 2, 3 and 4,...).
   (c) Each column of P makes an acute angle with the corresponding column of Q.
   (d) The first k columns of A are orthogonal to the remaining columns of A.

6. *Matrix Rank*:

   (a) Let $a \in \mathbf{R}^n$ be an n-dim vector. Show that the $n \times n$ matrix $A \triangleq aa^T$ is of rank 1.
   (b) Given two nonzero square matrices $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times n}$, argue that if $AB = 0$, then neither A nor B can be full rank.
   (c) Explain why the system $Ax = b$ has a solution if and only if rank(A) = rank $([A \;\; b])$.

7. *Ellipsoids:* Ellipsoid in $\mathbf{R}^n$ have two equivalent representations: (i) $E_1(P, x_c) = \{x \in \mathbf{R}^n : (x - x_c)P^{-1}(x - x_c) \leq 1\}$ and (ii) $E_2(A, x_c) = \{Au + x_c : \|u\|^2 \leq 1\}$. Given an eillipsoid $E_1(P, x_c)$ with P positive definite, its volume is $\nu_n\sqrt{det(P)}$ where ν_n is the volume of unit ball in $\mathbf{R}^n$, its semi-axes directions are given by the eigenvectors of P and the lengths of semi-axes are $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P.

   (a) Given an Ellipsoid $E_1(P, x_c)$, find the corresponding (A, b) (in terms of P and x_c) such that $E_2(A, b)$ represents the same ellipsoid as $E_1(P, x_c)$

   (b) Draw the ellipse $E_1(P, x_c)$ with $P = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and $x_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by hand.

   (c) Draw the ellipse in part (b) using Python

8. *Polyhedron:* Given two polyhedra $\mathcal{P}_1 = \{x \in \mathbf{R}^n : A_1x \leq b_1\}$ and $\mathcal{P}_2 = \{x \in \mathbf{R}^n : A_2x \leq b_2\}$

   - Find the expression of the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$
   - Check whether $\mathcal{P}_1$ intersects with the halfspace $a^Tx \leq 3$ using Python or by hand, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 5 & -2 \\ -1 & -2 \\ -4 & -2 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 7 \\ 36 \\ -14 \\ -26 \end{bmatrix}, \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$