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Traversal time in quantum scattering

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The quantum traversal time t_Ω is obtained as a matrix element of the classical functional in Feynman path-integral technique. The properties of t_Ω and its relation to the observables in scattering experiments are investigated. The general expression for the tunneling time is found in the one-dimensional case. Various traversal times of previous works are shown to be simply related to t_Ω as $\mathrm{Re}t_\Omega$, $\mathrm{Im}t_\Omega$, or $|t_\Omega|$. Other possible constructions of time parameters describing the motion of a quantum particle are discussed.

INTRODUCTION

The question of how much time a quantum particle with a given energy spends in a given region of space has often been discussed in literature.[1-12] There are a number of publications on the determination of collision times and lifetimes of intermediate states in the three-dimensional scattering problems.[2,3,6,8] Another group of papers deals with the time spent by a tunneling electron in the barrier region, so-called tunneling time.[4,5,9-12] This latter parameter has recently attracted much attention, since it is expected to define the character of tunneling through time-dependent barriers as well as the form of image forces and the rate of energy losses in many-particle tunneling.

In Refs. 2, 4, and 5 the incident particle is modeled by a wave packet and the traversal time is taken to be the difference between the times when the maximum of the packet crosses the boundary of the region. The authors of Refs. 8, 9, and 12 assume a small magnetic field in the region of interest and estimate the traversal time comparing the spin orientation of the incident and scattered particles, their method is widely known as the Baz' "clock method." In Ref. 11 the tunneling time is defined as the inverse of some characteristic frequency at which an electron passing through a time-dependent barrier starts "seeing" a static potential.

The often discussed contradictions arising from the results of the above-mentioned works lead one to doubt whether the conception of traversal time makes any sense beyond the framework of classical mechanics. It is clear, however, that a reasonable generalization of the classical traversal time should exist, when not unique, in quantum mechanics. The purpose of this paper is therefore to define such a quantity in the most general way, analyze its physical significance, and, if possible, clarify the controversies of previous works.

In Sec. I we define a quantity t_Ω having the meaning of the mean time spent by a quantum particle in a given region of space Ω as a matrix element of some classical functional in the Feynman path-integral technique.

In the Sec. II we consider a general relation between t_Ω and the spin orientation of the particle moving in a small magnetic field localized in Ω. We show that the spin of the particle is not only turned in the plane normal to the field, but also acquires a nonzero component in its direction. These two angles of rotation are proportional to the real and imaginary part of complex matrix element t_Ω, respectively, while the total angle is proportional to its modulus.

In Secs. III and IV we specify our general approach to central scattering and one-dimensional tunneling. We generalize Baz's result to the case when the momentum rather then the angular momentum of the incident particle is fixed, and for tunneling we define two complex parameters representing mean times spent in the barrier region by transmitted and reflected particles, respectively. We also briefly discuss the semiclassical limit.

In the Conclusion we compare our approach with that of other authors and show that various time parameters of Refs. 1–9 and 12 are simply related to the matrix element t_Ω being in most cases[8,9,12] $\mathrm{Re}t_\Omega$, $\mathrm{Im}t_\Omega$, or $|t_\Omega|$. Finally we discuss the generality of the method used and other possible constructions of time parameters describing the motion of a quantum particle.

I. A GENERAL EXPRESSION FOR QUANTUM TRAVERSAL TIME

We start the construction of quantum traversal time from the generalization of a well-known classical expression. Consider a particle moving along a classical path $\mathbf{r}(t)$ connecting the points $\mathbf{r}_1,t_1$, and $\mathbf{r}_2,t_2$ in some potential $V(\mathbf{r},t)$ (Fig. 1). The time the particle spends in an arbitrary region Ω is given by the integral

$$t_\Omega^{\mathrm{cl}} = \int_{t_1}^{t_2} \Theta_\Omega(\mathbf{r}(t))dt \ , \tag{1.1}$$

where $\Theta_\Omega(\mathbf{r})$ equals 1 if $\Omega \ni \mathbf{r}$ and 0 otherwise. The expression (1.1) is a functional on the path of the particle.

Then let a quantum particle emitted from $\mathbf{r}_1$ at t_1 be observed again in $\mathbf{r}_2$ at the time t_2. Now all the paths

connecting $\mathbf{r}_1$ and $\mathbf{r}_2$ contribute to the transition amplitude $g(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)$ and the mean value $\langle F\rangle$ of some functional $F[\mathbf{r}(\cdot)]$ is given by a path integral,[13]

$$\langle F(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)\rangle = g^{-1}(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1) \times \int D\mathbf{r}(\cdot)F[\mathbf{r}(\cdot)] \times\exp\left[\frac{i}{\hbar}S[\mathbf{r}(\cdot)]\right], \tag{1.2}$$

$$g(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)= \int D\mathbf{r}(\cdot)\exp\left[\frac{i}{\hbar}S[\mathbf{r}(\cdot)]\right],$$

where $S[\mathbf{r}(\cdot)]$ is the action,

$$S[\mathbf{r}(\cdot)]= \int_{t_1}^{t_2}[m\dot{\mathbf{r}}(\cdot)^2/2-V(\mathbf{r}(\cdot),t)]dt ,$$

and we have denoted an arbitrary path as $\mathbf{r}(\cdot)$ to avoid confusion with the classical path $\mathbf{r}(t)$ in (1.1).

The normalization in (1.2) is such that

$$\langle F\equiv 1\rangle = 1 . \tag{1.3}$$

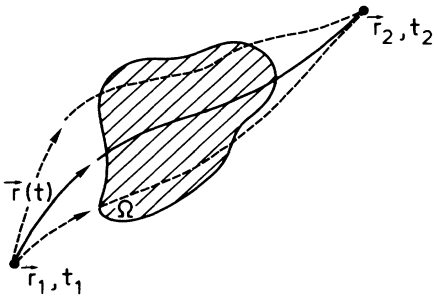

FIG. 1. Calculation of the matrix element t_Ω.

If a Hermitian operator corresponds to quantity F, (1.2) includes standard quantum-mechanical averaging procedure.[13] However, (1.2) holds for a more general case and we will use it to obtain the mean value of (1.1).

After rewriting (1.1) in a more suitable form,

$$t_\Omega^{\rm cl}[\mathbf{r}(\cdot)]= \int_{t_1}^{t_2}dt \int_\Omega d^3r\delta(\mathbf{r}-\mathbf{r}(\cdot)) , \tag{1.4}$$

we find with the help of (1.2) that

$$t_\Omega(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)\equiv\langle t_\Omega^{\rm cl}(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)\rangle =g^{-1}(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)\int_{t_1}^{t_2}dt\int_\Omega d^3r g(\mathbf{r}_2,\mathbf{r},t_2,t)g(\mathbf{r},\mathbf{r}_1,t,t_1) , \tag{1.5}$$

or, equivalently,

$$t_\Omega(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)=i\hbar\int_{t_1}^{t_2}dt\int_\Omega d^3r\frac{\delta\ln g(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)}{\delta V(\mathbf{r},t)} . \tag{1.6}$$

It is assumed in (1.2) that the initial and the final states of the particle are $\delta(\mathbf{r}-\mathbf{r}_1)$ and $\delta(\mathbf{r}-\mathbf{r}_2)$, respectively. To find the mean value of F for the transition between arbitrary initial and final states Ψ_i and Ψ_f one has to integrate over initial and final coordinates thus replacing the second right-hand term in (1.2) by

$$\langle\Psi_f\mid F\mid\Psi_i\rangle \equiv \int d^3r_2\int d^3r_1\Psi_f^*(\mathbf{r}_2) \times\int D\mathbf{r}(\cdot)F[\mathbf{r}(\cdot)] \times\exp\left[\frac{i}{\hbar}S[\mathbf{r}(\cdot)]\right]\Psi_i(\mathbf{r}_1) ,$$

and, to keep the normalization (1.3), g^{-1} by $\langle\Psi_f\mid 1\mid\Psi_i\rangle^{-1}$ (Ref. 13),

$$\langle\Psi_f\mid 1\mid\Psi_i\rangle = \int d^3r_2\, d^3r_1\Psi_f^*(\mathbf{r}_2) \times\int D\mathbf{r}(\cdot)\exp\left[\frac{i}{\hbar}S[\mathbf{r}(\cdot)]\right]\Psi_i(\mathbf{r}_i) .$$

Now it is straightforward to generalize (1.5) for the case when arbitrary Ψ_i and Ψ_f are given,

$$t_\Omega(\Psi_f,\Psi_i,t_2,t_1)=\langle\Psi_f\mid 1\mid\Psi_i\rangle^{-1}\langle\Psi_f\mid t_\Omega^{\rm cl}\mid\Psi_i\rangle , \tag{1.7}$$

where $t_\Omega^{\rm cl}$ is defined in (1.4). In the particular case when the final state Ψ_f is obtained from Ψ_i by evolution,

$$\Psi_f=\Psi_i(t)\mid_{t=t_2}\equiv\int d^3r_1 g(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1)\Psi_i(\mathbf{r}_1) , \tag{1.8}$$

(1.7) is considerably simplified, so that

$$t_\Omega(\Psi_i(t_2),\Psi_i,t_2,t_1)= \int_{t_1}^{t_2}dt\int_\Omega d^3r\mid\Psi(\mathbf{r},t)\mid^2 . \tag{1.9}$$

So far we have assumed that the states Ψ are normalized to unity and some comments should be made about the application of (1.9) in a time-independent scattering problem. In this case we must normalize the wave functions of continuous spectrum in some large volume V. The right-hand side of (1.9) is seen to depend on the volume chosen; however, to obtain the traversal time per particle we will have to divide t_Ω by the incoming flux (see Sec. III) and the final result is, as it should be, independent of V.

In the same way that (1.1) has its meaning in classical mechanics, matrix element (1.7) represents the mean time spent in Ω by a quantum particle moving between given initial and final states, and in the following sections we will refer to it as quantum traversal time, or simply traversal time. One sees that expressions

(1.5)–(1.7) are, in general, complex. This means that the quantum traversal time is not an observable in the usual sense. Yet in the following sections we will show that it still possess some properties of the classical time parameter (1.1) and that there are simple relations between t_Ω and certain observables, analogous to those in the classical case.

II. LARMOR PRECESSION AND THE TRAVERSAL TIME

In this section we establish a general relation between the angles of spin rotation and the mean time spent by a quantum particle in the region Ω where an infinitesimally small magnetic field $\mathbf{H}$ is localized. We expect the result to be of physical interest, since it relates to the definition of traversal time as the ratio between spin rotation angle φ and Larmor frequency ω_L, which is widely known as the "Baz' clock method."[8,9]

Again we start from the classical case. Let the particle move along the path $\mathbf{r}(t)$. If in the initial point $\mathbf{r}_1$ its spin (or angular momentum) is directed normal to the field $\mathbf{H}$, then in the final point $\mathbf{r}_2$, it is rotated in the plane normal to $\mathbf{H}$ by the angle

$$\varphi=\omega_L t_\Omega^{\mathrm{cl}}(\mathbf{r}_2,\mathbf{r}_1,\tau)\ , \tag{2.1}$$

where t_Ω^{cl} is classical traversal time as given by (1.1). (From now on we consider potential V to be time independent, hence all the quantities depend on $\tau=t_2-t_1$ only.)

Now consider the same problem in the quantum case. Let $\mathbf{H}$ be directed along z, and the axis and the spin of the particle ($\sigma=\frac{1}{2}$) be polarized in x direction at $t=0$. Thus we define the initial state to be

$$\Psi_i=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\delta(\mathbf{r}-\mathbf{r}_1)\ , \tag{2.2}$$

and the spin-dependent transition amplitude $\hat{g}$ takes the form,

$$\hat{g}(\mathbf{r}_2,\mathbf{r}_1,\tau)=\int D\mathbf{r}(\cdot)\exp\left[\frac{i}{\hbar}S[\mathbf{r}(\cdot)]-\frac{i}{\hbar}\mu H\int_0^\tau\Theta_\Omega(\mathbf{r}(\cdot))dt\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right]\ , \tag{2.3}$$

or in the limit of small $\mathbf{H}$,

$$\hat{g}(\mathbf{r}_2,\mathbf{r}_1,\tau)=g(\mathbf{r}_2,\mathbf{r}_1,\tau)\left[1-\frac{i}{\hbar}\mu H t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\begin{bmatrix}1&0\\0&-1\end{bmatrix}+O(H^2)\right]\ . \tag{2.4}$$

Note that in (2.3) and (2.4) S and g are the spatial parts of the action and transition amplitude as defined in Sec. I.

With the help of Green's function (2.4) we obtain the spin orientation in $\mathbf{r}_2$,

$$\Psi_f=\frac{1}{\sqrt{2}}\left[\begin{bmatrix}1\\1\end{bmatrix}-\frac{i}{\hbar}\mu H t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\begin{bmatrix}1\\-1\end{bmatrix}\right]\delta(\mathbf{r}-\mathbf{r}_2)\ . \tag{2.5}$$

It follows from (2.5) that the spin of the particle has nonzero y and z components $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ in $\mathbf{r}_2$;

$$\begin{aligned}\langle\sigma_y\rangle&=-\frac{2\mu H}{\hbar}\mathrm{Re}t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\ ,\\ \langle\sigma_z\rangle&=-\frac{2\mu H}{\hbar}\mathrm{Im}t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\ .\end{aligned} \tag{2.6}$$

The spin is therefore rotated in xy plane as well as in the plane parallel to the field by the angles $\varphi_\perp$ and $\varphi_\parallel$, respectively (Fig. 2). Finally, following Refs. 8 and 9 we consider the limit $H\to 0$ and as the angles $\varphi_{\perp,\parallel}$ become infinitesimal, obtain

$$\begin{aligned}\varphi_\perp&=\omega_L\mathrm{Re}t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\ ,\\ \varphi_\parallel&=\omega_L\mathrm{Im}t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\ ,\end{aligned} \tag{2.7}$$

and for the total angle of rotation φ_{tot} (Fig. 2),

$$\varphi_{\mathrm{tot}}=(\varphi_\perp^2+\varphi_\parallel^2)^{1/2}=\omega_L\,|\,t_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)\,|\ . \tag{2.8}$$

The main difference between the classical and quantum situations is that in the latter case the spin of the particle is rotated around two rather then around one z axis. It follows from (1.6) that t_Ω acquires an imaginary part if the transmission probability into the final state depends on the potential in Ω, and we will show that one indeed obtains a real traversal time in cases when there is no such dependence. Expressions (2.6)–(2.7) illustrate this fact since the presence of a magnetic field alters the

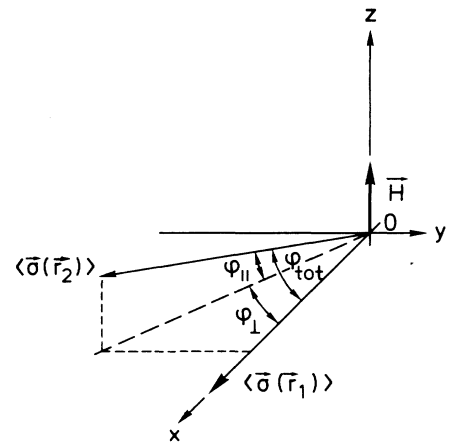

FIG. 2. The spin orientation of the particle in the initial and final points.

potential in Ω for the states with spin parallel and antiparallel to the field, which we superpose to obtain the initial polarization (2.2).

We want to emphasize here that by performing in this case the Baz' thought experiment,[8] i.e., measuring the expectation values of σ_y and then that of σ_z we obtain two real parameters $t_\perp=\varphi_\perp/\omega_L$ and $t_\parallel=\varphi_\parallel/\omega_L$. The use of the total rotation angle $\varphi_{\rm tot}$ adds the third parameter $t_{\rm tot}=\varphi_{\rm tot}/\omega_L$ and we are at a loss since we cannot say which one of them, or which of their combinations should be called the traversal time. We have shown that the natural construction uniting both $t_\perp$ and $t_\parallel$ is the matrix element t_Ω,

$$t_\Omega=t_\perp+it_\parallel \ .$$

III. TRAVERSAL TIME IN CENTRAL SCATTERING

Now we apply the approach developed in Secs I and II to determine the traversal times in the case when a particle is scattered by spherically symmetrical potential $V(r)$. This problem was discussed in Refs. 2, 3, 6, and 8 in order to obtain expressions for the collision times and lifetimes of the intermediate states in nuclear reactions. We consider two physically different situations in which an incident particle with energy E has a known (a) angular momentum l or (b) momentum $\hbar\mathbf{k}$.

(a) Let potential $V(r)$ by short range, $V(r)=0$, $r>R$. Following Ref. 8 we find the mean time a particle with a given value of l spends in the scattering region S_R $(r<R)$. We choose to describe the scattering with the radial wave function specified by its asymptotic behavior as $r\to\infty$,

$$\Psi_l(r)=A\left[\frac{e^{-ikr+i\pi l/2}}{r}-S_l(k)\frac{e^{ikr-i\pi l/2}}{r}\right],$$
$$k=\hbar^{-1}\sqrt{2mE}\ , \tag{3.1}$$
$$S_l(k)=\exp[2i\delta_l(k)]\ ,$$

where $\delta_l(k)$ is the scattering phase shift.

Now we can use (1.9) to immediately obtain

$$t_R(\Psi_l(\tau),\Psi_l,\tau)=\tau\int_0^R dr\,r^2\,|\,\Psi_l(r)\,|^2\ . \tag{3.2}$$

As the normalization of (3.1) implies, there is an incident flux of particles and (3.2) represents the mean time spent in S_R by all the particles that have entered this region during the time τ, and should therefore be divided by their number n_τ. (We note here that the method used in this case is close to that of Smith,[3] who estimated the time for which a particle is retarded in the scattering region by the rate of surplus density $|\Psi|^2$ around the scattering center.) Calculating n_τ as τ times the total flux of particles in the incoming wave [this quantity is the same for any sphere with radius larger or equal to R and can therefore be calculated in the asymptotic region with the help of the first right-hand term in (3.1)] we obtain the traversal time per particle as

$$t_R(E,l)=\frac{m}{\hbar k}\int_0^R dr\,r^2\,|\,\Psi_l\,|^2\ , \tag{3.3}$$

which is exactly the result obtained by Baz'[8] in his experiment with the Larmor precession. Finally we express the right-hand integral in (3.3) in terms of scattering phases δ_l. For $l=0$ we have, for instance,[8]

$$t_R(E,0)=\frac{m}{\hbar k}\left[\frac{\partial\delta_0(k)}{\partial k}+2R+\frac{1}{k}\sin[2kR+\delta_0(k)]\right]. \tag{3.4}$$

(b) Now let the momentum $\hbar\mathbf{k}$ of the incident particle be fixed. The corresponding wave function has the form as $r\to\infty$,

$$\Psi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{f(E,\Theta)}{r}e^{ikr}\ , \tag{3.5}$$

where

$$f(E,\Theta)=|\,f(E,\Theta)\,|\exp[i\delta(E,\Theta)]$$

is the scattering amplitude. Now our purpose is to determine the mean time $t_R(E,\Theta)$ a particle scattered at the angle Θ spends in the vicinity of the scattering center (Fig. 3). Since the state (3.5) describes all the particles scattered at different angles we cannot simply use (1.9), but must return instead to (1.6). The energy in (1.6) is undetermined in the general case and to fix it, we take the limit in which r_1 and τ go to infinity as their ratio stays finite and equal to the particle's velocity $v=\hbar k/m$, and $\mathbf{r}_2$ is large enough to use the asymptote (3.5) (see Fig. 3),

$$r_1\to\infty,\quad \tau\to\infty,\quad r_1/\tau=v,\quad kr_2\gg1\ . \tag{3.6}$$

We expand the Green's function g in the form

$$g(\mathbf{r}_2,\mathbf{r}_1,\tau)=\int d^3k\,\exp\left[-\frac{i\hbar k^2}{2m}\tau\right]\Psi_{\mathbf{k}}(\mathbf{r}_2)\Psi^*_{\mathbf{k}}(\mathbf{r}_1)\ , \tag{3.7}$$

and insert (3.5) into (3.7). The integration of the product of the first terms in (3.5) yields the free-particle Green's function; to evaluate the other three integrals we use the method of stationary phase. With the help of the relation

$$\lim_{r\to\infty}e^{i\mathbf{kr}}=\frac{2\pi i}{k}\left[\delta\left(\frac{\mathbf{k}}{k}+\frac{\mathbf{r}}{r}\right)\frac{e^{-ikr}}{r}-\delta\left(\frac{\mathbf{k}}{k}-\frac{\mathbf{r}}{r}\right)\frac{e^{ikr}}{r}\right],$$

and noting that only one of the three integrals has a sta-

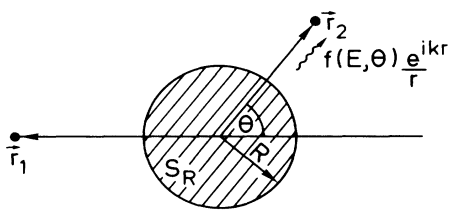

FIG. 3. Scattering on the central potential.

tionary point for $k>0$ as $\tau\to\infty$, we obtain the asymptotic behavior of g in the limit (3.6),

$$g(\mathbf{r}_2,\mathbf{r}_1,\tau)=g_0(\mathbf{r}_2,\mathbf{r}_1,\tau)+\frac{e^{ikr_2}}{r_2}f(E,\Theta)g_0(0,\mathbf{r}_1,\tau)+o(\tau^{-3/2}) , \tag{3.8}$$

where g_0 is a free-particle propagator,

$$g_0(\mathbf{r}_2,\mathbf{r}_1,\tau)=\left[\frac{2\pi m}{i\hbar\tau}\right]^{3/2}\exp\left[\frac{im\,|\,\mathbf{r}_2-\mathbf{r}_1\,|^2}{2\hbar\tau}\right] , \tag{3.9}$$

and

$$\Theta=\pi-\arccos\frac{\mathbf{r}_2\cdot\mathbf{r}_1}{r_1r_2} .$$

This provides us with the necessary decomposition of g; clearly the first term in (3.8) describes the free propagation from $\mathbf{r}_1$ to $\mathbf{r}_2$, while the second term corresponds to the scattering at an angle Θ and the energy of the particle is in both cases E. To obtain $t_\Omega(E,\Theta)$ we apply (1.6) to the second right-hand term in (3.8). This yields

$$t_R(E,\Theta)=-\hbar\int_{S_R}d^3r\frac{\delta(\delta(E,\Theta))}{\delta V(\mathbf{r})}+i\hbar\int_{S_R}d^3r\frac{\delta\ln|f(E,\Theta)|}{\delta V(\mathbf{r})} . \tag{3.10}$$

Unlike (3.3), the expression (3.10) is complex. As has already been mentioned, the imaginary part of the traversal time describes the dependence of the scattering probability on the form of the scattering potential V. We illustrate this again by taking the average of $\mathrm{Im}t_R(E,\Theta)$ with respect to the scattering angle Θ,

$$\langle\mathrm{Im}t_R(E,\Theta)\rangle_\Theta\equiv\sigma^{-1}(E)\int d\Omega\,|f(E,\Theta)|^2\mathrm{Im}t_R(E,\Theta)=\tfrac{1}{2}\int_{S_R}d^3r\frac{\delta\ln\sigma(E)}{\delta V(\mathbf{r})} . \tag{3.11}$$

Here $\sigma(E)$ is the total scattering cross section. Thus the mean value of $\mathrm{Im}t_R$ is proportional to the variation of the total number of particles scattered per unit time and per unit variation of the scattering potential. Note that in the spherical wave (3.1) the number of scattered particles equals the number of incident particles and does not depend on the form of $V(r)$. This explains the real value of $t_R(E,l)$ found by Baz'.[8]

Finally we want to establish the relation $t_R(E,\Theta)$ and $t_R(E,l)$ and express the first of the two parameters via the phase shifts δ_l. We use (3.3) and the stationary perturbation theory to show that, in accordance with (1.6),

$$t_R(E,l)=-i\hbar\int_{S_R}d^3r\frac{\delta\ln S_l(k)}{\delta V(\mathbf{r})} . \tag{3.12}$$

Thus taking the logarithmic variational derivative of the partial-wave expansion for the scattering amplitude,

$$f(E,\Theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}Y_{l0}(\cos\Theta)(S_l-1) ,$$

and inserting it into (3.10) we find

$$t_R(E,\Theta)=\frac{i}{2kf(E,\Theta)}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}Y_{l0}(\cos\Theta)S_lt_R(E,l) , \tag{3.13}$$

where $t_R(E,l)$ can be expressed via δ_l in the way it was done for $l=0$ in (3.4).[8]

To close this section we return to the Baz' experiment. From (2.7) and (3.8) it follows immediately that if a small magnetic field $\mathbf{H}$ is created around the scattering center, the spin of the particle scattered at the angle Θ undergoes rotation in the two planes as shown in Fig. 2,

$$\varphi_\perp(E,\Theta)=\omega_L\mathrm{Re}t_R(E,\Theta) ,\quad \varphi_\parallel(E,\Theta)=\omega_L\mathrm{Im}t_R(E,\Theta) , \tag{3.14}$$

and the Baz' definition of traversal time becomes ambiguous as was discussed at the end of Sec. II.

IV. THE TUNNELING TIME

The question of how to define traversal time in the case of tunneling has often been discussed, but no clear answer has been given. In this section we use our approach to show that in one-dimensional tunneling there exist two complex parameters $t_1(E)$ and $t_2(E)$, representing the mean times spent in the barrier region by the reflected and transmitted particles, respectively.

Let the barrier $V(x)$ have a finite width [$V(x)=0$, $|x|>a$] (see Fig. 4). For our analysis we will need two sets of functions corresponding to the flux of particles incident on the left and on the right side of the barrier,

$$\Psi_k^L(x)=\{e^{ikx}+A_1^Le^{-ikx},\ x<-a;\ \ A_2^Le^{ikx},\ x>a\} , \tag{4.1a}$$

$$\Psi_k^R(x)=\{A_2^Re^{-ikx},\ x<-a;\ \ e^{-ikx}+A_1^Re^{ikx},\ x>a\} ,$$

where $A_{1,2}^{L,R}=|A_{1,2}^{L,R}|e^{i\delta_{1,2}^{L,R}}$ are the reflection and transmission amplitudes, respectively. The amplitudes A^L and A^R are not independent; using the Wronskian relations between Ψ^L and Ψ^R one can show that

$$A_2^L=A_2^R=A_2,\quad A_1^LA_2^*+A_1^{R*}A_2=0 . \tag{4.1b}$$

Finally, we choose the particle to be incident on the

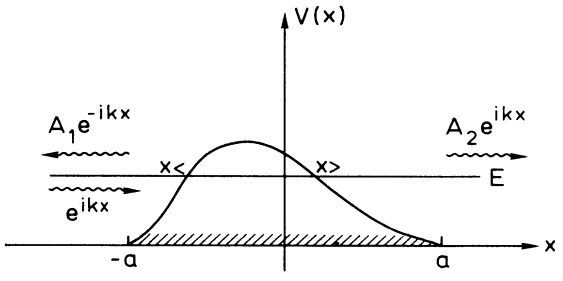

FIG. 4. One-dimensional tunneling.

left, and everywhere in the following index L will be assumed unless written explicitly. To distinguish between reflected and transmitted particles we consider the situation in which a particle emitted from a distant point to the left of the barrier is registered after a long time somewhere in the right half-space. Thus, allowing for the transmitted particle

$$x_1 \to -\infty, \quad \tau \to \infty, \quad |x_1|/\tau = v, \quad x_2 > a \ , \tag{4.2}$$

constructing the Green's function with the help of (4.1a), and following in the one-dimensional case the arguments of Sec. III, we find the transition amplitude,

$$g_o(x_2,x_1,\tau) = e^{ikx_2} A_2(E) g_0(0,x_1,\tau) + o(\tau^{-1/2}) \ . \tag{4.3}$$

Again the right-hand side of (4.3) contains a free-particle amplitude g_o multiplied by the transmission amplitude at the energy fixed by (4.2). The form of (4.3) illustrates the fact that the uncertainty principle $\Delta E \Delta t > \hbar$ is related to the moment of collision rather than to its duration.[8] (4.3) gives no information about when the particle crosses the barrier, while the traversal time can be obtained from (4.3) with the help of (1.6) with any accuracy.

Likewise for the reflected particle ($x_2 < 0$) we obtain

$$g_o(x_2,x_1,\tau) = g_0(x_2,x_1,\tau) + e^{-ikx_2} A_1(E) g_0(0,x_1,\tau) + o(\tau^{-1/2}) \ , \tag{4.4}$$

and application of (1.6) to (4.3) and (4.4) gives

$$t_{1,2}(E) = -\hbar \int_{-a}^{a} dx \frac{\delta(\delta_{1,2})}{\delta V(x)} + i\hbar \int_{-a}^{a} dx \frac{\delta \ln |A_{1,2}|}{\delta V(x)} \ . \tag{4.5}$$

We can express $t_1(E)$ and $t_2(E)$ in terms of $A_{1,2}$ and their energy derivatives with the help of perturbation theory. It is easy to see that

$$\delta A_{1,2} = -\exp[\mp ikx] \int_{-a}^{a} \delta V(x') g^+(x,x',E) \Psi_k^L(x') dx', \quad x \gtrless \pm a \tag{4.6}$$

where g^+ is the Green's function of time-independent Schrödinger equation, which only contains outgoing waves outside the barrier,

$$g^+(x,x',E) = -\frac{im}{\hbar^2 k A_2} \Psi_k^L(x_>) \Psi_k^R(x_<) \ ,$$

and $x_>$ ($x_<$) is the larger (the smaller) of the arguments x,x'. [Note that the right-hand side of (4.6) is independent of x.]

In (4.5) the integrals involving $\Psi^i \Psi^j$ can be expressed via the energy derivatives of their Wronskians[8] and using (4.1b) we get the desired result. In order to avoid lengthy expressions we introduce quantities $B_{1,2}$ related to conventional amplitudes $A_{1,2}$,

$$B_{1,2} = e^{2ika} A_{1,2} \ ,$$

to obtain[14]

$$t_1(E) = -\frac{im}{\hbar k} \left[\frac{1}{B_1} \frac{\partial B_1}{\partial k} + \frac{1}{2kB_1} (B_1^2 + B_2^2 - 1) \right] \ , \tag{4.7}$$

$$t_2(E) = -\frac{im}{\hbar k} \left[\frac{1}{B_2} \frac{\partial B_2}{\partial k} + \frac{B_1}{2k} (1 - e^{2i\delta_2}) \right] \ ,$$

and the expressions (4.7) allow the calculation of the traversal times $t_{1,2}$ for a barrier of arbitrary form, if amplitudes $A_{1,2}$ are known functions of energy E.

Equation (4.6) helps to establish a useful relation between t_1 and t_2. If n_1 and n_2 are the numbers of particles reflected and transmitted per second, respectively, then

$$t(E) \equiv (n_1 + n_2)^{-1} [n_1 t_1(E) + n_2 t_2(E)] = \frac{m}{\hbar k} \int_{-a}^{a} |\Psi_k^L(x)|^2 dx \ . \tag{4.8}$$

Equation (4.8) is analogous to (3.11), $t(E)$ is the mean time spent by an electron in the barrier regardless of which process, reflection or transmission, occurs. This mean time is real and equals the ratio of the mean number of particles in the barrier region to the incident flux. Note that this expression can be deduced from classical mechanics[3] but it gives no clue for finding t_1 and t_2 separately.

Consider the semiclassical limit of Eqs. (4.7). Using semiclassical expressions for $A_{1,2}$ we obtain

$$t_2(E)_{\hbar \to 0} = \int_{-a}^{a} \frac{m\,dx}{p} + O(\hbar), \quad p(x) = \begin{cases} |\sqrt{2m(E - V(x))}| & \text{if } E > V(x) \\ i\,|\sqrt{2m(E - V(x))}| & \text{if } E < V(x) \ . \end{cases} \tag{4.9}$$

Apparently for $E < V$, (4.9) is the value of the functional (1.1) on the real-valued semiclassical path $x(t)$, where p is imaginary in the classically forbidden region.[15] The role of this path becomes more clear if, instead of using (4.7) we apply (1.6) directly to the semiclassical asymptote of g as given by (see Ref. 15),

$$g = A \exp\left[\frac{i}{\hbar} S(E)\right] + O(\hbar) \ , \tag{4.10}$$

where the time-independent part of the action $S(E)$ is calculated on the above-mentioned path. So we find

$$t_2(E) = -\int_{-a}^{a} dx \frac{\delta S(E)}{\delta V(x)} + O(\hbar) \ , \tag{4.11}$$

which takes us back to expression (4.9).

For the reflected particles we obtain, as $\hbar \to 0$,

$$t_1(E) = 2 \int_{-a}^{x_<} \frac{m dx}{p} + O(\hbar) \ , \tag{4.12}$$

where $x_<$ is the left turning point (see Fig. 4).

It is easy to interpret the Baz'-Rybachenko experiment with magnetic field localized in the barrier.[9] Exactly as in the second part of Sec. III we obtain two angles of spin rotation for reflection or transmission,

$$\varphi_{\perp 1,2}=\omega_L \mathrm{Re}t_{1,2}(E) ,$$

$$\varphi_{\parallel 1,2}=\omega_L \mathrm{Im}t_{1,2}(E) ,$$

and again face the difficulties in defining the tunneling time in the Baz' method.[12]

CONCLUSION

In the quantum case we have found the parameter t_Ω representing the mean time spent by a particle in a given region of space. This parameter is most simply defined in the Feynman path-integral technique, the accuracy of its determination is not limited by $\Delta E \Delta t > \hbar$ uncertainty relation and it has the classical traversal time as its (semi) classical limit. The traversal time t_Ω is not an observable in the usual sense, but it still possesses some of the properties of t_Ω^{cl}; it is simply related to the angles of spin rotations in the magnetic field and to the number of particles in the region Ω (as given by $\int_\Omega |\Psi|^2 dr^3$) if we do not distinguish between the channels, the transition probability for which depends on the form of the potential in the region of interest—for instance between transmitted and reflected particles in the case of one-dimensional tunneling. At the same time we cannot expect t_Ω to play the role of the only relevant time parameter describing the motion of a quantum particle. We will return to this question at the end of this section and we now use our approach to explain the variety of traversal times arising from the results of Refs. 1–12.

First we establish the relation between t_Ω and the traversal time t_{pack} as defined by the wave-packet method.[2,4,5] The complete critical review of this method can be found in Refs. 9–12. We note, however, that, as is intuitively clear, t_{pack} should be a reasonable estimate when the size of the packet Δx is small compared to the size of the region of interest x_Ω. (Here we restrict ourselves to the one-dimensional case.) Let us also suppose that reflection is small, i.e., we have only one packet, whose maximum crosses the boundary of Ω at t_1 and t_2, respectively. Constructing wave packet of the form

$$\Psi(x,t)=\int dk\, c(k)\Psi_k(x)e^{-(i/\hbar)E(k)t} ,$$

$$\int |c(k)|^2 dk = 1 ,$$

and applying (1.9) we see that the integrand of the t integral is approximately unity if the packet is inside the region and it is zero if the packet is outside. Thus $t_{\mathrm{pack}}=t_2-t_1$ is approximately equal to the traversal time t_Ω

$$t_{\mathrm{pack}}=t_\Omega\left[1+O\left[\frac{\Delta x}{x_\Omega}\right]\right] ,$$

and expanding the limits of integration $t_{1,2}\to\pm\infty$ we see in addition that t_{pack} in this case is just the mean value of $t(E)$ as given by (4.8) taken with the weight function $|c(k)|^2 a$, which determines the shape of the packet in k space;

$$t_{\mathrm{pack}}\sim\int dk\,|c(k)|^2 t[E(k)] .$$

Now we consider the methods used in Refs. 3, 8, 9, and 12. In Ref. 8 Baz' obtained the value of traversal time t_Ω in the case when angular momentum of the particle is fixed and t_Ω has no imaginary part. Smith[3] arrived at a somewhat similar result, assuming the classical relation between the quantum lifetime and the number of particles in the region Ω. In Ref. 9 Rybachenko ignored the second angle of spin rotation $\varphi_\perp$ while applying the Baz' method to the transmitted particles in the tunneling problem and estimated therefore only $\mathrm{Re}t_\Omega$. In Ref. 12 the "clock method" used by Baz' was applied to tunneling through a rectangular barrier and both angles $\varphi_\parallel$ and $\varphi_\perp$ were taken into account. The three tunneling times that appear in Ref. 12 are in fact $\mathrm{Re}t_\Omega$, $\mathrm{Im}t_\Omega$, and $|t_\Omega|$. Note that the relations between the three times obtained in Ref. 12 are correct only for a symmetrical barrier.

However, since in general, all paths contribute to the mean value of t_Ω^{cl} in (1.5), one may expect that besides t_Ω there may exist other time parameters that equal t_Ω in the classical limit but are quite different in the quantum case. The detailed analysis of a situation when a particle moves in a small time-dependent field localized in Ω shows that this is indeed the case. The motion of the particle becomes adiabatic, i.e., the particle "sees" the static potential $V(\cdot)$, if the variation of the field is slow in the sense that

$$\omega |T_\Omega| \ll 1 ,$$

where ω is the frequency and T_Ω is given by

$$T_\Omega(\mathbf{r}_2,\mathbf{r}_1,\tau)=\frac{\left\langle \int_0^\tau (t-\tau)\Theta_\Omega(\mathbf{r}(\cdot))dt \right\rangle}{\left\langle \int_0^\tau \Theta_\Omega(\mathbf{r}(\cdot))dt \right\rangle} .$$

The quantities t_Ω and T_Ω may display quite different behavior as functions of the energy of the particle. For instance at the energies E_n for which there is no reflection from a rectangular barrier[12] t_Ω goes to infinity for the reflected particle, while T_Ω stays finite and proportional to the inverse level width Γ^{-1}.[16]

The distinction between the two parameters vanishes, however, in the semiclassical limit when (at least in one dimension) only one path, whether real or complex, contributes to the averages in the expressions for t_Ω and T_Ω. Putting x_2 at the boundary of Ω (otherwise T_Ω includes the time it takes the particle to travel from Ω to x_2) we obtain

$$\lim_{\hbar\to 0} T_\Omega(x_2,x_1,\tau)=\tfrac{1}{2}\lim_{\hbar\to 0} t_\Omega(x_2,x_1,\tau)=\tfrac{1}{2}\int_\Omega \frac{m\,dx}{p} ,$$

where p is real in the classically allowed and imaginary in the classically forbidden regions. This illustrates the well-known fact that the integral $\int dx/p$ plays the role of tunneling time for time-dependent semiclassical tunneling problems in one dimension.[11,17]

Thus our approach shows that, except in the semiclas-

sical case, one fails to provide a single time parameter describing the duration of quantum scattering process, firstly, because the natural generalization of the classical traversal time is given by a complex matrix element and, secondly, because this generalization is not unique. For this reason one has to analyze the type of parameters appearing in each particular quantum problem as well as their relation to the observables of interest. We have carried out such an analysis for the quantum traversal time t_Ω to show that this quantity possesses some degree of universality. Thus we may expect that in the quantum case the number of physically significant time parameters is not infinite and that the Feynman path-integral technique is an efficient tool for their construction and classification.

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