Tunnelling Times in Quantum Mechanics

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1 Larmor Precession

We consider the case of scattering in one dimension with particles of mass m, spin $\frac{1}{2}$ and kinetic energy $E = \frac{\hbar^2 k^2}{2m}$. The particles move along the y-axis with spins polarised with the x-axis and interact with a rectangular barrier,

$$V = \begin{cases} V_0 & -\frac{d}{2} < y < \frac{d}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

A small magnetic field $\vec{B_0}$ points along the z-axis and is confined to the barrier. As particles enter the barrier, the magnetic field induces a Larmor precession with frequency $\omega_L = \frac{g\mu B_0}{\hbar}$, where g is the gyromagnetic ratio, μ is the absolute value of the magnetic moment. The polarisations of the transmitted and reflected particles are compared with the polarisation of the incident particles.

Particles initially polarised in the x direction obtain a y and z components when tunnelling through the barrier. We know from the Stern-Gerlach experiment that particles polarised int eh x direction can be represented as combinations of particles with z polarisations, $|x;\pm\rangle = \frac{1}{\sqrt{2}}|z;+\rangle \pm \frac{1}{\sqrt{2}}|z;-\rangle$. Outside the barrier, particles have kinetic energy E, independent of their spin. Inside the barrier, the kinetic energy differs by the Zeeman contribution $\pm\frac{\hbar\omega_L}{2}$. The wavefunction inside the barrier will contain an exponentially decaying term $Exp(\kappa_\pm)$, where $\kappa_\pm = (k_0^2 - k^2 \pm \frac{m\omega_L}{\hbar})^{\frac{1}{2}}$, where $\kappa = (k_0^2 - k^2)^{\frac{1}{2}}$ and the sign indicates spin parallel or antiparallel to the field.

We can approximate this in the small ω_L limit as

$$\begin{aligned} \kappa_\pm &= \left(k_0^2 - k^2 \mp \frac{m\omega_L}{\hbar}\right)^{\frac{1}{2}} \\ &= \kappa\left(1 \mp \frac{m\omega_L}{\hbar\kappa^2}\right)^{\frac{1}{2}} \\ &\approx \kappa\left(1 \mp \frac{m\omega_L}{2\hbar\kappa^2}\right) \\ &= \kappa \mp \frac{m\omega_L}{2\hbar\kappa} \end{aligned}$$

Here we examine tunnelling through a barrier in a magnetic field. In this case our Hamiltonian is

$$H = \begin{cases} \left(\frac{p^2}{2m} + V_0\right)\mathbb{1} - \left(\frac{\hbar\omega_L}{2}\right)\sigma_z & |y| \leq \frac{d}{2} \\ \left(\frac{p^2}{2m}\right)\mathbb{1} & |y| \geq \frac{d}{2} \end{cases} \tag{2}$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

H acts on spinors

$$\psi = \begin{pmatrix} \psi_+(y) \\ \psi_-(y) \end{pmatrix} \tag{3}$$

As usual $|\psi_\pm(y)|^2 dy$ is the probability of finding a particle *upon measurement* with spin $\pm\frac{\hbar}{2}$ in the interval $y, y+dy$. We emphasise the 'upon measurement' here as this is an important point of distinction between the orthodox and pilot-wave interpretations addressed in this essay. The incident beam is polarised in the x direction,

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{iky} \tag{4}$$

i.e. ψ is an eigenvector of S_x

H is diagonal in the spinor basis so we can solve the scattering problem for particles with spin $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ separately. Our wavefunction is of the form

$$\psi = \begin{cases} A_{\pm}e^{iky} + B_{\pm}e^{-iky} & y \leq -\frac{d}{2} \\ C_{\pm}e^{\kappa_{\pm}y} + D_{\pm}e^{-\kappa_{\pm}y} & -\frac{d}{2} \leq y \leq \frac{d}{2} \\ F_{\pm}e^{iky} & y \geq \frac{d}{2} \end{cases} \tag{5}$$

We will soon set $A_{\pm} = 1$, corresponding to 1 particle per ??, but maintain it for now to aid a future calculation. Note there is no e^{-iky} term on the right of the barrier, as no particles are reflected.

The effect of the magnetic field B_0 is simply to adjust the height of the barrier, $V_0^{'} = V_0 \pm \frac{\hbar\omega_L}{2}$. Hence we can solve the scattering problem initially assuming no magnetic field, and then adjusting our solution by replacing κ in the field-free problem with $\kappa_{\pm}$. Our job now is to calculate the wavefunction coefficients A, B, C, D, F using the continuity of the wavefunction and its first derivative at the boundaries. This is a lengthy calculation but the results are used so frequently that it is necessary to include a derivation. The results are stated here and derived below:

$$\begin{aligned} F &= T^{\frac{1}{2}}e^{i\Delta\phi}e^{-ikd} & B &= R^{\frac{1}{2}}e^{-\frac{i\pi}{2}}e^{i\Delta\phi}e^{-ikd} \\ C &= \frac{\kappa + ik}{2\kappa}e^{\frac{ikd}{2}}e^{\frac{-\kappa d}{2}}F & D &= \frac{\kappa - ik}{2\kappa}e^{\frac{ikd}{2}}e^{\frac{\kappa d}{2}}F \end{aligned} \tag{6}$$

where T is t:he transmission probability and $R = 1 - T$ is the reflection probability.

First we introduce a new coordinate system so that the boundaries of our barrier become 0, d. Then, denoting our wavefunctions before, inside and after the barrier as ψ_1, ψ_2, ψ_3 respectively, we have:

$$\psi_1 = Ae^{iky} + Be^{-iky} \qquad \psi_1^{'} = Aike^{iky} - ikBe^{-iky} \tag{7}$$

$$\psi_2 = Ce^{-\kappa y} + De^{\kappa y} \qquad \psi_2^{'} = -\kappa Ce^{-\kappa y} + \kappa De^{\kappa y} \tag{8}$$

$$\psi_3 = Fe^{iky} \qquad \psi_3^{'} = ikFe^{iky} \tag{9}$$

Imposing continuity of the wavefunction and its first derivative at the barrier boundaries:

$$\psi_1(0) = \psi_2(0) \implies A + B = C + D \tag{10}$$

$$\psi_1^{'}(0) = \psi_2^{'}(0) \implies ikA - ikB = -\kappa C + \kappa D \tag{11}$$

$$\psi_2(d) = \psi_3(d) \implies Ce^{-\kappa d} + De^{\kappa d} = Fe^{ikd} \tag{12}$$

$$\psi_2^{'}(d) = \psi_3^{'}(d) \implies -\kappa Ce^{-\kappa d} + \kappa De^{\kappa d} = ikFe^{ikd} \tag{13}$$

$$ik(10) + (11) \implies 2ikA = C(ik - \kappa) + D(ik + \kappa) \tag{14}$$

$$ik(10) - (11) \implies 2ikB = C(ik + \kappa) + D(ik - \kappa) \tag{15}$$

$$\kappa(12) - (13) \implies 2\kappa Ce^{\kappa d} = Fe^{ikd}(\kappa - ik) \tag{16}$$

$$\kappa(12) + (13) \implies 2\kappa De^{\kappa d} = Fe^{ikd}(\kappa + ik) \tag{17}$$

Inserting equations (16) and (17) into equation (14) we arrive at:

$$
\begin{aligned}
2ikA &= -\frac{(ik-\kappa)^2}{2\kappa} F e^{(ik+\kappa)d} + \frac{(ik+\kappa)^2}{2p} F e^{(ik-\kappa)d} && (18)\\
\implies 4\kappa ikAe^{-ikd} &= F[(k^2-\kappa^2)(e^{\kappa d}-e^{-\kappa d}) + 2ik\kappa(e^{\kappa d}+e^{-\kappa d})] && (19)\\
&= F[2(k^2-\kappa^2)\sinh\kappa d + 4ik\kappa\cosh\kappa d] && (20)
\end{aligned}
$$

We hence arrive at our first result, the transmission probability $T = \frac{|F|^2}{|A|^2}$:

$$
T = [1 + \frac{(k^2+\kappa^2)^2\sinh^2\kappa d}{4k^2\kappa^2}]^{-1}.
$$

It is clear now why we left the coefficient A explicit, but from now on it will be set to A = 1. Rearranging (20) for F, we can compare with the final result in (6) and factorise out $T^{\frac{1}{2}}$:

$$
F = T^{\frac{1}{2}} e^{-ikd} \times \frac{(k^2-\kappa^2)i\sinh\kappa d + 2k\kappa\cosh(\kappa d)}{\sqrt{(k^2+\kappa^2)^2\sinh^2\kappa d + 4k^2\kappa^2}}
$$

This yields the identification

$$
\begin{aligned}
e^{i\Delta\phi} &= \frac{(k^2-\kappa^2)i\sinh\kappa d + 2k\kappa\cosh\kappa d}{\sqrt{(k^2+\kappa^2)\sinh^2(\kappa d) + 4k^2\kappa^2}} && (21)\\
&= \frac{(k^2-\kappa^2)i\tanh\kappa d + 2k\kappa}{\sqrt{(k^2-\kappa^2)^2\tanh^2\kappa d + 4k^2\kappa^2}} && (22)
\end{aligned}
$$

Expanding the left hand side into real and imaginary parts and comparing coefficients we deduce:

$$
\tan\Delta\phi = \frac{(k^2-\kappa^2)\tanh(\kappa d)}{2k\kappa}
$$

The result for B follows along similar lines and results for C and D follow immediately from equations (16) and (17).