

Pairwise Comparisons without Stochastic Transitivity: Model, Theory and Applications

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Abstract

Most statistical models for pairwise comparisons, including the Bradley-Terry (BT) and Thurstone models and many extensions, make a relatively strong assumption of stochastic transitivity. This assumption imposes the existence of an unobserved global ranking among all the players/teams/items and monotone constraints on the comparison probabilities implied by the global ranking. However, the stochastic transitivity assumption does not hold in many real-world scenarios of pairwise comparisons, especially games involving multiple skills or strategies. As a result, models relying on this assumption can have suboptimal predictive performance. In this paper, we propose a general family of statistical models for pairwise comparison data without a stochastic transitivity assumption, substantially extending the BT and Thurstone models. In this model, the pairwise probabilities are determined by a (approximately) low-dimensional skew-symmetric matrix. Likelihood-based estimation methods and computational algorithms are developed, which allow for sparse data with only a small proportion of observed pairs. Theoretical analysis shows that the proposed estimator achieves minimax-rate optimality, which adapts effectively to the sparsity level of the data. The spectral theory for skew-symmetric matrices plays a crucial role in the implementation and theoretical analysis. The proposed method's superiority against the BT model, along with its broad applicability across diverse scenarios, is further supported by simulations and real data analysis.

Keywords: Pairwise comparison, stochastic intransitivity, Bradley-Terry model, low-rank model, nuclear norm

1 Introduction

Pairwise comparison data have received intensive attention in statistics and machine learning, with diverse applications across domains. Such data often arise from tournaments, where each pairwise comparison outcome results from a match between two players or teams, or from crowdsourcing settings, where individuals are tasked with comparing two items, such as images, movies, or products. Specifically, the famous Thurstone (Thurstone, 1927) and Bradley-Terry (BT; Bradley and Terry, 1952) models have set a cornerstone in the field, followed by many extensions, including the parametric ordinal models proposed in Shah et al. (2016a), which broadens the class of parametric models. Oliveira et al. (2018) relax the assumption of a known link function and propose models that allow the link function to belong to a broad family of functions. Nonparametric approaches have also emerged, such as the work introduced in Shah and Wainwright (2018) based on the Borda counting algorithm, and the nonparametric Bradley-Terry models studied in Chatterjee (2015) and Chatterjee and Mukherjee (2019). Additionally, pairwise comparison models have been developed for crowdsourced settings, as discussed in Chen et al. (2013) and Chen et al. (2016), among many others. The models for pairwise comparisons have received a wide range of applications, including rank aggregation (Chen and Suh, 2015; Chen et al., 2019; Heckel et al., 2019; Chen et al., 2022b), predicting matches/tournaments (Cattelan et al., 2013; Tsokos et al., 2019; Macrì Demartino et al., 2024), testing the efficiency of betting markets (McHale and Morton, 2011; Lyócsa and Výrost, 2018; Ramirez et al., 2023), and refinement of large language models based on human evaluations (Christiano et al., 2017; Ouyang et al., 2022; Zhu et al., 2023).

While the models mentioned above have made significant contributions to the field, they rely on the assumption of stochastic transitivity, which implies a strict ranking among players/teams/items. However, this assumption may be unrealistic, particularly in settings involving multiple skills or strategies, where intransitivity naturally arises. Despite its practical importance, research on models that allow intransitivity remains limited. Some notable exceptions include the work of Chen and Joachims (2016) and Spearing et al. (2023), which extend the Bradley-Terry model by introducing additional parameters to describe intransitivity alongside parameters specifying absolute strengths based on Bradley-Terry probabilities. Spearing et al. (2023) propose a Markov chain Monte Carlo algorithm for parameter estimation under a full Bayesian framework. However, their Bayesian procedure is computationally intensive and impractical for high-dimensional settings involving many players or a relatively high latent dimension. Chen and Joachims (2016) treat the parameters as fixed quantities and estimate them by optimizing a regularized objective function. However, their objective function is non-convex, and their model is highly over-

parameterized. Consequently, their optimization is still computationally intensive and does not have a convergence guarantee. Moreover, no theoretical results are established in either work for their estimator.

Motivated by these challenges, we propose a general framework for modeling intransitive pairwise comparisons, assuming an approximately low-rank structure for the winning probability matrix. We propose an estimator for the probabilities, which can be efficiently solved by a convex optimization program. This estimator is shown to be optimal in the minimax sense, accommodating sparse data—a common issue when the number of players diverges. To our knowledge, this is the first framework to address intransitive comparisons with rigorous error analysis. The models presented in Chen and Joachims (2016) and Spearing et al. (2023), which assume a low-rank structure, can be seen as a special case of our framework. Furthermore, our method and computational algorithms scale efficiently to high-dimensional settings, making them suitable for applications with many players/teams/items. Empirical results on real-world datasets, including the e-sport *StarCraft II* and professional tennis, demonstrate the practical usefulness of our method, showing superior performance in intransitive settings and robust performance when transitivity largely holds.

Pairwise comparison data has been extensively studied in the statistics and machine learning literature, with numerous models and methods developed. We refer readers to Cattelan (2012) for a practical overview of the field. Theoretical properties of the BT model were first established in Simons and Yao (1999). These results were later extended to likelihood-based and spectral estimators, as well as other parametric extensions, with various losses and sparsity levels (Yan et al., 2012; Shah et al., 2016a; Negahban et al., 2017; Chen et al., 2019; Han et al., 2020; Chen et al., 2022a). More recently, Han et al. (2023) propose a general framework covering most parametric models satisfying strong stochastic transitivity, establishing uniform consistency results under sparse and heterogeneous settings.

Our development is also closely related to the literature on generalized low-rank and approximate low-rank models (Cai and Zhou, 2013; Davenport et al., 2014; Cai and Zhou, 2016; Chen et al., 2020; Chen and Li, 2022, 2024; Lee et al., 2024). While our asymptotic results and error bounds build on techniques from these works, the parameter matrix in the current work differs in that it has a skew-symmetric structure. This structure, which arises naturally from pairwise comparison data, leads to dependent data entries and distinguishes our setting from typical low-rank models. To address this, tailored analysis is performed to establish rigorous theoretical results.

The rest of the paper is organized as follows. Section 2 describes the setting, introduces the general approximate low-rank model, and proposes our estimator. Section 3 establishes

the theoretical properties of the proposed estimator, including results on convergence and optimality. In Section 4, we provide an algorithm for solving the optimization problem of the proposed estimator. Section 5 verifies the theoretical findings and compares the proposed model with the BT model using simulations. Section 6 applies the proposed method to two real datasets to explore the presence of intransitivity in sports and e-sports. Finally, we conclude with discussions in Section 7, and the appendix provides detailed proofs of our main results.

2 Generalized Approximate Low-rank Model for Pairwise Comparisons

2.1 Setting and Proposed Model

We consider a scenario with n subjects, such as players in a sports tournament. Let n_{ij} denote the total number of comparisons observed between subjects i and j , where $(n_{ij})_{n \times n}$ is a symmetric matrix. Let y_{ij} denote the observed counts where subject i beats subject j . Assuming no draws, we have $y_{ij} = n_{ji} - y_{ji}$ for $i, j \in \{1, \dots, n\}$.

Given the total comparisons n_{ij} , we model the observed counts y_{ij} using a Binomial distribution: $y_{ij} \sim \text{Binomial}(n_{ij}, \pi_{ij})$, where π_{ij} denotes the probability that subject i beats subject j . A fundamental property of the probabilities is that $\pi_{ij} = 1 - \pi_{ji}$ for all $i, j \in \{1, \dots, n\}$. This implies that the matrix $\Pi = (\pi_{ij})_{n \times n}$ is fully determined by its upper triangular part. Using the logistic link function $g(x) = (1 + \exp(-x))^{-1}$, we express the probabilities as $\pi_{ij} = g(m_{ij})$, where $M = (m_{ij})$ is a skew-symmetric matrix satisfying $M = -M^\top$. As a result, estimating the probabilities Π reduces to the problem of estimating M .

We say the model is stochastic transitive if there exists an unobserved global ranking among all the players, denoted by $i_1 \succ i_2 \succ \dots \succ i_n$, such that the pairwise comparison probabilities for the adjacent pairs satisfy $\pi_{i_1 i_2}, \pi_{i_2 i_3}, \dots, \pi_{i_{n-1} i_n} \geq 0.5$. In addition, $\pi_{ik} \geq \pi_{ij}$ whenever $j \succ k$, for all $i \neq j, k$. In other words, for two players, j and k , any player is more likely to win k than j if player j ranks higher than k . If stochastic transitivity does not hold, then we say a model is stochastic intransitive. For instance, stochastic intransitivity arises when there exists a triplet (i, j, k) , such that $\pi_{ik} \geq \pi_{ij}$ and $\pi_{jk} < 0.5$.

Most traditional models for pairwise comparison assume stochastic transitivity. For example, the BT model assumes $m_{ij} = u_i - u_j$, in which case, the global ranking of the players is implied by the ordering of u_i , $i = 1, \dots, n$. However, stochastic intransitivity naturally occurs in real-world competition data involving multiple strategies or skills. For example, in the professional competitions of the e-sport *StarCraft II*, players can choose

from a variety of combat units with differing attributes (e.g., building cost, attack range, toughness) during the game, leading to strategic decisions that can result in intransitivity. In fact, for the best predictive model that we learned for the *StarCraft II* data, more than 70% of the (i, j, k) triplets are estimated to violate the stochastic transitivity assumption, i.e., $\pi_{ik} \geq \pi_{ij}$ and $\pi_{jk} < 0.5$; see Section 6 for the details.

From the modeling perspective, stochastic transitivity is achieved by imposing strong monotonicity constraints on the parameter matrix M . To allow for stochastic intransitivity, we need to relax such constraints. Given $Y = (y_{ij})_{n \times n}$, the log-likelihood is

$$\begin{aligned}\mathcal{L}(M) &= \sum_{i=1}^n \sum_{j=1}^n y_{ij} \log(g(m_{ij})) \\ &= \sum_{i=1}^n \sum_{j>i} (y_{ij} \log(g(m_{ij})) + (n_{ij} - y_{ij}) \log(1 - g(m_{ij}))).\end{aligned}$$

To prevent overfitting while accommodating stochastic intransitivity, we impose a constraint on M to reduce the size of the parameter space. Specifically, we assume that M has an approximately low-rank structure enforced through a nuclear norm constraint:

$$\|M\|_* \leq C_n n, \quad (1)$$

where $\|\cdot\|_*$ denotes the nuclear norm, and $C_n > 0$ is a constant that may vary with n . The estimator is defined as:

$$\hat{M} = \arg \max_M \mathcal{L}(M) \text{ subject to } \|M\|_* \leq C_n n, M = -M^\top. \quad (2)$$

It is easy to see that the optimization in (2) is convex; see Section 4 for its computation.

2.2 Comparison with Related Work

We compare the proposed model with existing parametric models in the literature. Han et al. (2023) introduce a general framework for analyzing pairwise comparison data under the assumption of stochastic transitivity. In the current context, their model aligns with those proposed by Shah et al. (2016b) and Heckel et al. (2019), which are expressed as

$$\pi_{ij} = \Phi(u_i - u_j), \text{ and } \pi_{ji} = 1 - \Phi(u_i - u_j).$$

Here, $\Phi(\cdot)$ is any valid symmetric cumulative distribution function specified by the user, and $\mathbf{u} = (u_1, \dots, u_n)^\top$ is a latent score vector representing the strengths of the teams. This framework reduces to the Bradley-Terry (BT) model when $\Phi(\cdot) = g(\cdot)$, the logistic

function, and to the Thurstone model when $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Other models can be incorporated by specifying different forms of $\Phi(\cdot)$. The latent score \mathbf{u} is treated as a fixed parameter to be estimated, enabling the framework to handle a large number of players effectively. This parametric form, however, enforces a rank-2 structure on the parameter matrix, given by

$$\Pi = \Phi(\mathbf{u}\mathbf{1}_n^\top - \mathbf{1}_n\mathbf{u}^\top),$$

where $\mathbf{1}_n$ is an n -dimensional vector of ones.

Several attempts have been made in the literature to generalize this parametric form, allowing the rank of the underlying parameter matrix to exceed two and accommodate stochastic intransitivity. We should note that, since M is a skew-symmetric matrix, its rank must be even (e.g., Horn and Johnson, 2013). For instance, Chen and Joachims (2016) proposed a blade-chest-inner model, which can be expressed as

$$\Pi = g(AB^\top - BA^\top),$$

where A and B are $n \times K$ matrices. This model allows for a general rank- $2k$ parameter matrix, with the parameters in the frequentist sense. Similar to the parametrization in Chen and Joachims (2016), Spearing et al. (2023) propose a Bayesian model for pairwise comparison under stochastic intransitivity and further develop a Markov chain Monte Carlo algorithm for its computation. Both methods lack theoretical guarantees, such as convergence results or error bounds.

Our proposed method relaxes the requirement for an exact low-rank representation by only requiring an approximate low-rank structure specified by the nuclear norm. This offers a broad parameter space that covers the models proposed in Chen and Joachims (2016) and Spearing et al. (2023). In particular, if $\text{rank}(M) = 2k$ for some positive integer k , it follows that

$$\|M\|_* \leq \sqrt{2k}\|M\|_F \leq C_n n,$$

where $\|\cdot\|_F$ denotes the Frobenius norm, and C_n is a constant depending on the magnitude of the entries of M and its rank $2k$. The subscript n in C_n indicates that both the magnitude of the entries of M and its rank are allowed to grow with n . Moreover, the proposed model imposes no distributional assumptions on the parameter matrix M , making it more scalable for handling large numbers of players. Theoretical results, including convergence and error bounds, are presented in Section 3. As a remark, our estimation method and theoretical framework can be easily adapted when we replace the current assumption of the logistic form of the link function $g(\cdot)$ with other functions, such as the standard normal cumulative

distribution function used in the Thurstone model.

3 Theoretical Results

We establish convergence results and lower bounds for the estimator defined in (2) under settings with different data sparsity levels. For positive sequences $\{a_n\}$ and $\{b_n\}$, we denote $a_n \lesssim b_n$ if there exists a constant $\delta > 0$ that $a_n \leq \delta b_n$ for all n . Let \mathcal{K} denote the parameter space, defined as

$$\mathcal{K} = \{M \in \mathbb{R}^{n \times n} : \|M\|_* \leq C_n n, \quad M = -M^\top\}. \quad (3)$$

We impose the following conditions:

Assumption 1. *The true parameter $M^* \in \mathcal{K}$.*

Assumption 2. *For $j = 1, \dots, n$ and $i > j$, the variables n_{ij} are independent and follow a Binomial distribution, $n_{ij} \sim \text{Binomial}(T, p_{ij,n})$, where T is a fixed integer representing the maximum possible number of comparisons between subjects, and $p_{ij,n} = p_{ji,n}$ is the success probability, which may vary across different pairs (i, j) . Let $0 \leq p_n \leq q_n \leq 1$ denote the minimum and maximum comparison rates, respectively, such that $p_{ij,n} \in [p_n, q_n]$ for all $i \neq j \in \{1, \dots, n\}$. We assume that $p_n \asymp q_n$ and $p_n \gtrsim \log(n)/n$.*

Assumption 1 ensures that the true parameter exhibits an approximately low-rank structure specified by our model. Assumption 2 deserves more explanations. Under this assumption, the sparsity level of the data is characterized by the rate at which the success probabilities $p_{ij,n}$ converge to 0 as n grows. The condition $p_n \gtrsim \log(n)/n$ sets a lower bound on the sparsity level, which is the best possible threshold for pairwise comparison problems. Below this bound, the comparison graph becomes disconnected with high probability (Erdős and Rényi, 1960; Han et al., 2023). The condition $p_n \asymp q_n$ imposes homogeneity on $p_{ij,n}$, a common assumption in the literature (Simons and Yao, 1999; Chen et al., 2019; Han et al., 2020). The following theorem establishes the convergence rate of the proposed estimator.

Theorem 1. *Under Assumptions 1 and 2, let $\hat{\Pi} = (\hat{\pi}_{ij})_{n \times n}$, where $\hat{\pi}_{ij} = g(\hat{m}_{ij})$. Further let $\Pi^* = g(M^*)$. Then, with probability at least $1 - \kappa_1/n$,*

$$\frac{1}{n^2 - n} \|\hat{\Pi} - \Pi^*\|_F^2 \leq \kappa_2 C_n \sqrt{\frac{1}{p_n n}},$$

where κ_1 and κ_2 are constants that do not depend on n .

The following theorem addresses the optimality of Theorem 1 by establishing a lower bound.

Theorem 2. *Suppose $12 \leq C_n^2 \leq \min\{1, \kappa_3^2/T\}n$, where κ_3 is an absolute constant specified in (12). Consider any algorithm which, for any $M \in \mathcal{K}$, takes as input Y and returns \hat{M} . Then there exists $M \in \mathcal{K}$ such that with probability at least $3/8$, $\Pi = g(M)$ and $\hat{\Pi} = g(\hat{M})$, satisfy*

$$\frac{1}{n^2 - n} \|\Pi - \hat{\Pi}\|_F^2 \geq \min \left\{ \kappa_4, \kappa_5 C_n \sqrt{\frac{1}{np_n}} \right\} \quad (4)$$

for all $n > N$. Here $\kappa_4, \kappa_5 > 0$ and N are absolute constants.

A few technical assumptions are imposed in this theorem. The condition $C_n^2 \leq \min\{1, \kappa_3^2/T\}n$ is mild and naturally holds for sufficiently large n , provided that the rank of M does not grow at the same rate as n . We also require $C_n^2 \geq 12$ to avoid the parameter space being too small for packing set construction.

Since the rates in Theorems 1 and 2 match up to a multiplicative constant, the optimality of the proposed estimator is established.

4 Computation

To solve (2), we apply the nonmonotone spectral-projected gradient algorithm for closed convex sets proposed by Birgin et al. (2000), which guarantees convergence to a stationary point satisfying the constraints. Let Skew_n denote the space of $n \times n$ skew-symmetric matrices. Let \mathcal{V} be the bijective linear mapping that vectorizes the upper-triangular part of any matrix in Skew_n into $\mathbb{R}^{0.5n(n-1)}$. For any $\mathbf{m} \in \mathbb{R}^{0.5n(n-1)}$, define $f(\mathbf{m}) = \mathcal{L}(\mathcal{V}^{-1}(\mathbf{m}))$. Then, solving (2) is equivalent to solving the constrained optimization problem:

$$\hat{\mathbf{m}} = \arg \max_{\mathbf{m} \in \mathbb{R}^{0.5n(n-1)}} f(\mathbf{m}) \quad \text{subject to } \|\mathcal{V}^{-1}(\mathbf{m})\|_* \leq \tau, \quad (5)$$

where $\tau = C_n n$ if C_n is known. We will later discuss an algorithm for selecting τ in practical situations where C_n is unknown.

A key step in solving (5) involves the orthogonal projection operator $P_\tau(\cdot)$, defined as

$$P_\tau(\mathbf{m}) = \arg \min_{\mathbf{x} \in \mathbb{R}^{0.5n(n-1)}} \|\mathbf{x} - \mathbf{m}\|_2 \text{ subject to } \|\mathcal{V}^{-1}(\mathbf{x})\|_* \leq \tau.$$

It is well known that the projection is equivalent to singular value soft-thresholding. Let $0_{n \times n}$ denote a $n \times n$ zero matrix, and $\max\{\cdot, \cdot\}$ be applied entry-wise for matrix inputs. The detailed procedure is presented in Algorithm 1.

Algorithm 1 Projection algorithm

Input: Parameter vector \mathbf{m} and nuclear norm constraint parameter τ .

Compute $M = \mathcal{V}^{-1}(\mathbf{m})$.

Perform singular value decomposition and obtain $M = U\Sigma V^\top$, where U and V are $n \times n$ orthonormal matrices and $\Sigma = \text{diag}(\sigma_1, \sigma_1, \dots, \sigma_{n/2}, \sigma_{n/2})$ if n is even and $\Sigma = \text{diag}(\sigma_1, \sigma_1, \dots, \sigma_{\lfloor n/2 \rfloor}, \sigma_{\lfloor n/2 \rfloor}, 0)$ otherwise.

Compute λ , the smallest value for which $\sum_{i=1}^{\lfloor n/2 \rfloor} 2 \max\{\sigma_i - \lambda, 0\} \leq \tau$.

Compute projected matrix $P_\tau(M) = U \max\{\Sigma - \lambda I_n, 0_{n \times n}\} V^\top$

Output: Projection outcome $P_\tau(\mathbf{m}) = \mathcal{V}(P_\tau(M))$.

In the last step, the projection outcome is defined as $P_\tau(\mathbf{m}) = \mathcal{V}(P_\tau(M))$, which is only valid provided that $P_\tau(M)$ is a skew-symmetric matrix. The following proposition ensures that this is always the case:

Proposition 1. *For any matrix $M \in \text{Skew}_n$, the projection operator satisfies $P_\tau(M) \in \text{Skew}_n$.*

Proof. We consider the case where n is even; the proof for odd n is analogous. It is well known that M can be decomposed in the Murnaghan canonical form $M = QXQ^\top$ (Murnaghan and Wintner, 1931; Benner et al., 2000), where Q is orthogonal and X is block-diagonal of the form

$$X = \begin{pmatrix} 0 & \sigma_1 & 0 & 0 & \dots & 0 & 0 \\ -\sigma_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 & \dots & 0 & 0 \\ 0 & 0 & -\sigma_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_{n/2} \\ 0 & 0 & 0 & 0 & \dots & -\sigma_{n/2} & 0 \end{pmatrix},$$

where $\sigma_1, \dots, \sigma_{n/2}$ are the singular values of M . It can be verified that the projection operator $P_\tau(M)$ preserves the Murnaghan canonical form as $P_\tau(M) = QYQ^\top$, where

$$Y = \begin{pmatrix} 0 & \max\{\sigma_1 - \lambda, 0\} & 0 & \dots & 0 \\ -\max\{\sigma_1 - \lambda, 0\} & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & \max\{\sigma_{n/2} - \lambda, 0\} \\ 0 & 0 & 0 & -\max\{\sigma_{n/2} - \lambda, 0\} & 0 \end{pmatrix}.$$

Hence we have $P_\tau(M) \in \text{Skew}_n$. \square

We now introduce the spectral projected line search method, which uses the projection operator $\mathcal{P}_\tau(\cdot)$ to ensure that each iteration's outcome remains within the feasible set defined by the nuclear norm constraint. The procedure is outlined in Algorithm 2.

The method employs two types of line searches. The first type performs a projection once and searches along a linear trajectory $\mathbf{m}(\alpha)$. This approach is computationally efficient since the primary computational cost lies in the projection operation. If the linear search fails to converge, the algorithm switches to a curvilinear trajectory $\mathbf{m}^{\text{curve}}(\alpha)$, which requires projecting at each step. The Spectral-step length γ_{l-1} is decided using the method from Barzilai and Borwein (1988) in each iteration.

Algorithm 2 Spectral projected line search

Input: Parameter vector from last iteration $\mathbf{m}^{(l-1)}$, Matrix of comparison outcomes Y , nuclear norm constraint parameter τ and the spectral-step length γ_{l-1}

Compute gradient: $\mathbf{g}^{(l-1)} = \nabla f(\mathbf{m}^{(l-1)})$.

Compute search direction: $\mathbf{d}^{(l-1)} = P_\tau(\mathbf{m}^{(l-1)} - \gamma_{l-1}\mathbf{g}^{(l-1)}) - \mathbf{m}^{(l-1)}$

Perform line search along the linear trajectory: $\mathbf{m}(\alpha) = \mathbf{m}^{(l-1)} + \alpha\mathbf{d}^{(l-1)}$.

if Convergence is reached **then**

 Set $\mathbf{m}^{(l)}$ as the result from the line search.

else

 Perform line search along the alternative trajectory:

$$\mathbf{m}^{\text{curve}}(\alpha) = P_\tau(\mathbf{m}^{(l-1)} - \alpha\gamma_{l-1}\mathbf{g}^{(l-1)}).$$

 Set $\mathbf{m}^{(l)}$ as the result from the line search.

end if

Output: Updated parameter vector $\mathbf{m}^{(l)}$.

The final estimation procedure is detailed in Algorithm 3. The convergence criterion checks whether the optimality condition $P_\tau(\mathbf{m}^{(l)} - \nabla f(\mathbf{m}^{(l)})) = \mathbf{m}^{(l)}$ is approximately satisfied. Parts of the code are adapted from the SPGL1 package, originally implemented in MATLAB (Van Den Berg and Friedlander, 2008; Davenport et al., 2014). The proposed estimator is implemented in R, and the code is available at <https://github.com/ArthurLee51/PCWST>.

5 Simulation Results

We consider three distinct scenarios characterized by varying levels of sparsity. Specifically, we define p_n as $n^{-1} \log(n)$, $n^{-1/2}$, and $1/4$, corresponding to sparse, less sparse, and dense data, respectively. The parameter q_n is given by $4p_n$. Each $p_{ij,n}$ is then generated from a uniform distribution with range $[p_n, q_n]$.

Algorithm 3 Estimation Algorithm

Input: Matrix of comparison outcomes Y , nuclear norm constraint parameter τ .

Initialization: Set $l = 0$, $\mathbf{m}^{(0)} = \mathbf{0}_{0.5n(n-1)}$, the zero vector and set the spectral step-length $\gamma_0 = 1$.

while $l = 0$ **or** convergence criterion is not satisfied **do**

 Update $l \leftarrow l + 1$.

 Update $\mathbf{m}^{(l)}$ via line search using Algorithm 2 with inputs $\mathbf{m}^{(l-1)}$, Y , τ and γ_{l-1} .

 Update γ_l as proposed by Barzilai and Borwein (1988).

end while

Output: Estimated parameter matrix $\hat{M} = \mathcal{V}^{-1}(\mathbf{m}^{(l)})$.

The parameter matrix M is constructed as $\Theta J \Theta^\top$, where Θ is an $n \times 2k$ matrix, and J is a $2k \times 2k$ block diagonal matrix of the form

$$J = \begin{pmatrix} 0 & n & 0 & \dots & 0 \\ -n & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & n \\ 0 & 0 & 0 & -n & 0 \end{pmatrix}.$$

The matrix Θ is orthonormal, obtained via QR decomposition of a random matrix $Z \in \mathbb{R}^{n \times 2k}$, where each entry of Z is independently sampled from a standard normal distribution $N(0, 1)$. It can be verified that $\|M\|_* = 2kn$.

We conduct 50 simulations for $n = 500, 1000, 1500$, and 2000 , with k ranging from 1 to 10. Recall that the rank of M is $2k$. Additionally, the maximum number of comparisons, T , is fixed at 5, across all settings. We set $C_n = 2k$. The loss is computed as

$$\text{Loss} = (n^2 - n)^{-1} \|\hat{\Pi} - \Pi^*\|_F^2, \quad (6)$$

and the average loss across 50 simulations is reported for each model in Figure 1, considering different values of n , k , and sparsity levels.

The results in Figure 1 show that the mean loss of the proposed estimator decreases as n increases. Moreover, the mean loss is significantly lower as the data become denser, corresponding to an increase in p_n . These observations are consistent with the results from Theorem 1.

Notably, the proposed and BT models incur higher losses as the rank parameter k increases, which is expected due to increasing complexity. However, the proposed model consistently outperforms the BT model across all settings. Furthermore, while the BT model's performance remains relatively unchanged as n increases, the proposed method

Loss Comparison by Sparsity Level and Sample Size

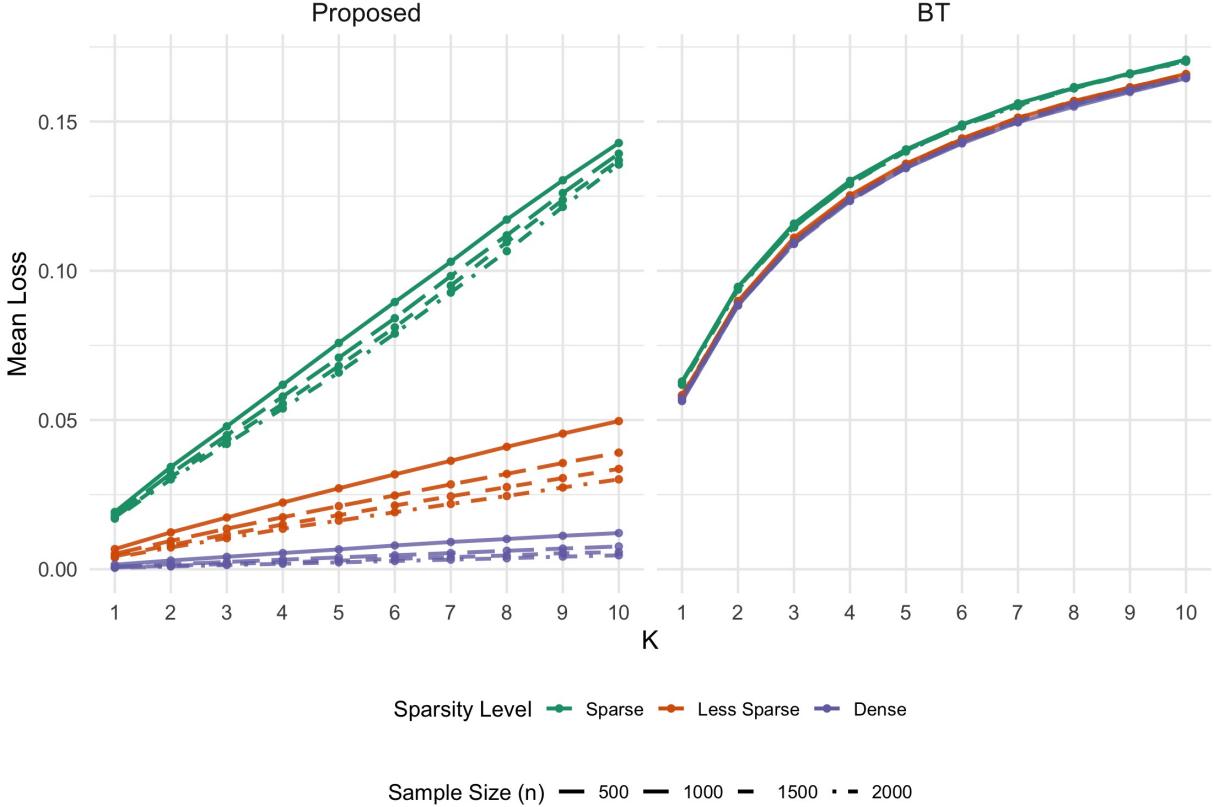


Figure 1: Comparison of loss between the proposed method and the Bradley-Terry (BT) model across different sparsity levels (sparse, less sparse, dense). The x-axis represents the rank parameter k , while the y-axis shows the mean loss, computed as the average of the losses defined in (6). Results are shown for varying sample sizes ($n = 500, 1000, 1500, 2000$)

continues to improve, showcasing its effectiveness in handling large datasets and capturing complex structures that stochastic transitivity assumptions cannot address.

6 Real Data Examples

In this section, we compare our model's performance with the celebrated BT model using two real datasets. Section 6.1 outlines the data preparation process and describes how the nuclear norm constraint parameter $\tau = C_n n$ is decided. Section 6.2 introduces the evaluation metrics used to compare the models. Finally, Sections 6.3 and 6.4 present detailed analyses of the results for the *StarCraft II* and tennis datasets, respectively.

6.1 Data Preparation and Parameter Tuning

The raw data consists of individual match records, with each comparison recorded as a separate entry. We reserve 30% of the match records for testing, while the remaining 70% is divided into 50% for training and 20% for validation.

The comparison data matrix is first constructed for the training set, with players absent from the training set removed from the validation set. The validation set is used to tune the nuclear constraint parameter C_n , as described in the sequel. After tuning, the training and validation sets are combined (including previously excluded entries), and the comparison data matrix is reconstructed from the combined dataset.

The test set is then evaluated against this combined dataset, excluding entries for players not present in the combined dataset. Although the proposed model can handle players who never lose or win any game, we still remove them in the training and combined dataset to ensure stabler results and a fair comparison with the BT model, as this is a common practice.

The nuclear norm of the parameter matrix M is unknown and is tuned on the training and validation sets using log-likelihood as the loss function. The nuclear constraint parameter $\tau = C_n n$ is determined by selecting C_n from 20 grid points, corresponding to powers of 10 evenly spaced between -1 and 1 . This results in $C_n = 10^{0.47} = 2.98$ for the *StarCraft II* dataset and $C_n = 10^{-0.36} = 0.43$ for the tennis dataset.

6.2 Evaluation Criteria

Let $Y^{(\text{test})} = (y_{ij}^{(\text{test})})_{n \times n}$ denote the observed comparison results from the test set. Given the estimated winning probabilities $\hat{\Pi} = (\hat{\pi}_{ij})_{n \times n}$, we evaluate the performance of the estimates using two criteria. The first criterion is the log-likelihood, given by

$$L(Y^{(\text{test})} | \hat{\Pi}) = \sum_{i=1}^n \sum_{j>i} \left(y_{ij}^{(\text{test})} \log(\hat{\pi}_{ij}) + y_{ji}^{(\text{test})} \log(1 - \hat{\pi}_{ij}) \right),$$

where a higher log-likelihood indicates a stronger agreement between the predicted probabilities and the observed results. The second criterion is the test accuracy, given by

$$A(Y^{(\text{test})} | \hat{\Pi}) = \frac{1}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}^{(\text{test})}} \sum_{i=1}^n \sum_{j>i} \left(y_{ij}^{(\text{test})} I(\hat{\pi}_{ij} \geq 0.5) + y_{ji}^{(\text{test})} I(\hat{\pi}_{ji} > 0.5) \right).$$

It measures the proportion of the comparison results correctly predicted, with higher values indicating better predictive performance. The results are presented in Table 1.

	<i>StarCraft II</i>		Tennis	
	Proposed	BT	Proposed	BT
Log-likelihood	−1,897,946	−2,137,115	−333,076	−322,483
Accuracy	0.766	0.713	0.652	0.658

Table 1: Comparison of model performance on StarCraft II and ATP datasets. The performance is evaluated using log-likelihood and accuracy for the proposed model and the BT model.

6.3 *StarCraft II* Data

StarCraft II is a military science fiction real-time strategy game developed and published by Blizzard Entertainment. The dataset comprises match results of professional *StarCraft II* players sourced from the website aligulac.com, covering the period from 2010 to 2016. The matches follow the most common competitive format, where two players face off against each other, and each game results in either a win or loss, with no possibility of a draw.

We specifically focus on matches played using the *StarCraft II: Heart of the Swarm* expansion, as different versions of the game are often treated as distinct games (Chen and Joachims, 2016). The training set includes 1,958 players, with 1.9% of all player pairs competing against each other at least once. The maximum number of matches between any pair of players is 30. The dataset is available at <https://www.kaggle.com/datasets/alimbekovz/starcraft-ii-matches-history>.

As seen in Table 1, the proposed model achieves a higher log-likelihood of −1,897,946 compared to −2,137,115 for the BT model. This suggests that our model provides a better fit for the observed test data. The test accuracy of the proposed model is also significantly higher at 0.766, compared to 0.711 for the BT model. Among the 1,249,168,756 distinct triplets in the data, stochastic transitivity is violated in 70% of cases, as indicated by the matrix of estimated probabilities $\hat{\Pi}$ under the proposed model. Specifically, this occurs when there exists an ordering of the three players, denoted as i , j , and k , such that $\hat{\pi}_{ik} \geq \hat{\pi}_{ij}$ and $\hat{\pi}_{jk} < 0.5$.

These results are consistent with previous findings by Chen and Joachims (2016), who analyzed a similar dataset over different time frames, suggesting that a strict ranking structure may not be appropriate in e-sports. In particular, intransititivity can naturally arise from game design, such as intransitive relationships among different unit types, which provide players with significant flexibility in choosing units and strategies. Moreover, the strong performance of our method on this dataset confirms its ability to effectively handle sparsity in real-world data, aligning with both simulation and theoretical results.

6.4 Tennis Data

We analyze the tennis dataset to evaluate the performance of our model in professional sports. The dataset contains the results of all men’s matches organized by the Association of Tennis Professionals (ATP) from 2000 to 2018. It includes matches from major tournaments such as the Grand Slams, the ATP World Tour Masters 1000, and other professional tennis series held during this period.

The training set consists of 723 players, with 6.4% of all player pairs having competed against each other at least once. The maximum number of matches between any pair of players is 23. The data is collected from <http://www.tennis-data.co.uk>.

From Table 1, the BT model achieves a marginally better performance, with a log-likelihood of $-322,483$ compared to $-333,076$ for the proposed model, and a slightly higher test accuracy (0.658 vs 0.652). This advantage may come from the BT model’s smaller parameter space, which is more efficient when the data aligns well with the stochastic transitivity assumption, where the level of intransitivity is minimal or absent. Nevertheless, the performance of the proposed model remains close to that of the BT model, demonstrating its robustness even in settings where transitivity holds. This flexibility is particularly useful when intransitivity is uncertain, as it maintains high accuracy without relying on strict ranking assumptions.

The lack of intransitivity in professional tennis may be due to several factors. Unlike e-sports, tennis offers limited gameplay flexibility, as adjustments to equipment like rackets and shoes have minimal impact compared to the choice of units in *StarCraft II*. Additionally, professional tennis players may be required to be well-rounded as weaknesses are quickly identified and exploited by opponents. In contrast, intransitivity may be more common at lower levels of competition, where skill imbalances are expected to be more significant. For example, a player with a strong serve but weak baseline play may be more likely to defeat one opponent while losing to another with a different style. Investigating intransitivity in lower-tier competitions remains an open question for future research.

7 Discussions

In this article, we propose a statistical framework for modeling stochastic intransitivity. The framework assumes an approximate low-rank structure in the parameter matrix, expressed through a nuclear norm constraint. Theoretical analysis demonstrates that the proposed estimator achieves optimal convergence rates under a wide range of data sparsity settings. Simulation and empirical analyses confirm that our model is superior to the Bradley-Terry model when the assumption of stochastic transitivity is violated.

Our framework stands apart from the existing literature by imposing an approximate

low-rank structure. To our knowledge, all existing methods for pairwise comparison data rely on exact low-rank models, even in the limited works that allow stochastic intransitivity. By accommodating a larger parameter space, our approach offers greater flexibility and applicability to a wider range of datasets. While this may lead to slightly reduced efficiency, our analysis of the tennis dataset demonstrates that the loss of efficiency is small when stochastic transitivity largely holds. Therefore, the proposed model may predict pairwise comparison results more accurately in many real-world applications. For example, for tournament data, the proposed may better predict the champion or the number of rounds each player can play, given historical data and the current tournament schedule.

The current research may be extended in several directions. Specifically, the current theoretical analysis focuses on the convergence of the loss $\|\hat{\Pi} - \Pi^*\|^2/(n^2 - n)$, which can be seen as a notion of convergence in an average sense (across entries of the comparison probability matrix). It can be strengthened by establishing the convergence results under the matrix max-norm loss $\|\hat{\Pi} - \Pi^*\|_\infty$, which may be achieved using the refinement techniques proposed in Chen and Li (2024). This notion of convergence ensures the consistency of each $\hat{\pi}_{ij}$. Moreover, it will be useful to further establish the asymptotic normality for each $\hat{\pi}_{ij} - \pi_{ij}^*$, which can be used to quantify the uncertainty associated with the estimated winning probabilities.

The proposed modeling framework also needs to be extended to accommodate more complex settings of pairwise comparisons. First, covariate information can be incorporated into the model to facilitate the prediction. For example, for many team sports tournaments (e.g., soccer and basketball), whether a team plays at their home court matters and should be included as a covariate. Second, pairwise comparison data are often collected over time, which is true for the *StarCraft II* and tennis data studied in Section 6. The current model ignores time information in data. To better predict future pairwise comparison results, it will be useful to model the comparison probabilities as a function of time. As a result, the estimation of these time-varying comparison probabilities will also differ substantially from the current procedure. Third, for pairwise comparison data produced by raters, which are commonly encountered in crowd-sourcing settings (e.g., Chen et al., 2013), characteristics of the raters, such as their reliability, affect the pairwise comparisons. In other words, the distribution of the comparison between two items depends not only on the pair of items but also on the rater who performs the comparison. In this regard, Chen et al. (2013) propose an extended version of the BT model that uses a rater-specific latent variable to account for raters' reliability. A similar extension can be made to the current model to simultaneously account for the raters' heterogeneity and the items' stochastic intransitivity.

A Proofs

The appendix presents the proofs of the main results. Section A.1 provides the proof of Theorem 1, while Section A.2 provides the proof of Theorem 2. Throughout this section, $\delta_0, \delta_1, \dots$ denote positive constants that do not depend on n . For two probability distributions \mathcal{P} and Q on a finite set A , $D(\mathcal{P}\|Q)$ will denote the Kullback-Leibler (KL) divergence,

$$D(\mathcal{P}\|Q) = \sum_{x \in A} \mathcal{P}(x) \log \left(\frac{\mathcal{P}(x)}{Q(x)} \right).$$

A.1 Proof of Theorem 1

For two scalars $x, z \in [0, 1]$, define the Hellinger distance as

$$d_H^2(x, z) = (\sqrt{x} - \sqrt{z})^2 + (\sqrt{1-x} - \sqrt{1-z})^2.$$

For $n \times n$ matrices $X = (x_{ij})_{n \times n}$ and $Z = (z_{ij})_{n \times n}$ where $X, Z \in [0, 1]^{n \times n}$, define

$$d_H^2(X, Z) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n d_H^2(x_{ij}, z_{ij}).$$

It is straightforward to show that $d_H^2(X, Z) \gtrsim \|X - Z\|_F^2 / (n^2 - n)$. Moreover, let $\|X\|_\infty = \max_{i,j} |x_{ij}|$ denotes the entry-wise infinity norm of X . We will first prove the theorem under an additional constraint that $\|M^*\|_\infty \leq \gamma$ and $\|\hat{M}\|_\infty \leq \gamma$ for some $\gamma > 0$, then send $\gamma \rightarrow \infty$ to recover Theorem 1. Formally, we prove the following theorem:

Theorem 3. *Under the conditions in Theorem 1, suppose in addition that $\|M^*\|_\infty \leq \gamma$. Let \hat{M} be a solution to (2) under the additional constraint that $\|\hat{M}\|_\infty \leq \gamma$. Then with probability at least $1 - \delta_1/n$,*

$$d_H^2(\hat{M}, M^*) \leq \delta_2 C_n \sqrt{\frac{1}{p_n n}},$$

where δ_1 and δ_2 are absolute constants.

Proof. Define $\bar{\mathcal{L}}(M) = \mathcal{L}(M) - \mathcal{L}(0_{n \times n})$. The following lemma is essential to proving Theorem 3:

Lemma 1. *Under the conditions in Theorem 3, we have*

$$P \left(\frac{1}{n^2} \sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))| \geq \delta_0 C_n \sqrt{\frac{T q_n}{n}} \right) \leq \frac{\delta_1}{n},$$

where δ_0 is an absolute constant, and $\mathcal{G} \subset \mathbb{R}^{n \times n}$ is defined as

$$\mathcal{G} = \{M \in \mathbb{R}^{n \times n} : \|M\|_* \leq C_n n, \|M\|_\infty \leq \gamma, M = -M^\top\}.$$

Before proving the lemma, we first show how Lemma 1 implies Theorem 3. For two scalars $x, z \in [0, 1]$, we abuse the notation of $D(\cdot \| \cdot)$ and define the divergence measure as

$$D(x \| z) = x \log\left(\frac{x}{z}\right) + (1-x) \log\left(\frac{1-x}{1-z}\right).$$

Similarly, for two matrices $X, Z \in [0, 1]^{n \times n}$, define

$$D(X \| Z) = \sum_{i=1}^n \sum_{j=1}^n D(x_{ij} \| z_{ij}).$$

For any choice of $M \in \mathcal{G}$, we have

$$\begin{aligned} & E(\bar{\mathcal{L}}(M) - \bar{\mathcal{L}}(M^*)) \\ &= E(\mathcal{L}(M) - \mathcal{L}(M^*)) \\ &= \sum_{i=1}^n \sum_{j>i} E \left(y_{ij} \log\left(\frac{g(m_{ij})}{g(m_{ij}^*)}\right) + (n_{ij} - y_{ij}) \log\left(\frac{1-g(m_{ij})}{1-g(m_{ij}^*)}\right) \right) \\ &= \sum_{i=1}^n \sum_{j>i} E \left(n_{ij} g(m_{ij}^*) \log\left(\frac{g(m_{ij})}{g(m_{ij}^*)}\right) + n_{ij} (1-g(m_{ij}^*)) \log\left(\frac{1-g(m_{ij})}{1-g(m_{ij}^*)}\right) \right) \\ &= -T \sum_{i=1}^n \sum_{j>i} p_{ij,n} D(g(m_{ij}^*) \| g(m_{ij})) \\ &\leq -0.5 T p_n D(\Pi^* \| \Pi). \end{aligned}$$

Note that $M^* \in \mathcal{G}$ by assumption. Therefore, for any $M \in \mathcal{G}$, we have

$$\begin{aligned} \bar{\mathcal{L}}(M) - \bar{\mathcal{L}}(M^*) &= E(\bar{\mathcal{L}}(M) - \bar{\mathcal{L}}(M^*)) + (\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))) - (\bar{\mathcal{L}}(M^*) - E(\bar{\mathcal{L}}(M^*))) \\ &\leq E(\bar{\mathcal{L}}(M) - \bar{\mathcal{L}}(M^*)) + 2 \sup_{X \in \mathcal{G}} |\bar{\mathcal{L}}(X) - E(\bar{\mathcal{L}}(X))| \\ &\leq -0.5 T p_n D(\Pi^* \| \Pi) + 2 \sup_{X \in \mathcal{G}} |\bar{\mathcal{L}}(X) - E(\bar{\mathcal{L}}(X))|. \end{aligned}$$

Moreover, from the definition of \hat{M} , we have $\hat{M} \in \mathcal{G}$ and $\mathcal{L}(\hat{M}) \geq \mathcal{L}(M^*)$. Therefore, we obtain

$$0 \leq -0.5 T p_n D(\Pi^* \| \hat{\Pi}) + 2 \sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|.$$

Applying Lemma 1, then with probability at least $1 - \delta_1/n$, we have

$$0 \leq \frac{-0.5Tp_n D(\Pi^* \|\hat{\Pi})}{n^2} + 2\delta_0 C_n \sqrt{\frac{Tq_n}{n}}.$$

This implies that

$$\frac{D(\Pi^* \|\hat{\Pi})}{n^2} \leq \frac{4\delta_0 C_n}{Tp_n} \sqrt{\frac{Tq_n}{n}} \lesssim \frac{4\delta_0 C_n}{\sqrt{Tp_n}} \sqrt{\frac{1}{n}}$$

by Assumption 2. Note that $d_H^2(\hat{\Pi}, \Pi^*) \leq n^{-2} D(\Pi^* \|\hat{\Pi})$ by Jensen's inequality combined with the fact that $(1-x) \leq \log(x)$. Hence Theorem 3 is proved. Theorem 1 then follows by the fact that $d_H^2(\hat{\Pi}, \Pi^*) \gtrsim \|\hat{\Pi} - \Pi^*\|_F^2/(n^2 - n)$ and taking the limit as $\gamma \rightarrow \infty$. \square

We now begin to prove Lemma 1.

Proof. For any $h > 0$, using Markov's inequality, we have

$$\begin{aligned} & P\left(\frac{1}{n^2} \sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))| \geq \delta_0 C_n \sqrt{Tq_n/n}\right) \\ &= P\left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h \geq \left(\delta_0 C_n n^{1.5} \sqrt{Tq_n}\right)^h\right) \\ &\leq \frac{E\left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h\right)}{\left(\delta_0 C_n n^{1.5} \sqrt{Tq_n}\right)^h}. \end{aligned} \tag{7}$$

The bound in Lemma 1 will be established by combining (7), deriving an upper bound on $E\left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h\right)$ and setting $h = \log(n)$. Note that we can write $\bar{\mathcal{L}}(M)$ as

$$\bar{\mathcal{L}}(M) = \sum_{i=1}^n \sum_{j>i} y_{ij} \log\left(\frac{g(m_{ij})}{g(0)}\right) + (n_{ij} - y_{ij}) \log\left(\frac{1-g(m_{ij})}{1-g(0)}\right).$$

By a symmetrization argument (Lemma 6.3 in Ledoux and Talagrand (1991)), we have

$$\begin{aligned} & E\left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h\right) \\ &\leq 2^h E\left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} \epsilon_{ij} \left\{ y_{ij} \log\left(\frac{g(m_{ij})}{g(0)}\right) + (n_{ij} - y_{ij}) \log\left(\frac{1-g(m_{ij})}{1-g(0)}\right) \right\} \right|^h\right), \end{aligned}$$

where $\epsilon_{i,j}$ are i.i.d. Rademacher random variables for $i, j = 1, \dots, n$. To bound the latter term, we apply a contraction principle (Theorem 4.12 in Ledoux and Talagrand (1991)).

From the assumption that $\|M\|_\infty \leq \gamma$, conditional on n_{ij} , for $n_{ij} \geq 1$,

$$n_{ij}^{-1} \left(y_{ij} \log \left(\frac{g(m_{ij})}{g(0)} \right) + (n_{ij} - y_{ij}) \log \left(\frac{1 - g(m_{ij})}{1 - g(0)} \right) \right)$$

is a contraction that vanish at 0. Thus, we have

$$\begin{aligned} E \left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h \right) &\leq (2^h)(2^h) E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} \epsilon_{ij} m_{ij} \right|^h \right) \\ &= 4^h E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} \epsilon_{ij} m_{ij} \right|^h \right). \end{aligned} \quad (8)$$

To bound $E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} \epsilon_{ij} m_{ij} \right|^h \right)$, we apply the skew-symmetric property of M and the fact that $n_{ij} = n_{ji}$ for $i, j \in \{1, \dots, n\}$. For any $M \in \mathcal{G}$, we have

$$\sum_{i=1}^n \sum_{j=1}^n n_{ij} \epsilon_{ij} m_{ij} = \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} - \epsilon_{ji}) m_{ij}.$$

On the other hand, for $h > 1$, by the convexity of $|\cdot|^h$, we have

$$\begin{aligned} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} \epsilon_{ij} m_{ij} \right|^h &= \left| 0.5 \left\{ \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} - \epsilon_{ji}) m_{ij} + \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} + \epsilon_{ji}) m_{ij} \right\} \right|^h \\ &\leq 0.5 \left(\left| \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} - \epsilon_{ji}) m_{ij} \right|^h + \left| \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} + \epsilon_{ji}) m_{ij} \right|^h \right). \end{aligned}$$

Since ϵ_{ji} and $-\epsilon_{ji}$ have identical distribution, after taking expectation, we have

$$\begin{aligned} E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} \epsilon_{ij} m_{ij} \right|^h \right) &\leq E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j>i} n_{ij} (\epsilon_{ij} - \epsilon_{ji}) m_{ij} \right|^h \right) \\ &= E \left(\sup_{M \in \mathcal{G}} \left| \sum_{i=1}^n \sum_{j=1}^n n_{ij} \epsilon_{ij} m_{ij} \right|^h \right) \\ &= E \left(\sup_{M \in \mathcal{G}} |\langle \mathcal{E} \circ \mathcal{N}, M \rangle|^h \right). \end{aligned} \quad (9)$$

Here, $\mathcal{E} = (\epsilon_{ij})_{n \times n}$, $\mathcal{N} = (n_{ij})_{n \times n}$ and $\mathcal{E} \circ \mathcal{N}$ represents the hadamard product between \mathcal{E} and \mathcal{N} , and $\langle X, Z \rangle = \sum_{i=1}^n \sum_{j=1}^n x_{ij} z_{ij}$ for any $n \times n$ matrices X and Z . Note that

$|\langle X, Z \rangle| \leq \|X\|_{op} \|Z\|_*$, where $\|\cdot\|_{op}$ is the Euclidean operator norm. Hence we have

$$\begin{aligned} E \left(\sup_{M \in \mathcal{G}} |\langle \mathcal{E} \circ \mathcal{N}, M \rangle|^h \right) &\leq E \left(\sup_{M \in \mathcal{G}} \|\mathcal{E} \circ \mathcal{N}\|_{op}^h \|M\|_*^h \right) \\ &\leq (C_n n)^h E \left(\|\mathcal{E} \circ \mathcal{N}\|_{op}^h \right). \end{aligned} \quad (10)$$

We can write $\mathcal{E} \circ \mathcal{N} = \sum_{i=1}^n \sum_{j=1}^n \epsilon_{ij} n_{ij} E_{ij}$, where E_{ij} is a $n \times n$ matrix with 1 at the (i, j) th entry and 0 otherwise. Following arguments similar to Section 4.3 of Tropp et al. (2015), and applying Theorem 4.1.1 in Tropp et al. (2015), for $t > 0$, we set $s = -t^{2/h}/(2 \max_j \{\sum_{i=1}^n n_{ij}^2\})$ such that

$$\begin{aligned} E(\|\mathcal{E} \circ \mathcal{N}\|_{op}^h | \mathcal{N}) &= \left(\int_0^\infty P(\|\mathcal{E} \circ \mathcal{N}\|_{op}^h \geq t) dt \right) \\ &\leq \left(\int_0^\infty 2n \exp \left(\frac{-t^{2/h}}{2 \max_j \{\sum_{i=1}^n n_{ij}^2\}} \right) dt \right) \\ &= \left(\int_0^\infty 2n \left(\frac{h}{2} (2 \max_j \{\sum_{i=1}^n n_{ij}^2\})^{h/2} s^{h/2-1} \right) \exp(-s) ds \right) \\ &= \left((nh) (2 \max_j \{\sum_{i=1}^n n_{ij}^2\})^{h/2} \int_0^\infty s^{h/2-1} \exp(-s) ds \right) \\ &= nh \Gamma(h/2) (2 \max_j \{\sum_{i=1}^n n_{ij}^2\})^{h/2}, \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. Taking expectation, we have

$$E(\|\mathcal{E} \circ \mathcal{N}\|_{op}^h) \leq nh \Gamma(h/2) 2^{h/2} E \left(\max_j \left\{ \sum_{i=1}^n n_{ij}^2 \right\}^{h/2} \right). \quad (11)$$

We aim to find a bound for $E(\max_j \{\sum_{i=1}^n n_{ij}^2\}^{h/2})$. Using Bernstein's inequality, for each j and all $t > 0$, we have

$$\begin{aligned} P \left(\left| \sum_{i=1}^n (n_{ij}^2 - E(n_{ij}^2)) \right| > t \right) &\leq 2 \exp \left(\frac{-t^2/2}{\sum_{i=1}^n \{E(n_{ij}^4) - (E(n_{ij}^2))^2\} + T^2 t / 3} \right) \\ &\leq 2 \exp \left(\frac{-t^2/2}{n T^4 q_n + T^2 t / 3} \right). \end{aligned}$$

In particular, for $t \geq 6nT^2q_n$, we have

$$P \left(\left| \sum_{i=1}^n (n_{ij}^2 - E(n_{ij}^2)) \right| > t \right) \leq 2 \exp(-t/T^2) = 2P(U_j > t/T^2),$$

where U_1, \dots, U_n are independent and identically distributed exponential random variables. Hence, we have

$$\begin{aligned}
& \left(E \left(\max_j \left\{ \sum_{i=1}^n n_{ij}^2 \right\}^{h/2} \right) \right)^{1/h} \\
&= \left(E \left(\max_j \left| \sum_{i=1}^n n_{ij}^2 - E(n_{ij}^2) + E(n_{ij}^2) \right|^{h/2} \right) \right)^{1/h} \\
&\leq 2 \left(E \left(\max_j \left| \sum_{i=1}^n n_{ij}^2 - E(n_{ij}^2) \right|^{h/2} \right) \right)^{1/h} + 2 \left(E \left(\max_j \left| \sum_{i=1}^n E(n_{ij}^2) \right|^{h/2} \right) \right)^{1/h} \\
&\leq 2\sqrt{nT^2q_n} + 2 \left(E \left(\max_j \left| \sum_{i=1}^n n_{ij}^2 - E(n_{ij}^2) \right|^h \right) \right)^{1/2h} \\
&= 2\sqrt{nT^2q_n} + 2 \left(\int_0^\infty P \left(\max_j \left| \sum_{i=1}^n n_{ij}^2 - E(n_{ij}^2) \right|^h \geq t \right) dt \right)^{1/2h} \\
&\leq 2\sqrt{nT^2q_n} + 2 \left\{ (6nT^2q_n)^h + \int_{(6nT^2q_n)^h}^\infty P \left(\max_j \left| \sum_{i=1}^n n_{ij}^2 - E(n_{ij}^2) \right|^h \geq t \right) dt \right\}^{1/2h} \\
&\leq 2\sqrt{nT^2q_n} + 2 \left\{ (6nT^2q_n)^h + 2 \int_{(6nT^2q_n)^h}^\infty P \left(\max_j \{U_j\}^h \geq t/T^{2h} \right) dt \right\}^{1/2h} \\
&\leq 2\sqrt{nT^2q_n} + 2 \left\{ (6nT^2q_n)^h + 2E \left(\max_j \{T^2U_j\}^h \right) \right\}^{1/2h} \\
&= 2\sqrt{nT^2q_n} + 2 \left\{ (6nT^2q_n)^h + 2T^{2h}E \left((\max_j \{U_j\})^h \right) \right\}^{1/2h}.
\end{aligned}$$

By standard computations for exponential random variables, we can obtain the inequality $E((\max_j \{U_j\})^h) \leq 2h! + \log^h(n)$. Thus, we have

$$\begin{aligned}
& \left(E \left(\max_j \left\{ \sum_{i=1}^n n_{ij}^2 \right\}^{h/2} \right) \right)^{1/h} \leq 2\sqrt{nT^2q_n} + 2 \left\{ (6nT^2q_n)^h + 2T^{2h}(2h! + \log^h(n)) \right\}^{1/2h} \\
&\leq 2T(1 + \sqrt{6})\sqrt{nq_n} + 2T(2)^{1/2h}(\sqrt{\log(n)} + 2^{1/2h}\sqrt{h}) \\
&\leq 2T(1 + \sqrt{6})\sqrt{nq_n} + 2T(2 + \sqrt{2})\sqrt{\log(n)}
\end{aligned}$$

using the choice $h = \log(n)$ in the final line. Combining this result with (11), we have

$$\begin{aligned} E(\|\mathcal{E} \circ \mathcal{N}\|_{op}^h)^{1/h} &\leq (nh\Gamma(h/2))^{1/h} \sqrt{2} \{2T(1 + \sqrt{6})\sqrt{nq_n} + 2T(2 + \sqrt{2})\sqrt{\log(n)}\} \\ &\leq \delta_3 T \sqrt{nq_n} \end{aligned}$$

for some constant $\delta_3 > 0$ by Assumption 2. Combining this with (8), (9) and (10), we obtain

$$E \left(\sup_{M \in \mathcal{G}} |\bar{\mathcal{L}}(M) - E(\bar{\mathcal{L}}(M))|^h \right)^{1/h} \leq (4T)(C_n n)(\delta_3) \sqrt{nq_n}.$$

Plugging this into (7), the probability in (7) is upper bounded by

$$\left\{ \frac{(4T)(C_n n)(\delta_3) \sqrt{nq_n}}{\delta_0 C_n n^{1.5} \sqrt{Tq_n}} \right\}^h \leq \left(\frac{4\sqrt{T}\delta_3}{\delta_0} \right)^{\log(n)} \leq \frac{\delta_1}{n},$$

provided that $\delta_0 \geq 4\sqrt{T}\delta_3/e$, which establishes the lemma. \square

A.2 Proof of Theorem 2

We first quote the following lemma from Davenport et al. (2014):

Lemma 2. *Suppose $x, z \in (0, 1)$. Then*

$$D(x\|z) \leq \frac{(x-z)^2}{z(1-z)}.$$

The following lemma constructs a packing set $\mathcal{X} \subset \mathcal{K}$ such that, for any distinct $X^{(a)}, X^{(b)} \in \mathcal{X}$, $\|X^{(a)} - X^{(b)}\|_F^2$ is large:

Lemma 3. *Let \mathcal{K} be defined as in (3), and k a positive integer. Let $\gamma \leq 1$ be such that k/γ^2 is an integer, and suppose $k/\gamma^2 \leq n$. Then, there exists a set $\mathcal{X} \subset \mathcal{K}$ satisfying*

$$|\mathcal{X}| \geq \exp \left(\frac{kn}{25600\gamma^2} \right)$$

with the following properties:

1. For all $X = (x_{ij})_{n \times n} \in \mathcal{X}$, each entry of X satisfies $|x_{ij}| \leq C_n \gamma / \sqrt{2k}$.
2. For all $X^{(a)} \neq X^{(b)} \in \mathcal{X}$,

$$\|X^{(a)} - X^{(b)}\|_F^2 > \frac{C_n^2 \gamma^2 n^2}{16k}.$$

Proof. We use a probabilistic argument. The set will be constructed by drawing

$$|\mathcal{X}| = \left\lceil \exp \left(\frac{kn}{25600\gamma^2} \right) \right\rceil$$

matrices independently from the following distribution. Set $B = k/\gamma^2$. Each matrix in \mathcal{X} is constructed of the form $S - S^\top$, where $S = (s_{ij})_{n \times n}$ consists of blocks of dimension $B \times n$, stacked vertically. The entries of the first block are independent and identically distributed symmetric random variables taking values $\pm C_n \gamma / (2\sqrt{2k})$. Then S is filled out by copying this block as many times as it fits. That is,

$$s_{ij} = s_{i'j}, \text{ where } i' = i \pmod{B} + 1.$$

Now we argue that with nonzero probability, this set will have all the desired properties. For $X \in \mathcal{X}$, it is easy to verify that $X = -X^\top$. Moreover, we have

$$\|X\|_\infty \leq 2\{C_n \gamma / (2\sqrt{2k})\} \leq C_n / \sqrt{2k}.$$

Further, since $\text{rank}(X) \leq 2\text{rank}(S) \leq 2B$,

$$\|X\|_* \leq \sqrt{2B} \|X\|_F \leq \sqrt{2k/\gamma^2} n (C_n \gamma / \sqrt{2k}) = C_n n.$$

Thus $\mathcal{X} \subset \mathcal{K}$, and it remains to show that \mathcal{X} satisfies property 2 in Lemma 3. Let $p = \lfloor n/B \rfloor$. Consider the submatrix of S containing the first B rows, denoted by $S_{[1:B,:]}$. This can be written as

$$S_{[1:B,:]} = (S_1, S_2, \dots, S_p, S_{p+1}),$$

where S_1, \dots, S_p are matrices of dimension $B \times B$, and S_{p+1} accounts for the remaining part of $S_{[1:B,:]}$. If n is divisible by B , then S_{p+1} is an empty matrix. For $X^{(a)} = S^{(a)} - (S^{(a)})^\top$ and $X^{(b)} = S^{(b)} - (S^{(b)})^\top$, drawn from the above distribution, define

$$\Theta_i = \frac{\sqrt{2k}}{C_n \gamma} \left(S_i^{(a)} - S_i^{(b)} \right), \quad \text{for } i = 1, \dots, p.$$

Each Θ_i is a $B \times B$ matrix, and we write $\Theta_i = (\theta_{i,sl})_{B \times B}$, where each $\theta_{i,sl}$ is independent and identically distributed random variables such that for each $s, l \in \{1, \dots, B\}$, we have

$$P(\theta_{i,sl} = 1) = P(\theta_{i,sl} = -1) = 0.25 \text{ and } P(\theta_{i,sl} = 0) = 0.5.$$

Hence we can write

$$\begin{aligned}
\|X^{(a)} - X^{(b)}\|_F^2 &\geq \sum_{i=1}^p \sum_{j=1}^p \|S_i^{(a)} - (S_j^{(a)})^\top - S_i^{(b)} + (S_j^{(b)})^\top\|_F^2 \\
&= \frac{C_n^2 \gamma^2}{2k} \sum_{i=1}^p \sum_{j=1}^p \|\Theta_i - \Theta_j^\top\|_F^2 \\
&= \frac{C_n^2 \gamma^2}{2k} \sum_{i=1}^p \sum_{j=1}^p (\|\Theta_i\|_F^2 + \|\Theta_j^\top\|_F^2 - 2\text{tr}(\Theta_i \Theta_j)) \\
&= \frac{C_n^2 \gamma^2}{2k} \left\{ 2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2\text{tr} \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \right\}.
\end{aligned}$$

The trace can be expanded as:

$$\begin{aligned}
\text{tr} \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) &= \sum_{s=1}^B \sum_{k=1}^B \left(\sum_{i=1}^p \theta_{i,sl} \right) \left(\sum_{i=1}^p \theta_{i,ls} \right) \\
&= 2 \sum_{s=1}^B \sum_{k>s} \left(\sum_{i=1}^p \theta_{i,sl} \right) \left(\sum_{i=1}^p \theta_{i,ls} \right) + \sum_{s=1}^B \left(\sum_{i=1}^p \theta_{i,ss} \right)^2.
\end{aligned}$$

Hence we can write

$$\begin{aligned}
&2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2\text{tr} \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \\
&= 2p \sum_{i=1}^p \sum_{s=1}^B \sum_{k=1}^B \theta_{i,sl}^2 - 4 \sum_{s=1}^B \sum_{k>s} \left(\sum_{i=1}^p \theta_{i,sl} \right) \left(\sum_{i=1}^p \theta_{i,ls} \right) - 2 \sum_{s=1}^B \left(\sum_{i=1}^p \theta_{i,ss} \right)^2 \\
&= 2 \sum_{s=1}^B \left\{ p \sum_{i=1}^p (\theta_{i,ss}^2) - \left(\sum_{i=1}^p \theta_{i,ss} \right)^2 \right\} + 2 \sum_{s=1}^B \sum_{k>s} \left\{ p \sum_{i=1}^p (\theta_{i,sl}^2 + \theta_{i,ls}^2) - 2 \left(\sum_{i=1}^p \theta_{i,sl} \right) \left(\sum_{i=1}^p \theta_{i,ls} \right) \right\}.
\end{aligned}$$

Taking expectation, we have

$$\begin{aligned}
E \left(2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2\text{tr} \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \right) &= 2B \{p(0.5p) - 0.5p\} + \frac{2B(B-1)p^2}{2} \\
&= Bp^2 - Bp + B^2p^2 - Bp^2 \\
&= B^2p^2 - Bp.
\end{aligned}$$

Using the fact that $p = \lfloor n/B \rfloor \geq n/2B$ and $p \leq n/B$, we have

$$\begin{aligned}
& P \left(2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \leq n^2/8 \right) \\
&= P \left(-2p \sum_{i=1}^p (\|\Theta_i\|_F^2) + 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) + B^2 p^2 - Bp \geq -n^2/8 + B^2 p^2 - Bp \right) \\
&\leq P \left(-2p \sum_{i=1}^p (\|\Theta_i\|_F^2) + 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) + B^2 p^2 - Bp \geq -n^2/8 + n^2/4 - n \right) \\
&\leq P \left(-2p \sum_{i=1}^p (\|\Theta_i\|_F^2) + 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) + B^2 p^2 - Bp \geq n^2/16 \right),
\end{aligned}$$

where the last inequality holds as long as $n \geq 16$. Using McDiarmid's inequality, we can obtain the bound

$$\begin{aligned}
P \left(2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \leq n^2/8 \right) &\leq \exp \left(-\frac{2(n^2/16)^2}{\sum_{s=1}^B \sum_{k=1}^B \sum_{i=1}^p (10p)^2} \right) \\
&= \exp \left(-\frac{n^4}{12800 B^2 p^3} \right) \\
&\leq \exp \left(-\frac{nB}{12800} \right).
\end{aligned}$$

Using Union bound, we have that

$$P \left(\min_{X^{(a)} \neq X^{(b)} \in \mathcal{X}} 2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) \leq n^2/8 \right) \leq \binom{|\mathcal{X}|}{2} \exp \left(-\frac{nB}{12800} \right),$$

which is less than 1 given the size of \mathcal{X} . Thus the event that

$$2p \sum_{i=1}^p (\|\Theta_i\|_F^2) - 2tr \left(\left(\sum_{i=1}^p \Theta_i \right) \left(\sum_{i=1}^p \Theta_i \right) \right) > n^2/8$$

for all $X^{(a)} \neq X^{(b)} \in \mathcal{X}$ has non-zero probability. In this event,

$$\|X^{(a)} - X^{(b)}\|_F^2 > \frac{C_n^2 \gamma^2}{2k} (n^2/8) = \frac{C_n^2 \gamma^2 n^2}{16k}.$$

The proof of the lemma is thus complete. \square

We now proceed to prove the following theorem, which concerns the lower bound treating n_{ij} as given.

Theorem 4. Suppose $12 \leq C_n^2 \leq \min\{1, \kappa_3^2/T\}n$. For any given n_{ij} , $i, j \in \{1, \dots, n\}$, $j > i$, consider any algorithm which, for any $M \in \mathcal{K}$, takes as input Y and returns \hat{M} . Then there exists $M \in \mathcal{K}$ such that with probability at least $3/4$, $\Pi = g(M)$ and $\hat{\Pi} = g(\hat{M})$ satisfy

$$\frac{1}{n^2 - n} \|\Pi - \hat{\Pi}\|_F^2 \geq \min \left\{ \kappa_4, \kappa_3 C_n \sqrt{\frac{n}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}}} \right\}. \quad (12)$$

for all $n > N$. Here $\kappa_3, \kappa_4 > 0$ and N are absolute constants.

Proof. Let $c = g'(-1) = g(-1)(1 - g(-1))$, and let $c' = g(-1)$. Note that for all $x \in [-1, 1]$, we have $g'(x) \geq c$ and $c' \leq g(x) \leq 1 - c'$. We begin by choosing ϵ so that

$$\epsilon^2 = \min \left\{ \frac{c}{64}, \kappa_3 C_n \sqrt{\frac{n}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}}} \right\}, \quad (13)$$

where κ_3 is an absolute constant to be determined. Let $k = 6$ and choose γ so that k/γ^2 is an integer and

$$4\sqrt{2} \frac{\epsilon \sqrt{2k}}{C_n c} \leq \gamma \leq \frac{8\epsilon \sqrt{2k}}{C_n c}.$$

This is possible since by assumption $C_n \geq \sqrt{12}$, $\epsilon \leq c/8$ and $c = 0.197$. One can check that γ satisfies the assumptions of Lemma 3. Note that for $X^{(i)} \neq X^{(j)} \in \mathcal{X}$,

$$\|g(X^{(i)}) - g(X^{(j)})\|_F^2 \geq c^2 \|X^{(i)} - X^{(j)}\|_F^2 > c^2 C_n^2 \gamma^2 n^2 / 16k \geq 4\epsilon^2 n^2. \quad (14)$$

Now suppose for the sake of a contradiction that there exists an algorithm such that for any $X \in \mathcal{K}$, it returns an \hat{X} such that

$$\frac{1}{n^2} \|g(X) - g(\hat{X})\|_F^2 \leq \epsilon^2. \quad (15)$$

with probability at least $1/4$. Define

$$X^* = \arg \min_{X^{(a)} \in \mathcal{X}} \frac{1}{n^2} \|g(X^{(a)}) - g(\hat{X})\|_F^2.$$

If (15) holds, then (14) implies that $X^* = X$. Thus, if (15) holds with probability at least $1/4$ then

$$P(X \neq X^*) \leq 3/4.$$

However, by a variant of Fano's inequality, we have

$$P(X \neq X^*) \geq 1 - \frac{n^2 \max_{X^{(a)} \neq X^{(b)}} D(Y | X^{(a)} \| Y | X^{(b)}) + 1}{\log |\mathcal{X}|}. \quad (16)$$

Since $y_{ij} + y_{ji} = n_{ij}$ (with n_{ij} given), the value of y_{ji} is determined by y_{ij} . Moreover, y_{ij} are independent for $i = 1, \dots, n, j > i$. Therefore,

$$D(Y | X^{(a)} \| Y | X^{(b)}) = \sum_{i=1}^n \sum_{j>i} D(y_{ij} | x_{ij}^{(a)} \| y_{ij} | x_{ij}^{(b)}).$$

Using Lemma 2, we have

$$\begin{aligned} D(y_{ij} | x_{ij}^{(a)} \| y_{ij} | x_{ij}^{(b)}) &\leq \frac{(g(C_n \gamma / \sqrt{2k}) - g(-C_n \gamma / \sqrt{2k}))^2}{g(C_n \gamma / \sqrt{2k})(1 - g(C_n \gamma / \sqrt{2k}))} \\ &\leq \frac{4(g'(\xi))^2 C_n^2 \gamma^2 / (2k)}{g(C_n \gamma / \sqrt{2k})(1 - g(C_n \gamma / \sqrt{2k}))} \\ &= \frac{4\{g(\xi)(1 - g(\xi))\}^2 C_n^2 \gamma^2 / (2k)}{g(C_n \gamma / \sqrt{2k})(1 - g(C_n \gamma / \sqrt{2k}))} \end{aligned}$$

for some $|\xi| \leq C_n \gamma / \sqrt{2k}$. Since $c' < g(x) < 1 - c'$ for $|x| < 1$, $g(\xi) \leq g(C_n \gamma / \sqrt{2k})$, and that

$$C_n \gamma / \sqrt{2k} \leq C_n \frac{8\epsilon \sqrt{2k}}{C_n c \sqrt{2k}} = \frac{8\epsilon}{c} \leq 1,$$

we have

$$D(y_{ij} | x_{ij}^{(a)} \| y_{ij} | x_{ij}^{(b)}) \leq \frac{4(1 - c')}{c'} \frac{64\epsilon^2}{c^2} = \delta_4 \epsilon^2,$$

where $\delta_4 = 256(1 - c')/(c' c^2)$. Thus, from (16), we have

$$\begin{aligned} \frac{1}{4} &\leq \frac{\delta_4 (\sum_{i=1}^n \sum_{j=1}^n y_{ij}) \epsilon^2 + 1}{\log(|\mathcal{X}|)} \leq \frac{25600 \gamma^2}{kn} \{ \delta_4 (\sum_{i=1}^n \sum_{j=1}^n y_{ij}) \epsilon^2 + 1 \} \\ &\leq \frac{3276800}{c^2} \epsilon^2 \left(\frac{\delta_4 (\sum_{i=1}^n \sum_{j=1}^n y_{ij}) \epsilon^2 + 1}{n C_n^2} \right). \end{aligned}$$

We now argue that this leads to a contradiction. Specifically, if $\delta_4 (\sum_{i=1}^n \sum_{j=1}^n y_{ij}) \epsilon^2 \leq 1$, then together with (13) implies that $n C_n^2 \leq 409600/c$. Since $C_n^2 \geq 2k$ by assumption, if we set $N > 204800/(kc)$, this would lead to a contradiction. Thus, suppose now that

$\delta_4(\sum_{i=1}^n \sum_{j=1}^n y_{ij})\epsilon^2 > 1$, in which case we have

$$\epsilon^2 \geq \frac{cC_n\sqrt{n}}{5120\sqrt{\delta_4(\sum_{i=1}^n \sum_{j=1}^n y_{ij})}}.$$

Thus setting $\kappa_3 \leq c/(5120\sqrt{\delta_4})$ in (13) leads to a contradiction, and hence (15) must fail to hold with probability at least $3/4$, which completes the proof. \square

We now apply Theorem 4 to prove Theorem 2. For any $\epsilon > 0$, Hoeffding's inequality allows us to derive that

$$\begin{aligned} & P\left(\sqrt{\frac{n}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}}} \geq \epsilon \sqrt{\frac{1}{np_n}}\right) \\ &= P\left(\sum_{i=1}^n \sum_{j=1}^n y_{ij} \leq \frac{n^2 p_n}{\epsilon^2}\right) \\ &= 1 - P\left(\sum_{i=1}^n \sum_{j>i} (n_{ij} - Tp_{ij,n}) \geq \frac{n^2 p_n}{\epsilon^2} - T \sum_{i=1}^n \sum_{j>i} p_{ij,n}\right) \\ &\geq 1 - P\left(\sum_{i=1}^n \sum_{j>i} (n_{ij} - Tp_{ij,n}) \geq \frac{n^2 p_n}{\epsilon^2} - \frac{Tn(n-1)q_n}{2}\right) \\ &\geq 1 - \exp\left(\frac{-2[(n^2 p_n/\epsilon^2) - \{Tn(n-1)q_n\}/2]^2}{T^2 n(n-1)/2}\right) \\ &= 1 - \exp\left(\frac{-\{(2n^2 p_n/\epsilon^2) - Tn(n-1)q_n\}^2}{T^2 n(n-1)}\right). \end{aligned}$$

To apply Theorem 4, it suffices to find ϵ such that

$$1 - \exp\left(\frac{-\{(2n^2 p_n/\epsilon^2) - Tn(n-1)q_n\}^2}{T^2 n(n-1)}\right) \geq 0.5$$

for sufficiently large n . From Assumption 2, we have $p_n \asymp q_n$ and $q_n \gtrsim \log(n)/n$. Consequently, there exists $\delta_5, \delta_6 > 0$ such that $p_n \geq \delta_5 q_n$ and $q_n \geq \delta_6 \log(n)/n$. Taking

$\epsilon = \sqrt{\delta_5/T}$, we have

$$\begin{aligned}
P\left(\sqrt{\frac{n}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}}} \geq \sqrt{\frac{\delta_5}{T}} \sqrt{\frac{1}{np_n}}\right) &\geq P\left(\sqrt{\frac{n}{\sum_{i=1}^n \sum_{j=1}^n y_{ij}}} \geq \sqrt{\frac{p_n}{Tq_n}} \sqrt{\frac{1}{np_n}}\right) \\
&\geq 1 - \exp\left(-\frac{(2Tn^2q_n - Tn(n-1)q_n)^2}{T^2n(n-1)}\right) \\
&= 1 - \exp\left(-\frac{T^2n^2q_n^2(n+1)^2}{T^2n(n-1)}\right) \\
&\geq 1 - \exp(-q_n^2(n+1)^2) \\
&\geq 1 - \exp(-\delta_6^2(\log(n))^2) \\
&\geq 1/2
\end{aligned}$$

for sufficiently large n . Therefore, the proof of Theorem 2 is complete by setting $\kappa_5 = \kappa_3 \sqrt{\delta_5/T}$.

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