More Math on Markov Chains

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Definition 1. The n-step transition probability, p_{ij}^n, is defined as

$$p_{ij}^n = \Pr\left(X_n = j | X_0 = i\right)$$

0.1 Irreducibility

Definition 2. We say that state i **leads to** state j (written as $i \to j$) if $\exists n \geqslant 0$ such that $p_{ij}^n > 0$.

Therefore, i communicates with j iff $i \to j$ and $j \to i$, written as $i \longleftrightarrow j$. This relation partitions the state space into disjoint sets of states, called communication classes.

Definition 3. A Markov Chain is **irreducible** if it only has one communication class.

0.2 Transient and Recurrent States

We define the first passage time to state i by $T_i = \min_n \{n \geqslant 1 : X_n = i\}$. Then we define

$$f_{ij}^m \equiv \Pr_i\left(T_j = m\right) = \Pr\left(X_m = j, X_{m-1} \neq j, \ldots, X_2 \neq j, X_1 \neq j | X_0 = i\right)$$

and

$$f_{ij} = \sum_{m=1}^{\infty} f_{ij}^m$$

Definition 4. A state i is **recurrent** if $f_{ii} = 1$ and **transient** if $f_{ii} < 1$.

In a finite irreducible Markov chain, all the states are recurrent.

0.3 Periodicity

Definition 5. State i is said to have period $d(i)$ if $d(i)$ is the greatest common divisor of all integers $n \geqslant 1$ for which $p_{ii}^n > 0$. If $\forall n, p_{ii}^n = 0$ then $d(i) = \infty$. State i is **aperiodic** if $d(i) = 1$.

More importantly, if $i \longleftrightarrow j$ then $d(i) = d(j)$.

0.4 Stationary Distribution

For finite Markov chains, the number of stationary distributions depends on the number of positive recurrent classes k:

- $k = 0$: no stationary distribution
- $k = 1$, unique stationary distribution
- $k \geqslant 2$, infinitely many distributions

In $k = 1$ case, the distribution would converge in the long run, iff the chain is aperiodic.