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MATH 112 - Discrete Mathematics L01

Definition of an even number -
An integer n is even if and only if there exists an integer k such that $n = 2k$

Therefore 0 is even $0 = 2(0)$

It is key to understand the way / form a definition is presented. It must contain "if" or "if and only if". Definitions are also difficult to make specific. The object being defined in the definition should be underlined.

e.g Definition -
A plastic object is recyclable iff it is plastic and a bottle.

Chapter 2 - Sets

A set is a well-defined collection of objects These objects are called elements of a set.

Def 2.2
$A \subseteq B$, A is a subset of B if and only if $x \in A$ implies $x \in B$ $(x \in A) \Rightarrow (x \in B)$

Sets can be defined as:

- $A = \{ a_1, a_2, a_3, \ldots \}$ (non-negative $\neq$ positive)
- $B = \{ x \in A \mid P(x) \}$

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Definition (2.3 and 2.4) - (Double Inclusion)
Let A and B be sets. We say A and B are equal ($A = B$) if and only if $A \subseteq B$ and $B \subseteq A$

Lemma (2.5) -
Any two empty sets are equal.

Proof:
Let A, B be empty sets. Recall Def 2.2
$A \subseteq B$ iff $(x \in A) \Rightarrow (x \in B)$
Therefore in this case False => False = True
Therefore $A \subseteq B$ and $B \subseteq A$ similarly

Definition 2.6 -
For two sets A and B some operations:
$A \cup B$ OR Union
$A \cap B$ AND Intersection
$A \setminus B$ set-theoretic difference ($A - B$)

Definition 2.8 -
Let A be a set. The power set of A, denoted $P(A)$ is the set of all subsets of A.

$$P(A) = \{ B \mid B \subseteq A \}$$

Example of power sets -

1) $A = \emptyset$ $P(A) = \{\emptyset\}$
2) $B = \{b\}$ $P(B) = \{\emptyset, \{b\}\}$
3) $C = \{1, 2\}$ $P(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Key Rules to know

Cartesian product -

Cartesian product of A and B is

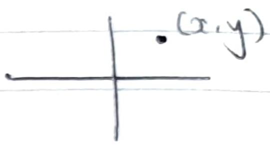
$$A \times B := \{(a,b) \mid a \in A, b \in B\}$$

$:=$ this equality is not something concluded but by definition. EQUAL BY DEFINITION

$A \times B$ is the set of ordered pairs of elements

Example -

1) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$



2) $A = \{a,b\}$ $B = \{0,1,2\}$

$$A \times B = \begin{Bmatrix} (a,0), (a,1), (a,2) \\ (b,0), (b,1), (b,2) \end{Bmatrix}$$

Def 2.12 Symmetric difference -

The symmetric difference of A and B is

$$A \Delta B := (A \cup B) \setminus (A \cap B)$$ This means XOR

Lemma 2.13

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$ Still XOR

Proof:

Let $x \in A \Delta B = (A \cup B) \setminus (B \cap A)$ $x \in A$ or $x \in B$ but $x \notin (A \cap B)$

a) $x \in A$ but $x \notin B$ $\quad x \in A \setminus B$

b) $x \in B$ but $x \notin A$ $\quad x \in B \setminus A$

Hence $x \in (A \setminus B) \cup (B \setminus A)$

Let $x \in (A \setminus B) \cup (B \setminus A)$ If $x \in A$ $x \in A \cup B$

If $x \in B$ $x \in A \cup B$ so $x \notin A \cap B$ $= A \Delta B$ $\square$

Def 2.14 -

Let $\mathcal{U}$ be a universal set and let $A \subseteq U$.
The complement of A in $\mathcal{U}$ is: NOT

$$A^c := \{x \mid x \in \mathcal{U} \text{ and } x \notin A\} = \mathcal{U} \setminus A$$

Def 2.16 -

Let I be a set. A collection of sets $\mathcal{A}$ is an I-indexed family of sets if there is a one-to-one correspondence between $\mathcal{A}$ and I.

$$\mathcal{A} = \{A_i \mid i \in I\}$$

Example -

$A = \{$sets of students at LU$\}$
$I = \{$set of majors$\}$
Let $I = \{1, 2, 3, 4\}$ and $A_i = [i-1, i+1]$
$= \{x \in \mathbb{R} \mid i-1 \leq x \leq i+1\}$

$A_1 = [0, 2]$ $A_2 = [1, 3]$ $A_3 = [2, 4]$
$A_4 = [3, 5]$

$\mathcal{A} = \{A_1, A_2, A_3, A_4\} = \{[0,2], [1,3], [2,4], [3,5]\}$ is a $\{1, 2, 3, 4\}$-indexed family of sets.

Formal definition of a way to write down a list of sets.

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Def 2.17 - Let $\mathcal{A} = \{A_i \mid i \in I\}$ be an I-indexed family of sets

there exists

$\bigcup_{i \in I} A_i := \{x \mid i \in I$ such that $x \in A_i\}$ is the union of $\mathcal{A}$

for every

$\bigcap_{i \in I} A_i := \{x \mid i \in I, x \in A_i\}$ is the intersection of $\mathcal{A}$

These allow us to show unions and intersections of multiple sets easier

Example -

$\mathcal{A} = \{A_1, A_2\}$ is a $\{1,2\}$-indexed family of sets.

$$\bigcap_{i \in I} A_i = \{x \mid (\forall i \in I)(x \in A_i)\}$$
$$= \{x \mid x \in A_1 \text{ and } x \in A_2\}$$
$$= A_1 \cap A_2$$

Example - (From L02)

$I = \{1,2,3,4\}$ $A_i = [i-1, i+1] = \{x \in \mathbb{R} \mid i-1 \leq x \leq i+1\}$

$A_1 = [0,2]$ $A_2 = [1,3]$ $A_3 = [2,4]$ $A_4 = [3,5]$

$\bigcup_{i \in I} A_i = [0,5]$ $\bigcap_{i \in I} A_i = \emptyset$

Proof:

Suppose $x \in \bigcup_{i \in I} A_i$ There exists $j \in I$ such $x \in A_j \subseteq [0,5]$ so $x \in [0,5]$ Hence $\bigcup_{i \in I} A_i \subseteq [0,5]$.

On the other hand,

Suppose $x \in [0,5]$. Consider

$\lfloor x \rfloor$ = largest integer smaller than x

$\lceil x \rceil$ = smallest integer bigger than x

if $1 \leq \lfloor x \rfloor < 5$:

then $j = \lfloor x \rfloor$ $x \in [j, j+1) \subseteq A_j \subseteq \bigcup_{i \in I} A_i$

if $\lfloor x \rfloor = 5$, $x = 5 \in A_4 \subseteq \bigcup_{i \in I} A_i$

if $\lfloor x \rfloor = 0$:

$x \in [0,1] \subseteq A_1 \subseteq \bigcup_{i \in I} A_i$

Hence $[0,5] \subseteq \bigcup_{i \in I} A_i$

Double Inclusion $[0,5] = \bigcup_{i \in I} A_i$ $\square$

To show $\bigcap_{i \in I} A_i = \emptyset$

clearly $\emptyset \subseteq \bigcap A_i$

$\bigcap_{i \in I} A_i \subseteq A_j \cap A_k$ for $j \neq k$

$A_1 = [0,2]$, $A_4 = [3,5]$ so $A_1 \cap A_4 = \emptyset$

Hence $\bigcap_{i \in I} A_i = \emptyset$ $\square$

Proposition 2.20 -

If $A \subseteq B$ $A \cup B = B$

If $A \subseteq B$ $A \cap B = A$

All of these can be proved

Corollary 2.21 -

$\emptyset \cup A = A$

$\emptyset \cap A = \emptyset$

$A \cup A = A$

$A \cap A = A$

Proposition 2.22

Remember $A^c = \neg A$

$\emptyset^c = U$

$U^c = \emptyset$

$(A^c)^c = A$

If $A \subseteq B$. $B^c \subseteq A^c$

Proof

Proposition 2.23

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$ Association

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distribution

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Theorem 2.25 -

Let $\mathcal{A} = \{A_i \mid i \in I\}$ be an I-indexed family of sets and B be any set

can be proved

a) $B \cup \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cup A_i)$

b) $B \cap \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cap A_i)$

c) $B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$

d) $B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i)$

Almost similar to distribution law.

Proof of d) -

We show $B \cup \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} (B \cup A_i)$

Let $x \in B \cup \bigcap_{i \in I} A_i$. Then $x \in B$ or $x \in \bigcap_{i \in I} A_i$

Suppose $x \in B$, $x \in B \cup A_i$ for each i since $B \subseteq B \cup A_i$. Hence $x \in \bigcap_{i \in I} (B \cup A_i)$

Suppose $x \notin B$, then $x \in \bigcap_{i \in I} A_i$, so $x \in A_i$ for each $i \in I$. Hence $x \in B \cup A_i$ for each i since $A_i \subseteq B \cup A_i$. Hence $x \in \bigcap_{i \in I} (B \cup A_i)$.

Therefore $B \cup \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} (B \cup A_i)$

Conversely, we show $\bigcap_{i \in I} (B \cup A_i) \subseteq B \cup \bigcap_{i \in I} A_i$.

Let $x \in \bigcap_{i \in I} (B \cup A_i)$. Thus, $x \in B \cup A_i$ for each $i \in I$. $x \in B$ or $x \in A_i$

Suppose $x \in B$, $x \in B \cup \bigcap_{i \in I} A_i$.

Suppose $x \notin B$, $x \in A_i$. Hence $x \in \bigcap_{i \in I} A_i \subseteq B \cup \bigcap_{i \in I} A_i$

Therefore $\bigcap_{i \in I} (B \cup A_i) \subseteq B \cup \bigcap_{i \in I} A_i$. Proved by double inclusion $\square$

Theorem 2.26 - (De Morgan's Laws)

$\mathcal{U}$ = Universal Set

a) $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$

b) $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$

Proof of a)

Let $x \in \mathcal{U}$. Then $x \in (\bigcup_{i \in I} A_i)^c \iff x \notin \bigcup_{i \in I} A_i$ complement

$\iff \neg((\exists i \in I)(x \in A_i))$ union definition

$\iff (\forall i \in I)(x \notin A_i)$ negation

$\iff (\forall i \in I)(x \in A_i^c)$ complement

$\iff x \in \bigcap_{i \in I} (A_i)^c$ intersection definition $\square$

Cardinality - 2.3

We assume all sets are finite in this chapter

$|A| :=$ number of elements in A

2.28 Proposition - (inclusion, exclusion for two sets)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Think of a venn-diagram

2.31 Proposition -

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

inclusion - exclusion three sets

2.33 Theorem - (inclusion, exclusion formula)

Let A_i be a finite set for $1 \leq i \leq n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{\substack{I = \{1, \dots, n\} \\ |I| = k}} \left|\bigcap_{i \in I} A_i\right| \right)$$

Proof:

Proceed by induction. By 2.28 prop the statement is true when $n=2$. Suppose $n>2$ and true for $n-1$

Let $B = \bigcup_{i=1}^{n-1} A_i$, $\bigcup_{i=1}^{n} A_i = B \cup A_n$

Hence $\left|\bigcup_{i=1}^{n} A_i\right| = |B \cup A_n| = |B| + |A_n| - |B \cap A_n|$ 2.28

By induction then,

$$|B| = |\bigcup_{i=1}^{n-1} A_i| = \sum_{k=1}^{n-1} (-1)^{k-1} \left(\sum_{|I|=k} |\bigcap_{i \in I} A_i| \right)$$

By 2.25,

$$B \cap A_n = \bigcup_{i=1}^{n-1} A_i \cap A_n = \bigcup_{i=1}^{n-1} (A_i \cap A_n)$$

2.25

By induction

$$|B \cap A_n| = |\bigcup_{i=1}^{n-1} (A_i \cap A_n)| = \sum_{k=1}^{n-1} (-1)^{k-1} \left(\sum_{|J|=k} |\bigcap_{j \in J} (A_j \cap A_n)| \right)$$

Therefore

$$|\bigcup_{i=1}^{n} A_i| = |B| + |A_n| - |B \cap A_n|$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \left(\sum_{|I|=k} |\bigcap_{i \in I} A_i| \right) + |A_n| - \sum_{k=1}^{n-1} (-1)^{k-1}$$

continuing on from line above

$$\cdots \left(\sum_{|J|=k} |\bigcap_{j \in J} A_j) \cap A_n| \right)$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \left(\sum_{|I|=k} |\bigcap_{i \in I} A_i| \right) \quad \square$$

Theorem 2.34

Let A_i be a finite set $1 \leq i \leq n$

$$\max_i |A_i| \leq |\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|.$$

Proposition 2.36

a) $|P(A)| = 2^{|A|}$

b) $|A \times B| = |A||B|$

Proof of a)

Induction $n=0, A=\emptyset$ $P(\emptyset) = \{\emptyset\}$ so $|P(A)| = 2^0 = 1$. True.

Let $x \in A$ $B = A \setminus \{x\}$

$|B| = n-1$ $|P(B)| = 2^{n-1}$ consider $X \subseteq A$, $x \in X$ or $x \notin X$. If $x \notin X$ $X \subseteq B$ and 2^{n-1} sets.

If $x \in X$ $X \setminus \{x\} \subseteq B$ and 2^{n-1} sets. Hence

$$|P(A)| = 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n \quad \square$$

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For curiosity. Broke Set Theory.

2.4 Russell's Paradox

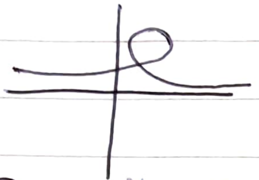
Let $S = \{\text{sets that are not elements of themselves}\}$

$A \in S \iff A \notin A$

Question: Does $S \in S$?

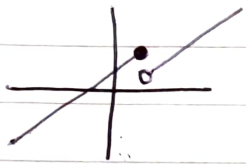
Suppose $S \in S$ then $S \notin S$ which is a paradox

Does there exist a function whose graph is



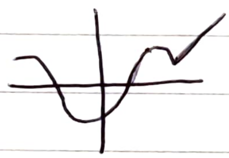
Yes - a few No - a lot

Does there exist a function:



Yes - almost all No - almost nobody

Does there exist a function:



Yes - most

- Does there exist a function which assigns every number but 0 it's square and assigns 0 to 1. Yes - most
- Does there exist a function where all values are equal to each other. Yes - most
- Does there exist a function whose values for non-integers are integral and values for integers are non-integral 50/50
- What is a function?

Chapter 3: Functions

Let X, Y be sets. Recall $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

Definition 3.1 -

Γ = gamma

Let X, Y be non-empty sets. A function can be defined $f: X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that for each element ~~...~~ $x \in X$ there exists a unique element $y \in Y$. Such that $(x,y) \in \Gamma$
Forbids $1 \to$ Many.

Notation and Terminology -

- Write $f(x)$ for the unique element of Y such that $(x, f(x)) \in \Gamma$. $f(x)$ is called the image of x under the application of f.
- X is called the domain of f
- Y is called the co-domain or target of f
- Γ is called the graph of f.

A function $f: X \to Y$ consists of 3 pieces of info. Domain X, co-domain Y, the "rule" $x \mapsto f(x)$

$\exists!$ = there exists a unique

Example -

① The set $\Gamma(f) = \{(a, a-1) \mid a \in \mathbb{R}\} \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ defines a function $f: \mathbb{R} \to \mathbb{R}$ given $f(x) = x - 1$
For each $x \in \mathbb{R}$ $\exists!$ $y \in \mathbb{R}$ such that $(x,y) \in \Gamma(f)$ ie $y = x - 1$.

② The set $\Gamma(g) = \{(a, a^2) \mid a \in \mathbb{Z} \setminus \{0\}\}$ defines a function $g: \mathbb{Z} \to \mathbb{N}$ given by $g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$
for each $x \in \mathbb{Z}$ there exists a unique $y \in \mathbb{N}$ such that $(x,y) \in \Gamma(g)$.

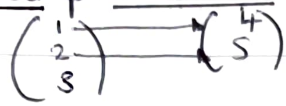
Example 3.4 -

The rule $g(x) = \sqrt{x}$ doesn't define a function $g: \mathbb{R} \to \mathbb{R}$ because it's undefined for negative real numbers

Not unique What about $h(x) = \sqrt{x}$, $h: \mathbb{R}_{\geq 0} \to \mathbb{R}$ e.g $\sqrt{4} = \pm 2$

- What about $k(x) = |\sqrt{x}|$, $k_{\geq 0} \to \mathbb{R}$. Does define a function. ✓

Example 3.5 -



$\Gamma_1 = \{(1,4),(2,5)\}$

$f_1: \{(1,2,3) \to (4,5)\}$? Does this define a function. No as existence fails for $(3,y)$

Remark -

Not everything in the co-domain need be the image of something in the domain. That is we do not require the following to hold.

- for each $y \in Y$ $\exists x \in X$ such that $(x,y) \in \Gamma$ or $f(x) = y$.

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Definition 3.7 -

Let $f: X - Y$ be a function. The image (range) of f is the set.

$\mathrm{Im} f := \{ y \in Y \mid \text{there exists } x \in X \text{ such } y = f(x) \}$

Examples -

- $f: \{1,2,3\} \to \{4,5\}$ defined by $f(1) = f(2) = f(3) = 4$ has image $\{4\} \subseteq \{4,5\}$
- $g: \{1,2,3\} \to \{4,5\}$ defined by $g(1) = g(3) = 4, g(2) = 5$
  $\mathrm{Im} g = \{4,5\} \subseteq \{4,5\}$
- $h: \{\mathbb{R} \to \mathbb{R}\}$ defined by $h(x) = x^2$
  $\mathrm{Im}\, h = \mathbb{R}_{\geq 0}$

  Proof:

  Let $y \in \mathbb{R}_{\geq 0}$. Set $x = |\sqrt{y}|$ then $h(x) = y$. $y \in \mathrm{Im} h$. Therefore $\mathbb{R}_{\geq 0} \subseteq \mathrm{Im} h$. Let $y \in \mathrm{Im} h$, then $y = x^2 \geq 0$ for some $x \in \mathbb{R}$, so $\mathrm{Im}\, h \subseteq \mathbb{R}_{\geq 0}$

3.2 Restriction and Extension -

$f: \mathbb{R} \to \mathbb{R}$ $\quad$ $g: \mathbb{Z} \to \mathbb{R}$ $\quad$ $h: \mathbb{Z} \to \mathbb{Z}$

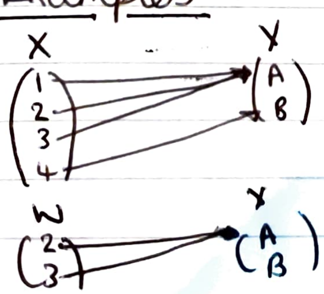
$x \mapsto x^2$ $\quad$ $x \mapsto x^2$ $\quad$ $x \mapsto x^2$

Definition 3.9 -

Restriction

Let $f: X \to Y$ be a function and $W \subseteq X$. The (domain) restriction of f to W is the function $f|_W : W \to Y$ whose graph is $\Gamma(f|_W) = \{(x,y) \in \Gamma(f) \mid x \in W\}$

Examples -



$f: \mathbb{R} \to \mathbb{R}$, $\quad$ $g: \mathbb{Z} \to \mathbb{R}$

$x \mapsto x^2$ $\quad$ $x \mapsto x^2$

$g = f|_{\mathbb{Z}}$

Extension

Definition 3.11 -

Let $f: X \to Y$ be a function with graph Γ and let Z be a superset of Y, ie $Y \subseteq Z$. The (co-domain) extension of f to Z,

$f|^Z : X \to Z$

is the function whose graph is $\Gamma(f|^Z) = \Gamma(f) \subseteq X \times Y$
$\subseteq X \times Z$

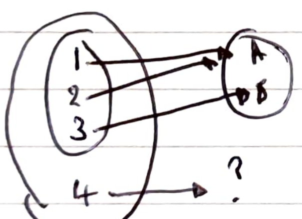
Examples -

$Y = \{A, B\}$

$h: \mathbb{Z} \to \mathbb{Z}$ $\quad$ $g: \mathbb{Z} \to \mathbb{R}$ $\quad$ $g|^{\mathbb{R}} = h$
$x \mapsto x^2$ $\quad$ $x \mapsto x^2$

Remark 3.12 -

- We can always do domain restriction and co-domain extension but domain extension and co-domain restriction are much more difficult and can't be done generally



Domain Extension?

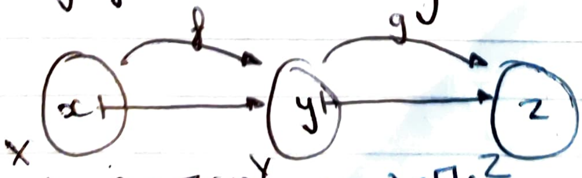
Co-domain restriction?

what happens to B?

Definition 3.13 -

Let $f: X \to Y$, $g: Y \to Z$ be functions. The composition of f and g written $g \circ f : X \to Z$ is the function whose graph is

$\Gamma(g \circ f) = \{(x,z) \mid \exists y \in Y \text{ s.t } (x,y) \in \Gamma(f) \text{ and } (y,z) \in \Gamma(g)\}$



$(x,y) \in \Gamma(f)$ $\quad$ $(y,z) \in \Gamma(g)$

Lemma 3.14 -

Let $f: X \to Y$, $g: Y \to Z$ be functions. The composition $g \circ f : X \to Z$ is a function.

Proof:

Check definition: for each $x \in X$ we show a unique $z \in Z$ such that $(x, z) \in \Gamma(g \circ f)$

Let $x \in X$, then there exists a unique $y \in Y$ such that $(x, y) \in \Gamma(f)$. For this y, there exists a unique $z \in Z$ such that $(y, z) \in \Gamma(g)$.

But by the definition of $g \circ f$ we have $(x, z) \in \Gamma(g \circ f)$

The element z is unique such that $(x, z) \in \Gamma(g \circ f)$ because y is uniquely determined by x and z is uniquely determined by y. $\square$

Remarks -

1) Composition is not necessarily commutative. In general $g \circ f \neq f \circ g$

   Unless $X = Z$. If $X = Z$ $f: X \to Y$, $g: Y \to X$ so

   $g \circ f = X \to X$ and $f \circ g = Y \to Y$

   unless $Y = X$, $g \circ f \neq f \circ g$

   Even if $X = Y = Z$, $g \circ f \neq f \circ g$ necessarily

2) $g \circ f$ means "do f first then g".

Proposition 3.17 -

Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be functions.

Then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

That is composition of functions is associative

Proof:

Domain of $h \circ (g \circ f)$ is A which is the domain of $(h \circ g) \circ f$.

Co-domain of $h \circ (g \circ f)$ is D which is the co-domain of $(h \circ g) \circ f$.

Let $x \in A$ then,

$(h \circ (g \circ f))(x) = h(g \circ f(x)) = h(g(f(x)))$

$= (h \circ g)(f(x)) = ((h \circ g) \circ f)(x)$

Hence, rules, domains and co-domains are the same so $h \circ (g \circ f) \equiv (h \circ g) \circ f$. $\square$

3.4 Injective, surjective and bijective functions - Definitions 3.18 → 3.20

Let $f : X \to Y$ be a function

Injective:

if for all $x_1, x_2 \in X$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Surjective:

if for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$

Bijective:

if f is both injective and surjective.

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Look at end of L06, Injective, Surjective, Bijective

Contraposition of injective: if $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$

Examples -

(a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

f is not injective: $1 \neq -1$ but $f(1) = f(-1)$

f is not surjective: $-1 \in \mathbb{R}$ but squares of real numbers must be non-negative so $\nexists x \in \mathbb{R}$ with $f(x) = -1$

(b) $k: \mathbb{R}_{\geq 0} \to \mathbb{R}$, $k(x) = |\sqrt{x}|$

k is injective: suppose $x_1, x_2 \in \mathbb{R}_{\geq 0}$ are such that $k(x_1) = k(x_2)$ Then $|\sqrt{x_1}| = |\sqrt{x_2}|$

Squaring both sides $x_1 = x_2$.

$-1 \in \mathbb{R}$ but $k(x) \geq 0$ $\forall x \in \mathbb{R}_{\geq 0}$ so $\nexists x \in \mathbb{R}_{\geq 0}$ such that $k(x) = -1$.

Definition 3.23 -

Let $f: X \to Y$ be a function and let $y \in Y$. The set

$$f^{-1}(y) := \{x \in X \mid f(x) = y\}$$

This is called the pre-image of y in X

If $Z \subseteq Y$ then

$$f^{-1}(Z) := \{x \in X \mid f(x) \in Z\}$$

is called the pre-image of Z in X.

Note: $f^{-1}(y)$ is a set. f doesn't need to be reversible

Lemma 3.24 -

Let X be a non-empty set. Then there exists a bi-jection $I_X : X \to X$ defined by $I_X(x) = x$ for all $x \in X$.

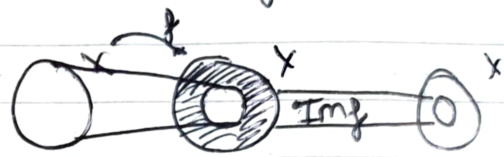
We call I_X the identity map on X

Lemma 3.25 -

Let $f : X \to Y$ be an injection. Then the co-domain restriction of f to $\mathrm{Im}f$, $g : X \to \mathrm{Im}f$, $x \mapsto f(x)$ is a bijection.

Idea:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & \mathrm{Im}f \arrow[ur, "\text{Inclusion}"'] & \end{tikzcd}$$



g is co-domain restriction of f to $\mathrm{Im}f$. Since $\mathrm{Im}f = \mathrm{Im}g$, g is surjective. And since f is injective so is g.

Lemma 3.27 -

Let $f : X \to Y$ be a function. Then there exists a set Z and functions $g : X \to Z$, $h : Z \to Y$ such that g is surjective, h is injective and $f = h \circ g$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z = \mathrm{Im}f \arrow[ur, "\text{Inclusion function}"'] & \end{tikzcd}$$

Proof -

Take $Z = \mathrm{Im}f \subseteq Y$ and $g : X \to Z$. The co-domain restriction of f which is surjective. $\mathrm{Im}f = \mathrm{Im}g$

Now let $h : Z \to Y$, $z \mapsto z$ be inclusion map.

This is injective. Now $(h \circ g)(x) = h(g(x)) = h(f(x)) = f(x)$ $\square$

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Proposition 3.28 -

Let $f: X \to Y$, $g: X \to Y$ be functions. $g: Y \to Z$

a) if f and g are injective $g \circ f$ is injective

b) if f and g are surjective $g \circ f$ is surjective

c) if $g \circ f$ is injective f is injective

d) if $g \circ f$ is surjective g is surjective

Proof:

a) Want to show $g \circ f$ is injective. Suppose $x_1, x_2 \in X$ are such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since g is injective, $g(f(x_1)) = (g \circ f)(x_1) = (g \circ f)(x_2) = g(f(x_2))$ $\Rightarrow f(x_1) = f(x_2)$. Since f is injective $x_1 = x_2$.

b) Want to show $g \circ f$ is surjective. for each $z \in Z$ there exists $x \in X$ such that $g \circ f(x) = z$.
Let $z \in Z$. Since g is surjective for this $y \in Y$ there exists $x \in X$ such that $f(x) = y$. By composition:
$g \circ f(x) = g(f(x)) = g(y) = z$
$g \circ f$ is surjective.

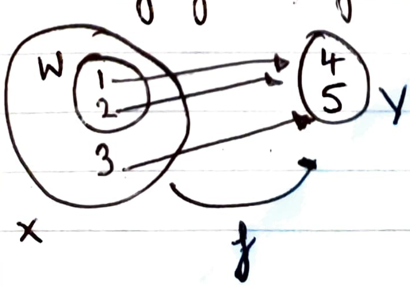
Corollary 3.29 -

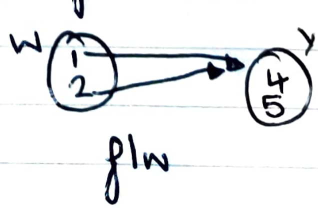
The composition of two bijections is a bijection.

Remark 3.30 -

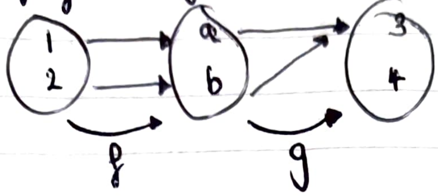
Let $f: X \to Y$ be a function and let $W \subseteq X$

* If f is injective so is $f|_W$
* If f is surjective then $f|_W$ need not be.





If f is injective, then $g \circ f$ need not be injective



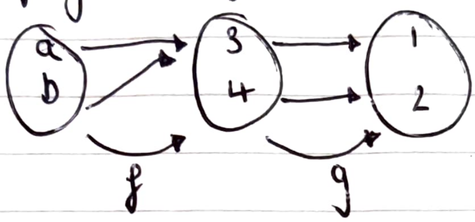
Let $f: \{1,2\} \to \{a,b\}$

$f(1) = a$, $f(2) = b$

$g: \{a,b\} \to \{3,4\}$ $g(a) = g(b) = 3$

$g \circ f(1) = 3$ $g \circ f(2) = 3$

If g is surjective, then $g \circ f$ need not be surjective



3.5 Inverses

Lemma 3.31 -

Let $f: X \to Y$, $g: Y \to X$ be functions. Then the following conditions are equivalent.

a) For each $x \in X$ and $y \in Y$, $f(x) = y$ iff $g(y) = x$

b) For each $x \in X$ $(g \circ f)(x) = x$, for each $y \in Y$, $(f \circ g)(y) = y$

Definition 3:31 -

Let $f: X \to Y$ be a function. We say that f is invertible if there exists a function $g: Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X (resp I_Y) is the identity map on X (resp Y) we call g an inverse for f.

Example 3.33 -

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x+3$ is invertible.
Find a candidate for the inverse: set $y = x+3$
then $x = y-3$.
Define $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x-3$. We claim g is an inverse of f:

$$(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3)-3 = x = I_{\mathbb{R}}(x)$$
$$(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)+3 = x = I_{\mathbb{R}}(x)$$

Hence g is an inverse of f.

Proposition 3.34 -

Let $f : X \to Y$ be an invertible function. Then there exists a unique inverse function for f.

Proof:

Suppose $g_1 : Y \to X$, $g_2 : Y \to X$ are two inverses of f. The domains and co-domains of g_1, g_2 are equal. We only need to check $g_1(y) = g_2(y)$ $\forall y \in Y$.

$$\begin{aligned}
g_1(y) &= g_1(I_Y(y)) && \text{(Identity map on } Y) \\
&= (g_1 \circ I_Y)(y) && \text{(composition)} \\
&= (g_1 \circ (f \circ g_2))(y) && \text{(inverse)} \\
&= ((g_1 \circ f) \circ g_2)(y) && \text{(associativity)} \\
&= (I_X \circ g_2)(y) && \text{(inverse)} \\
&= I_X(g_2(y)) && \text{(composition)} \\
&= g_2(y) && \text{(Identity map on } X)
\end{aligned}$$

Hence $g_1 = g_2$

Theorem 3.35 -

Let $f: X \to Y$ be a function. Then f is invertible iff f is a bijection.

Proof -

Suppose f is invertible with inverse $g: Y \to X$.
Then $g \circ f = I_X$ and $f \circ g = I_Y$. Both I_X and I_Y are bijections.

$g \circ f$ bijective $\Rightarrow$ $g \circ f$ injective $\Rightarrow$ f injective (3.28)
$f \circ g$ bijective $\Rightarrow$ $f \circ g$ surjective $\Rightarrow$ f surjective (3.28)

Hence f is bijective.

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Recap:

Theorem 3.35 -

Let $f: X \to Y$ be a function. Then f is invertible if and only if f is a bijection.

Proof:

($\Rightarrow$) Suppose f is invertible with inverse $g: Y \to X$. Then $g \circ f = I_X$ and $f \circ g = I_Y$. Both I_X and I_Y are bijections.

$g \circ f$ bijective $\Rightarrow$ $g \circ f$ injective $\Rightarrow$ f is injective

$f \circ g$ bijective $\Rightarrow$ $f \circ g$ surjective $\Rightarrow$ f is surjective

Hence f is a bijective.

($\Leftarrow$) Suppose $f: X \to Y$ is a bijection. We need to construct an inverse $g: Y \to X$.

Consider the graph $\Gamma(f) \subseteq X \times Y$. We define g using the graph $\Gamma(g) = \{(y, x) \in Y \times X \mid (x, y) \in \Gamma(f)\}$

We need to check $g: Y \to X$ defined in this way is a function. Once we have seen g is a function we need to check that g is an inverse for f.

That is ① $g \circ f = I_X$

② $f \circ g = I_Y$.

To see g is a function observe that for each $y \in X$, there exists $x \in X$ such that $f(x) = y$ because f is surjective. That is, for each $y \in X$, there exists $x \in X$ such that $(x, y) \in \Gamma(f)$.

But this means that $(y, x) \in \Gamma(g)$. Hence for each $y \in Y$ there exists a unique $x \in X$ such that $(y, x) \in \Gamma(g)$.

Suppose there exists $x_1, x_2 \in X$ such that $(y, x_1), (y, x_2) \in \Gamma(g)$. By definition of $\Gamma(g)$, we have $(x_1, y), (x_2, y) \in \Gamma(f)$. That is $f(x_1) = y = f(x_2)$.

But f is injective, so $x_1 = x_2$. There, for each $y \in X$, there exists a unique $x \in X$ such that $(y, x) \in \Gamma(g)$. Therefore g is a function.

To check $g \circ f = I_X$ we need to check $(g \circ f)(x) = x$ for each $x \in X$.

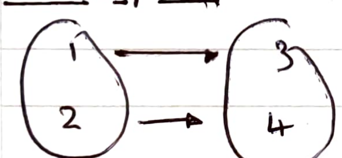
Let $x \in X$. Now $y = f(x) \iff (x, y) \in \Gamma(f)$
$$\iff (y, x) \in \Gamma(g)$$
$$\iff g(y) = x$$

Then $(g \circ f)(x) = g(f(x)) = g(y) = x$.

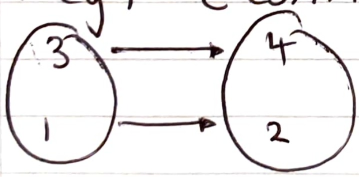
Check $f \circ g = I_Y$ - Exercise! $\square$

Instead of memorising the entire proof memorise the structure. Existence uses surjectivity. Uniqueness uses injectivity.

<u>Examples</u>:



$\Gamma(f) = \{(1,3), (2,4)\}$
$\Gamma(g) = \{(3,1), (4,2)\}$



<u>Proposition 3.38</u> -

Let $f: X \to Y$, $g: Y \to Z$ be invertible functions with inverses $f^{-1}: Y \to X$ and $g^{-1}: Z \to Y$. Then $g \circ f$ is invertible with inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

<u>Proof</u>

Proof:
We claim $f^{-1} \circ g^{-1}$ is an inverse for $g \circ f$.
Consider $(g \circ f) \circ (f^{-1} \circ g^{-1}) =$

$g \circ (f \circ f^{-1}) \circ g^{-1}$ By associativity
$= g \circ I_Y \circ g^{-1}$
$= g \circ g^{-1}$
$= I_Z$

Theorem 3.40 - (Schröder - Bernstein)
Next Lecture

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MATH 112 ~~L09~~ L10

3.6 The Schröder-Bernstein Theorem - (3.40) -

Let A and B be sets and $f: A \to B$ and $g: B \to A$ be functions. If both f and g are injective, then there exists a bijection $h: A \to B$

Example:

Consider $f, g: \mathbb{Z} \to \mathbb{Z}$ given by
$f(n) = 2n$ and $g(n) = 2n+1$

Both functions are injective but neither is surjective.

Definition 3.41 -

Let A be a set. An I-indexed family of subsets of A, $\{A_i \mid i \in I\}$ is a partition of A if:

1) $\bigcup_{i \in I} A_i = A$ and
2) $A_i \cap A_j = \emptyset$ for $i \neq j$

Example:

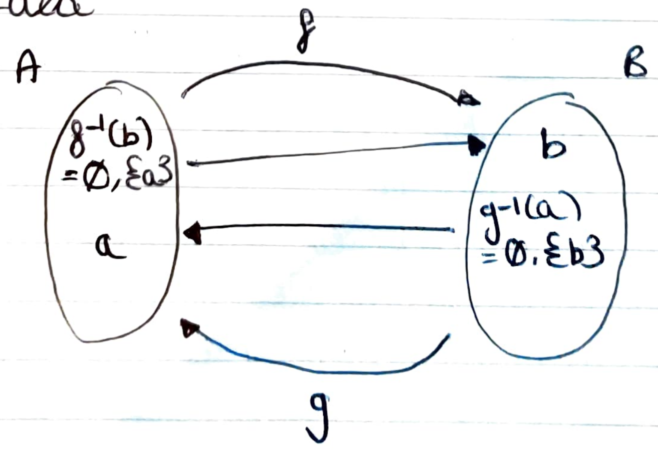
$A = \{1, 2, 3, \dots 10\}$, $\{\underset{A_1}{\{1,2,3\}}, \underset{A_2}{\{4,5,6,10\}}, \underset{A_3}{\{7\}}, \underset{A_4}{\{8,9\}}\}$

$\bigcup_{i=1}^{4} A_i = A$

But $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_1 \cap A_4 = \emptyset$
$A_2 \cap A_3 = \emptyset$, $A_2 \cap A_4 = \emptyset$, $A_3 \cap A_4 = \emptyset$

Proof of the Schröder-Bernstein Theorem -

Idea



Step 1:

For each $a \in A$, there exists at most $b \in B$ such that $g(b) = a$

Proof:

Let $a \in A$, suppose $b_1, b_2 \in B$ are such that $g(b_1) = a = g(b_2)$. But g is injective so $b_1 = b_2$. If such a b exists we call it an ancestor of a and write $b = g^{-1}(a)$

Step 2:

For each $b \in B$, there exists at most $a \in A$ such that $f(a) = b$. We call such an a if exists an ancestor of b and write $a = f^{-1}(b)$.

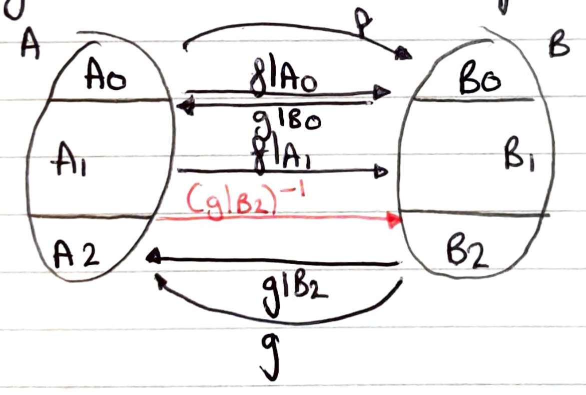
Define subsets of A as follows

$A_0 = \{a \in A \mid$ the chain of ancestors of a continues forever $\}$

$A_1 = \{a \in A \mid$ the chain of ancestors of a stops in $A\}$

$A_2 = \{a \in A \mid$ the chain of ancestors of a stops in $B\}$

Define subsets B_0, B_1, B_2 of B similarly



Step 3:

$\{A_0, A_1, A_2\}$ is a partition of A. We have $A_0 \cap A_1 = \emptyset$, $A_0 \cap A_2 = \emptyset$ and $A_1 \cap A_2 = \emptyset$. Clearly $A_0 \cup A_1 \cup A_2 \subseteq A$.

Let $a \in A$. There are two cases

i) a has no ancestor in which case chain of ancestors stops at A, at a itself. So $a \in A_1$.

ii) a does have an ancestor in which case one of the three possibilities must occur.

Therefore, $a \in A_0 \cup A_1 \cup A_2$ so $A \subseteq A_0 \cup A_1 \cup A_2$
$A = A_0 \cup A_1 \cup A_2$

Step 4:
$\{B_0, B_1, B_2\}$ is a partition of B.

Step 5:
($f|_{A_0}: A_0 \to B$)
The restrictions $f|_{A_0}: A_0 \to B_0$, $f|_{A_1}: A_1 \to B_1$ and $g|_{B_2}: B_2 \to A_2$ are bijections.
We show $g|_{B_2}: B_2 \to A_2$ is a bijection. We need to check 3 things.

- $g|_{B_2}$ is a function (restricted co-domain too)
- $g|_{B_2}$ is injective
- $g|_{B_2}$ is surjective

Suppose $b \in B_2$, we need to check $g|_{B_2}(b) \in A_2$.
Let $b \in B_2$ and let $a = g(b) = g|_{B_2}(b)$. Then b is the unique ancestor of a, so the ancestors of a are the same as the ancestors of b and eventually stops in B. Hence $a \in A_2$.

Injectivity:
$g|_{B_2}$ is the domain restriction of an injective function so it is injective.

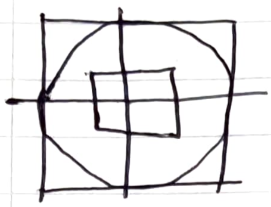
Surjectivity:
Let $a \in A_2$. Then the chain of ancestors of a stops in B. Thus, there exists unique $b \in B$ such that $g(b) = a$. But since the chain of ancestors of b is the same as for a, we have $b \in B_2$. Hence $g|_{B_2}$ is surjective.

Step 6:
Define a function $h: A \to B$ by $h(a) = \begin{cases} f(a) & \text{if } a \in A_0 \cup A_1 \\ g|_{B_2}^{-1}(a) & \text{if } a \in A_2 \end{cases}$
Then h is a bijection

Example:

Let $D = \{(x,y) \in \mathbb{R} \mid x^2+y^2 \leq 1\}$ and $S = \{(x,y) \in \mathbb{R} \mid -1 \leq x, y \leq 1\}$.

Then there is a bijection $h: D \to S$.



Notice $D \subseteq S$

Define a function $f: D \to S$

$(x,y) \mapsto (x,y)$ inclusion function

f is injective.

Define a function $g: S \to D$

$(x,y) \mapsto (\frac{x}{2}, \frac{y}{2})$

Clearly injective $(\frac{x_1}{2}, \frac{y_1}{2}) = (\frac{x_2}{2}, \frac{y_2}{2}) \Rightarrow (x_1, y_1) = (x_2, y_2)$

MATH 112 L13

L11 and L12 didn't take place due to strike

Chapter 4: Cardinality revisited - countability

Definition 4.1 -

a) Let A and B be sets. We say A and B have the same cardinality and write $|A| = |B|$ if there exists a bijection $f: A \to B$

b) We say that A has smaller cardinality than b and write $|A| \leq |B|$ if there exists an injection $f: A \to B$

c) We say that A has strictly smaller cardinality than B and write $|A| < |B|$ if there exists an injection $f: A \to B$ but there does not exist a surjection / bijection $g: A \to B$. (equivalently $|A| \leq |B|$ but $|A| \neq |B|$).

Schröder - Bernstein Theorem (Reformulation) -

Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

This $\leq$ and $\leq$ can be useful method in proof. In this case this theorem extends to infinite cardinality

Example:

The sets $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
$S = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x, y \leq 1\}$

have the same cardinality

For each $n \in \mathbb{N}$, define $C_n = \{r \in \mathbb{N} \mid 1 \le r \le n\} = \{1, \dots n\}$ $C_0 = \emptyset$

Definition 4.2 -

Let A be a set. We say

a) A is finite and write $|A| = n$ if there exists $n \in \mathbb{N}_0$ and a bijection $f: C_n \to A$, otherwise we say A is infinite.

b) A is countably infinite if there exists a bijection $f: \mathbb{N} \to A$.

c) A is countable if A is finite or A is countably infinite

Countable can mean different things to different authors so you need to distinguish their meaning.

d) A is uncountable (infinite) if there exists no bijection between a subset of $\mathbb{N}$ and A.

Proposition 4.3 -

The set $\mathbb{N}_0$ is countably infinite (P08)

Proof:

Define $f: \mathbb{N} \to \mathbb{N}_0$ via $f(n) = n-1$

Exercise: Check this is a bijection by either injectivity + surjectivity or checking this has an inverse.

Proposition 4.4 -

The set $\mathbb{Z}$ is countably infinite.

Proof:

$f: \mathbb{N} \to \mathbb{Z}$. However inverse of bijection $\mathbb{Z} \to \mathbb{N}$ is easier to prove. $f: \mathbb{Z} \to \mathbb{N}$

$$i \mapsto \begin{cases} 2i & \text{if } i > 0 \\ 2|i|+1 & \text{if } i \le 0 \end{cases}$$

Exercise: Check this is a bijection

Proposition 4.5 -

The set $\mathbb{Q}$ is countably infinite

Proof :

Construct bijection $\mathbb{N} \to \mathbb{Q}$ or $\mathbb{Q} \to \mathbb{N}$.

Let $\mathbb{Q}_{>0} = \{q \in \mathbb{R} \mid q > 0\}$

$\mathbb{Q}_{\leq 0} = \{q \in \mathbb{R} \mid q \leq 0\}$

First construct bijection $h : \mathbb{Q}_{>0} \to \mathbb{N}_0$ as follows

$\mathbb{Q}_{>0} = p/q$ where $p \geq 0$ and $q > 0$

Now define $f : \mathbb{Q} \to \mathbb{Z}$ via

$$f(q) = \begin{cases} h(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0 \\ -h(-q) & \text{if } q < 0 \end{cases}$$

7/12/2021

MATH 112 L14 Beginning of Combinatorics

Proposition 4.6 -
The set $\mathbb{R}$ is uncountably infinite
Proof: (Cantor's Diagonal argument)
We argue by contradiction. We show that $[0,1)$ $= \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ is uncountably infinite.
If $\mathbb{R}$ were countable there would be a bijection $f|_{[0,1)} = [0,1) = \operatorname{Im} f|_{[0,1)} \to A \subseteq \mathbb{N}$
Therefore $[0,1)$ would be countable. The contrapositive is $[0,1)$ uncountable $\Rightarrow \mathbb{R}$ uncountable.
Suppose there is a bijection $f: \mathbb{N} \to [0,1)$ and write $f(i) = a_i$. Represent finite decimal expansions with infinite sequences of 0s

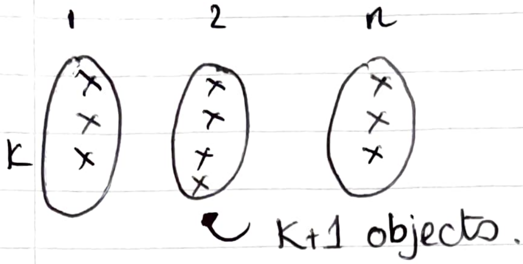
Chapter 5: Combinatorics

5.1 Basic techniques of counting -
The rule of sum -
If a set of objects can be divided into n disjoint subsets such that the i^{th} subset has n_i elements, then we have $\sum_{i=1}^{n} n_i$ objects.
The rule of product -
If an activity can be composed from m different objects of which the i^{th} can be chosen in n_i ways then we have $\prod_{i=1}^{m} n_i$ ways of composing the activity.

3/12/2021

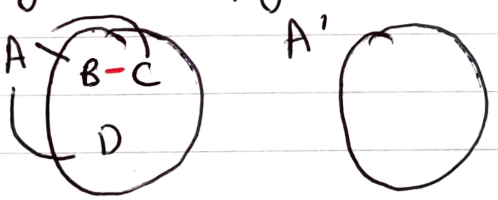
MATH 112 L15 + L16

Extended pigeonhole principle. Assign $kn+1$ objects to n pigeonholes, must be one pigeonhole with at least $K+1$ objects in it.



In a room with 6 people handshaking there will either be
i) 3 people each of whom have shaken hands with other 2
ii) 3 people " " have shaken hands with none of other two.

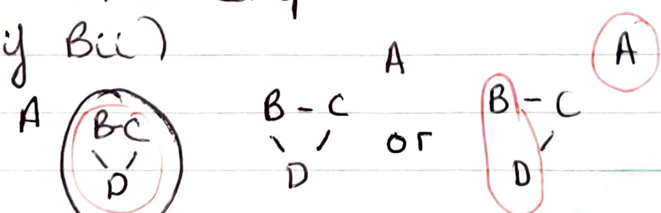
Person A, 5 remaining people. $5 = 2(2)+1$
$n=2, K=2, K+1=3$ $= K(n)+1$
Define two pigeonholes, shaken with A and not A.



So there are 2 cases:
B i) A contains 3 people
B ii) none have shaken hands
A' contains 3 people

if - initial case i) is complete. else, initial case ii) is complete.

if B ii)



Set of three ii)

Set of three i)

Pigeonhole principle

If more than n objects to be distributed among n pigeonholes, there exists at least one pigeonhole with 2 or more objects.

=

If A and B are finite sets such $|A| > |B|$ then there are no injective functions $f: A \to B$
$\exists b \in B$ such $|f^{-1}(b)| > 1$

Extended Pigeonhole principle:

From before ←

If A and B are finite sets such $|A| > k|B|$ for some $k \in \mathbb{N}$. then $f: A \to B$ there exists $b \in B$ such $|f^{-1}(b)| \geq k+1$

5.2 Permutations + Combinations: (Picking elements out of a bag)

Gambling Theory

Four cases:

Selection

	With repetition (Placing item back in)	Without repetition
Ordered	n^k	$\frac{n!}{(n-k)!}$
Unordered	$\frac{(n+k-1)!}{k!(n-1)!}$	$\frac{n!}{k!(n-k)!}$

Ordered Selection with Repetition.

Theorem 5.1: Of k objects, $= n^k$.

Example: Number of 3 letter words $= 26^3$

Ordered Selection without Repetition.

Theorem 5.3: Of k objects from a set of n objects is $\frac{n!}{(n-k)!}$ Permutations.

Unordered selection without repetition · Combinations

Theorem 5.4. Unordered selections of k objects from a set of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

n choose k from Binomial Theorem.

Proof:

ordered selections = # unordered selections · k objects # of orderings

$$\underbrace{\frac{n!}{(n-k)!}}_{\text{Thm } 5.3} = \frac{n!}{k!(n-k)!} \cdot \underbrace{k!}_{\text{thm } 5.1} \quad \square$$

Unordered selection with repetition.

Theorem 5.6. Unordered selections of k objects from a set of n objects is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

Proof:

Let the set of size n be $\{a_1, \dots a_n\}$. Since unordered we can list selection by "type"

$$\underbrace{a_1 \dots a_1}_{r_1} \dots \underbrace{a_n \dots a_n}_{r_n} \qquad \sum_{i=1}^{n} r_i = k \text{ objects.}$$

$1 \dots 1 \quad 0 \quad 1 \dots 1$

string of 0s and 1s where 0 is when type is changed.

e.g $a_1a_1a_2a_2a_2a_4a_5a_5 = 11011100101 1$ .. Boolean

= sequence of length $n-1+k$ as k 1s and $n-1$ 0s

Therefore $\binom{n+k-1}{k}$

5.2.5 Multinomials and rearrangements -

e.g How many rearrangements of the word BOB are there? Initially I think two, OBB and BBO.
Answer: Distinguish Bs as B_1 and B_2. There are now 6 rearrangements:

B_1B_2O, B_1OB_2, B_2B_1O, B_2OB_1, OB_1B_2, OB_2B_1

There are 3! ways of ordering distinctly. 2 are indistinguishable with 2! ways of ordering.

$= \frac{3!}{2!} = 3$

e.g MISSISSIPPI rearrangements

$= \frac{11!}{4!4!2!}$

Theorem 5.8 (Multinomial Theorem) -

Generalisation of Binomial

The number of rearrangements of n objects which are divided into n_i objects of type i for $1 \le i \le k$

$$= \binom{n}{n_1, n_2 \dots n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$$

Theorem 5.9 -

abstract symbols

Let $x_1, x_2 \dots x_k$ be indeterminates. Then the coefficient of $x_1^{r_1}, x_2^{r_2} \dots x_k^{r_k}$ is $\binom{n}{r_1, r_2 \dots r_k}$

Proof:

Coefficient of $x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$ is the number of ways of choosing x_1 from r_1 factors

x_2 from r_2 factors ...

x_k from r_k factors

of $(x_1 + x_2 \dots x_k)^n$. This is equal to no. of rearrangements of the word $\underbrace{x_1 \dots x_1}_{r_1} \dots \underbrace{x_k \dots x_k}_{r_k}$

$$= \binom{n}{r_1, r_2 \dots r_k}$$

Example:

① coefficient of $x^3y^2z^3$ in $(x+y+z)^8$

② coefficient of x^3y^2 in $(x+y-2)^8$

① $= \frac{8!}{3!\,2!\,3!} = 560\,x^3y^2z^3$

② = Let -2 be z

$x^3y^2z^3 = 560$

$z^3 = -2 \quad , \quad -2^3 = -8$

$x^3y^2 = \cancel{4480} = -8 \cdot 560 = -4480$

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Theorem 5.3 -
If repetition is not allowed then the number of ordered selections of k objects from n is:
$$\frac{n!}{(n-k)!}$$

Special Case - $k=n$. In this case there are $n!$ ordered selections. This is called a permutation.

Permutations -

Definition 5.11 -
Let X be a set. A permutation of X is a bijection $X \to X$. We denote the set of permutations of X by $\mathrm{Sym}(X)$

Notation -
If $X = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ we write $S_n = \mathrm{Sym}(X)$. If $f \in S_n$, write
$$f = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f(1) & f(2) & f(3) & \cdots & f(n) \end{pmatrix}$$

Example -
Suppose $X = \{1,2,3,4\}$ and $f, g \in S_4$ defined by
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$
(right to left)

$$g \circ f = \text{same method } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

NOTE $f \circ g \neq g \circ f$ in general.

Definition 5.13 -

A permutation $f \in S_n$ is called an r-cycle if there is a subset $\{a_1, \dots, a_r\} \subseteq \{1, 2, \dots, n\}$ such that

$f(a_i) = a_{i+1}$ for $1 \le i < 1$

$f(a_r) = a_1$

$f(x) = x$ for all $x \notin \{a_1, .. a_r\}$

We write $f = (a_1, a_2 a_3, .. a_r)$ which cycles continously.

Example -

Let $f = (1\ 3\ 2\ 4) \in S_4$ (from same example)

means $1 \to 3 \to 2 \to 4 \to 1$ etc.

Lemma 5.15 -

Let $f \in S_n$. Then there exists $g_1, \dots g_t \in S_n$ such that $f = g_1 \circ g_2$ " "... g_t. Because $g_1, \dots g_t$ are cycles and no symbol from $\{1 \dots n\}$ occurs in more than one of the cycles. We say the cycles are disjoint.

Example -

$f = (1\ 2\ 3\ 4)$ $g = (1\ 4)(2\ 3) \in S_4$

Proof ...

Idea: $|Z| < n$, so by induction any permutation of z can be written as a product of disjoint cycles

claim $f|_z \in \mathrm{Sym}(z)$

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Proof of Lemma 5.15 -

Let $f \in S_n$. Then there exist $g_1, \dots g_t \in S_n$ such that $f = g_1 \circ g_2 \circ \dots g_t$, $g_1 \dots g_t$ are cycles and no symbol from $\{1 \dots n\}$ occurs in more than one of the cycles; the cycles are disjoint.

Proof:

We proceed by induction on n. If $n=1$ then there is nothing to show. So suppose $n>1$.

Claim: If $m>0$ is the smallest integer such that $f^m(1)$ $\quad f^m = \underbrace{f \circ \dots \circ f}_{m \text{ times}}$ $\quad f^m(1) \in \{1, f(1), f^2(1), \dots f^{m+1}(1)\}$ then $f^m(1) = 1$. Exercise.

Let $Y = \{f^m(1) \mid m \in \mathbb{N}_0\} \subseteq \{1, \dots n\}$.

If $|Y| = n$, then by the claim f is already a cycle.

$1 \to f(1) \to f^2(1) \to \dots \to f^{m-1}(1) \to f^m(1) \to 1$

If $|Y| = m < n$, then by the claim f is a cycle

$1 \to f(1) \to f^2(1) \to \dots f^{m-1}(1) \to f^m(1) \to 1$

Let $g(1) = (1 \ f(1) \ f^2(1) \dots f^{m-1}(1)) \in S_n$

Observe $g(1)|_Y = f|_Y \in \mathrm{Sym}(Y)$

Let $Z = \{1, \dots n\} \setminus Y$

$|Z| < n$, so by induction any permutation of Z can be written as a product of disjoint cycles.
We then build f out of a permutation on Z and g_1.

Claim: $f|_Z \in \mathrm{Sym}(Z)$ Three things to check

i) $f|_Z$ is well defined

ii) $f|_Z$ is injective true of i) as bijective

iii) $f|_Z$ is surjective

Suppose $f(z) \notin Z$. Therefore, $f(z) \in Y$ so that $f(z) = f^i(1)$ for some $i \in \mathbb{N}$. Since f is bijective $z = f^{-1} \cdot f(z) = f^{-1} f^i(1) = f^{i-1}(1)$. so $z \notin Z$.
contrapositive: claim is true.

Remark 5.16 -

The composition of two disjoint cycles $g_1 g_2 \in S_n$ does commute,

$$g_1 \circ g_2 = g_2 \circ g_1$$

This is because any element of X is acted on by at most one of the cycles Either $g_1(x) = x$ or not

If $g_1(x) \neq x$, $g_2(x) = x$ $g_1(g_2(x)) = g_1(x)$

Example -

$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ $f = (1\ 3\ 2)\cancel{(4)}$ in disjoint notation

$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $g = (14)(2\ 3)$

$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 4 & 1 & 2 & 3 \end{pmatrix}$ $h = (1\ 5)(2\ 7\ 3\ 6)\cancel{(4)}$

because a one cycle.

Example -

$f = (1234)$, $g = (1\ 4)(2\ 3) \in S_4$

$f \circ g = (1234)(14)(23) = (1)(2\ 4)(3) = (2\ 4)$

$g \circ f = (1\ 4)(23)(1234) = (1\ 3)(2)(4) = (1\ 3)$

Example -

$f = (1\ 3\ 2\ 4) \in S_4$

$f^2 = (1324)(1324) = (1\ 2)(3\ 4)$

$f^3 = f^2 \circ f = f \circ f^2 = (1324)(12)(34) = (1\ 4\ 2\ 3)$

$f^4 = f^3 \circ f = f \circ f^3 = (1423)(1324) = (1)(2)(3)(4)$

$= I_{\{1,2,3,4\}}$

Lemma 5.19 -

Let $X = \{1, \dots n\}$. Suppose $g \in S_n$ is an r-cycle. Then $g^m = I_X$ if and only if $r \mid m$

Proof:

If $g = (a_1\ a_2 \dots a_r)$ then $g^m(a_i) = a_{i+m \bmod r}$ for each $m \in \mathbb{N}$ and $1 \leq i \leq r$. $\square$

Definition 5.20 -

Let $f \in S_n$. Suppose $f = g_1 \circ g_2 \circ \dots \circ g_r$ is a product (i.e composition) of disjoint cycles. The $\boxed{\text{order}}$ of f is the lowest common multiple of the lengths of $g_1, \dots g_r$.

Example -

$f = (132) \in S_3$ has order 3.
$g = (14)(23) \in S_7$ has order 2.
$h = (15)(2736) \in S_7$ has order 4.

Definition 5.22 -

A cycle of length 2 is called a transposition

Theorem 5.23 -

Every permutation can be written as a product (composition) of transpositions.
For $f \in S_n$, there exist transpositions $\sigma_1, \dots \sigma_s \in S_n$ such that $f = \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_s$

Proof:

By Lemma 5.15 every permutation is a product of disjoint cycles. It is enough to show the statement for cycles. Each r-cycle can be written as a product of r-1 transpositions. Let $g = (a_1 \cdots a_r)$ be an r-cycle. $g = (a_1\, a_2 \cdots a_r) = (a_1\, a_r)(a_1\, a_{r-1}) \dots (a_1\, a_3)(a_1\, a_2)$. Hence every r-cycle can be written as a product of transpositions $\square$

Definition 5.24 -

Let $f \in S_n$. We say f is even if f can be written as a product of an even number of transpositions. Likewise, f is $\boxed{\text{odd}}$ if f can be written as a product of an odd number of transpositions.

Example 5.25 -

Consider $(1\,2\,3) \in S_3$,

$(1\,2\,3) = (1\,3)(1\,2)$

$(1\,2\,3) = (1\,2)(2\,3)$

Theorem 5.26 -

τ = Tau

The parity of a permutation is well defined, i.e, if

$f = \sigma_1 \circ \dots \circ \sigma_s = \tau_1 \circ \dots \circ \tau_t$

are two expressions for f with σ_i and τ_j transpositions for $1 \leq i \leq s$, $1 \leq j \leq t$, then either both s and t are even or both s and t are odd

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Key Point: Don't use bubble diagrams for mapping functions. Instead write as: Let $X = \{1,2,3\}$, $Y = \{a,b\}$ and let $f: X \to Y$, $f(1) = f(2) = a$ and $f(3) = b$.

Proof of Theorem 5.26.

Idea: Consider a polynomial $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$ and ask what happens if we swap x_k with x_l but leave the other x_i unchanged.

Formally: Let $f \in S_n$ act on the set $\{x_1, \dots x_n\}$ via

$$f \cdot (x_i) = x_{f(i)}$$

so that f becomes a permutation of $\{x_1 \dots x_n\}$.

Example.

Let $n = 4$. Then $\Delta = \prod_{1 \le i < j \le 4} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$

$I_{\{1,2,3,4\}} \cdot x_i = x_i$ $\qquad (2\ 4) \cdot x_1 = x_1$

$(1\ 2\ 3) \cdot x_1 = x_2$ $\qquad (2\ 4) \cdot x_2 = x_4$

$(1\ 2\ 3) \cdot x_2 = x_3$ $\qquad (2\ 4) \cdot x_3 = x_3$

$(1\ 2\ 3) \cdot x_3 = x_1$ $\qquad (2\ 4) \cdot x_4 = x_2$.

$(1\ 2\ 3) \cdot x_4 = x_4$

opposite sign to corresponding

$I \cdot \Delta = \Delta$ $\qquad$ 2 and 4 swap

$(2\ 4) \cdot \Delta = (x_1 - x_4)(x_1 - x_3)(x_1 - x_2)(x_4 - x_3)(x_4 - x_2)(x_3 - x_2)$

same factors, different order.

$(2\ 4) \cdot \Delta = -\Delta$

Chapter 6: Graphs

Different to a graph of a function

Definition 6.1 -
A graph G consists of a non-empty set $V(G)$ of vertices of G and a list $E(G)$ of unordered pairs of elements of $V(G)$ called edges.

Examples -

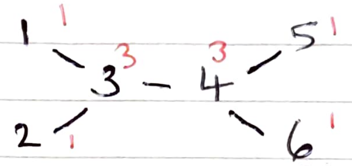
SIMPLE
NOT REGULAR

1) G_1 with $V(G_1) = \{1,2,3\}$, $E(G_1) = [(1,2),(2,3)]$

$1 - 2 - 3$

2) G_2 with $V(G_2) = \{1,2,3,4,5,6\}$, $E(G_2) =$
$[(1,3),(2,3),(3,4),(4,5),(4,6)]$

NOT REGULAR
SIMPLE



Degree sequence = $(1,1,3,3,1,1)$

NOT SIMPLE

3) G_3 with $V(G_3) = \{1,2\}$ $E(G_3) = [(1,1)(1,2)(1,2)]$

loop → $C1 = 2$

4) G_4 with $V(G_4) = \{1,2,3,4\}$ $E(G_4) = [(1,2),(2,3),$
$(3,4),(1,4)]$

$$\begin{array}{ccc} 1 & - & 2 \\ | & & | \\ 4 & - & 3 \end{array}$$

SIMPLE = At most 1 edge between vertices and no loops
REGULAR = Each vertex has same number of edges incident.

Lemma 6.4 - (handshaking)
Let G be a finite graph. Then $\sum_{v \in V(G)} d(v) = 2e(G)$
Proof: Every edge adds 2 to the degree sum $\square$

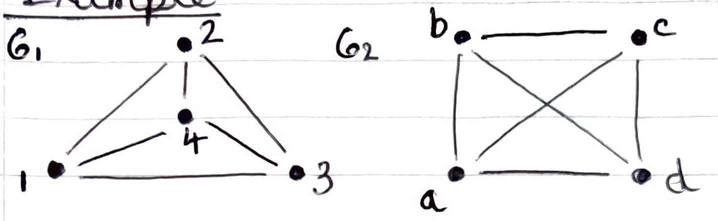
Corollaries 6.5 and 6.6 -
Let G be a finite graph
* degree sum of G, $\sum_{v \in V(G)} d(v)$ is even
* number of vertices of odd degree is even.

Definition 6.10 -

Let G_1 and G_2 be graphs. We say that G_1 and G_2 are isomorphic and write $G_1 \cong G_2$ if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ with the property that:

$$(v_1, v_2) \in E(G_1) \text{ iff } (\varphi(v_1), \varphi(v_2)) \in E(G_2)$$

Example -

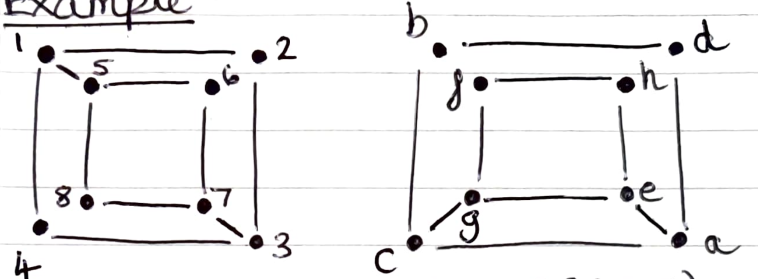


$\varphi : V(G_1) \to V(G_2)$

$1 \mapsto a$
$2 \mapsto b$
$3 \mapsto c$
$4 \mapsto d$

Example -

G_3



(3, 2, 3, 2, 3, 2, 3, 2)
1 2 3 4 5 6 7 8

(3, 2, 3, 2, 3, 2, 3, 2)
a b c d e f g h

1 is incident with 3 edges so 1 must be sent to a, c, e, g

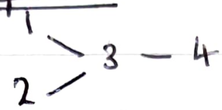
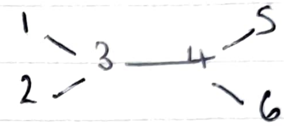
1 is adjacent to 2, 4, 5 with degrees (2, 2, 3) resp.
Vertices adjacent to $\varphi(1)$ must also have degrees (2, 2, 3)
At a, c, e, g adjacent vertices degrees = (3, 3, 2)
Hence $G_3 \ncong G_4$

Test isomorphism of graphs using invariants, properties of graphs preserved under isomorphism:
edges
vertices
degree sequence.

Definition 6.12 -

Let G be a graph. A subgraph H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

Examples -

1)  is a subgraph of 

2) 1-2-3 is a subgraph of 1-2-3

3) C1 = 2 is a subgraph of C1 ≡ 2)

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Definitions -

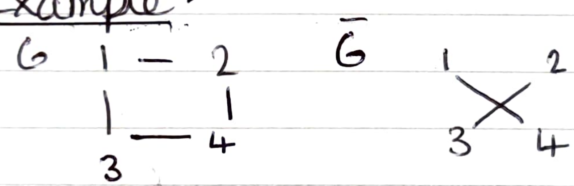
- The null graph N_m on m vertices has $|V(N_m)| = m$ and empty edge list
- A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. The complete graph on m vertices is denoted K_m

Definition -

Let $G = (V(G), E(G))$ be a simple graph. Define the complement of G to be the simple graph $\bar{G}$ having vertex set $V(\bar{G}) = V(G)$ and edges:

$$(v_i, v_j) \in E(\bar{G}) \iff (v_i, v_j) \notin E(\bar{G})$$

Example -



Examples of Regular Graphs -

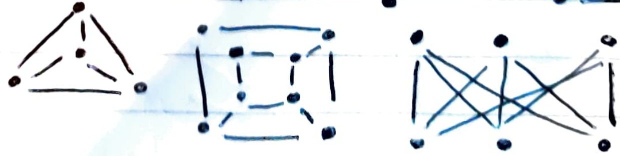
0-regular : N_m $\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{m \text{ times}}$

1-regular : K_2 $\bullet$ —— $\bullet$ (or m disjoint copies of K_2)

Any finite 2-regular graph is a cycle.

2-regular :

3-regular :



d-regular : K_{d+1}

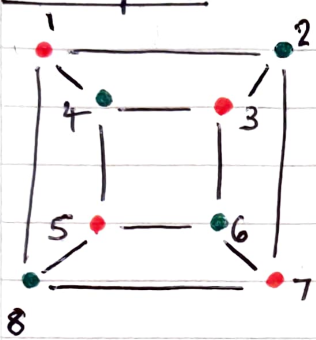
Platonic Graphs - (3D)

Definition -

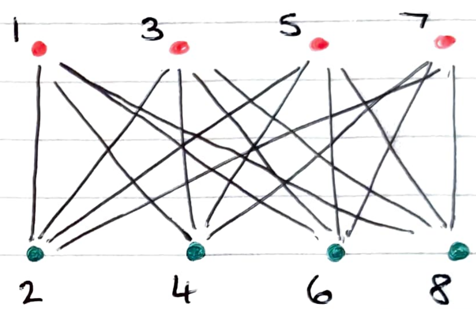
A graph $G = (V(G), E(G))$ is bipartite if there exists a partition of $V(K)$ into two non-empty sets X and Y such that:

- vertices in X are not joined by an edge
- Vertices in Y are not joined by an edge

Examples -



$\cong$



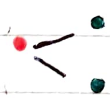
Definition :

A bipartite graph with $(V(G) = X \sqcup Y$ is complete bipartite if every X is joined to every vertex of Y.

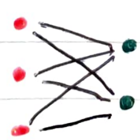
If $|X| = n$ and $|Y| = m$ then it is denoted $K_{n,m}$

Examples -

$K_{1,2}$



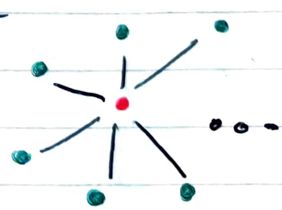
$K_{3,2}$



$\cong$

$K_{2,3}$

$K_{1,n}$



$K_{n,m}$ is regular iff $n = m$

Definitions -

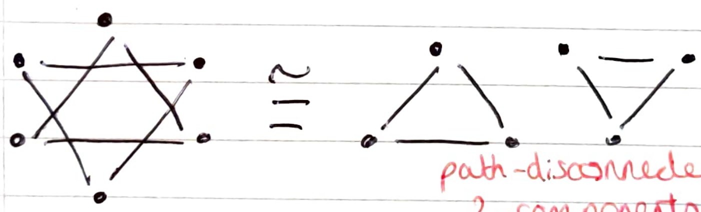
- A walk of length k is a sequence of k edges of a graph G of the form $(v_{i1}, v_{i2})(v_{i2}, v_{i3}) \cdots (v_{ik-1}, v_{ik})$
- A walk all of whose edges are different is a trail
- A trail all of whose vertices are different is a path
- A walk or trail is closed if $v_{i1} = v_{ik}$
- A closed trail with all vertices different except the first and last is called a cycle.

Definition 6.16 -

A graph G is path-connected if for any vertices $v_1, v_2 \in V(G)$ there exists a path between v_1 and v_2. Otherwise, path-disconnected

A path-disconnected graph consists of a number of path-connected subgraphs called (path) components

Example -



path-disconnected
2-components

Definition 6.17 -

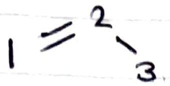
Don't remove repetition when combining lists.

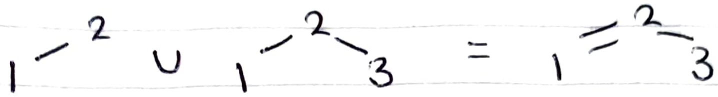
Let G_1 and G_2 be two graphs; their (disjoint) union is defined as the graph G with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge list $E(G) = E(G_1) \cup E(G_2)$

Definition 6.18 -

A graph G is connected if it be written as a (disjoint) union of two subgraphs. Otherwise, disconnected.

Example -

 is connected



If unions are not disjoint.

If the union did not mean disjoint union then the graph above would be disconnected according to definition

Proposition 6.19 -

A finite graph G is connected iff it is path-connected

6.4 Trees -

Definition 6.21 -

- A forest is a graph containing no cycles
- A tree is a connected forest.

Theorem 6.22 -

Let T be a graph with n vertices, The graph T is a tree if and only if it is a connected graph with $n-1$ edges.

Proof:

$\Rightarrow$ Suppose T is a tree. If $n=1$, $T=N$ with 0 edges. True for $n=1$. Assume $n>1$ and any tree with $m<n$ has $m-1$ edges.

etc ...

Corollary 6.23 -

Let F be a forest with n vertices and K components. Then F has $n-K$ edges.