MATH 113 L01

Lecture 1 (Real Numbers) -

A real number is a decimal
A decimal can also be written as an infinite sum
e.g $1/6 = 0.1666666\dot{6}$

$1/6 = \frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \dots$

Lecture 2 (Rational Numbers) -

Rational numbers are fractions of integers e.g a/b.
Two fractions are equivalent ie
$a/b = c/d$ if $ad = bc$

Proposition 1 -
Let $a, b, c, d, m, n, p, q \in \mathbb{Z}$ such $b, d, n, q \neq 0$.
Assume $a/b = m/n$ and $c/d = p/q$. Then $\frac{ad+bc}{bd} = \frac{mq+np}{nq}$

Proposition 2 -
Any periodic decimal represents a rational number.
e.g $1/7 = 0.\overline{142857}$
$= 10, 30, 20, 60, 40, 50, 10, 30 \dots$

4/1/2022

MATH 113 L02

Lecture 3 (Convergence of Rational Numbers) -

Convergence - (Rational)

A sequence of rational numbers converges to a rational number q if they are getting closer to q.

e.g $1/7$ $\left[1/10, \; 14/100, \; 142/1000, \; \frac{1428}{10{,}000} \right]$ = Converges to $1/7$

$= 0.1428\ldots$

Mathematic Convergence Definition -

A sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ converges to a rational number q if for any $\epsilon > 0$ there exists some $N > 0$ such if $n \geq N$ then $|q_n - q| \leq \epsilon$.

Notation $q_n \to q$. ϵ is chosen as a bound.

No matter how small ϵ there exists some N.

Smaller ϵ, Larger N.

e.g $\{1/n\}_{n=1}^{\infty}$ $\quad \frac{1}{n} \leq \epsilon$. When $N > \frac{1}{\epsilon}$ then converges to 0.

Lecture 4 (Cauchy Sequences) -

Definition :

A sequence of rational numbers form a Cauchy sequence if they are getting closer and closer to each other. (No target number).

Key Def

Mathematical Cauchy Definition -

Sequence of rationals $\{q_n\}_{n=1}^{\infty}$ if for any $\epsilon > 0$ there exists some $N > 0$ such if $n, m \geq N$ then $|q_n - q_m| \leq \epsilon$

e.g the $1/7$ example was Cauchy.

Theorem 1 :

Any sequence of rational numbers that converges to a rational number is a Cauchy sequence. The inverse is not true.

Proof :

Triangle Inequality : For any triple of numbers a, b, c
$|a-b| \leq |a-c| + |b-c|$. This means 2 lions close to a tiger means the lions are close to each other.

Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers.
Because this converges to a rational number.
$\epsilon > 0$ and $N > 0$ such $n \geq N$ and $|q_n - q| \leq \epsilon/2$.
Therefore by Triangle Inequality:
if $n, m \geq N$ then
$$|q_n - q_m| \leq |q_n - q| + |q_m - q| \leq \epsilon/2 + \epsilon/2 \leq \epsilon$$
And so q_n and q_m are close so this is Cauchy □

MATH 113 L03

Lecture 5 - (Properties of Cauchy sequences) -

Proposition:
Let $\{q_n\}_{n=1}^{\infty}$ be Cauchy and r be rational. For infinite n $q_n \leq r$ and for infinite n $q_n \geq r$. Then q_n converges to r.

e.g $1+1/2, 1-1/2, 1+1/4, 1-1/4 \ldots$ converges to 1.

Proof:
This matches the definition of Cauchy. For any $\epsilon > 0$ and $N > 0$ such $|q_n - q_m| \leq \epsilon$ if $n, m \geq N$. if $n \geq N$ $|q_n - r| \leq \epsilon$ and converges to r. □

Definition -
A sequence $\{q_n\}_{n=1}^{\infty}$ of rationals is <u>bounded</u> if there exists $M > 0$ such for all $n \geq 1$, $|q_n| \leq M$.

e.g $0, 1, 0, 1, 0, 1$ when $M = 2$
e.g $1, 2, 3, 4 \ldots$ not bounded.

Proposition 2: Cauchy sequences are always bounded
(Therefore from Lecture 4 Theorem 1 convergent sequences are also bounded.

Proof:
$\{q_n\}_{n=1}^{\infty}$ is a Cauchy of rationals. Then there exists $N \geq 0$ such $|q_n - q_N| \leq 1$ if $n \geq N$. Therefore if $n \geq N$ by Triangle inequality $|q_n| \leq |q_N| + 1 = A$.
B = maximum of set $\{|q_1|, |q_2| \ldots |q_N|\}$ and $M = \max\{A, B$
Then for $n \geq 1$ $|q_n| \leq M$.

Theorem :

Let $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be Cauchy of rationals. Then $\{q_n + r_n\}_{n=1}^{\infty}$ and $\{q_n r_n\}_{n=1}^{\infty}$ are Cauchy of rationals.

Lecture 6 : (Limit values) -

Let $x = a.d_1 d_2 d_3$ be a decimal. $a.d_1$ is called the first truncation of x. $(x)_1 = a.d_1$. $(x)_k = k^{th}$ truncation. $|(x)_k - (x)_l| \leq \frac{1}{10^k}$ if $l > k$ so $\{(x)_k\}_{k=1}^{\infty}$ is Cauchy.

e.g $(\pi)_1 = 3.1$ $(\pi)_2 = 3.14$ $(\pi)_3 = 3.141$ ...

Limit value Theorem :

Let $\{q_n\}_{n=1}^{\infty}$ be a Cauchy of rationals. Then there exists decimal r such $q_n - (r)_n \to 0$. $\{q_n - (r)_n\}_{n=1}^{\infty}$ tends to zero. r is called the limit value of the Cauchy sequence.

3/11/2022

MATH 113 L04

Lecture 7 : (Real numbers as decimals)

Proposition 1 : For two sequences of rational numbers Let $q_n \to 0$ and $r_n \to 0$ Then $q_n + r_n \to 0$

Two cauchy sequences are equivalent if $q_n - r_n \to 0$ for $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$.

Proposition 2 : If the decimal x is limitvalue (L03) of $\{q_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty} = \{r_n\}_{n=1}^{\infty}$ then x is limitvalue of $\{r_n\}_{n=1}^{\infty}$

Real numbers

Mathematical Definition of Real Numbers :
Real numbers can be defined as the equivalence classes of Cauchy sequences of rational numbers.

This is a universal definition and equivalence classes are in a bijective relation with decimals.

Sum and product of decimals :
Let x and y be decimals. Then $\{(x)_n\}_{n=1}^{\infty}$ and $\{(y)_n\}_{n=1}^{\infty}$ are Cauchy sequences with limitvalues x and y respectively.

$x + y =$ limitvalue of $\{(x)_n + (y)_n\}_{n=1}^{\infty}$

Similarly can define xy.

Lecture 8: (Properties of Real Numbers) -

If $\{a_n\}_{n=1}^{\infty} \equiv \{x_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} \equiv \{y_n\}_{n=1}^{\infty}$

Then $\{a_n + b_n\}_{n=1}^{\infty} \equiv \{x_n + y_n\}_{n=1}^{\infty}$

and $\{a_n b_n\}_{n=1}^{\infty} \equiv \{x_n y_n\}_{n=1}^{\infty}$

The main advantage of Cauchy sequence definition is it allows for us to prove the famous identities of rational numbers for real numbers simply. /quicker

- (Commutativity) $p+q = q+p$
- (Associativity) $(p+q)+r = p+(q+r)$
- (Distributivity) $p(q+r) = pq + pr$

etc.

The Archimedean Law -

For any positive real number r, there is some integer n such that $nr \geq 1$

This is because if r is positive then for some $k \geq 1$ $(r)_k$ is a positive rational number. So $n(r)_k \geq 1$ and $nr \geq 1$.

Real version of Archimedean Law -

If a is positive and $0 < b < c$ then $ab < ac$

MATH 113 L05

Lecture 9 (Cauchy sequences of real numbers) -

Definition 1: A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ converges to a real number x if for any $\epsilon > 0$ there exists $N > 0$ such if $n \geq N$ then $|x - x_n| \leq \epsilon$

$\lim_{n \to \infty} x_n = x$ $x_n \to x$

Definition 2: A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists $N > 0$ such if $n, m \geq N$ then $|x_n - x_m| \leq \epsilon$

Theorem 1: Every convergent sequence is Cauchy. Every Cauchy sequence is bounded.
Proof is the same as for rationals.

Theorem 2: Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers. Assume $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$.

(Sum) $\lim_{n \to \infty} x_n + y_n = x + y$

(Subtraction) $\lim_{n \to \infty} x_n - y_n = x - y$

(Product) $\lim_{n \to \infty} x_n y_n = xy$

and when $y \neq 0$ and for all $n \geq 1$ $y_n \neq 0$

$$\lim_{n \to \infty} \frac{x_n}{y_n} = \frac{x}{y}$$

Sum Rule Proof:

Fix $\epsilon > 0$ and let $N > 0$ such if $n \geq N$ then $|x_n - x| \leq \epsilon/2$ $|y_n - y| \leq \epsilon/2$

Then by triangle inequality

$|(x_n + y_n) - (x + y)| \leq \epsilon$

Hence sum rule proved $\square$

Subtraction rule can be proved the same way.

Product Rule Proof:

Let $M > 0$ be a positive number such for any $n \geq 1$ $|x_n| \leq M$, $|y_n| \leq M$ fix $\epsilon > 0$ and $N > 0$ such if $n \geq n$ then $|x_n - x| \leq \epsilon/2M$ and $|y_n - y| \leq \epsilon/2M$

$$|x_n y_n - xy| = |x(y_n - y) + y_n(x_n - x)| \leq$$
$$\leq |x||y_n - y| + |y_n||x_n - x| \leq 2M \, \epsilon/2M = \epsilon \quad \square$$

Lecture 10: (Properties of Convergent Sequences) -

Theorem 1: (Sandwich Rule)

Let $\lim_{n\to\infty} x_n = a$ $\lim_{n\to\infty} y_n = a$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence such for all $n \geq 1$ $x_n \leq z_n \leq y_n$

Then $\lim_{n\to\infty} z_n = a$.

Theorem 2: (Monotonicity)

Let $\lim_{n\to\infty} x_n = a$ $\lim_{n\to\infty} y_n = b$ such for all $n \geq 1$ $x_n \leq y_n$ then $a \leq b$.

Crucial Theorem 3: Cauchy sequences of reals are convergent

Proof: If $\{q_n\}_{n=1}^{\infty}$ is a Cauchy of rationals then there exists a decimal x by which it converges to.

Let $\{y_n\}_{n=1}^{\infty}$ be cauchy. Then $\{q_n\}_{n=1}^{n=\infty}$ such $|y_n - q_n| \leq \frac{1}{n}$ and $y_n - q_n \to 0$

Then $\{q_n\}_{n=1}^{\infty}$ is equivalent to $\{y_n\}_{n=1}^{\infty}$ and converges to x. $\square$

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MATH 113 LO6 + LO7

Lecture 11 (Subsequences) -

Definition 1: If $n_1 < n_2 < n_3 < \dots$ then $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of the original sequence $\{x_n\}_{n=1}^{\infty}$

e.g 0, 1, 0, 1, 0, 1, 0, 1
has subsequences 0, 0, 0, 0 ... and 1, 1, 1, 1 ...

e.g $\{x_n\}_{n=1}^{\infty}$ Let $x_n = 0$ if n is not a positive power of a prime number. Let $x_{p^k} = \frac{1}{k}$ if p is a prime and $k \geq 1$
$1/k, 1/k, 1/k \dots$ is a subsequence. $1/2, 1/3, 1/4$ is a subsequence

VERY IMPORTANT

Lecture 12 (Bolzano - Weierstrass Theorem) -

Bolzano - Weierstrass Theorem -
Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then it has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$

Proof:
Integer Interval $[a, a+1] = \{x \mid a \leq x \leq a+1\}$
1/10 Interval $[a/10, \frac{a+1}{10}] = \{x \mid a/10 \leq x \leq \frac{a+1}{10}\}$
$1/10^k$ Interval $[a/10^k, \frac{a+1}{10^k}] = \{x \mid a/10^k \leq x \leq \frac{a+1}{10^k}\}$
for all when a is an integer.

Claim 1: There exists an integer interval I_1 containing infinitely many elements of our sequence $\{x_n\}_{n=1}^{\infty}$
Infinite Pigeonhole principle: Infinite objects in finite pigeonholes, at least one pigeonhole contains infinitely many objects

Claim 2 : There exists a $1/10$ interval $I_2 \subseteq I_1$ containing infinitely many elements of our sequence $\{x_n\}_{n=1}^{\infty}$

Claim 3 : There exists a $1/100$ interval $I_3 \subseteq I_2$ containing infinitely many elements of our sequence $\{x_n\}_{n=1}^{\infty}$.

Therefore $I_1, I_2 \ldots I_k$

$I_1 \supseteq I_2 \supseteq I_3 \supset \ldots I_k$

I_k is a $\frac{1}{10^k}$ interval

For all $k \geq 1$, I_k contains infinitely many elements of $\{x_n\}_{n=1}^{\infty}$

So we can pick a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ in such a way that $x_{n_k} \in I_k$ and $n_k < n_L$ if $k < L$.

Therefore if $k < L$ then x_{n_k} and x_{n_L} are in I_k

since $I_L \subseteq I_k$ $|x_{n_k} - x_{n_L}| \leq \frac{1}{10^k}$

Hence $\{x_{n_k}\}_{k=1}^{\infty}$ is Cauchy and therefore convergent so this Theorem follows. $\square$

1/2/2022

MATH 113 L08 (Lecture 13)

Upper Bounds + Lower Bounds -
t_{max} = largest value t_{min} = smallest value.

Theorem 1 (Least Upper Bound principle) -
The least upper bound is the minimum of the set of the upper bounds.
The greatest lower bound is similar and opposite.

Suprenum and Infinim -
Supremum = Least upper bound of a bounded sequence
Infimim = Greatest lower bound of a bounded sequence

If a sequence is convergent then supremum >= limit and infimum <= limit.

Proposition:
A sequence of positive real numbers $\{x_n\}_{n=1}^{\infty}$ tends to infinity if and only if $\lim_{n \to \infty} \frac{1}{x_n} = 0$.
This makes sense because $\frac{1}{\infty} = 0$ as a general rule.

MATH 113 L09

Lecture 15 (The Race to Infinity) -

Proposition:
Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and $\{y_n\}_{n=1}^{\infty}$ converge to 0. Then $\lim_{n\to\infty} x_n y_n = 0$.

Which sequence converges to infinity quickest?

$\{n^{100}\}_{n=1}^{\infty}$

$\{n^{\frac{1}{100}}\}_{n=1}^{\infty}$

$\{\log(n)\}_{n=1}^{\infty}$

Definition:
Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of positive numbers. We say $\{x_n\}_{n=1}^{\infty}$ $\boxed{\text{beats}}$ $\{y_n\}_{n=1}^{\infty}$ if $\{\frac{x_n}{y_n}\}_{n=1}^{\infty}$ tends to infinity as well.

e.g. $\{n^{100}\}_{n=1}^{\infty}$ beats $\{n^{10}\}_{n=1}^{\infty}$ and $\{n^{\frac{1}{1000}}\}_{n=1}^{\infty}$ beats $\{\log(n)\}_{n=1}^{\infty}$

Proposition: $\{n!\}_{n=1}^{\infty}$ beats $\{k^n\}_{n=1}^{\infty}$ for any $k > 1$.
(Factorials beat exponentials)

Proposition: $\{(1+x)^n\}_{n=1}^{\infty}$ beats $\{n^k\}_{n=1}^{\infty}$ for any $x > 0$, $k > 1$. (Exponential beats polynomials)

Proposition: Let $\{x^1_n\}_{n=1}^{\infty}, \{x^2_n\}_{n=1}^{\infty}, \{x^3_n\}_{n=1}^{\infty}, \dots$ be a sequence of sequences such that all of them tend to infinity. Then, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ beating all of them.

Interesting Point: There are only countable many sequences which can be generated by finite computer codes. Therefore by the proposition there exists a sequence that beats all the computer generated sequences.

LECTURE 16 (Worth coming back to as a revision of the first 9 Lectures).

1/2/2022

MATH 113 L10

Lecture 17 - (Limit points)

Definition -
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say x is a limit point of $\{x_n\}_{n=1}^{\infty}$ if a subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x. The set of limit points is denoted by $LIM(\{x_n\}_{n=1}^{\infty})$

Proposition: $LIM(\{x_n\}_{n=1}^{\infty})$ is empty iff $\{|x_n|\}_{n=1}^{\infty}$ tends to infinity.

Proposition: If $\{x_n\}_{n=1}^{\infty}$ is bounded, $LIM(\{x_n\}_{n=1}^{\infty})$ has exactly one element iff $\{x_n\}_{n=1}^{\infty}$ is convergent.

e.g. 0, 1, 0, 1, 0, 1 has exactly 2 limit points
e.g. 0, 1, 2, 0, 1, 2 has exactly 3 limit points
It is possible for a sequence to have infinite limit points.

Proposition:
There exists no sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ such that $LIM(\{x_n\}_{n=1}^{\infty}) = \mathbb{Q}$

Definition: Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers. Then $x_1, y_1, x_2, y_2 \ldots$ is called the combing of the two sequences

Definition: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. A sequence $\{y_n\}_{n=1}^{\infty}$ is a blow-up of the sequence $\{x_n\}_{n=1}^{\infty}$ if for any $n \geq 1$ there exists infinitely many m's such $y_m = x_n$. On the other hand, for $n \geq 1$ there exists at least one l such $y_n = x_l$. That is, $\{y_n\}_{n=1}^{\infty}$ contains the same real numbers as $\{x_n\}_{n=1}^{\infty}$, infinitely many times.

Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers converging to x. If $\{y_n\}_{n=1}^{\infty}$ is a blow-up of $\{x_n\}_{n=1}^{\infty}$ then $LIM(\{y_n\}_{n=1}^{\infty}) = \{x, x_1, x_2, x_3 \ldots$

Lecture 18: (The Art of Sequences) -

Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence such that each element of the sequence $\{y_n\}_{n=1}^{\infty}$ is a limit point of $\{x_n\}_{n=1}^{\infty}$. Then any limit point of $\{y_n\}_{n=1}^{\infty}$ is a limit point of $\{x_n\}_{n=1}^{\infty}$.

Proposition: The set $\{0, 1/2, 1/4, 1/8 \ldots\}$ is a limit point set of a sequence. On the other hand, the set $1/2, 1/4, 1/8 \ldots$ can not be the limit point set of a sequence.

Definition: A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is increasing if for any $n \geq 1$, $x_n \leq x_{n+1}$. Similarly, a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is decreasing if for any $n \geq 1$, $x_n \geq x_{n+1}$.

e.g $1, 2, 3, 4$ is increasing $1, 1/2, 1/4$ is decreasing.

Proposition : A bounded increasing sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is always convergent.

Proposition : A bounded sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ contains an increasing or decreasing subsequence (or both).

MATH 113 L11

Lecture 19 : (Limsup and Liminf) -

Definition : Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of reals
$\limsup x_n = \sup LIM(\{x_n\}_{n=1}^{\infty})$
$\liminf x_n = \inf LIM(\{x_n\}_{n=1}^{\infty})$

e.g If $\{x_n\}_{n=1}^{\infty}$ is convergent, $\lim_{n\to\infty} x_n = \limsup x_n = \liminf x_n$
If $\{x_n\}_{n=1}^{\infty} = 0,1,2,0,1,2\ldots$ $\limsup x_n = 2$ $\liminf x_n = 0$
If $\{x_n\}_{n=1}^{\infty} = 1,1,1/2,1,1/2,1/4,1\ldots$ $\limsup x_n = 1$ $\liminf x_n = 0$
Suppose $\{q_n\}_{n=1}^{\infty}$ is a sequence containing rationals between 0 and 1. $LIM(\{q_n\}_{n=1}^{\infty} = [0,1]$ hence $\limsup q_n = 1$, $\liminf q_n = 0)$.

Proposition : $\limsup x_n$ and $\liminf x_n$ are limit points of $\{x_n\}_{n=1}^{\infty}$

Proposition : Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and $L = \limsup x_n$. Then there exists a subsequence converging to L. For any $\epsilon > 0$ there exists some $N > 0$ such if $n \geq N$ then $x_n \leq L + \epsilon$.

Theorem 1 : Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences then $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$.

Proof : There exists a subsequence $\{x_{n_k} + y_{n_k}\}_{n=1}^{\infty}$ such $\lim_{k\to\infty} (x_{n_k} + y_{n_k}) = \limsup (x_n + y_n)$.
$\{x_{n_m} + y_{n_m}\}_{m=1}^{\infty}$ tends to $\lim_{k\to\infty} (x_{n_k} + y_{n_k})$ as well.
Hence, $x + y = \limsup (x_{n_m} + y_{n_m}) = \limsup (x_n + y_n)$.
Since limsup is the max limit point, $x \leq \limsup x_n$ and $y \leq \limsup y_n$. $\square$.

Lecture 20: (Closed sets):

Definition: A set $S \subseteq \mathbb{R}$ is closed if for any convergent sequence $\{s_n\}_{n=1}^{\infty}$ $\lim_{n\to\infty} s_n \in S$ provided for any $n \geq 1$ $s_n \in S$.

e.g Empty set is closed, set of all $\mathbb{R}$ is closed. By monotonicity of limits $[a,b]$ is closed. A set of a single number $\{x\}$ is closed. Set $\{0, 1/2, 1/3, 1/4\}$ is closed. Set $S = [0,1)$ is not closed as it converges to 1, set $S = (0, \infty)$ is not closed.

Proposition: Let $\{F_n\}_{n=1}^{\infty}$ be closed sets. Then, their intersection $\bigcap_{n=1}^{\infty} F_n$ is a closed set as well.

Proposition: Let F and G be closed sets. Then $F \cup G$ is a closed set as well.

Proposition: Finite union of closed sets is still a closed set. On the other hand, if $F_1, F_2, \dots$ is an infinite sequence of closed sets then $\bigcup_{n=1}^{\infty} F_n$ is not always a closed set.

Proposition: For $k \geq 1$. Let $F_k \subseteq [k, k+1]$. Then $\bigcup_{k=1}^{\infty} F_k$ is closed.

Theorem: Let $[0,1] \supseteq F_1 \supseteq F_2 \supseteq \dots$ be a nested sequence of non-empty closed sets. Then their intersection $\bigcap_{n=1}^{\infty} F_n$ is non-empty as well.

Proof :

For $n \geq 1$, let $s_n \in F_n$ be an arbitrary point. The sequence $\{s_n\}_{n=1}^{\infty}$ is bounded, hence by the Bolzano - Weierstrauss Theorem it contains a convergent subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ tending to s.

Since all sets F_n are closed and if $m \geq n$ then $F_m \subseteq F_n$. Then we have $s \in F_{n_k}$ for each $k \geq 1$. Consequently $s \in F_m$ for any $m \geq 1$. Thus, $s \in \bigcap_{n=1}^{\infty} F_n$. Therefore intersection is non-empty. $\square$

12/2022

MATH 113 L12 + 13

Lecture 21: (Limit point sets are closed sets) -

Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Then it's limit point set $LIM(\{x_n\}_{n=1}^{\infty})$ is closed.

Theorem: Let M be an integer and $S \subseteq [0,1]$ be a closed subset. Then, there exists a bounded sequence $\{x_n\}_{n=1}^{\infty}$ such that $LIM(\{x_n\}_{n=1}^{\infty}) = S$.

Proof: define the base sequence $\{y_n\}_{n=1}^{\infty}$. The first ten elements are 0, 0.1, 0.2 .. $1/10$ type elements.
The next 100 are $1/100$ type elements and The next 1000 are $1/1000$ type elements.
Therefore the limit point set of the base sequence is the interval $[0,1]$. Let $\{y_{n_k}\}_{k=1}^{\infty}$ be the subsequence of the detectos.
$LIM(\{y_{n_k}\}_{k=1}^{\infty}) = S$
If $s \in S$ then set is an element of this limit set.
If all elements of S are in intervals then hence there will be no detectos in the interval $[s - \frac{1}{10^k}, s + \frac{1}{10^k}]$. Therefore, s is not a limit point of the sequence of detectos. □

Lecture 22: (Open sets) -

Definition: Let $U \subseteq \mathbb{R}$ be a subset of the real numbers. We say U is open, if for any $x \in U$ there exists some $\epsilon > 0$ such if $|y - x| < \epsilon$ then $y \in U$.
That is, if a point in the set, points very close to the point are still in the set.

e.g real numbers are an open set. The empty set is an open set. Half lines (a, ∞) and Intervals (a, b) are open sets.

Proposition: Let U and V be open sets. Then, $U \cap V$ is an open set as well.

Proposition: Let $U_1, U_2, U_3 \ldots$ be open sets, then $\bigcup_{n=1}^{\infty} U_n$ is an open set as well.

Theorem: Let $U \subseteq \mathbb{R}$ be a subset of the real numbers. Then, the complement U^c of U is closed if and only if U is open.

Proof: Assume U is open. Let $\{s_n\}_{n=1}^{\infty} \subseteq U^c$ be a sequence tending to s. We need to show $s \in U^c$.
Suppose $s \notin U^c$ then $s \in U$.
So there exists $\epsilon > 0$ such if $|x - s| < \epsilon$ then $x \in U$.
Therefore for any $n \geq 1$ we have $|s_n - s| \geq \epsilon$.
Assume U^c is closed.
Let $x \in U$ then $\epsilon > 0$ such if $|y - x| < \epsilon$ then $y \in U$. If ϵ doesn't exist then $\{s_n\}_{n=1}^{\infty} \subseteq U^c$ tending to s. However this contradicts U^c is closed $\square$.

Proposition: Let U be an open set and F be a closed set. Then $U \setminus F$ is an open set.

MATH 112 L14

Lecture 25: (Continuity) -

Definition: A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at a point $x \in \mathbb{R}$ if $f(y)$ is close to $f(x)$ provided that y is very close to x.

Definition: A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at a point $x \in \mathbb{R}$ if for all sequences $x_n \to x$ we have $f(x_n) \to f(x)$

If f is continuous at x we call it a point of continuity for f. Otherwise, a point of discontinuity.

Theorem:
* Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions such that f and g are continuous at x. Then $f+g$ and fg are continuous at x.
* Let $h: \mathbb{R} \to \mathbb{R}$ be a continuous function such $h > 0$ and h is continuous at x. Then $1/h$ is continuous at x.
* Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at y and let $g: \mathbb{R} \to \mathbb{R}$ be continuous at x. Assume $g(x) = y$, then $f \circ g$ is continuous at x. Note: $f \circ g(z) = f(g(z))$

Proof:
* If $x_n \to x$ then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. Hence by sum rule, $f+g(x_n) \to f(x) + g(x)$. Therefore $f+g$ is continuous at x. $\square$
* If $x_n \to x$ then $h(x_n) \to h(x)$. Hence quotient rule $\frac{1}{h(x_n)} \to \frac{1}{h(x)}$. Therefore continuous at x. $\square$
* If $x_n \to x$ then $g(x_n) \to g(x) = y$. Hence, $f(g(x_n)) \to f(g(x))$. Therefore $f \circ g$ continuous at x $\square$

Examples :

$3x+2$ = Continous

$5x^2+4x-2$ = Continous

All polynomials are continous

$a_n x^n$ = continous

Lecture 26 : (The (ϵ, δ) definition of continuity) -

Definition : A function $f: \mathbb{R} \to \mathbb{R}$ is continous at a point $x \in \mathbb{R}$ if for any $\epsilon > 0$ there exists some $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

* Theorem : The sequential definition of continuity and (ϵ, δ) definition are equivalent.

Proof :

If f is not continous (ϵ, δ) : $\forall \epsilon \, \exists \delta$ such if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$

Negation : $\exists \epsilon \, \forall \delta$ there is some y such $|x-y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$

So for any particular, there exists y_n such that for any $n \geq 1$

$|x - y_n| < \frac{1}{n}$ but $|f(x) - f(y_n)| \geq \epsilon$.

Hence we made a sequence $\{y_n\}_{n=1}^{\infty}$ such $y_n \to x$ but $\lim_{n \to \infty} f(y_n) \neq f(x)$.

Therefore not sequentially continous. If not continous by (ϵ, δ) then not continous by sequential.

Continued proof: (other way sequential → (ϵ, δ))

If f is not continuous (sequential)

$\forall$ sequences $x_n \to x$ we have $f(x_n) \to f(x)$

Negation: $\exists$ a sequence $x_n \to x$ such $f(x_n) \not\to f(x)$.

This means there exists a subsequence:

$\{x_{n_k}\}_{k=1}^{\infty}$ and some $\epsilon > 0$ such $|f(x_{n_k}) - f(x)| \geq \epsilon$

Hence for any $\delta > 0$ there exists some x_{n_k} such

$|x_{n_k} - x| < \delta$ but $|f(x_{n_k}) - f(x)| \geq \epsilon$.

Therefore when sequential is non-continuous so is (ϵ, δ).

Therefore proved both ways and they are equivalent.

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MATH 112 L15

Lecture 27 : (Uniform Continuity) -

Definition: Let S be a closed subset of the real numbers. Let $f: S \to \mathbb{R}$ be a function. Then f is continuous at $s \in S$ iff for any sequence $s_n \to s$, $f(s_n) \to f(s)$.

e.g Let S consist of finitely many points. Then all the functions on S are continuous.
e.g Let S be the set of natural numbers. Then all the functions on the set S are continuous.
e.g Let $S = \{0, 1, 1/2, 1/4, 1/8 \ldots\}$ Then a function f on S is continuous iff $f(\frac{1}{2^n}) \to f(0)$.

Definition: A function is uniformly continuous on a closed set $S \subseteq \mathbb{R}$ if for any $\epsilon > 0$ there exists $\delta > 0$ such for all $x \in S$, $|f(x) - f(y)| < \epsilon$ provided $y \in S$ and $|y - x| < \delta$.

e.g $f: x \to \sin(x)$ is uniformly continuous on $\mathbb{R}$
e.g If $S \subseteq \mathbb{R}$ then $f: x \to \sin(x)$ is still uniformly continuous on $\mathbb{R}$
e.g $f: x \to x^2$ is continuous but NOT uniformly continuous on $\mathbb{R}$.

Theorem (Uniform Continuity Theorem):
Let $S \subseteq \mathbb{R}$ be a bounded, closed set. That is, $S \subseteq [-M, M]$ for some $M > 0$. Let $f: S \to \mathbb{R}$ be a continuous function. Then, f is uniformly continuous.

Proof : (Contradiction)

f is not uniformly continuous on S:

$\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in S$ we have $|f(x) - f(y)| < \epsilon$ provided $|x-y| < \delta$

Negation:

$\exists \epsilon > 0$, $\forall \delta > 0$ $\exists x$ and $\exists y$ such that $|x-y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$

(1) $|f(x_n) - f(y_n)| \geq \epsilon$

By Bolzano - Weierstrass there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ tending to x.

By the sum rule $\{y_{n_k}\}_{k=1}^{\infty}$ tends to x as well.

By sum rule, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$ in contradiction with (1). $\square$.

Lecture 28 : (Discontinuity) -

Theorem : There exists a function $f: \mathbb{R} \to \mathbb{R}$ such that f is not continuous at any point.

Theorem : There exists a function $g: \mathbb{R} \to \mathbb{R}$ which is continuous only at 0 and nowhere else.

Theorem : There exists a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous at x iff x isn't an integer.

Theorem : There exists a function $h: \mathbb{R} \to \mathbb{R}$ which is continuous at x iff x is an integer.

Theorem : Let $U \subseteq \mathbb{R}$ be an open set. Then, there exists a function $f: \mathbb{R} \to \mathbb{R}$ such that g is continuous at x iff $x \in U$.

Theorem: If $F \subseteq \mathbb{R}$ is a closed set, then there exists, $g: \mathbb{R} \to \mathbb{R}$ which is continuous at x iff $x \in F$

Theorem: There exists a function $f: \mathbb{R} \to \mathbb{R}$ such that f is continuous at x iff x is an irrational number.

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Lecture 30 (The Big search):

Proposition:
Let $f : S \to \mathbb{R}$ be a continious function where $S \subseteq [0,1]$ is a closed set. Then f is bounded. That is, there exists a positive number $M > 0$ such for all $s \in S$, $|f(s)| \leq M$.

Proposition:
Let $f : S \to \mathbb{R}$ be a continious function, where $S \subseteq [0,1]$ is a closed set. Then there is an $s \in S$ where f attains its supremum / least upper bound hence it is a maximal value. For any $t \in S$, $f(t) \leq f(s)$.

Theorem (The Intemediate value Theorem):
Let $f : [a,b] \to \mathbb{R}$ be a continious function. Let $f(a) \leq s \leq f(b)$. Then, $a \leq c \leq b$ such $f(c) = s$

Proof:
Let $a_1 = a$, $b_1 = b$. Let $c_1 = \frac{a+b}{2}$. If $f(c_1) \geq s$, Let $a_2 = a$, $b_2 = c_1$ Then, $f(a_2) \leq s \leq f(b_2)$. But $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$.
If $f(c_1) \leq s$, let $a_2 = c_1$, $b_2 = b_1$. Then $f(a_2) \leq s \leq f(b_2)$. But $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$
Inductively taking averages as above we can construct an increasing sequence a_1, a_2, a_3 and a decreasing sequence b_1, b_2, b_3 in such a way, $f(a_n) \leq s \leq f(b_n)$ and $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$
Hence $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$ and $f(\lim_{n \to \infty}(a_n)) \leq s$, $f(\lim_{n \to \infty}(b_n)) \geq s$. Therefore if $c = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$. Then $f(c) = s$.

Theorem :

Let $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + \dots a_{2n+1} x^{2n+1}$ be a polynomial of odd degree, where $a_{2n+1} > 0$. Then there exists at least one value y such $P(y) = 0$. So, y is a root of the polynomial P.

Lecture 31 : (Invertible functions) -

Definition : Let $f: [a, b] \to [c, d]$ be a function. We say f is invertible if it is injective and surjective. That is,

(1) If $x \neq y \in [a, b]$ then $f(x) \neq f(y)$

(2) For any $z \in [c, d]$ there exists $x \in [a, b]$ such that $f(x) = z$.

Examples of invertible functions:

* $f: [0, 1] \to [0, 1]$ $f(x) = x^{k}$
* $f: [0, 1] \to [1, e]$ $f(x) = \exp(x)$
* $f: [0, \pi/2] \to [0, 1]$ $f(x) = \sin(x)$

Theorem : Let $f: [a, b] \to [c, d]$ be a continuous function. Then $f^{-1}: [c, d] \to [a, b]$ is also continuous.

Definition : A function $f: [a, b] \to [c, d]$ is called strictly monotonic if $f(x) < f(y)$ provided $x < y$.

Proposition : Let $f: [a, b] \to [c, d]$ be strictly monotonic ~~if f(x) < f~~ and continuous such $f(a) = c$ and $f(b) = d$. Then f is invertible and its inverse is strictly monotonic.

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MATH 113 L17, 18, 19 (End of Course)

Lecture 32: (Closed sets and continuous functions) -

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function, then

$$\{x \in \mathbb{R} \mid f(x) = 0\}$$

is called the set of zeroes of f.

Proposition: For any continuous function f the set of zeroes of f is closed.

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and $S \subseteq \mathbb{R}$ be a closed subset. Then $C = f^{-1}(S)$ $= \{x \in \mathbb{R} \mid f(x) = S\}$ is a closed set as well.

Theorem: Let $S \subseteq [0,1]$ be a closed subset and $f: [0,1] \to \mathbb{R}$ be a continuous function. Then, $f(S)$ is a closed set as well.

Theorem: Let $U \subseteq \mathbb{R}$ be an open set and $f: \mathbb{R} \to \mathbb{R}$ be an open set. Then $f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$ is an open set as well.