MATH 114 L01

Lecture 1 : (Geometric Series) -

Definition : Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ is called a series.
Also, $p_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k$ is called the k-th partial sum of the series.
We say the series is convergent if $\lim_{k\to\infty} p_k$ exists.
If the series is convergent then $\lim_{k\to\infty} p_k = S$ is called the sum of the series. So,

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} p_k$$

If the sequence of partial sums is not convergent, the series is divergent.

* $$\sum_{n=0}^{k} q^n = \frac{1-q^{k+1}}{1-q} \quad \text{(RULE)}$$

Proposition (Convergence test of Geometric series) -
Let q be an arbitrary real number. Then the Geometric Series $\sum_{n=1}^{\infty} q^{n-1}$ is convergent iff $|q| < 1$.
If $|q| < 1$ then,

$$\sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q}$$

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MATH 113 L02

Paradox of Achilles and the Tortoise -

Achilles can run 100 x as fast as the tortoise.
Tortoise gets a mile headstart.
Argument that Achilles can never overtake the tortoise
By the time Achilles reaches starting point X_1 of the tortoise, the tortoise is now at X_2 and so on.
However in a finite time frame it takes Achilles t minutes to reach X_1 and $\frac{1}{100}t$ for X_2 etc.
So after $t\sum_{i=0}^{\infty}\left(\frac{1}{100}\right)^i = \frac{100}{99}$ minutes they are at the same position.

MATH 114 L03

Lecture 3 : (Easy convergence tests)

Proposition : (n^{th} term test) :
Let $\sum_{n=1}^{\infty} a_n$ be a convergent series where $\{a_n\}_{n=1}^{\infty}$ are real numbers. Then, $\lim_{n\to\infty} a_n = 0$.

Exercise : Show the following series is divergent.

$$1 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/16 + \ldots$$

That is, $\sum_{n=1}^{\infty} a_n$ is divergent where $a_1 = 1$ and $a_n = \frac{1}{2^k}$ if $2^{k-1} \leq i \leq 2^k - 1$ and $k \geq 2$

Let p_k be the k^{th} partial sums. $p_1 = 1$ $p_3 = 3/2$ $p_7 = 2$ $\{p_k\}_{k=1}^{\infty}$ tends to infinity and hence this series is divergent.

Proposition (Finite modification) -
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers. Suppose for large enough n, a_n equals to b_n. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the series $\sum_{n=1}^{\infty} b_n$ is convergent as well.

MATH 114 L04

Lecture 4 : (Easy convergence tests 2) -

Proposition (Subseries) -

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such $\sum_{n=1}^{\infty} a_n$ is convergent. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence. Then, $\sum_{k=1}^{\infty} a_{n_k}$ is convergent as well.

Proposition (Simple Comparison) -

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-negative real numbers. Suppose for any $n \geq 1$ $a_n \leq b_n$. Then, if $\sum_{n=1}^{\infty} b_n$ is convergent so is $\sum_{n=1}^{\infty} a_n$.

Proposition (Cauchy Test) -

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then, $\sum_{n=1}^{\infty} a_n$ is convergent iff for any $\epsilon > 0$ there exists $L > 0$ such for any $M > L$

$$\left| \sum_{n=L}^{M} a_n \right| \leq \epsilon$$

Proposition (Small Tail Test) -

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then, $\sum_{n=1}^{\infty} a_n$ is convergent iff for any $\epsilon > 0$ there exists $L > 0$ such that $\sum_{n=L}^{\infty} a_n$ is convergent and

$$\left| \sum_{n=L}^{\infty} a_n \right| \leq \epsilon$$

These tests are important in checking if series are convergent or divergent.

MATH 114 L04 Extra (Continued)

Lecture 5: (Alternating Series) -

Definition:
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: Any absolutely convergent series is convergent.

Theorem: (Alternating Series Test) -
Let $a_1 \geq a_2 \geq a_3$ be a sequence of positive real numbers tending to zero. Then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent. This such series is an alternating series.

Examples -

- $1 - 1/2 + 1/4 - 1/8 + \dots$ is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^{n-1}}$ is absolutely convergent.
- $1 - 1/2 + 1/2 - 1/3 + 1/3$ is convergent
- $1 + 1/2 + 1/2 + 1/3 + 1/3 + 1/3$ is divergent.

Lecture 6: (Further convergent tests)

Proposition (Ratio Test)
Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Assume there exists some $N \geq 1$ and $0 < r < 1$ such that $\frac{|a_{n+1}|}{|a_n|} \leq r$ provided $n \geq N$. Then the series is convergent.

Proposition : (Root Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Assume $N \geq 1$ and $0 < r < 1$ such $|a_n|^{\frac{1}{n}} \leq r$ provided $n \geq N$. Then the series is convergent.

Proposition : (Cauchy Condensation Test)

Let $a_1 \geq a_2 \geq a_3 \geq \dots$ be a decreasing sequence of positive numbers. Then, the series $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges

Theorem (p-series)

For any $p > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Lecture 7 + lecture 8 are Revision / Recap.

[?]/3/2022

MATH 114 LO5

Lecture 9 : (Rearrangements 1) -

Rearrangement examples -

* For any $n \geq 1$, let $\sigma(2n-1) = 2n$ and $\sigma(2n) = 2n-1$. So σ is a bijection of natural numbers which maps odd numbers to even and vice versa.
* Divide natural numbers into two sets $\mathbb{N} = A \cup B$.
  Let $A = \{n_k\}_{k=1}^{\infty}$ an increasing sequence and
  Let $B = \{m_k\}_{k=1}^{\infty}$ an increasing sequence
  Define $\sigma(n_k) = m_k$ $\sigma(m_k) = n_k$ is a bijection.

Proposition (Rearrangement of sequences) :

Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence tending to x. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\{x_{\sigma(n)}\}_{n=1}^{\infty}$ also tends to x.

Theorem :

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive numbers. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent as well and their sums are equal.

Lecture 10 : (Rearrangements 2) -

Theorem :

Let $a_1 \geq a_2 \geq a_3$ be a sequence of positive real numbers. Such that $\sum_{n=1} a_n$ diverges. By the Alternating Series Theorem $\sum_{n=1} (-1)^n a_n$ is convergent. However, for any real number $r \in \mathbb{R}$ there exists a bijection. $\sigma : \mathbb{N} \to \mathbb{N}$ so that the sum of the rearranged series $\sum_{n=1}^{\infty} (-1)^{\sigma(n)} a_{\sigma(n)}$ equals to r.

Proposition:

Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of positive numbers. Assume $a_n \to 0$. Let $0 < s < \infty$ be a real number and $0 < \epsilon < s$. Then, for any $N \geq 1$ we have $n \geq N$ and $k \geq 1$ such,

$$s - \epsilon < \sum_{i=n}^{n+k} a_i < s$$

Theorem:

Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of positive numbers. Assume $a_n \to 0$. Then for any $0 < r < \infty$ we have a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k} = r$.

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Lecture 11: (Integral Test)

Proposition (Integral Test)

Let $f: [0,\infty] \to \mathbb{R}$ be a positive, decreasing function. Then, the improper integral $\int_1^\infty f(x)\,dx$ exists iff the series $\sum_{n=1}^{\infty} f(n)$ converges.

Lecture 12: (Euler Number)

Proposition:

The sequence $\{a_n = (1+1/n)^n\}_{n=1}^{\infty}$ is bounded and increasing.

THIS IS PROVEN USING BINOMIAL THEOREM

Definition:

The limit of this sequence is referred to as the Euler's number (e). It follows from the proof $2 < e < 3$ (2.718...) it models growth rates.

Theorem:

e is equal to the sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$

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MATH 114 L07 + L08

Lecture 13 : (Sequences of Complex Numbers) -

$z = a \pm bi$

$|z| = \sqrt{a^2+b^2}$

Properties :

* $|z+w| \leq |z| + |w|$
* If $\lambda > 0$ is a real number $|\lambda z| = \lambda |z|$
* $|zw| = |z||w|$

Definition (Complex Calculus) -

A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is converging to a complex number z if for any $\epsilon > 0$ there exists $N > 0$ such if $n \geq N$ then $|z - z_n| \leq \epsilon$.

A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists $N > 0$ such if $n, m \geq N$ then $|z_n - z_m| \leq \epsilon$.

Proposition :

A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is convergent iff both $\{Re(z_n)\}_{n=1}^{\infty}$ and $\{Im(z_n)\}_{n=1}^{\infty}$ are convergent. Also, $\{z_n\}_{n=1}^{\infty}$ is Cauchy iff both Real and Imaginary parts are Cauchy.

Lecture 14 : (Bounded Sequences. Closed sets and series of complex numbers) -

$$\frac{|Re(z-w)| + |Im(z-w)|}{2} \leq \max\{|Re(z-w)|, |Im(z-w)|\}$$

$$\leq |z-w| \leq |Re(z-w)| + |Im(z-w)|$$

Definition :

A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is called bounded if there exists $M > 0$ such that for any $n \geq 1$, $|z_n| \leq M$.

Proposition : (Complex Bolzano-Weierstrauss).

Any bounded sequence of complex numbers contains a convergent subsequence.

A subset $L \subseteq \mathbb{C}$ of the complex numbers is closed if for any convergent sequence $\{z_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} z_n \in L$ provided $z_n \in L$ for any $n \geq 1$.

Definition :

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Then, the series $\sum_{n=1}^{\infty} z_n$ is convergent if the sequence of partial sums $\{p_k\}_{k=1}^{\infty}$ is convergent. Also, $\sum_{n=1}^{\infty} z_n := \lim_{k\to\infty} p_k$.

Proposition : If $\sum_{n=1}^{\infty} z_n$ is convergent then $z_n \to 0$.

Proposition : If $\sum_{n=1}^{\infty} |z_n|$ is convergent so is $\sum_{n=1}^{\infty} z_n$.

MATH 114 L09

Lecture 17: (Sequences of continuous functions 1)

Definition: Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then, the norm of f is defined as:

$$\|f\| := \max_{x \in [a,b]} |f(x)|$$

Proposition: Let $f, g: [a,b] \to \mathbb{R}$ be continuous functions and $\lambda > 0$ be a positive number: Then,

* $\|f + g\| \leq \|f\| + \|g\|$
* $\|\lambda f\| = \lambda \|f\|$

Definition: A sequence of continuous functions $\{f_n: [a,b] \to \mathbb{R}\}$ is converging to a continuous function $f: [a,b] \to \mathbb{R}$ if for any $\epsilon > 0$ there exists $N \geq 1$ such $\|f - f_n\| \leq \epsilon$ provided $n \geq N$. Also a sequence of continuous functions $\{f_n: [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists $N \geq 1$ such that $\|f_n - f_m\| < \epsilon$ provided $n, m \geq N$.

Proposition: Let $\{f_n: [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions converging to a continuous function $f: [a,b] \to \mathbb{R}$. Then the sequence is Cauchy.

Lecture 18: (Sequences of continuous Functions 2) -

<u>Theorem</u>: Any Cauchy sequence of continuous functions $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ is convergent

<u>Proof</u>:

Fix $x \in [a,b]$. For any $n, m \geq 1$ we have

1) $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$

Pick $\epsilon > 0$ Then there exists $N \geq 1$ such $\|f_n - f_m\| < \epsilon/3$ provided $n, m \geq N$. (used $\epsilon/3$ instead of ϵ)

Therefore by 1)

2) $|f_n(x) - f_m(x)| < \epsilon/3$

Hence $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence and therefore is converging to some number $f(x)$.

Now we define function $f : [a,b] \to \mathbb{R}$ by

$\lim_{n \to \infty} f_n(x) := f(x)$

Taking $n \to \infty$ in 2) we have:

3) $|f_N(x) - f(x)| \leq \epsilon/3$

By definition:

$|f_N(x) - f_n(x)| < \epsilon/3$ and $f_n(x) \to f(x)$

We now need to use a lemma to prove f is continuous.

Lemma:

The function $f : [a,b] \to \mathbb{R}$ is continuous

<u>Proof</u>:

Since f_N is continuous it is uniformly continuous hence there exists $\delta > 0$ such for all $x \in [a,b]$

4) $|f_N(x) - f_N(y)| < \epsilon/3$ provided $|x-y| < \delta$

Therefore if $|x-y| < \delta$:

$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

So this is uniformly continuous $\square$

$\square$

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MATH 114 L10 + L11 + L12

Lecture 20 : (Power Series) -

Theorem : Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Then exactly one of the following occurs.

a) The series converges absolutely for all $x \in \mathbb{R}$

b) $\exists R > 0$ such that the series is absolutely convergent wherever $|x| < R$ and divergent wherever $|x| > R$

Proof : Let $E = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n$ is convergent$\}$ and $R = \sup\{x : x \in E\} \in [0, \infty]$ where $R = \infty$ means this above set isn't bounded.

By the n^{th} term test $a_n x_0^n \to 0$ as $n \to \infty$.

Hence $\{|a_n x_0^n|\}_{n=1}^{\infty}$ is a bounded sequence.

We claim if $|x| < |x_0|$ then $x \in E$:

$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M\left|\frac{x}{x_0}\right|^n$ where $\{|a_n x_0^n|\}_{n=1}^{\infty}$ is bounded by M.

Then $\sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n$ is convergent being a geometric series with common ratio < 1. Hence $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent and $x \in E$. $\square$

Lecture 19 : (Series of continuous functions) -

Definition :

Let $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions. Then the series $\sum_{n=1}^{\infty} f_n$ is convergent if $\{p_k\}_{k=1}^{\infty}$ is a convergent sequence where p_k is the k^{th} partial sum, $p_k = \sum_{n=1}^{k} f_n$. The sum of a convergent series, $s = \sum_{n=1}^{\infty} f_n$ is defined as $\lim_{k \to \infty} p_k$.

Proposition (Sum Rule for sequences of continuous functions) -
Let $\{h_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ and $\{k_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ be convergent sequences of continuous functions tending to $h : [a,b] \to \mathbb{R}$ and $k : [a,b] \to \mathbb{R}$. Then $\{h_n + k_n\}_{n=1}^{\infty}$ is a convergent sequence of continuous functions tending to $h+k$.

Proposition :
Let $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ and $\{g_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ be sequences of convergent functions such that $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} g_n$ are both convergent series. Then, $\sum_{n=1}^{\infty} (f_n + g_n)$ is a convergent series and :

$$\sum_{n=1}^{\infty} (f_n + g_n) = \sum_{n=1}^{\infty} f_n + \sum_{n=1}^{\infty} g_n .$$

Definition : Let $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of convergent functions. We say that $\sum_{n=1}^{\infty} f_n$ is an absolutely convergent series if $\sum_{n=1}^{\infty} \|f_n\|$ is a convergent series of real numbers.

CRUCIAL THEOREM : All absolutely convergent series of continuous functions are convergent. So it's sum is a continuous function.

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MATH 114 L13

Lecture 23 : (Step Functions) -

Definition : A partition P of the interval $[a,b]$ is a sequence of $n+1$ real numbers $p_0 < p_1 < \dots p_n$ such that $p_0 = a$ and $p_n = b$.
e.g $P = \{0, 1/3, 1/2, 1\}$ is a partition of interval $[0,1]$.

Definition : Let R be a partition of the interval $[a,b]$ containing the partition P. Then we call R the refinement of P.

Definition : Let P and R be partitions of the interval $[a,b]$. Then, a partition 6 of the interval $[a,b]$ is joint refinement of P and R if it is a refinement of both P and R.

Definition : Let $a \leq c \leq d \leq b$ be two numbers. Then the primitive step-function $1_{[c,d)}$ is defined in the following way:
- $1_{[c,d)}(x) = 0$ if $x < c$ or $d \leq x$
- $1_{[c,d)}(x) = 1$ if $c \leq x < d$

Particularly, $1_{[c,d)}(c) = 1$ $\quad 1_{[c,d)}(d) = 0$.

Definition : Let P be a partition of the interval $[a,b]$ then a step-function $s(x)$ for P is a function in the form of $\sum_{i=1}^{n} c_i 1_{[p_{i-1}, p_i]}$

Lecture 24: (The Integral of a Step function) -

Proposition: Let P be a partition of the interval $[a,b]$ and R be a refinement of P. Then any step function for P is automatically a step function for R.

Definition: Let $s = \sum_{i=1}^{n} c_i 1_{[p_{i-1}, p_i)}$ be a step function for the partition $P = \{p_0, p_1, \cdots p_n\}$ of the interval $[a,b]$. Then $\int_a^b s(x)\,dx$ is defined as $\sum_{i=1}^{n} c_i(p_i - p_{i-1})$.

Note: $\int_a^b s(x)\,dx$ is the Area under the curve for step function s.

Proposition: The sum of two step functions for the partition P of the interval $[a,b]$ is still a step function for the partition P.

Proposition: The sum of two step functions on the interval $[a,b]$ is still a step function on the interval $[a,b]$.

Theorem: Let s, t be step functions on the interval $[a,b]$. Then,

$$\int_a^b s(x)\,dx + \int_a^b t(x)\,dx = \int_a^b s(x) + t(x)\,dx$$

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Lecture 25 : (The Integral of continuous Functions) -

Theorem: For any continuous function $f:[a,b]\to\mathbb{R}$ we have

$$I_{step}(f) = I^{step}(f)$$

Therefore we define the integral of f as

$$I_{step}(f) = I^{step}(f) := \int_a^b f(x)\,dx$$

Proof: Let $f:[a,b]\to\mathbb{R}$ be a continuous function.
Prove for any $\epsilon>0$ there exist step functions $s\le f$ and $f\le t$ on interval $[a,b]$ such for any $x\in(a,b)$ we have

$$t(x)-s(x)\le\epsilon.$$

If $t-s$ is a step function on $[a,b]$ such for any $x\in(a,b)$, $t(x)-s(x)\le\epsilon$ then

$$\int_a^b t(x)-s(x)\,dx \le \epsilon[a,b].$$

Therefore, $I^{step}(f) - I_{step}(f) \le \epsilon[a,b]$.
and so, $I^{step}(f) = I_{step}(f)$.

f is uniformly continuous so fix $\epsilon>0$ and pick $n\geq 1$ such $|f(x)-f(y)|<\epsilon$ if $|x-y|<1/n(b-a)$

Let P be the following partition of $[a,b]$

$$p_0=a \quad p_n=b \quad p_i=a+i\cdot\frac{b-a}{n}$$

For $1\le i\le n$ define $c_i=\max_{x\in[p_{i-1},p_i]} f(x)$ and $d_i=\min_{x\in[p_{i-1},p_i]} f(x)$.

Claim:

* $t(x)=\sum_{i=1}^{n} c_i 1_{[p_{i-1},p_i]}(x) \geq f(x)$ for any $x\in(a,b)$
* $s(x)=\sum_{i=1}^{n} d_i 1_{[p_{i-1},p_i)}(x) \le f(x)$ for any $x\in(a,b)$
* $t(x)-s(x)\le\epsilon$ for any $x\in(a,b)$.

First two parts follow from definitions and if $x,y\in[p_{i-1},p_i)$ then $|f(x)-f(y)|<\epsilon$.
So Theorem follows $\square$.

Lecture 26: (Properties of the Integral) -

Definition: Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of step functions on $[a,b]$. We say that $\{s_n\}_{n=1}^{\infty}$ is an approximating sequence for f such that

* For any $x\in[a,b)$ and $n\geq 1$, $s_n(x)\leq f(x)$
* For any $\epsilon>0$ there exists $N\geq 1$ such that if $n\geq N$ we have $f(x)-s_n(x)\leq\epsilon$.

Theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\{s_n\}_{n=1}^{\infty}$ be an approximating sequence for f then $\lim_{n\to\infty}\int_a^b s_n(x)\,dx=\int_a^b f(x)\,dx$.

Proposition: Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions then,

$$\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.$$

Proposition: Let $f:[a,c]\to\mathbb{R}$ be a continuous function. Suppose, that $a<b<c$ then:

$$\int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx$$

Proposition: Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Assume for any $x\in[a,b]$ we have $L\leq f(x)\leq M$. Then $L(b-a)\leq\int_a^b f(x)\,dx\leq M(b-a)$.

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Lecture 27 : (Further Properties of the Integral) -

Proposition : Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions such that for any $x \in [a,b]$ we have $f(x) \leq g(x)$. Then,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Proposition : Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then,

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

Theorem : Let $\{f_n\}_{n=1}^{\infty}$ be continuous functions on the interval $[a,b]$ converging to a continuous function $f : [a,b] \to \mathbb{R}$. Then,

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proposition : The integral function $F : [a,b] \to \mathbb{R}$ is continuous.

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MATH 114 L16

Lecture 29 : (The Basic Definition) -

$\lim_{h \to 0} f(x+h) = f(x)$ is the (ϵ, δ) definition in disguise. That is for any $\epsilon > 0$ there exists some $\delta > 0$ such that $|f(x+h) - f(x)| < \epsilon$ provided that $|h| < \delta$.

Definition : Let $f : [a,b] \to \mathbb{R}$ be a function. We say that f is differentiable at $x \in [a,b]$ with derivative $f'(x)$ if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$ First Principles.

Proposition : The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is differentiable at every x with derivative $f'(x) = 2x$.

Proposition : If $f : [a,b] \to \mathbb{R}$ is differentiable at x, then it is continuous at x as well.

Proposition : If $f : [a,b] \to \mathbb{R}$ is differentiable at x with derivative $f'(x)$ and $g : [a,b] \to \mathbb{R}$ is differentiable at x with derivative $g'(x)$. Then,

* $f + g$ is differentiable at x with derivative $f'(x) + g'(x)$.
* fg is differentiable at x with derivative $f(x)g'(x) + f'(x)g(x)$ Product Rule.

Lecture 30 : (Further Properties of the Derivatives) -

Proposition : Let $g : [a,b] \to \mathbb{R}$ be a positive function differentiable at $x \in [a,b]$, then $1/g$ is also differentiable at x and

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g^2(x)}.$$

Proposition: Let $g: [a,b] \to \mathbb{R}$ be a positive function differentiable at $x \in [a,b]$ and $f: [a,b] \to \mathbb{R}$ be function differentiable at $x \in [a,b]$ then f/g is also differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$ Rule

Theorem (Inverse Function Rule) -

Let $f: [a,b] \to [c,d]$ be a continuous invertible function. Assume that for some $x \in (a,b)$ f is differentiable and $f(x) = y \in (c,d)$. Then f^{-1} is differentiable at y and

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Proof:

$f^{-1}(y+h) = x+h'$ where $\lim_{h \to 0} h' = 0$ so,

$$\frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \frac{(x+h') - x}{f(x+h') - f(x)}$$

Indeed $y+h = f(x+h')$ since $\lim_{h \to 0} h' = 0$ we have

$$\lim_{h \to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \lim_{h \to 0} \frac{(x+h') - x}{f(x+h') - f(x)} = \frac{1}{f'(x)}$$

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MATH 114 L17 Mean-Value

Lecture 31: (Invertible Functions) -

Definition: Let $f:[a,b] \to [c,d]$ be a function. We say f is invertible if it is injective and surjective:
a) If $x \neq y \in [a,b]$ then $f(x) \neq f(y)$
b) For any $z \in [c,d]$ there exists $x \in [a,b]$ such that $f(x) = z$.

Theorem: Let $f:[a,b] \to [c,d]$ be a continuous function. Then $f^{-1}:[c,d] \to [a,b]$ is continuous as well.

Definition: A function $f:[a,b] \to [c,d]$ is called strictly monotonic if $f(x) < f(y)$ provided that $x<y$.

Proposition: Let $f:[a,b] \to [c,d]$ be strictly monotonic and continuous such that $f(a) = c$ and $f(b) = d$. Then f is invertible and it's inverse is strictly monotonic.

Applying the Theorem

* $\arcsin:[0,1] \to [0,\pi/2]$ is continuous (as the inverse of sin).
* $\log:[1,\infty) \to [0,\infty)$ is continuous (as inverse of exponential).
* $g(x) = \sqrt{x}:[0,\infty) \to [0,\infty)$ is continuous (as inverse of $f(x) = x^2$.)

Lecture 32: (The Mean-value Theorem) -

Theorem (Rolle's Theorem) -

Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at all points $x \in (a,b)$.
Assume that $f(a) = f(b)$. Then there exists $t \in (a,b)$ such that $f'(t) = 0$.

Theorem (Mean Value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at all points $x \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(t)$$

Proof:

Let $m = \frac{f(b) - f(a)}{b-a}$

Define $k(x) = f(a) + m(x-a)$

Then k is differentiable and it's derivative $= m$ at all points $x \in (a,b)$.

So the function defined $g(x) = f(x) - k(x)$ is continuous at $[a,b]$, $g(a) = g(b) = 0$ and is differentiable at all $x \in (a,b)$.

Therefore by Rolle's Theorem there exists $t \in (a,b)$ such $g'(t) = 0$.

Then,

$$f'(t) = g'(t) + k'(t) = m = \frac{f(b) - f(a)}{b-a} \quad \square.$$

Proposition: Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at all $x \in (a,b)$. Then the two statements are equivalent:

a) f is constant on the interval $[a,b]$

b) For all $t \in (a,b)$, $f'(t) = 0$.

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Lecture 33 : (Darboux's Theorem) -

Theorem (Intermediate Value) -

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Let $f(a) < u < f(b)$. Then there exists $a < c < b$ such that $f(c) = u$.

Theorem (Darboux's)

Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at all values $x \in [a,b)$. Let $a < x < y < b$ and $f'(x) < u < f'(y)$. Then there exists $x < c < y$ such that $f'(c) = u$.

Proof :

Consider the function $g(t) = f(t) - ut$.

Claim: The function g has no minimum at x on the interval $[x,y]$. If g has minimum at x on $[x,y]$ then:

$\frac{g(t) - g(x)}{t - x}$ is non-negative on $(x, y]$. Then, as t tends to x we obtain that $g'(x) \geq 0$. On the other hand $g'(x) = f'(x) - u < 0$.

Claim: The function g has no minimum at y on the interval $[x,y]$. Indeed if g has minimum at y on $[x,y]$ then,

$\frac{g(t) - g(y)}{t - y}$ is non-positive on $[x, y)$. Then as t tends to y we obtain that $g'(y) \leq 0$.

Hence g attains it's minimum at $x < c < y$. Therefore by Fermat's principle $g'(c) = f'(c) - u = 0$. Thus, $f'(c) = u$. □