MATH 115 L01 (2)

Vector Addition -

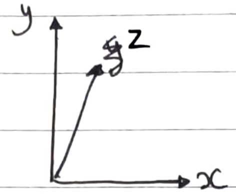
To distinguish vectors from scalars vectors are written

$$U = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \lambda U = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \end{bmatrix} \quad V = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \quad U + V = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$$

so addition is commutative, associative and distributive over scalar multiplication

Standard Basis for Euclidean Space -

$u = ai + bj + ck$



3D

Lines -

$U = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $V = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ For $V \neq 0$ the line L through u in the direction v has

vector equation $L: X = U + tV$ or equivalent

parametric form $L: x = a + td, \; y = b + et, \; z = c + ft$ or

symmetric form $L: \dfrac{x-a}{d} = \dfrac{y-b}{e} = \dfrac{z-c}{f}$

where d, e, f are called the direction numbers

if $d = 0$ we have $x = a$ for all $t \in \mathbb{R}$ and likewise for e and f.

Line through two points -

To find the line through A and B introduce a point $U = A$ in the line and a direction $V = B - A$ when $\vec{AB}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -8 \end{bmatrix}$$

for example $A = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$

The Dot (scalar) product -

$U = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $V = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ $U \cdot V = ad + be + cf$

The length of U (norm, length, modulus)

$\|U\| = \sqrt{a^2+b^2+c^2}$

If $\|U\| = 1$ then we say U is a unit vector

Properties of the Dot product -

Let $U, V, W \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ Then

i) $U \cdot V = V \cdot U$ ii) $U \cdot (V + W) = U \cdot V + U \cdot W$

iii) $U \cdot (\lambda V) = \lambda (U \cdot V) = (\lambda U) \cdot V$

iv) $U \cdot U \geq 0$ and $U \cdot U = 0$ iff $U = 0$ for example

$U \cdot U = a^2 + b^2 + c^2$ $U = 0$ iff $a = b = c = 0$.

Parallel and Perpendicular -

i) The zero vector 0 is parallel to all vectors

ii) Non-zero vectors U and V are parallel if there exists $t \in \mathbb{R}$ such $U = tV$.

iii) Vectors U and V are perpendicular if $U \cdot V = 0$.

Area of a Parallelogram -

vertices $(0,0)$ (a,b) $(a+c, b+d)$ (c,d) Then P has area $ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

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MATH 115 LO2

1.12 **Primer on Determinants**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Vector or Cross Product -

The cross product is special to 3D $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $v = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf - ce \\ -af + cd \\ ae - bd \end{bmatrix} = \begin{bmatrix} i & a & d \\ j & b & e \\ k & c & f \end{bmatrix}$$

The Cross Product as a determinant -

$$u \times v = \begin{bmatrix} bf - ce \\ -(af - cd) \\ ae - bd \end{bmatrix} = (bf - ce)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (af - cd)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (ae - bd)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{vmatrix} b & e \\ c & f \end{vmatrix} i - \begin{vmatrix} a & d \\ c & f \end{vmatrix} j + \begin{vmatrix} a & d \\ b & e \end{vmatrix} k = \begin{bmatrix} i & a & d \\ j & b & e \\ k & c & f \end{bmatrix}$$

where we expand the determinant about the first column.

Example of Cross Product $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$u \times v = \begin{bmatrix} i & 1 & 4 \\ j & 2 & 5 \\ k & 3 & 6 \end{bmatrix} = i\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} - j\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + k\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}(12 - 15) - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}(6 - 12) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}(5 - 8) = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

Properties of Cross Product

i) $u \times v = -v \times u$ ii) $u \times (v + w) = u \times v + u \times w$

iii) for all scalars λ $u \times (\lambda v) = \lambda(u \times v) = (\lambda u) \times v$

iv) Cross product $u \times v$ is perpendicular to u and v so

$(u \times v) \cdot u = 0 = (u \times v) \cdot v$

v) $u \times v = 0$ iff u and v are parallel

Properties of Cross Product: Orthogonality -

iv) $(U \times V) \cdot U = \begin{bmatrix} bf - ce \\ -af + cd \\ ae - bd \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = abf - ace - abf + bcd + ace - bcd = 0.$

Cross Product is not associative -

i) Cross product satisfies Lagrange's formula

$$U \times (V \times W) = (U \cdot W)V - (U \cdot V)W$$

ii) Cross product satisfies Jacobi's identity where we cyclically permute the vectors U, V, W

$$U \times (V \times W) + W \times (U \times V) + V \times (W \times U) = 0$$

iii) Cross product is not associative so there exist U, V, W such

$$(U \times V) \times W \neq U \times (V \times W)$$

Norm of the Cross Product

i) $\|U \times V\|^2 + (U \cdot V)^2 = \|U\|^2 \|V\|^2$

ii) There exists $\theta \in [0, \pi]$, angle between U and V such

$$U \cdot V = \|U\| \|V\| \cos\theta$$

iii) $\|U \times V\|$ is the area of the parallelogram with vertices $\{0, U, V, U+V\}$ so

$$\|U \times V\| = \|U\| \|V\| |\sin\theta|$$

Example - $U = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $V = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $U \times V = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

$U \cdot U = 14$ $V \cdot V = 77$ $U \cdot V = 32$ $(U \times V) \cdot (U \times V) = 54$

$\|U \times V\|^2 + (U \cdot V)^2 = 54 + 32^2 = 1078.$

$\cos\theta = \frac{U \cdot V}{\|U\| \|V\|}$

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MATH 115 L03 + L04.

Find angle between vectors $U = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $V = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

$U \cdot U = 6 \quad U \cdot V = 4 \quad V \cdot V = 11$

$$\cos\theta = \frac{4}{\sqrt{6}\sqrt{11}}$$

Parallel and perpendicular vectors -

i) $|U \cdot V| \leq \|U\|\|V\|$ with equality iff U and V are parallel
ii) $U \cdot V = 0$ iff U and V are perpendicular
iii) $U \times V = \vec{0}$ iff U and V are parallel

Parallelogram Law -

Vectors U and V together with 0 and $U+V$ are vertices of a parallelogram such

i) Diagonals of parallelogram satisfy
$$\|U+V\|^2 + \|U-V\|^2 = 2\|U\|^2 + 2\|V\|^2$$
ii) Area of parallelogram is
$$\|U \times V\| = \|U\|\|V\||\sin\theta|$$
iii) Length of diagonals satisfy
$\|U+V\| \leq \|U\| + \|V\|$ with equality iff U and V are parallel and point in the same direction

Example Area of a Triangle -

$A = (1,1,-1) \quad B = (2,-1,0) \quad C = (0,2,7)$

$$U = \overline{AB} = B - A = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$V = \overline{AC} = C - A = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 8 \end{bmatrix}$$

$$U \times V = \begin{vmatrix} i & 1 & -1 \\ j & -2 & 1 \\ k & 1 & 8 \end{vmatrix} = \begin{bmatrix} -17 \\ -9 \\ -1 \end{bmatrix}$$

so Area $= \frac{1}{2}\|U \times V\| = \frac{1}{2}\sqrt{17^2+9^2+1^2} = \frac{1}{2}\sqrt{371}$

Further Examples in L03 . . .

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MATH 115 L05

1.34 Plane through the Origin -

Let U and V be vectors that are not parallel

i) Then U and V lie in a unique plane passing through origin

$$P = \{X : X = sU + tV : s,t \in \mathbb{R}\}$$

ii) Vector $N = U \times V$ is perpendicular to all vectors in P, so P is given by $X \cdot N = 0$.

iii) Any vector W can be expressed uniquely as

$$W = aU + bV + c(U \times V) \quad \text{for some } a,b,c \in \mathbb{R}$$

1.36 Linear Independence -

Vectors $U, V, U \times V$ can be written as a linear combination of the standard basis vectors i, j and k. Conversely by successively choosing W as i, j, k we can find a, b, c in each case and express $\{i, j, k\}$ in terms of $\{U, V, U \times V\}$

Example -

Find equation of plane through origin in $\mathbb{R}^3$ where $u = (1,-1,2)$ and $v = (0,3,-1)$

Take normal direction to be $N = U \times V$ where $U = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $V = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

$$N = U \times V = \begin{bmatrix} i & 1 & 0 \\ j & -1 & 3 \\ k & 2 & -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} \quad \text{so } P = -5x + y + 3z = 0$$

1.38 General Planes -

Let U and V be non-parallel vectors and Z any vector. Then the plane P passes through Z and contains directions U and V is

$$P = \{X : X = Z + sU + tV : s,t \in \mathbb{R}\}$$

which has non-zero normal $N = U \times V$. Equivalently

$$P: \quad X \cdot N = Z \cdot N$$

1.40 The plane through 3 points -

A plane P can be specified by

i) A point on P and normal direction to P

ii) A point on P and two non-parallel directions in P.

iii) Three distinct points on P

1.43 Two Planes -

Let P and Q be two planes. Then either

i) $P = Q$ ii) P doesn't intersect Q iii) P and Q intersect in a line

1.44 Planes intersecting in lines -

The planes are $P: X \cdot N = c$ $Q: X \cdot M = d$

a) suppose N and M are parallel so $N = \lambda M$ for some λ Then

$P: X \cdot N = c$ $Q: X \cdot N = \lambda d$

If $c = \lambda d$ Then P and Q are i) otherwise ii)

b) otherwise N and M are not parallel so

$X = \alpha N + \beta M + \gamma (N \times M)$ where α, β and γ are scalars.

Then X lies on both P and Q if:

$X \cdot N = \alpha N \cdot N + \beta M \cdot N = c$ $X \cdot M = \alpha N \cdot M + \beta M \cdot M = d$

so α and β are determined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{(N \cdot N)(M \cdot M) - (N \cdot M)^2} \begin{bmatrix} M \cdot M & -M \cdot N \\ -M \cdot N & N \cdot N \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

1.45 Angle between planes

Let P and Q be planes where P has normal direction N and Q has normal direction M. Suppose P and Q intersect in a line. Then acute angle between P and Q is angle between N and M or N and $-M$

1.46 Volume determined by three vectors -

$U = (a, b, c) \quad V = (d, e, f) \quad W = (g, h, j)$

$$(U \times V) \cdot W = [U, V, W] = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & j \end{vmatrix}$$

is called the scalar triple product of U, V and W.

1.47 Determinant as a volume -

Let U, V, W be any triple of vectors. Then the parallelepiped P generated by U, V and W has volume equal to

$$\mathrm{vol}(P) = \pm (U \times V) \cdot W$$

Either $[U, V, W] \neq 0$ and the volume is nonzero or $[U, V, W] = 0$ and $\{U, V, W\}$ lie in some plane that passes through 0.

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MATH 115 L06

Chapter 2: Space Curves

2.1 Lines in the Plane -

$y = mx + c$ with gradient $\neq 0$ and parametrize it by $x = t$ and $y = mt + c$ for parameter (time) $t \in \mathbb{R}$ then:

$$L: \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ m \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$N: \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -m \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$$

so N has gradient $-1/m$ and intersects L perpendicularly at $(0, c)$

2.2 Plane Curves -

Let J be an interval in $\mathbb{R}$, let time be $t \in J$ and let $x, y : J \to \mathbb{R}$ be differentiable functions; write $x'(t) = dx/dt$ etc.
Recall a planar curve γ is a function.

$$\gamma : J \to \mathbb{R}^2 : \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with Tangent Direction $\gamma' : J \to \mathbb{R}^2 : \gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$
Assume $\|\gamma'(t)\| > 0$ for all t. If $\gamma'(t) = 0$ then the curve can have a corner or cusp at $\gamma(T)$.

2.3 Tangent and Normal to Plane Curve -

Now the tangent line to γ at $t = a$ is the line through $\gamma(a)$ in the direction of $\gamma'(a)$ so has vector form:

$$X = \gamma(a) + \lambda \gamma'(a)$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x(a) \\ y(a) \end{bmatrix} + \lambda \begin{bmatrix} x'(a) \\ y'(a) \end{bmatrix}$$

or

$$\frac{X - x(a)}{x'(a)} = \frac{Y - y(a)}{y'(a)}$$

and the normal line through $\gamma(a)$ has vector equation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x(a) \\ y(a) \end{bmatrix} + \lambda \begin{bmatrix} -y'(a) \\ x'(a) \end{bmatrix}$$

2.5 Space Curves -

Let J be an interval in $\mathbb{R}$ let time be $t \in J$ and let $x, y, z:$ $J \to \mathbb{R}$ be differentiable functions. Then $\gamma : J \to \mathbb{R}^3$ is called a differentiable space curve where

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

With $x' = \frac{dx}{dt}$

$$\gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} \qquad \gamma''(t) = \begin{bmatrix} x''(t) \\ y''(t) \\ z''(t) \end{bmatrix}$$

So t = time $\gamma(t)$ = point moving along a curve $\gamma'(t)$ = velocity of the point $\gamma''(t)$ = acceleration.

e.g instantaneous speed

$$\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} > 0$$

2.8 Example of Parabolic Trajectory -

A ball weighing 1kg is thrown from a height h above (x, y) plane subject to g; initial velocity is u in x-direction and v in y-direction

$$\gamma(0) = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} \quad \gamma'(0) = \begin{bmatrix} u \\ 0 \\ v \end{bmatrix}$$

$F = ma$ so

$$\gamma''(t) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = g$$

a is constant so integrate to time

$$\int_0^t \gamma''(s)\,ds = \int_0^t g\,ds$$

$$\gamma'(t) = \gamma'(0) + tg$$

$$\int_0^t \gamma'(s)\,ds = \int_0^t (\gamma'(0) + sg)\,ds$$

so $\gamma(t) = \gamma(0) + t\gamma'(0) + (t^2/2)g$

$$\gamma(t) = \gamma(0) + t\gamma'(0) + \left(t^2/2\right) g$$

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} + t\begin{bmatrix} u \\ 0 \\ v \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

ball hits ground when $z(t) = 0$

so $h + tv - gt^2/2 = 0$

so $t = \dfrac{-v \pm \sqrt{v^2+2gh}}{-g} = \dfrac{v + \sqrt{v^2+2gh}}{g}$ since $t > 0$

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MATH 115 L07

2.12 Arclength on Space Curves -

The distance moved along γ from time $t = a$ to $t = b$ is

$$S(a,b) = \int_a^b \|\gamma'(t)\| \, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

Example -

For $r, w > 0$, let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be the space curve

$\gamma(t) = \begin{bmatrix} r\cos wt \\ r\sin wt \\ b\cos t \end{bmatrix}$ so by differentiating $\gamma'(t) = \begin{bmatrix} -wr\sin wt \\ wr\cos wt \\ -b\sin t \end{bmatrix}$

so $\|\gamma'(t)\| = w^2 r^2 \sin wt + w^2 r^2 \cos^2 wt + b^2 \sin^2 t = w^2 r^2 + b^2 \sin^2 t$

$(\sin^2 + \cos^2 = 1)$

so $\|\gamma'(t)\| = \sqrt{w^2 r^2 + b^2 \sin^2 t}$

and

$$S = \int_0^{2\pi} \|\gamma'(t)\| \, dt = \int_0^{2\pi} \sqrt{w^2 r^2 + b^2 \sin^2 t} \, dt$$

2.16 Convex Graphs -

We compare certain graphs to straight lines. Let f be a continuous real function. Say that f is convex if for all $a < b$.

$$f((1-t)a + tb) \leq (1-t)(f(a) + tf(b))$$

The points

$A = \begin{bmatrix} a \\ f(a) \end{bmatrix}$ $B = \begin{bmatrix} b \\ f(b) \end{bmatrix}$ lie on the graph of f. The equation $X = (1-t)A + tB$ or

$\begin{bmatrix} x \\ y \end{bmatrix} = (1-t)\begin{bmatrix} a \\ f(a) \end{bmatrix} + b\begin{bmatrix} b \\ f(b) \end{bmatrix}$ gives the chord between A and B. Therefore, function is convex if the chord always lies above the graph.

Examples -

i) Let $f(x) = x^2/2$ so $f'(x) = x$ $f''(x) = 1 > 0$ so $f'(x)$ is increasing and f is convex

ii) $f(x) = x^4/4$ so $f'(x) = x^3$ $f''(x) = 3x^2 \geq 0$ so f' is increasing and f is convex

iii) $f(x) = x\log x$ for $x > 0$ $f'(x) = \log x + 1$ $f''(x) = 1/x$ $x > 0$ so f is convex on $(0, \infty)$

2.18 Legendre's Transform -

We consider s fixed and the straight line $y = sx$ through the origin with gradient s. Then the function

$$h(x) = sx - f(x)$$

which is the height of this straight line above the graph of f at x.

Definition: Suppose f is continuously differentiable of $[a, b]$ let $c = f'(a)$ and $d = f'(b)$; suppose $c < d$. Then Legendre's transform of f is:

$$g(s) = \sup\{sx - f(x) ; a < x < b\} \quad (c < s < d)$$

Proposition:

i) There is an inequality $sx \leq f(x) + g(s)$ $(a < x < b, c \leq s \leq d)$

ii) For all $s \in (c, d)$ there exists $x \in (a, b)$ such $sx = f(x) + g(s)$ and $f'(x) = s$

iii) The function g is convex

Examples -

i) Let $f(x) = x^2/2$ $f'(x) = x$ then $h(x) = sx - f(x)$; $h'(x) = s - x$ so for $h'(x) = 0$ take $x = s$ and

$g(s) = sx - f(s) = s^2 - s^2/2 = s^2/2$

ii) Let $f(x) = x^2/4$ $f'(x) = x^3$ then $h(x) = sx - f(x)$; $h'(x) = s - x^3$

so $h'(x) = 0$ take $x = s^{1/3}$ suppose $s \geq 0$

$g(s) = sx - f(s) = s^{4/3} - s^{4/3}/4 = 3s^{4/3}/4$

so $sx \leq \frac{x^4}{4} + \frac{3s^{4/3}}{4}$

iii) Let $f(x) = x\log x$ for $x > 0$ $f'(x) = \log x + 1$ so

$h(x) = sx - x\log x$ $h'(x) = s - \log x - 1$

so for $h'(x) = 0$ $x = e^{s-1}$ and

$g(s) = sx - x\log x = se^{s-1} - (s-1)e^{s-1} = s^{s-1}$

so $sx \leq x\log x + e^{s-1}$ $(x > 0,\ s \in (-\infty, \infty))$

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MATH 115 L08

2.20 Curvature of a circle -

Consider $r, w > 0$ and the plane curve $\gamma : \mathbb{R} \to \mathbb{R}^2$

$$\gamma(t) = \begin{bmatrix} r\cos(wt) \\ r\sin(wt) \end{bmatrix}$$

The point $\gamma(t)$ moves round the circle with centre O and radius r since :

$$\gamma(t) \cdot \gamma(t) = r^2\cos^2(wt) + r^2\sin^2(wt) = r^2$$

and $\gamma(\frac{2\pi}{w}) = \gamma(0)$. This w is the angular velocity which measures the rate at which the angle between $\gamma(t)$ and $\gamma(0)$ changes with time t. The derivatives :

$$\gamma'(t) = \begin{bmatrix} -rw\sin(wt) \\ rw\cos(wt) \end{bmatrix} \quad \gamma''(t) = \begin{bmatrix} -w^2r\cos(wt) \\ -w^2r\sin(wt) \end{bmatrix}$$

so the speed is $\|\gamma'(t)\| = rw$ $\gamma''(t)$ points towards the center of the circle and $\|\gamma''(t)\| = rw^2$. By taking $rw = 1$ or $w = 1/r$ we ensure that γ has unit speed. Then $\|\gamma''(t)\| = 1/r$ this is the curvature of γ.

2.21 The osculating plane of a space curve -

Consider a hawk flying around. At position $\gamma(t)$ the beak points along the velocity vector $\gamma'(t)$ and acceleration is $\gamma''(t)$

Case of Interest : when γ' and γ'' are not parallel so the hawk isn't flying in a straight line.

$\gamma(t)$ = position $\gamma'(t)$ = velocity $\gamma''(t)$ = acceleration

Definition : Suppose γ' and γ'' are not parallel. Then the plane through the point $\gamma(t)$ is called the osculating plane. The triple $\{\gamma', \gamma'', \gamma' \times \gamma''\}$ called the moving frame. The moving frame consists of velocity, acceleration and the normal to the osculating plane at γ. The frame moves as $\gamma(t)$ changes with time.

2.22 Moving frame $\{T, N, B\}$ of a curve of unit speed -

Suppose γ is a space curve such that $\|\gamma'(t)\| = 1$ for all t then we say the curve has unit speed. The moving frame has the following properties:

Definition (Serret-Frenet): Let $T(t) = \gamma'(t)$ be the tangent so $T'(t) = \gamma''(t)$. We define the curvature by $\kappa(t) = \|T'(t)\|$.

If $\kappa(t) = 0$ for all t then the curve is a straight line.

Otherwise for $\kappa(t) > 0$ we introduce the principal normal:

$$N(t) = \frac{1}{\kappa(t)} T'(t)$$

and the binomal $B(t) = T(t) \times N(t)$

and $\{T, N, B\}$ is the moving frame of the curve.

2.23 Serret-Frenet on a curve of unit speed -

Proposition (Serret-Frenet). Suppose that γ is a curve of unit speed which is not a straight line. Then the vectors $\{T, N, B\}$ all have unit norm and are mutually orthogonal.

Proof: In this case γ'' is perpendicular to γ' so:

$\gamma'(t) \cdot \gamma'(t) = \|\gamma'(t)\|^2 = 1$ so when differentiated

$\gamma''(t) \cdot \gamma'(t) + \gamma'(t) \cdot \gamma''(t) = 0$

so $\gamma''(t) \cdot \gamma'(t) = 0$ That is $T \cdot N = 0$ and T and N are orthogonal. By hypothesis $\|T\| = \|\gamma'\| = 1$ and by κ $\|N\| = 1$ so T and N are unit vectors. Then B is orthogonal to T and N since $B = T \times N$ and B is a unit vector.

2.24 The moving frame for a helix -

To find the above $\{T, N, B\}$ for the helix (corkscrew)

$$\gamma(t) = \begin{bmatrix} \cos(t/2) \\ \sin(t/2) \\ \sqrt{3}t/2 \end{bmatrix} \quad \gamma'(t) = \begin{bmatrix} -(1/2)\sin(t/2) \\ (1/2)\cos(t/2) \\ \sqrt{3}/2 \end{bmatrix} \quad \gamma''(t) = \begin{bmatrix} -(1/4)\cos(t/2) \\ -(1/4)\sin(t/2) \\ 0 \end{bmatrix}$$

Then $\|\gamma'(t)\|^2 = (1/4)\sin^2(t/2) + (1/4)\cos^2(t/2) + 3/4 = 1$

so the curve has unit speed and $T(t) = \gamma'(t)$.

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2.27 Co-ordinates on a unit sphere -

Consider a sphere of radius 1 and centre O; choose a point P(North Pole) on the sphere and let $S = -P$; polar axis is SOP.

A great circle is the circle on the sphere that arises as the intersection of the sphere with a plane through O. In particular the equatorial plane is the plane through O that is perpendicular to SOP and the equator is the circle of intersection of this plane and SOP (sphere).

Given X on the sphere with $X \neq P$ or S there exists a unique plane through OXP; the meridian through X is the great circle arising from the intersection of OXP and sphere. The colatitude θ of X is the angle XOP measured along the meridian where $0 < \theta < \pi$

Example -

We call / choose a point G on the sphere distinct from P and S and call the meridian through G the G meridian. Then the longitude ψ of X is the angle between the plane through POX and the plane through POG. So $-\pi < \psi \leq \pi$

2.28 Polar Co-ordinates on the Earth -

Continuing on from 2.27 and now modelling the Earth as a sphere, Let X be a point on Earth with Cartesian co-ordinates (x, y, z)

x-axis passes from O to the point on the Equator meeting the Greenwich meridian.

y-axis is in the Equatorial plane

z-axis is from O to north pole

with colatitude θ longitude ψ

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\psi \\ r\sin\theta\sin\psi \\ r\cos\theta \end{bmatrix}$$

(because $x = r\cos\theta$ $y = r\sin\theta$)

2.29 Frames on a meridian -

From 2.28, fix the longitude ψ, fix $r=1$ and consider

$$X(\theta) = \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix}$$

such that $X(\theta)$ moves along the meridian at longitude ψ from the North Pole $X(0)$ to the South Pole $X(\pi)$

Proposition -

For the curve given by a meridian on the unit sphere the moving frame is

$$T = \begin{bmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ -\sin\theta \end{bmatrix} \xrightarrow{\text{integrate}} N = \begin{bmatrix} -\sin\theta\cos\psi \\ -\sin\theta\sin\psi \\ -\cos\theta \end{bmatrix} \quad B = \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix}$$

2.30 Frames on the meridian -

Check that X is a unit vector

$$X \cdot X = \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix}$$

$$= \sin^2\theta\cos^2\psi + \sin^2\theta\sin^2\psi + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1$$

so $X(\theta)$ lies on the unit sphere.

$$X'(\theta) = \begin{bmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ -\sin\theta \end{bmatrix} \quad X''(\theta) = \begin{bmatrix} -\sin\theta\cos\psi \\ -\sin\theta\sin\psi \\ -\cos\theta \end{bmatrix}$$

$X' \cdot X' = 1$ so the curve has unit speed. The tangent $T = X'$ then.

2.31 Calculation of Frames on the meridian -

$X'' = -X$ so X and X'' are unit vectors.

$\therefore$ curvature $\kappa = 1$ and principal normal $N = X''$ so compute the binormal.

$$B = T \times N = \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix}$$

COROLLARY : In this moving frame, T points along the meridian from North Pole to the South Pole, N points towards the centre of the earth and B points along a parallel of constant colatitude

2.32 Parallels on the sphere -

$T = X'$ and $N = -X$ so we need to describe B

$$X(\theta, \psi) = \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix}$$

the curve $\psi \mapsto X(\theta, \psi)$ gives a parallel of constant colatitude pointing in the direction →

$$\frac{\partial X}{\partial \psi} = \begin{bmatrix} -\sin\theta\sin\psi \\ \sin\theta\cos\psi \\ 0 \end{bmatrix} = \sin\theta \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix}$$

$= \sin\theta \, (B)$

This is parallel to B so B points along the parallel of constant colatitude. A parallel is a circle on the sphere given by fixing the colatitude and changing the longitude.

2.33 Navigation - (Navigating by the Stars) -

i) To move along a meridian the navigator heads in the direction of the Pole Star and goes Northwards

ii) To move along a parallel the navigator moves perpendicular to the meridians and keeps the Pole Star at a constant angle to the Zenith (directly above their heads).

2.34 Surfaces in $\mathbb{R}^3$

Definition : A surface in $\mathbb{R}^3$ consists of a rectangle $R \subset \mathbb{R}^2$ and a differentiable function $X : R \to \mathbb{R}^3$ given by $X(s,t)$ such $\frac{\partial x}{\partial s}$ is not parallel to $\frac{\partial x}{\partial t}$; the unit normal N is :

$$N = \frac{\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}}{\left\| \frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t} \right\|}$$

The function x is called a co-ordinate chart and $\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial t}$ are tangents to the surface.

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Chapter 3: Derivatives of functions of two or three variables

3.1 Example of the gradient -

To find the gradient of $f(x,y,z) = xy + z\sin x + \cos(xy)$

The gradient vector has components given by the first order partial derivative of f, as in:

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

$$\nabla f(x,y,z) = \begin{bmatrix} y + z\cos x - y\sin(xy) \\ x - x\sin(xy) \\ \sin x \end{bmatrix}$$

3.2 Scalar and vector fields -

Definition i) A scalar field is a function $f : \mathbb{R}^3 \to \mathbb{R}$

ii) A vector field is a function $V : \mathbb{R}^3 \to \mathbb{R}^3$ often written

as $V = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$ where $f = f(x,y,z)$, $g = g(x,y,z)$, $h = h(x,y,z)$

3.3 Graph of a functions of two variables -

Let (x,y) be the co-ordinates of a point in the plane. Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ that assigns to each pair (x,y) a value $z = f(x,y)$. Then f is a function of x and y which can vary independently. In the 3-dimensional axis, the domain of f is a subset of the (x,y) plane and the codomain is in the z axis. The graph of f is $z = f(x,y)$ which gives a surface.

$$P = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

3.4 The surface from a graph -

$$\frac{\partial P}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{bmatrix} \qquad \frac{\partial P}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\frac{\partial P}{\partial x} \times \frac{\partial P}{\partial y} = \begin{vmatrix} i & 1 & 0 \\ j & 0 & 1 \\ k & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} = \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial x} \\ 1 \end{bmatrix}$$

hence $\frac{\partial P}{\partial x} \times \frac{\partial P}{\partial y} \neq 0$ so the graph gives a surface

3.5 continuous functions of two variables -

Definition : i) Say that $(x,y) \to (a,b)$ if $\|(x,y)-(a,b)\| \to 0$. Equivalently $(x,y) = (a+h, b+k)$ where $h \to 0$, $k \to 0$.

ii) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function. We say f is continuous at (a,b) if $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$ in any way that is $\forall \epsilon > 0 \; \exists \delta > 0$ such that $|f(x,y) - f(a,b)| < \epsilon$ provided $\sqrt{(x-a)^2+(y-b)^2} < \delta$

Example - $f(x,y) = x^2+y^2$

$|f(x,y) - f(a,b)| = |x^2+y^2-a^2-b^2| \leq |x^2-a^2| + |y^2-b^2|$

$= |x-a||x+a| + |y-b||y+b| \leq |x-a|(|x-a|+2|a|) + |y-b| \times (|y-b|+2|b|)$;

so given $\epsilon > 0$ take $\delta = \epsilon / (4(1+\epsilon+2|a|+2|b|))$.

3.7 Paraboloid -

The surface S in the space $Oxyz$ lies above Oxy plane. At height $z > 0$ section through surface is a circle $z = x^2+y^2$. If $y=0$ we have a parabola $z = x^2$ where S is a surface of revolution. You can fix y and vary x to form $\frac{\partial f}{\partial x}$ or fix x and vary y and form $\frac{\partial f}{\partial y}$. We can form a column of partial derivatives -

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

iff $(x,y) = (0,0)$ $\nabla f(x,y) = 0$ then $(0,0)$ is the only stationary point and the point where $z = f(x,y)$ takes it's min value.

3.8 Differentiation in 1D

From 101 and 114 the notion of the derivative as a difference quotient. A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a if there exist $X \in \mathbb{R}$ and

$$\frac{f(a+h) - f(a)}{h} \to X \qquad (h \to 0)$$

and write $A = \frac{df}{dx}(a)$. We introduce $\epsilon(a,h) \in \mathbb{R}$ by

$$\epsilon = \frac{f(a+h) - f(a)}{h} - X$$

such that $f(a+h) = f(a) + (X + \epsilon)h$
and $\epsilon \to 0$ as $h \to 0$.

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3.9 The Gradient in 2D -

Definition: Let $f:\mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives near to (x,y).

i) We determine the gradient of f to be $\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

ii) The gradient field is the function $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$

iii) If $\nabla f(a,b) = 0$ we say f has a stationary point at (a,b)

iv) $f:\mathbb{R}^2 \to \mathbb{R}$ is a scalar field

3.10 Differentiation in 2D -

Definition: A function $f:\mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if there exist real numbers X and Y and $\epsilon(a,b,h,k)$ such that $f(a+h, b+k) = f(a,b) + Xh + Yk + \epsilon$
where $\epsilon / \|(h,k)\| \to 0$ as $(h,k) \to 0$. Equivalently there exist $\epsilon_1(a,b,h,k)$ and $\epsilon_2(a,b,h,k)$ such that
$f(a+h, b+k) = f(a,b) + (X+\epsilon_1)h + (Y+\epsilon_2)k$
and $\epsilon_1 \to 0$ as $(h,k) \to 0$.

Example -

$f(x,y) = (x^2+y+1)^{-1}$ is differentiable for $y>0$ and $x \in \mathbb{R}$.

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y+1)^2} \qquad \frac{\partial f}{\partial y} = \frac{-1}{(x^2+y+1)^2}$$

where $x^2+y+1 > 1$ for all $y>0$ so the partial derivatives are continuous. $\therefore$ f is differentiable.

3.11 Differentiation and the Gradient -

i) Suppose $f:\mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b). Then f is continuous at (a,b) and has partial derivatives
$X = \frac{\partial f}{\partial x}(a,b) \qquad Y = \frac{\partial f}{\partial y}(a,b)$

ii) Suppose conversely that f has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in a neighbourhood of (a,b). Then f is differentiable at (a,b) with. $\nabla f(a,b) = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

3.12 Partial Differentiation and Differentiability -

i) From the definition of differentiability

$f(a+h, b+k) = f(a,b) + (X+\epsilon_1)h + (Y+\epsilon_2)k \longrightarrow f(a,b)$

as $(h,k) \to (0,0)$. To find X consider $k=0$,

$$\frac{f(a+h, b) - f(a,b)}{h} = X + \epsilon_1 \longrightarrow X \quad (h \to 0)$$

so $X = \frac{\partial f}{\partial x}(a,b)$

and likewise when $h=0$ to find Y.

3.13 Differentiation and the gradient -

ii) Since f has partial derivatives we can write

$$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a,b+k) - f(a,b)$$
$$= \left(\frac{\partial f}{\partial x}(a, b+k) + \epsilon_1\right)h + \left(\frac{\partial f}{\partial y}(a,b) + \epsilon_2\right)k$$

and because partial derivatives are continuous

$$f(a+h, b+k) - f(a,b) = \left(\frac{\partial f}{\partial x}(a,b) + \epsilon_1 + \epsilon_3\right)h + \left(\frac{\partial f}{\partial y}(a,b) + \epsilon_2\right)k$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ as $h, k \to 0$ hence f is differentiable

3.14 The Geometrical interpretation of the gradient -

$f(a+h, b+k) = f(a,b) + (h,k) \cdot \nabla f(a,b) + \epsilon$ where

$\epsilon / \|(h,k)\| \to 0$ as $(h,k) \to 0$ differentiable function definition

i) Fix (a,b) and c then: $(h,k) \cdot \nabla f(a,b) = c$ gives the equation of a line perpendicular to $\nabla f(a,b)$.

ii) $|(h,k) \cdot \nabla f(a,b)| \leq \|(h,k)\| \, \|\nabla f(a,b)\|$ by Cauchy-Schwarz

iii) Suppose (a,b) isn't a stationary point and $(h,k) = \delta \nabla f(a,b)$. Then $f(a+h, b+k) = f(a,b) + \delta \|\nabla f(a,b)\|^2 + \epsilon$

This makes f increase as fast as possible for $\delta > 0$

the gradient points in the direction where f increases quickest

3.15 The Gradient Field is perpendicular to contours -

At (x,y) draw the vector $\nabla f(x,y)$. If $\nabla f(a,b) = 0$ gradient field is 0 and (a,b) is a stationary point. It is perpendicular to the contour $\{(x,y) : f(x,y) = k\}$ at (x,y).

Example - Let $f(x,y) = x^2+y^2$

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Then $\nabla f(x,y)$ points away from the origin. Also $f(x,y) = k$ gives a circle centre O with radius $\sqrt{k}$.

3.16 Chain Rule for differentiating along curves -

THEOREM (Chain Rule 1)

Let J be an interval in $\mathbb{R}$ and $\gamma : J \to \mathbb{R}^2$ be a differentiable curve and $f : \mathbb{R}^2 \to \mathbb{R}$ a differentiable function. Then $f \circ \gamma : J \to \mathbb{R}$ is a differentiable function with derivative

$$\frac{d}{dt} f(\gamma(t)) = (\nabla f)(\gamma(t)) \cdot \gamma'(t)$$

where $\gamma'(t)$ is the velocity vector and $(\nabla f)(\gamma(t))$ is the gradient of f at $\gamma(t)$.

Example - Apply Chain Rule 1 to $f(x,y) = \exp(-x^2-xy)$ and $\gamma : [0, 2\pi] \to \mathbb{R}^2$ $\gamma(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$

The curve γ has co-ordinates $x = a\cos t$ $y = b\sin t$ so $x^2/a^2 + y^2/b^2 = 1$ so γ is an ellipse

$$\frac{d\gamma}{dt} = \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix} \text{ and } \frac{\partial f}{\partial x} = (-2x-y)\exp(-x^2-xy)$$
$$\frac{\partial f}{\partial y} = -x\exp(-x^2-xy)$$

$$\nabla f(x,y) = \begin{bmatrix} (-2x-y)\exp(-x^2-xy) \\ -x\exp(-x^2-xy) \end{bmatrix}$$

$$\frac{d}{dt} f(\gamma(t)) = (\nabla f)(\gamma(t)) \cdot \gamma'(t)$$

$$= \begin{bmatrix} (-2a\cos t - b\sin t)\exp(-a^2\cos^2 t - ab\cos t\sin t) \\ -a\cos t\exp(-a^2\cos^2 t - ab\cos t\sin t) \end{bmatrix} \cdot \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix}$$

$= \ldots$ (hard to simplify)

3.20 Differentiation in 3D -

Similar to 2D with 3 unknowns.

A function $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at (a,b,c) if there exist real numbers X, Y and Z and $\epsilon(a,b,c,h,k,m)$ such

$$f(a+h, b+k, c+m) = f(a,b,c) + Xh + Yk + Zm + \epsilon$$

where $\epsilon / \|(h,k,m)\| \to 0$ as $(h,k,m) \to 0$.

to find X consider $k=0$, $m=0$

$$\frac{f(a+h,b,c) - f(a,b,c)}{h} \to X \qquad (h \to 0)$$

so $X = \frac{\partial f}{\partial x}(a,b,c)$

and the same for Y and Z

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3.23 Partial Differentiation in 3D -

i) Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at (a,b,c). Then f has partial derivatives

$$X = \frac{\partial f}{\partial x}(a,b,c) \quad Y = \frac{\partial f}{\partial y}(a,b,c) \quad Z = \frac{\partial f}{\partial z}(a,b,c)$$

ii) Suppose conversely that f has continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ for (x,y,z) in a neighborhood of (a,b,c). Then f is differentiable at (a,b,c) with X, Y, Z.

Example -

$f(x,y,z) = x^2 + \sin(yz) + e^{xz^2}$ fix 2 variables and differentiate with respect to the other.

$$\frac{\partial f}{\partial x} = 2x + z^2 e^{xz^2} \quad \frac{\partial f}{\partial y} = z\cos(yz) \quad \frac{\partial f}{\partial z} = y\cos(yz) + 2xe^{xz^2}$$

and then write as a column vector called the gradient.

$$\nabla f = \begin{bmatrix} 2x + z^2 e^{xz^2} \\ z\cos(yz) \\ y\cos(yz) + 2xze^{xz^2} \end{bmatrix}$$

3.25 Gradient in 3D -

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $A = (a,b,c)$ Gradient of f =

$$\nabla f(A) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

The equation $X = A + tY$ gives a line through A in the direction of Y. Then we can write

$$f(A+tY) = f(A) + tY \cdot \nabla f(A) + \epsilon \quad \text{where } \epsilon/t \to 0 \text{ as } t \to 0$$

Then the derivative of f in the direction of Y at A is

$$\lim_{t \to 0} \frac{f(A+tY) - f(A)}{t} = Y \cdot \nabla f(A)$$

Fix A and D and consider Y as a variable in $\mathbb{R}^3$. Then $Y \cdot \nabla f(A) = d$ gives the equation of a plane with normal direction $\nabla f(A)$.

3.26 Properties of the Gradient -

Suppose $f = f(x,y,z)$ and $g = g(x,y,z)$ are differentiable functions on $\mathbb{R}^3$ and C is a constant

i) $\nabla(Cf) = C\nabla f$ ii) $\nabla(f+g) = \nabla f + \nabla g$

iii) $\nabla(fg) = (\nabla f)g + f\nabla g$ iv $\nabla f(x,y,z) = 0$ for all (x,y,z) iff f is constant.

$$\{\text{constants}\} \longrightarrow \{\text{scalar fields}\} \xrightarrow{\nabla} \{\text{vector fields}\}$$

3.29 Example : Radial functions -

Let $x = (x,y,z)$ $r = \|x\| = (x^2+y^2+z^2)^{1/2}$

so $\nabla r = \frac{x}{r}$ $(r \neq 0)$

Now consider a differentiable real function $f : \mathbb{R} \to \mathbb{R}$ and form $f(r)$ which we call a radial function of (x,y,z).

Then by 1D Chain Rule :

$\frac{\partial f(r)}{\partial x} = f'(r)\frac{\partial r}{\partial x}$ and $\frac{\partial f(r)}{\partial x} = f'(r)\frac{x}{r}$ $\frac{\partial f(r)}{\partial y} = f'(r)\frac{y}{r}$ $\frac{\partial f(r)}{\partial z} = f'(r)\frac{z}{r}$

3.31 Chain rule along space curves -

THEOREM (Chain Rule 1) -

For J an interval in $\mathbb{R}$ let $\gamma : J \to \mathbb{R}^3$ be a differentiable space curve and $f : \mathbb{R}^3 \to \mathbb{R}$ a differentiable function. Then $f \circ \gamma : J \to \mathbb{R}$ is a differentiable function with derivative

$$\frac{d}{dt} f(\gamma(t)) = (\nabla f)(\gamma(t)) \cdot \gamma'(t)$$

where $\gamma'(t)$ = velocity $(\nabla f)(\gamma(t))$ = gradient at $\gamma(t)$

3.33 The Jacobian -

Change old variables (u,v) to new variables (s,t) so that $u = u(s,t)$ and $v = v(s,t)$ are differentiable functions. We introduce the Jacobian matrix of derivatives

$$J(u,v;s,t) = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{bmatrix}$$ and the corresponding Jacobian (determinant)

$$\frac{\partial(u,v)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{vmatrix}$$

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3.34 Independent functions -

Usually we suppose that $J(u,v,s,t)$ has an inverse matrix or equivalently $\frac{\partial(u,v)}{\partial(s,t)} \neq 0$. Note that

$\begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{bmatrix}$, $\begin{bmatrix} \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{bmatrix}$ are the gradient vectors of u and v with respect to (s,t). Usually we wish these gradients not to be parallel so we require

Example -

Let $x = 7u + 2v$ $y = 3u + v$ Then we invert $u = x - 2y$ $v = -3x + 7y$

The Jacobian matrices are $J(u,v,x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$

and for the inverse change of variables

$J(x,y,u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ where $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} =$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Note $\frac{\partial u}{\partial x} \neq 1 / \frac{\partial x}{\partial u}$

3.36 Polar Co-ordinates in the plane -

Let $x = r\cos\theta$ $y = r\sin\theta$ for $r > 0$ and $\theta \in [0, 2\pi)$ assume $(x,y) \neq (0,0)$. Then the Jacobian matrix is

$J((x,y,r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$ with

determinant $\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r$

3.37 Chain Rule II Change of variables in the gradient -

Proposition: Change old variables (u,v) to new variables (s,t) so that $u = u(s,t)$ and $v = v(s,t)$ are differentiable functions. Let f be a differentiable function. Then,

$$\begin{bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{bmatrix}$$

3.39 Chain Rule II : Repeated change of variables -

Proposition : Change old variables (u,v) to new variables (s,t) : so that $u=u(s,t)$ and $v=v(s,t)$ are differentiable functions. Let f and g be differentiable functions.

i) Then the matrices of partial derivatives satisfy

$$\begin{bmatrix} \frac{\partial f}{\partial s} & \frac{\partial g}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial g}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} \end{bmatrix}$$

ii) Their determinants satisfy

$$\frac{\partial(f,g)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(s,t)} \frac{\partial(f,g)}{\partial(u,v)}$$

3.40 Changing variables in partial derivatives -

Let old variables (s,t) be changed to new variables (u,v) where $u=st$ $v=s^2-t^2$

$$\frac{\partial u}{\partial s} = t \qquad \frac{\partial v}{\partial s} = 2s \qquad \frac{\partial u}{\partial t} = s \qquad \frac{\partial v}{\partial t} = -2t$$

$$\begin{bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{bmatrix} = \begin{bmatrix} t & 2s \\ s & -2t \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{bmatrix}$$

3.44 Inverting changes of variables and their Jacobians

$u=u(s,t)$ $v=v(s,t)$ are differentiable functions.

Proposition : Suppose the change of variables is invertible so $s=s(u,v)$ and $t=t(u,v)$. Then

$J(u,v;s,t)\,J(s,t;u,v) = I$ in 2×2 form

and the determinants satisfy

$$\frac{\partial(u,v)}{\partial(s,t)} \frac{\partial(s,t)}{\partial(u,v)} = 1$$

3.46 Implicit functions in 2D -

Implicit function Theorem : Let R be a rectangle with $(a,b) \in R$, let $K \in \mathbb{R}$ and let $f: R \to \mathbb{R}$ be a differentiable function such that $f(a,b) = K$ and $\nabla f(a,b) \neq 0$. Then there is a curve Γ in R called the contour passing through (a,b) at height K such that $f(x,y) = K$ for all (x,y) on Γ and $\nabla f(a,b)$ is normal to Γ at (a,b). The tangent line to Γ at (a,b) has equation:

$$\nabla f(a,b) \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \nabla f(a,b) \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

3.49 Curves versus Graphs -

Let $\gamma(t)$ determine the curve Γ with velocity $\gamma'(t)$ tangent to Γ

$\gamma(t) = \begin{bmatrix} t \\ g(t) \end{bmatrix}$ $\gamma'(t) = \begin{bmatrix} 1 \\ g'(t) \end{bmatrix}$ so $\nabla f(t, g(t))$ is normal to $\gamma'(t)$

so $\nabla f(a,b)$ is normal to Γ at (a,b)

3.52 Crossing tangents at stationary points -

If (a,b) is a stationary point of f, so $\nabla f(a,b) = 0$ then we cannot usually define the tangent line to the curve at (a,b).

i) For example, consider $y^2 - x^2 = (y+x)(y-x) = 0$ at $(0,0)$ this consists of two intersecting lines pointing in different directions namely $y = x$ and $y = -x$ crossing at $(0,0)$

ii) Consider the function $f(x,y) = 3y^2 - x^2(1-x)$. The equation $f(x,y) = 0$ gives Tschirnhausen's cubic $C: 3y^2 = x^2(1-x)$.

Also $\nabla f(x,y) = \begin{bmatrix} 3x^2 - 2x \\ 6y \end{bmatrix}$

3.55 Implicit function theorem in 3D -

THEOREM: Suppose Ω is a cuboid in $\mathbb{R}^3$ with $(a,b,c) \in \Omega$ let $k \in \mathbb{R}$ and let $g : \Omega \to \mathbb{R}$ be a differentiable function such that $g(a,b,c) = k$ and $\nabla g(a,b,c) \neq 0$. Then there is a surface S passing through (a,b,c) such that.

i) $g(x,y,z) = k$ for all (x,y,z) on S

ii) S has a tangent plane at (a,b,c) and $\nabla g(a,b,c)$ is perpendicular to this tangent plane.

$$\nabla g(a,b,c) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \nabla g(a,b,c) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

3.59 Hessian and Discriminant :

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable function if the second order partial derivatives exist and are continuous.

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}$$

Such an f has a Hessian

$$\mathrm{Hess} f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

and a discriminant

$$\Delta = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = \det \mathrm{Hess} f(x,y)$$

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Chapter 4: Vector Calculus

4.1 Local maximum and minimum -

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a continuous function.

i) We say that $A = (a,b,c)$ is a local maximum for f if $f(X) \leq f(A)$ for all $X = (x,y,z)$ sufficiently close to $A = (a,b,c)$

ii) Likewise we define $A = (a,b,c)$ to be a local minimum if $f(X) \geq f(A)$ for all $X = (x,y,z)$ sufficiently close to $A = (a,b,c)$

4.2 Constrained Local max and min -

Let $f: S \to \mathbb{R}$ be a continuous function defined on a surface S.

i) We say $A = (a,b,c)$ on S is a local maximum for f if $f(X) \leq f(A)$ for all $X = (x,y,z)$ on S sufficiently close to $A = (a,b,c)$

ii) Likewise we define $A = (a,b,c)$ on S to be a local minimum ...

4.3 Lagrange Multiplier Theorem -

Suppose that $f, g: \mathbb{R}^3 \to \mathbb{R}$ are differentiable functions s.t

i) The surface $S = \{X = (x,y,z) : g(X) = k\}$ is non-empty and $\nabla g(X) \neq 0$ for all X on S;

ii) f restricted to S has a local max or min at $A = (a,b,c)$.

Then there exists λ called the Lagrange multiplier,

$$* \quad \nabla f(A) = \lambda \nabla g(A)$$

The equation $*$ is called the Lagrange multiplier equation and $g(X) = k$ is the constraint equation

4.6 Constraints and side conditions -

Remarks

i) In the Lagrange multiplier theorem $\nabla f(A)$ and $\nabla g(A)$ are parallel. The hypotheses on f and g are different since g determines S and f is the function we try to optimise on S

ii) We talk about maximising f subject to the constraint $g=k$, sometimes called a side condition.

iii) The value of λ, Lagrange multiplier is related to eigenvalues in matrix theory.

Eliminating the Lagrange multiplier - Following methods

Method a) Divide one equation by another

Method b) Take the dot product with another vector to form a scalar equation e.g $\nabla f(x) \cdot X = \lambda \nabla g(x) \cdot X$

Method c) Build a system of linear eqs and form a determinant equation satisfied by λ

Example -

Let $g(x,y) = x^2 + xy + y^2$ consider ellipse $x^2 + xy + y^2 = 4$

Maximise $f(x,y) = xy$ over points on the ellipse.

Consider $h(x,y) = f(x,y) - \lambda g(x,y) = xy - \lambda(x^2+xy+y^2)$

$\nabla h = \begin{bmatrix} y - \lambda(2x+y) \\ x - \lambda(x+2y) \end{bmatrix}$ $0 = \nabla h = \begin{bmatrix} -2\lambda & 1-\lambda \\ 1-\lambda & -2\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Here $g(0,0) = 0$ doesn't satisfy the constraint so

$0 = \begin{vmatrix} -2\lambda & 1-\lambda \\ 1-\lambda & -2\lambda \end{vmatrix} = 4\lambda^2 - (1-\lambda)^2$

so either i) $2\lambda = 1-\lambda$ $\lambda = 1/3$

$\begin{bmatrix} -2/3 & 2/3 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ so $x = y$, $x^2+x^2+x^2 = 4$ $x = y = 2/\sqrt{3}$

$f(2/\sqrt{3}, 2/\sqrt{3}) = 4/3$ or $x = y = -2/\sqrt{3}$ $f(\dots) = 4/3$

or $2\lambda = -1+\lambda$ and $\lambda = -1$ so

$0 = \nabla h = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ so $x+y = 0$ and $x = -y = 2$

$f(2,-2) = -4$ or $x = -y = -2$

$f(-2,2) = -4$

Hence the max is 4/3 at $(2/\sqrt{3}, 2/\sqrt{3})$ and $(-2/\sqrt{3}, -2/\sqrt{3})$

4.12 Every real symmetric matrix has a real eigenvalue -

Let A be a real symmetric 3×3 matrix. Then A has a real eigenvalue λ with corresponding real eigenvector X so

$$AX = \lambda X.$$

4.15 The determinant -

To find λ we can use the equation

$$\begin{bmatrix} a-\lambda & b & c \\ b & d-\lambda & e \\ c & e & f-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

hence λ is a real eigenvalue of A with corresponding real eigenvector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $\lambda = \lambda X \cdot X = X \cdot AX$ is the max or min value of F on the sphere.

Example from 4.16 - 4.18

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4.19 Divergence -

Let $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions. Then

$$V = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}$$

is a vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ where to each point (x,y,z) we associate the vector V with the given components.

Definition - The divergence of V is the scalar field

$$\nabla \cdot V = \operatorname{div} V = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

The divergence is used in theory of electrodynamics.

Calculating divergence -

$$V = \begin{bmatrix} x^2+y \\ yz+\sin y \\ x+z^2 \end{bmatrix} \quad \nabla \cdot V = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} x^2+y \\ yz+\sin y \\ x+z^2 \end{bmatrix} = 2x + z + \cos y + 2z$$

4.21 Properties of Divergence -

Suppose V and W are vector fields and c is a constant. Then

i) $\nabla \cdot (cV)(x,y,z) = c(\nabla \cdot V(x,y,z))$

ii) $\nabla \cdot (V+W)(x,y,z) = \nabla \cdot V(x,y,z) + \nabla \cdot W(x,y,z)$

iii) div grad is the Laplacian

$$\operatorname{div} \nabla f(x,y,z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\{\text{scalar fields}\} \xrightarrow{\nabla} \{\text{vector fields}\} \xrightarrow{\nabla \cdot} \{\text{scalar fields}\}$$

4.23 Curl of a vector field -

The curl of a differentiable vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\operatorname{curl} V = \begin{bmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{bmatrix} = \nabla \times V = \begin{vmatrix} i & \frac{\partial}{\partial x} & f \\ j & \frac{\partial}{\partial y} & g \\ k & \frac{\partial}{\partial z} & h \end{vmatrix}.$$

The right hand equation is a popular symbolic notation which is useful for calculations. Interpret ∇ as a vector.

4.24 Example.

4.25 Grad, curl, div -

$\{\text{scalar fields}\} \xrightarrow{\nabla} \{\text{vector fields}\} \xrightarrow{\nabla\times} \{\text{vector fields}\} \xrightarrow{\nabla\cdot} \{\text{scal...}$

i) A scalar field ϕ has a gradient $\nabla\phi$ which is a vector field

ii) A vector field V has a curl $\nabla \times V$ which is a vector field

iii) A vector field V has a divergence $\nabla \cdot V$ is a scalar field

4.27 Properties of a curl -

Proposition: Suppose $V = V(x,y,z)$ and $W = W(x,y,z)$ are differentiable vector fields and C is a constant, then

i) $\nabla \times (CV) = C(\nabla \times V)$ ii) $\nabla \times (V+W) = \nabla\times V + \nabla \times W$

iii) The curl of a gradient is zero so for all scalar fields ϕ
$\operatorname{curl} \nabla\phi = 0$.

iv) Conversely if V is differentiable vector field such that $\operatorname{curl} V = 0$, then there exists ϕ such that $V = \nabla\phi$. Also ϕ is unique up to an additive constant.

4.36 The Divergence of a curl is zero -

Proposition: All twice continously differentiable vector fields V satisfy $\operatorname{div} \operatorname{curl} V = 0 = \nabla \cdot (\nabla \times V)$

Chapter 5 : Double Integrals, volumes and surface areas

5.1 Double Integrals -

Let R be a region in the (x,y) plane, f a function of x and y. The double integral $J = \iint_R f(x,y)\,dx\,dy$ is the volume between the region R and the surface $z = f(x,y)$ with any volume below (x,y) plane negative. This is a definite integral.

5.2 Repeated Integrals -

consider R as a rectangle given by $a \le x \le b$, $c \le y \le d$.

For a fixed value of y the cross section of area under $z = f(x,y)$

$\int_a^b f(x,y)\,dx$ integration respect to x

Hence, $J = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$ (1)

$J = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$ (2)

(1) and (2) are called repeated integrals

$\iint_R 1\,dx\,dy$ = area of R.

5.3 Double Integrals over triangular and general regions -

Let $y_1(x)$ and $y_2(x)$ be functions of x with $y_1(x) \le y_2(x)$ for $a \le x \le b$. Then,

$R = \{(x,y) : y_1(x) \le y \le y_2(x) ; a \le x \le b\}$

To find the area of this region

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

5.5 Specifying triangles -

Example. T = triangle bound by $x = 0$, $y = a$ and $y = x$

$\iint_T \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

given double integral $= \int_0^a \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ and $\int_0^a \int_x^a \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

5.7 Integration in polar co-ordinates

Suppose R is a region $(x,y) \in A$ or in polar $(r,\theta) \in A'$

$x = r\cos\theta$ $y = r\sin\theta$ and $\theta \in [0, 2\pi)$

$\iint_A f(x,y)\,dx\,dy = \iint_{A'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

where A' is the same region as A.

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5.12 General change of variable for double integrals -

Let (x,y) be the usual Cartesian co-ordinates. Let (u,v) be new co-ordinates so $x = x(u,v)$ $y = y(u,v)$. We need to estimate the area in (x,y) plane of rectangle (u,v) $(u+h,v)$ $(u,v+k)$ and $(u+h,v+k)$. Transform the rectangle with vertices $(0,0)$, $(h,0)$, $(0,k)$, (h,k) to a parallelogram.

$$\begin{bmatrix} ah+ck \\ bh+dk \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$$

and the parallelogram with vertices

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} ah \\ bh \end{bmatrix}$, $\begin{bmatrix} ck \\ dk \end{bmatrix}$, $\begin{bmatrix} ah+ck \\ bh+dk \end{bmatrix}$ has area $(ad-bc)hk$ Now R is a curvilinear rectangle with vertices.

(x,y), $(x+\delta x, y)$, $(x, y+\delta y)$ and $(x+\delta x, y+\delta y)$

where $\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} h \\ h \end{bmatrix}$

by Chain Rule II

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix}$$

THEOREM : Let R be a bounded region in the (x,y) plane that corresponds bijectively with the region S in the (u,v) plane under the change of variables $x = x(u,v)$ and $y = y(u,v)$. Let $f : R \to \mathbb{R}$ be continuous. Then

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

5.15 Hyperbolic and parabolic curves -

Let R be the region bounded by the curves

$xy = a \quad xy = b \quad y^2 = px \quad y^2 = qx$

where $0 < a < b$ and $0 < p < q$. We let $u = xy$ and $v = y$ so R corresponds to S by $a \le u \le b$ and $y^3 = pxy$ becomes $v^3 = pu$ so $(pu)^{1/3} \le v \le (qu)^{1/3}$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & 0 \\ x & 1 \end{vmatrix} = y$$

so $$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{v}.$$

5.18 Volumes of Solids by Slicing -

Let R be a bounded region and $f: R \to \mathbb{R}$ a continuous function $V = \iint_R f(x,y)\,dx\,dy$ equals volume between region R and surface $z = f(x,y)$ so volumes below (x,y) plane are negative. This can be used to find the volume of simple bodies and their rearrangements. e.g a loaf of bread in parallel slices.

5.19 Cavalieri's Slicing Principle -

THEOREM: Let B be a body and P_z a family of parallel planes such that:

i) for $u \le z \le v$ the plane P_z lies between P_u and P_v

ii) B lies between P_u and P_v

iii) The area of the slice of B that is cut by P_z is $A(z)$

Then the volume of B is

$$\mathrm{vol}(B) = \int_u^v A(z)\,dz$$

5.19 Vol of tetrahedron

5.20 Vol of sphere

5.21 Volumes of revolution -

Corollary (Pappus) Let $f : [a,b] \to \mathbb{R}$ be a positive and continuous function and let B be the body obtained by rotating the graph of $f(z)$ for $z \in [a,b]$ about the z-axis.

Then, $\text{vol}(B) = \pi \int_a^b f(z)^2 \, dz$

The plane at height z cuts the body B in a disc of radius $f(z)$ so we can apply Cavalieri's with $A(z) = \pi f(z)^2$.

5.22 Cone Example

5.23 Volume of a Torus (doughnut) -

Proposition (Kepler) minor radius a, major radius b. Then volume $= 2\pi^2 a^2 b$.

5.25 Surface areas of solids of revolution -

Proposition (Pappus) Let $\gamma(t) = (x(t), z(t))$ be a curve in the (x,z) plane for $t \in [u,v]$ where $x(t)$ and $z(t)$ are continuously differentiable and $x(t) > 0$ for all $t \in [u,v]$ and such that $\gamma(u) = \gamma(v)$. B = Volume of revolution

$SA(B) = 2\pi \int_u^v x(t) \sqrt{x'(t)^2 + z'(t)^2} \, dt$

5.26 Surface Area of a Torus -

Proposition (Kepler) minor radius a, major radius b. Then surface area $= 4\pi^2 ab$.