12/10/21

L01

MATH 101

Polynomials, Partial Fractions, Induction Proof

The Division Algorithm

Let $f(x)$, $d(x)$ be non-zero polynomials
Then there exists
$q(x)$ and $r(x)$
quotient remainder
$f(x) = q(x) \cdot d(x) + r(x)$ where
$r(x) = 0$
or $\deg r(x) < \deg d(x)$

e.g. $f(x) = x^2+3x-4$
$d(x) = 2x - 1/2$ $\quad \deg d(x) = 1$

$$
\begin{array}{r|l}
 & \tfrac{1}{2}x + \tfrac{13}{8} \\
\hline
2x - \tfrac{1}{2} & x^2 + 3x - 4 \\
 & x^2 - \tfrac{1}{4}x \\
\hline
 & \tfrac{13}{4}x - 4 \\
 & \tfrac{13}{4}x - \tfrac{13}{16} \\
\hline
 & -\tfrac{51}{16}
\end{array}
\qquad \deg r(x) = 0
$$

deg = degrees of x $\quad 2 = x^2$ etc.

$q(x) = 1/2x + 13/8 \qquad r(x) = -51/16$

$\bullet$ = multiplication

Fundamental theorem of Algebra

Proposition: Let $f(x)$ be a polynomial of degree $n > 0$.

Then $f(x) = g(x) \cdot (x-a_1)(x-a_2)\dots(x-a_r)$

where $a_1 \dots a_r$ are the real roots of $f(x)$ and $g(x)$ has no real roots. (irreducible)

$$\frac{f(x)}{g(x)} = q(x) + \left[\frac{r(x)}{g(x)}\right]$$ Rewritten Division Algorithm

$\deg r(x) < \deg g(x)$

"proper form"
deg on top less than bottom

This can be rewritten as a product of linear factors

$$g(x) = (x-a_1)^{s_1} \dots (x-a_r)^{s_r} \cdot (x^2 + \beta_k x + \gamma_k)^{t_k}$$
$$\dots (x^2 + \beta_1 x + \gamma$$

Furthermore $\frac{r(x)}{g(x)}$

$$\frac{r(x)}{g(x)} = S_1 + \dots + S_r + T_1 \dots T_K$$

where

$$S_i = \frac{\sigma_{i1}}{x - \alpha_i} + \frac{\sigma_{i2}}{(x-\alpha_i)^2} + \dots + \frac{\sigma_{is_i}}{(x-\alpha_i)^{s_i}}$$

and

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LO1

$T_i = \varepsilon_i X$ $\quad \dfrac{\text{constant}}{\text{linear}}$

$\dfrac{\text{linear}}{\text{quadratic}}$

e.g. $\dfrac{x}{(x-1)^3}$

1) Check deg numerater < deg denominater for proper form

2) Denominater is linear

$$\frac{\delta_1}{x-1} + \frac{\delta_2}{(x-1)^2} + \frac{\delta^3}{(x-1)^3}$$

Cross multiply

$$\frac{\delta_1 (x-1)^2 + \delta^2 (x-1) + \delta^3}{(x-1)^3}$$

$$= \frac{\delta_1 x^2 - 2\delta_1 + \delta_1 + \delta 2x - \delta_2 + \delta 3}{(x-1)^3}$$

3) Compare coefficients

$$\delta_1 (x^2) + \delta 2 (x) + \delta_1 - \delta 2 + \delta 3$$
$$= x$$

$\delta_1 = 0$

$-2\delta_1 + \delta 2 = 1 \quad \delta 2 = 1$

$\delta_1 - \delta 2 + \delta 3 = 0 \quad \delta 3 = 1$

$$\therefore \frac{x}{(x-1)^3} = \cancel{\frac{0}{x-1}} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

e.g. $\dfrac{2x^2}{(x^2+1)^2}$ $\qquad$ $\dfrac{\text{linear}}{\text{quadratic}}$

1) check proper form (deg) $\left(\frac{1}{4}\right)$

$$= \frac{\varepsilon_1 x + \zeta_1}{x^2+1} + \frac{\varepsilon_2 x + \zeta_2}{(x^2+1)^2}$$

Cross multiply

$$= \frac{(\varepsilon_1 x + \zeta_1)(x^2+1) + \varepsilon_2 x + \zeta_2}{(x^2+1)^2}$$

Compare coefficients

$$2x^2 = \begin{array}{ll} x^3 & \varepsilon_1 = 0 \\ x^2 & \zeta_1 = 2 \\ x & \varepsilon_1 + \varepsilon_2 = 0 \\ 1 & \varepsilon_2 + \zeta_1 = 0 \end{array}$$

$\varepsilon_1 = 0$, $\varepsilon_2 = 0$

$\zeta_1 = 2$ $\quad$ $\zeta_2 = -2$

$\therefore$

$$\frac{2x^2}{(x^2+1)^2} = \frac{2}{(x^2+1)} - \frac{2}{(x^2+1)^2}$$

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LO2

MATH 101 LO3 (Lecture 2)

* Proof of Induction

Prove $P(n)$ for all $n \in \mathbb{N}$
- Show base case is true $P(1)$
- Assume statement is true $P(k)$
- Show $P(k+1)$ is true using $P(k)$

} Induction step

The Induction step works as an iterating process meaning if $P(1)$ is true so are all natural numbers

e.g $3 + 7 + 11 + \ldots + (4n-1) = n(2n+1) \; \forall n \in \mathbb{N}$

$\forall$ = all set of

Let $P(n)$: $3+7+11+\ldots+(4n-1) = n(2n+1)$

Basis of induction: Consider $P(1)$

$3 = 1(3)$ which is true

Induction step: Assume $P(k)$ holds for k

Consider $P(k+1)$

$$\underbrace{3+7+11+\ldots+(4k-1)}_{= k(2k+1)} + (4(k+1)-1)$$

as it is k step

$2k^2 + k + 4k + 3$ (expanded)

$2k^2 + 5k + 3$

[we want: $(k+1)(2k+3)$]

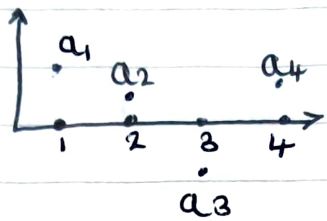
$2k^2 + 5k + 3 = (k+1)(2k+3)$

Therefore $P(k)$ holds by induction $\square$

Indicates end of proof

Sequences and Series

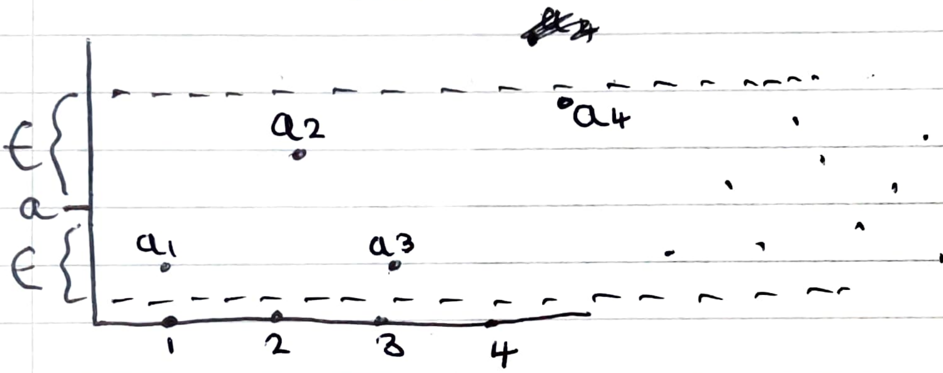
Sequences - (a_n)



$\mathbb{N} \to \mathbb{R}$
$n \to a_n$

$\lim_{n \to \infty} a_n = a$ means

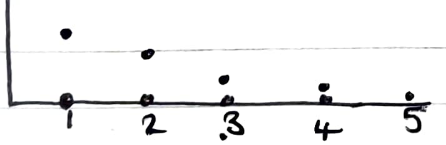
(for all) $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ (there exists) such that $\forall n > N : |a_n - a| < \epsilon$



From some point ~~on~~ you can guarantee some n for all a_n ~~t~~ be within an ϵ strip.

e.g $\lim_{n \to \infty} \frac{1}{n} = 0$

This graph approaches zero.



Proof: Let $\epsilon > 0$

[we want an N such that ~~for all~~ $\forall n > N : \frac{1}{n} < \epsilon$]

Choose $N \geq \frac{1}{\epsilon}$ Then $\frac{1}{N} \leq \epsilon$
and $\frac{1}{n} < \epsilon$ $\forall n > N$

An example

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Proof :

Let $\epsilon > 0$

Choose $N \geq \frac{1}{\epsilon^2}$ [Then $\frac{1}{\sqrt{N}} \leq \epsilon$] what you aim for

and $\frac{1}{\sqrt{n}} < \epsilon \quad \forall n > N \qquad \square$

Limit Laws :

$\lim_{n\to\infty} a_n = a \qquad \lim_{n\to\infty} b_n = b$

$\lim_{n\to\infty} (a_n + b_n) = a + b$ (see lecture notes)

e.g $\lim_{n\to\infty} \frac{n^4 + 3n}{3n^3 + 5n^2}$ You would expect this to go to infinity due to n^4

Divide by highest power to get a constant as a limit

$$\lim_{n\to\infty} \frac{n + 3n^{-2}}{3 + 5n^{-1}} = \frac{\lim_{n\to\infty} n + \lim_{n\to\infty} 3n^{-2}}{3 + \lim_{n\to\infty} 5n^{-1}}$$

$\lim_{n\to\infty} 3n^{-2} \to 0$ goes to zero

$\lim_{n\to\infty} 5n^{-1} \to 0$ goes to zero

$$= \frac{\lim_{n\to\infty} n}{3}$$

$\therefore$ goes to ∞ and diverges

Series $\sum_{k=1}^{\infty} a_k$

$(a_1, a_1+a_2, a_1+a_2+a_3, \ldots)$

partial sums

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LO3

MATH 101 L01

* Pre-recorded lectures

Functions and Continuity -

Function $f: A \to B$

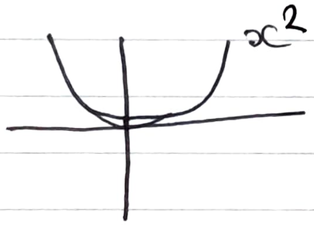
Every element in A is assigned by f to a unique element in B

A = Domain

B = Co-Domain

Range, $R = \{ y \in B \mid y = f(x) \text{ for some } x \in A \}$

e.g $f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$ Range $= \mathbb{R}_{\geq 0} = \{ x \in \mathbb{R} \mid x \geq 0 \}$

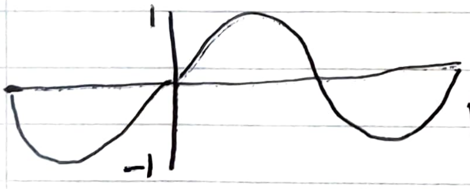


Domain = x

Range = y

e.g $f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin x \end{cases}$ Range $= [-1, 1]$

everything included from -1 to 1



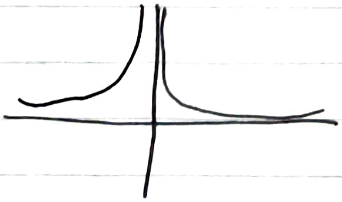
Domain = x

Range = y

[ ] = Includes

e.g $f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{1}{1+x^2} \end{cases}$ Range $= (0, 1]$

Doesn't Include 0

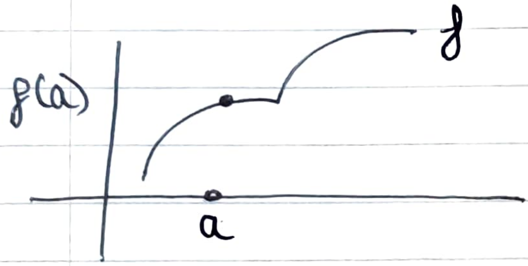


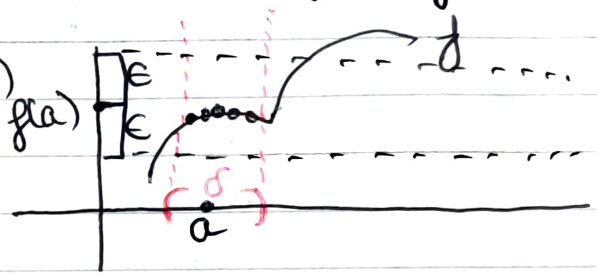
Theorem? The following are equivalent statements from a function $f: A \to B$; $a \in A$ for continuity

i) The function f is continuous at a.

ii) For every sequence $(a_n)_{n\in\mathbb{N}}$ in A with $\lim_{n\to\infty} a_n = a$, we have $\lim_{n\to\infty} f(a_n) = f(a)$

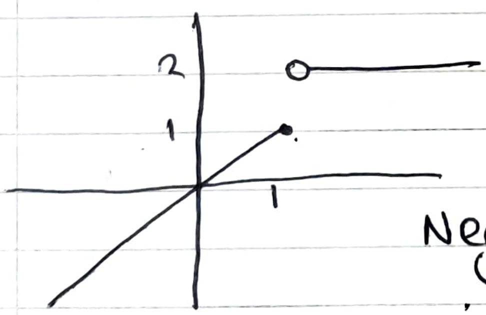
iii) $\lim_{x\to a} f(x) = f(a) \overset{\text{def}}{\Longleftrightarrow} (\forall \epsilon > 0)(\exists \delta > 0)$ such th
by definition
$(\forall x \in A)$ with $|x-a| < \delta$ we have $|f(x) - f(a)| < \epsilon$

ii) 

iii) 

ϵ = small distance around $f(a)$
δ = neighborhood around a
All function values are within this square
If continuous you should be able to do this

e.g consider $f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x & \text{for } x \leq 1 \\ x \mapsto 2 & \text{for } x > 1 \end{cases}$



Continuous except at $x = 1$
Negate (ii) to show not continuous

use ii) Find a sequence that converges to 1
claim f is not continuous at 1.

Consider the sequence $(1 + \frac{1}{n})_{n\in\mathbb{N}}$

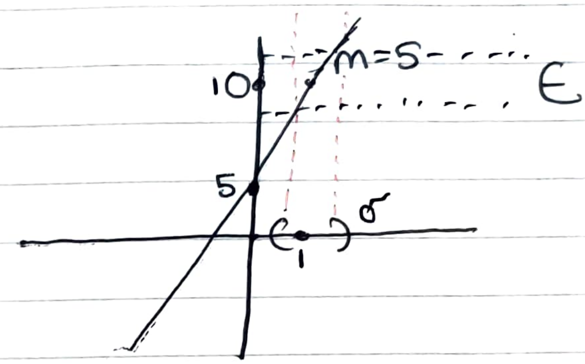
$(1+\frac{1}{n})_{n\in\mathbb{N}}$ This converges to 1

$\lim_{n\to\infty}(1+\frac{1}{n}) = 1$ and $\lim_{n\to\infty}(f(1+\frac{1}{n})) = 2$

$2 \neq f(1)$ Therefore there is a contradiction $\square$

You could also use (iii) to prove this by trying to find an ϵ and δ neighborhood where this doesn't work. This happens when a small ϵ is chosen

e.g. $f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 5x+5 \end{cases}$



Claim is that f is continuous at $x=1$

Use (iii)

Proof:

[we want: $\delta > 0$ such $|x-1|<\delta \Rightarrow |fx - f(1)| = |5x+5-10| < \epsilon$]

Let $\epsilon > 0$

$|5x+5-10| = |5x-5| = 5|x-1| < \epsilon$

Assume $|x-1|<\delta$, $|5x+5-10| < \epsilon$

$\therefore \delta_\epsilon = \frac{\epsilon}{5}$

Then:

$$\begin{aligned} |x-1| < \delta_\epsilon \Rightarrow |f(x) - f(1)| &= |(5x+5)-10| \\ &= |5x-5| \\ &= 5|x-1| \\ &< 5\delta_\epsilon \\ &= 5(\tfrac{\epsilon}{5}) = \epsilon \\ &< \epsilon \end{aligned}$$

This proves it is less than ϵ in this condition and points are in this interval. $\square$

Alternative proof using (ii)
Claim f is continuous at 1

Proof :

arbitrary limit

If $\lim_{n\to\infty} x_n = 1$ then we need to show that

$\lim_{n\to\infty} (f(x_n)) = 10$ Because $5(x_n)+5$ limit

$\square$

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LO4

MATH 101 LO4

Quiz1 Recap -
3) Not Equivalent and (1) is true C
4) a_n hasn't been stated as convergent and could be 0. C
5) C
6) A

Functions and Limits -

$f : \mathbb{R} \mapsto \mathbb{R}$

$\lim_{x \to a} f(x) = L$ by definition

$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 \ \forall x \in \mathbb{R} : |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$
depends on ϵ

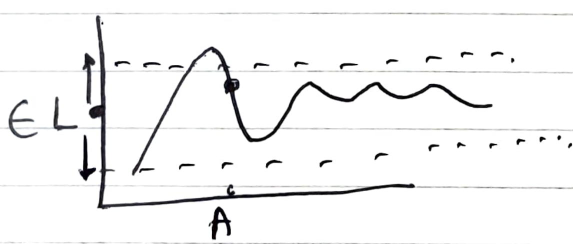
Same definition as LO3

f is continuous at a $\overset{def}{\iff}$ $\lim_{x \to a} f(x) = f(a)$

def $\iff$ = by definition

$\lim_{x \to \infty} f(x) = L$ $\overset{def}{\iff}$ $\forall \epsilon > 0 \ \exists A_\epsilon : \forall x > A_\epsilon$
$\Rightarrow |f(x) - L| < \epsilon$

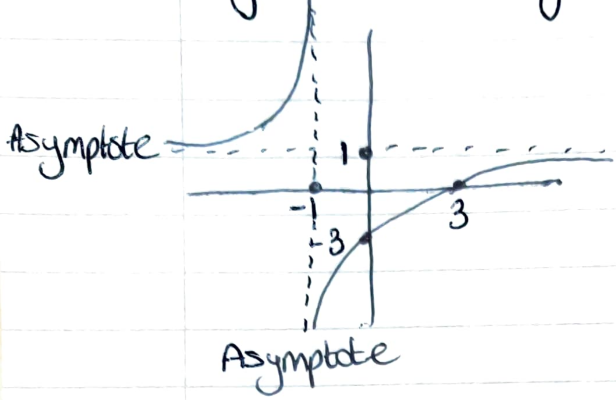
Similar definition



From A it converges etc.

All real numbers but -1

e.g Sketch the graph of $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{x-3}{x+1} \end{cases}$



$x = 0 \Rightarrow f(x) = -3$

$x = 3 \Rightarrow f(x) = 0$

Proper form deg(denominator) > deg(numerator)

$\dfrac{x-3}{x+1}$

$$\begin{array}{r} 1 \\ x+1 \overline{\smash{)}\, x-3} \\ \underline{x+1} \\ -4 \end{array}$$

$\dfrac{x-3}{x+1}$ ← $-4 = \deg(0)$

$= 1 - \dfrac{4}{x+1}$ Remainder -4

$$\lim_{x \to \infty} \frac{x-3}{x+1} = \lim_{x \to \infty} \left(1 - \frac{4}{x+1}\right) = 1$$

$$\lim_{x \to -\infty} \frac{x-3}{x+1} = \lim_{x \to -\infty} \left(1 - \frac{4}{x+1}\right) = 1$$

What happens when $x = -1$

$- =$ approach left $\lim_{x \to (-1)_-} \dfrac{x-3}{x+1} = \infty$

$+ =$ approach right $\lim_{x \to (-1)_+} \dfrac{x-3}{x+1} = -\infty$

Learn

Exponential function $e^x = \exp(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ ✓ Converges

Logarithmic function is the inverse of the exponential function $\exp(x)$

reflect in $y = x$

Look at Lecture Notes

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LO5

MATH 101 LO5 - Derivatives, Differentials

Assessment 2 Q2 (ϵ, δ)

$\frac{1}{x}$ is continous $(0, \infty)$

show the following.

Fix $a \in (0, \infty)$ show $\frac{1}{x}$ is continous at a.

Let $\epsilon > 0$, want $\delta > 0$ such $|x - a| < \delta \Rightarrow$

$f(x) - f(a) < \epsilon \Rightarrow \left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$ ...

$f : A \to B$

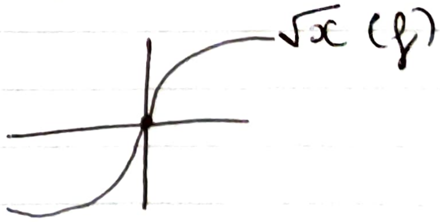
$\frac{f(a+h) - f(a)}{h}$ Rate of Change

Instantaneous rate of change as $h \to 0$ can be found by a tangent at a. This gradient is called the deriative.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \text{ if the limit exists}$$

Difference Quotient (First Principles)

e.g $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sqrt{x} & \text{if } x \geq 0 \\ \quad -\sqrt{-x} & \text{if } x < 0 \end{cases}$

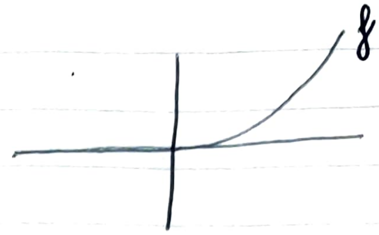


$$\lim_{h \to 0} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}} = \infty$$ so the limit doesn't exist at 0.

f is not differentiable at 0.

eg $f: \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$



Both sides seem differentiable but investigate 0

$$\frac{f(h) - f(0)}{h} = \begin{cases} 0 & \text{if } h < 0 \\ h & \text{if } h > 0 \end{cases}$$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$$

Therefore this has a limit so f is differentiable at all points.

If a function is differentiable then it is also continous.

If f is differentiable at a point a in it's domain then f is also continous at a.

Proof: (show f is continous > $\lim_{x \to a} f(x)$)

$$\lim_{h \to 0} f(a+h) = f(a)$$

$$f(a+h) = \left(\frac{f(a+h) - f(a)}{h} \right)(h) + f(a) \text{ (unsimplified)}$$

$h \to 0$

$$f'(a) \cdot 0 + f(a) = f(a) \text{ Therefore}$$

$$\lim_{h \to 0} f(a+h) = f(a)$$ so f is continous and differentiable.

$\square$

REMEMBER if continous at a it does not mean it is differentiable at 0.

Inverse Function rule -

Let $y = f(x)$

Let g be the inverse function of f, so $x = g(y)$

If f and g are differentiable then

$$\boxed{dy/dx = \frac{1}{\frac{dx}{dy}}}$$

e.g Find $\frac{d}{dx}(x^{\frac{1}{n}})$ if f and g are differentiable

$y = x^{\frac{1}{n}}$ Then $x = y^n$

$$dy/dx = \frac{1}{\frac{dx}{dy}} = \frac{1}{n(y^{n-1})} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}}$$

$$\frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{-(\frac{n-1}{n})} = \frac{1}{n}x^{\frac{1}{n}-1}$$

A Level

Growth/Decay Differential Equation (CP2 8)

$(*)$ $\frac{df}{dx} = K \cdot f$ for some constant K

$f(0) = A.$

Then $(*)$ has a unique solution: $f(x) = Ae^{Kx}$

e (euler's number)

Proof in Lecture Notes

e.g Bacteria in petri dish increase proportionally to the current population. 100s $10^6 \to 2(10^6)$

$t \to 10^7$?

$A = 10^6$

$\frac{dN}{dx} = KN$ $N(0) = A$ $N(x) = Ae^{Kx}$ $N(100) = 2(10^6)$

$= 10^6 e^{K100}$

$e^{100K} = 2$, $K = \frac{\ln 2}{100}$ $N(x) = 10^7$

$10^7 = 10^6 e^{(\frac{\ln 2}{100})100x}$

$= \frac{\ln 10(100)}{\ln 2} = \sim 332$ seconds

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LOG

MATH 101 LOG

Quiz 2 Q1 Recap -

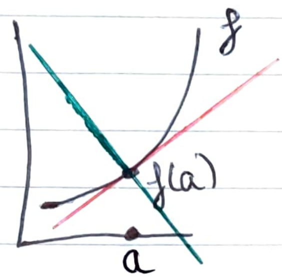
The range is the same as the domain as you can see when sketched. The answer would be D. ✓

Quiz 2 Q2 Recap -

Limits only apply if $f(x)$ and $g(x)$ exist.

e.g if $g(x) = \frac{1}{x}$ the limit $f(x)g(x) = 1$

Answer = D   A ×



tangent to f at a

$f'(a)$ = gradient of tangent

$$\therefore f'(a) = \frac{y - f(a)}{x - a}$$

$$y = f'(a)x + f(a) - f'(a)a$$ equation of tangent

normal to f at a

$$-\frac{1}{f'(a)} = \text{gradient of normal}$$

$$y = \frac{-1}{f'(a)}x + \frac{a}{f'(a)} + f(a)$$ equation of normal

e.g find equation of the normal to $y = -x + \sqrt{x^2+1}$ at $x = 1$

$$dy/dx = -1 + (\text{chain rule})\ \tfrac{1}{2}(x^2+1)^{-1/2}\, 2x$$

$$= -1 + \frac{x}{\sqrt{x^2+1}}$$

at $x = 1$ $y(1) = -1 + \sqrt{2}$

at $x = 1$ $y'(1) = \frac{1 - \sqrt{2}}{\sqrt{2}}$

Then use normal equation

$$y = (2+\sqrt{2})x + \frac{\sqrt{2}}{1-\sqrt{2}} + \sqrt{2} - 1 \quad (\text{can simplify})$$

Chain Rule

Snowball has radius 20cm. The radius shrinks by 1cm/hr. Find $\frac{dV}{dt}$ (rate of change of volume)

$r(t) = 20 - t$ radius at t $V(r) = \frac{4}{3}\pi r^3$ volume with r

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = \underbrace{4\pi r^2}_{\text{surface area}} \cdot \frac{dr}{dt}$$

$$= -4\pi(20-t)^2$$

L'Hospital's Rule -

f, g are differentiable. g' is non-zero around a. (except possibly at $x = a$)

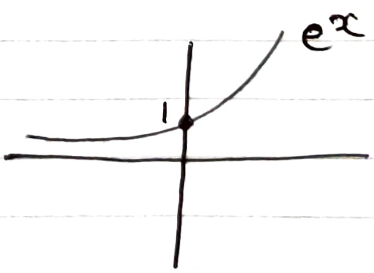
If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ if this limit exists (or is $\pm\infty$)

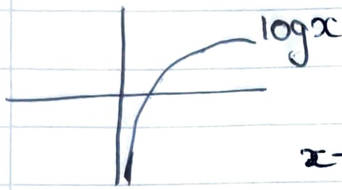
e.g 1) $\lim_{x \to 1} \frac{e^{x-1} - 1}{x - 1}$



Numerator $\to 0$
Denominator $\to 0$

$\lim_{x \to 1} \frac{e^{x-1}}{1} = 1$ so the original limit is also 1.

e.g 2) $\lim_{x \to 0+} x^2 \log x = \lim_{x \to 0+} \dfrac{\log x \to -\infty}{\frac{1}{x^2} \to +\infty}$



$$\lim_{x \to 0+} \frac{\frac{1}{x}}{-2\frac{1}{x^3}} = \lim_{x \to 0+} \frac{x^2}{-2} = 0$$

Therefore original limit is also 0.

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L07

MATH 101 ~~L09~~ L07

Assessment 2 :

Claim $f : \begin{cases} (0,\infty) \to \mathbb{R} \\ x \mapsto \frac{1}{x} \end{cases}$ is continuous

Proof :

Fix $x_0 \in (0,\infty)$

Let $\epsilon > 0$ (we want $\delta > 0$ such that $|x - x_0| < \delta$ $\Rightarrow |\frac{1}{x} - \frac{1}{x_0}| < \epsilon$ so find δ)

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{x \cdot x_0}$$

(want $\frac{1}{x \cdot x_0}$ < constant)

(want $\frac{1}{x \cdot x_0}$ to be small so $x \cdot x_0$ to be large)

choose this value

Assume $\delta < \frac{x_0}{2}$ then $x > \frac{x_0}{2}$

$\Rightarrow x \cdot x_0 > \frac{x_0^2}{2}$

$\Rightarrow \frac{1}{x \cdot x_0} < \frac{2}{x_0^2}$

Choose $\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^2}{2}\epsilon\right\}$

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{x \cdot x_0} < \frac{2}{x_0^2}|x - x_0| < \frac{2}{x_0^2}\delta$$

$$\frac{2}{x_0^2}\delta \leq \epsilon \quad \square$$

Assessment 3 :

Q2) Capacity of mobile phone at time x

$R(x) = (\text{full charge}) - \underbrace{[(\text{current charge})]}_{Q(x)}$

$R(x) = 100 - Q(x)$

$\frac{dR}{dx} = -kR(x)$ ...

A Level
CP2 Chapter 2

Taylor Series - (Theorem)

Let f be suitably differentiable.
Then: (Let a be in the domain of x

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \dots \frac{f^{n}(a)}{n!}(x-a)^n + R_n(x)$$

Remainder Term

where $R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$

for some c between a and x

If $R_n(x) \to 0$ as $n \to \infty$ then
$$f(x) = f(a) + f'(a)(x-a) + \dots \frac{f^{n}(a)}{n!}(x-a)^n + \dots$$
$$= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$$

Note: If $|f^{n}(x)| < k$ for some constant k and all $n \in \mathbb{N}$ and all x "near to a"
then $\lim_{n \to \infty} R_n(x) = 0$ for all x "near to a"

Maclaurin Series

If $R_n(x) \to 0$ as $n \to \infty$ for x "near 0"
then for all x "near 0":

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \dots \frac{f^{n}(0)}{n!}x^n \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!}x^n$$

e.g find Maclaurin for sin(x)

$f(x) = \sin(x)$ $f(0) = 0$
$f'(x) = \cos(x)$ $f'(0) = 1$
$f''(x) = -\sin(x)$ $f''(0) = 0$
$f'''(x) = -\cos(x)$ $f'''(0) = -1$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

Sign test

f function

$f'(a) = 0$ $f''(a) > 0 \Rightarrow$ local minimum
$f'(a) = 0$ $f''(a) < 0 \Rightarrow$ local max
$f'(a) = 0$ $f''(a) = 0$ $f'''(a) \neq 0$ = inflection

Can then be proved

+/11/2021

L08

MATH 101 L08

Quiz 3 Question 2 Recap -

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{here } a = 0$$

$$\lim_{h \to 0} \frac{h|h| - 0}{h} = 0 \quad (A)$$

Quiz 3 Question 4 Recap -

If f is differentiable at $a \Rightarrow f$ is continuous at a.

$$\lim_{x \to a} f(x) = f(a) \quad (B)$$

Sign Test

$f'(a) = 0$ $f''(a) > 0$ $\Rightarrow$ local min at a

$f'(a) = 0$ $f''(a) < 0$ $\Rightarrow$ local max at a

$f'(a) = 0$ $f''(a) = 0$ $f'''(a) \neq 0$ $\Rightarrow$ inflection point

Proof:

$$\underbrace{f(a+h)}_{=x} = f(a) + \cancel{f'(a)h} + -\frac{f''(a)}{2!}h^2 \ldots\ldots \text{(Taylor Series)}$$

$$h = x - a$$

As h gets smaller (very small h) $f(a+h) \approx f(a) + -\frac{f''(a)}{2}h^2$

This term determines max, min or inflection.

Theorem (min/max):

Of all rectangles of a given area, the square has least perimeter.

Proof:

area $= c$

perimeter $= p = 2x + 2y$

$c = xy \qquad y = c/x$

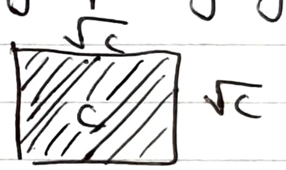
$p(x) = 2x + 2c/x$

$p'(x) = 2 - \dfrac{2c}{x^2} \quad [= 0 \text{ when } x = \sqrt{c}\,]$

$p''(x) = \dfrac{4c}{x^3} \quad [> 0 \text{ when } x = \sqrt{c}\,]$

$x = \sqrt{c}$ is a local minimum for p (by sign test)

$y = c/x \qquad y = c/\sqrt{c} \qquad y = \sqrt{c}$



CP1 1
CP2 1

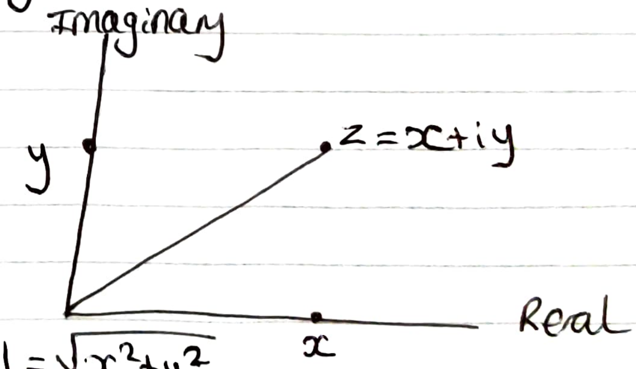
Complex Numbers

we define i to be the number with $i^2 = -1$

$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

$x + iy$

real   imaginary



Length of $z = |z|$

modulus z $\qquad |z| = \sqrt{x^2 + y^2}$

$\bar{z}$ = complex conjugate $= x - iy \qquad |z|^2 = z \cdot \bar{z} \in \mathbb{R}$

Keep in mind

$(x+iy)+(w+iz) = (x+w)+i(y+z)$ Addition

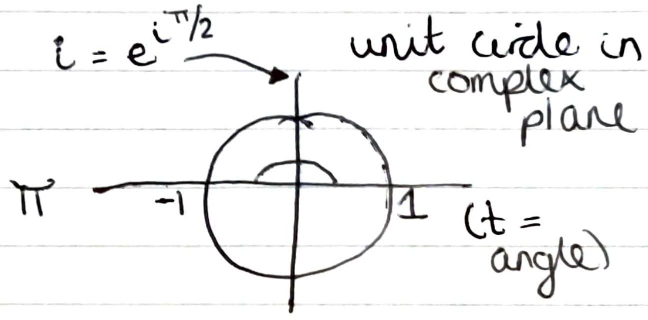
$(x+iy)\cdot(w+iz) = (xw-yz)+i(xz+yw)$ Multiply

example) $\dfrac{2-i}{3+2i} = \dfrac{2-i}{3+2i} \cdot \dfrac{3-2i}{3-2i}$

$= \dfrac{4}{13} - \dfrac{7}{13} i$ Division (Rationalise)

Euler's formula:

$e^{it} = \cos t + i\sin t$



Proof:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$$

Taylor Series of $\cos t = 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} \dots$

of $\sin t = t - \dfrac{t^3}{3!} + \dfrac{t^5}{5!} \dots$

Therefore these sums of series are equivalent $\square$

any complex number can be written as $r \cdot e^{it}$

Polar co-ordinates

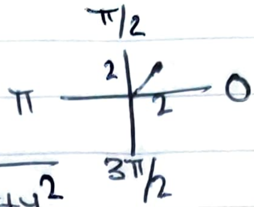
Cartesian $\equiv$ Polar

$z = x+iy$ (x, y) cartesian

$z = r\cdot e^{it}$ (r, t) polar

Need to be able to convert.

e.g. $z = 2 + 2i$



$r = |z|$

$r = \sqrt{8}$ $\quad \sqrt{x^2+y^2}$

$t(\theta) = \pi/4$

$z = \sqrt{8}\, e^{i\pi/4}$

Conversely $z = \sqrt{8}\, e^{i\pi/4}$ convert to cartesian

$x = r\cos t$ $\quad y = r\sin t$

and then calculate

__Theorem:__ $re^{i\theta} \cdot se^{i\phi} = rs\, e^{i(\theta+\phi)}$

(De Moivre)

$(e^{i\theta})^n = e^{in\theta}$

You then need to know how to express

1) $\cos(n\theta)$ or $\sin(n\theta)$ as powers of cos and sin

2) $(\cos\theta)^n$ or $(\sin\theta)^n$ as $(\cos n\theta)$ and $\sin(n\theta)$

L09

MATH 101 L09

1) Express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

Binomial Theorem

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ Real

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ Imaginary

2) Express $\sin^5\theta$ in terms of $\sin(n\theta)$ for some n.

$(2i\sin\theta = z - 1/z)$

$(2\cos\theta = z + 1/z)$

$(2i\sin\theta)^5 = 2^5 i\sin^5\theta$

$= (e^{i\theta} - e^{-i\theta})^5$

$= e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}$

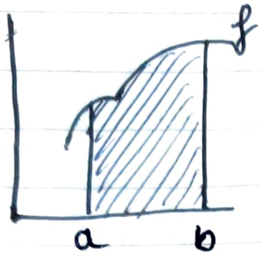
Then

$(e^{5i\theta} - e^{-5i\theta}) + (5e^{-3i\theta} - 5e^{3i\theta}) + (10e^{i\theta} - 10e^{-i\theta})$

$= 2i\sin 5\theta \quad - 10i\sin 3\theta \quad + i20\sin\theta$

$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\sin^5\theta = 1/16\sin 5\theta - 5/16\sin 3\theta + 5/8\sin\theta$



$h_n = \frac{b-a}{2^n}$ (Similar to trapezium rule FP2 chp 11)

Let f be a continous function (so the integral can exist).
Then the definite integral of f from a to b is:

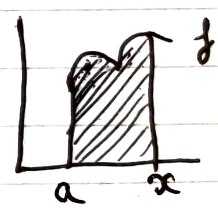
$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{j=1}^{2n} h_n m_j = \lim_{n\to\infty} \sum_{j=1}^{2n} h_n m_j$$

m_j denotes a rectangle (trapezium rule)

<u>Fundamental Theorem of Calculus :</u>

common sense 'Area = Graph

Let f be continous and $F(x) = \int_a^x f(t)\,dt + c$
Then $F'(x) = f(x)$



Note : $\underbrace{\int f(x)\,dx}_{\text{Indefinite Integral}} = F(x) + c$

Collection of all derivative functions

(Part Two):

Let f be continious. Let $F'(x) = f(x)$ Then

$$\int_a^b f(t)\,dt = F(b) - F(a) = [F(t)]_a^b$$

Example:

$f(r) = \pi r^2$ (Area)
$f'(r) = df/dr = 2\pi r$ (circumference)

Conversely
$f(r) = 2\pi r$
$\int_0^r f(t)dt = [\pi t^2]_0^r = \pi r^2$

Remember +C always
Examples can be found on Integral

Integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

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L10

MATH 101 L10

Quiz 4 Recap -

1)

2) Answer is D because $f(0) = T(0)$. B is true as it describes how the maclaurin series works. Therefore C is then true.

3) $|z| < 1$ $|z - (1+i)| < 1$ Intersection at infinitely many points as they define a region not a circle. D)

Integration by Substitution: FP2 11

$$\int f(a) \frac{da}{dx} dx = \int f(a)\, da$$

$$\int_a^b f(u) \frac{du}{dx} dx = \int_{u(a)}^{u(b)} f(u)\, du \quad \text{(With limits)}$$

e.g. $\int_1^2 \frac{dx}{(3-5x)^2}$:

Choose $u(x) = 3 - 5x$ $\quad du/dx = -5$

x	1	2
u	-2	-7

Look at limits of integration

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} -\tfrac{1}{5} \frac{1}{u^2}\, du$$

$$\int_{-2}^{-7} -\tfrac{1}{5} \frac{1}{u^2}\, du = \left[-\tfrac{1}{5}\left(-\tfrac{1}{u}\right)\right]_{-2}^{-7}$$

$$= \left[\tfrac{1}{5u}\right]_{-2}^{-7} = \tfrac{1}{14}$$

This completes material from lecture notes where it is more in-depth

Need to know formulae for the exams.

Past Papers can be found online

2017 101 Past Paper workthrough

1) Let $\cosh x = (e^x + e^{-x})/2$ $\sinh x = (e^x - e^{-x})/2$

show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

RHS =

$= \frac{1}{2}(e^x+e^{-x})\frac{1}{2}(e^y+e^{-y}) + \frac{1}{2}(e^x-e^{-x})\frac{1}{2}(e^y-e^{-y})$

$= \frac{1}{2}(e^{x+y} + e^{-x-y})$

$= \cosh(x+y)$ = LHS

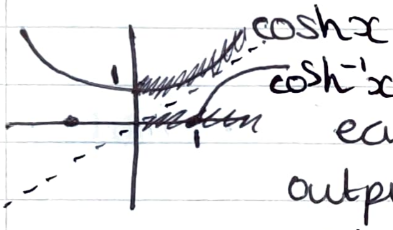
Always start with one side only.

Show that $\cosh 2x = 1 + 2\sinh^2 x$

Let $x = y$

$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x$

ii) sketch $\cosh x$ and inverse on same graph



This has no inverse here as each x value can map to multiple outputs making it not a function.

so: we make it injective

∴ we focus on positive only

iii) calculate $d/dx \cosh^{-1} x$

$y = \cosh^{-1} x$ $x = \cosh y$ $dx/dy = \sinh y$

$dy/dx = 1/dx/dy = \frac{1}{\sinh y}$

$\cosh^2 y - \sinh^2 y = 1$

$\sinh^2 y = \cosh^2 y - 1$

$= \frac{1}{\sqrt{x^2-1}}$

Notice we take positive root as $\cosh^{-1} x$ is increasing.

2) i) show $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} \dots + \frac{1}{n \cdot (n+2)}$

$= \frac{1}{2}(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n-2})$

This can be done using

$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$ and then simplify all terms.

ii) Deduce the value of the sum

$$\frac{1}{1\cdot3}+\frac{1}{2\cdot4}$$

This means find the limit of the series as $n\to 0$ which equals 0.

$\therefore \;= \tfrac{1}{2}(1+\tfrac{1}{2}) = \tfrac{3}{4}$

3) Prove by induction

$$\frac{d^n}{dx^n}(1-x)^{-1} = \frac{n!}{(1-x)^{n+1}}$$

$$P_1 = \frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2} = \frac{1!}{(1-x)^2} = \frac{1}{(1-x)^2} \text{ True}$$

Induction Step: Assume P_n holds

Show P_{n+1} holds

$$\frac{d^{n+1}}{dx^{n+1}}(1-x)^{-1} \underset{P_n}{=} \frac{d}{dx}\left(\frac{n!}{(1-x)^{n+1}}\right) = \frac{(n+1)n!}{(1-x)^{n+2}} \text{ True}$$

4) Find first few terms of a maclaurin series for $f(x) = (1-x)\cos x$

5) By integrating by parts $\int_0^{\pi} x\sin x \, dx$

Let $x = f \quad \sin x = g'$

$f' g \qquad g''$

$= [-x\cos x] + \int_0^{\pi} \cos x \, dx$

$= [-x\cos x]_0^{\pi} + [\sin x]_0^{\pi}$

$= \pi$

$\boxed{\text{CP2 1}}$ 6) Show $\sin^4\theta = \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$