# MATH 102 L01

## Chapter 1 - Further Integration

### 1.1 Tangent substitutions $\tan x/2$ -

Here we consider integrals of the form
$\int R(\sin x, \cos x)$ R is a rational function

**Lemma**

The substitution $t = \tan x/2$ converts a rational function of $\sin x$ and $\cos x$ into a function of $t$

$$dx/dt = \frac{2}{1+t^2} \quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

**Proof** :

$$dt/dx = \tfrac{1}{2}\sec^2 x/2 = \tfrac{1}{2}(1+\tan^2 x/2) = \tfrac{1}{2}(1+t^2)$$

$$dx/dt = 1/dt/dx = \frac{2}{1+t^2}$$

$$\sin x = 2\sin x/2 \cos x/2 = 2\tan x/2 \cos^2 x/2 = \frac{2\tan x/2}{\sec^2 x/2}$$

$$= \frac{2t}{1+t^2}$$

$$\cos x = 2\cos^2 x/2 - 1 = \frac{2}{\sec^2 x/2} - 1 = \frac{2}{1+t^2} - 1$$

$$= \frac{2-1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

**Theorem 1.3** -

Let $R(\cos x, \sin x)$ then
$\int R(\cos x, \sin x)$ reduces to the integral of a rational function by the substitution $t = \tan x/2$

**Example** -

$$\int \sec x\, dx = \int \frac{1}{\cos x}\, dx = \int \frac{1+t^2}{1-t^2}\frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{1-t^2} = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$$

$$= -\log|1-t| + \log|1+t| + C$$

$$= -\log|1-\tan x/2| + \log|1+\tan x/2| + C.$$

## 1.5 Continuing Wright's Integral -

Note: $\frac{1+\tan^2 x/2}{1-\tan^2 x/2} = \sec x + \tan x$

$$\frac{1+\tan^2 x/2}{1-\tan^2 x/2} = \frac{\cos^2 x/2 + \sin^2 x/2}{\cos^2 x/2 - \sin^2 x/2} = \frac{(\cos^2 x/2 + \sin^2 x/2)^2}{\cos^2 2x/2 - \sin^2 2x/2}$$

$$= \frac{\cos^2 2x/2 + 2\sin^2 x/2 \cos^2 x/2 + \sin^2 2x/2}{\cos x} = \frac{1+\sin x}{\cos x}$$

$= \sec x + \tan x$ (Double angle formulae)

Therefore $\int \sec x \, dx = \log|\sec x + \tan x| + C$

## 1.6 Integral's associated with conics -

Consider three curves

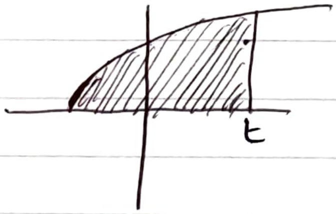
a) $y^2 = ax + b$ parabola
b) $x^2 + y^2 = a^2$ circle
c) $x^2 - y^2 = a^2$ hyperbola

Consider $\int_0^t y(x)\,dx$ (above $x$-axis)

a) For the parabola sub $u = ax+b$

$\int_0^t (ax+b)^{1/2} dx \quad du = a\,dx \quad dx = 1/a\, du$

$\int_0^t (ax+b)^{1/2} dx = \int_b^{at+b} u^{1/2} \cdot 1/a \, du = \left[\frac{2}{3a} u^{3/2}\right]_b^{at+b}$



$= \frac{2}{3a}((at+b)^{3/2} - b^{3/2})$

# 1.16 Improper Integrals - (Unbounded / Infinite)

Suppose $f$ is continuous on $[a, \infty)$ $\int_a^R f(x)\,dx$

Improper Integral =

$$\int_a^\infty f(x)\,dx = \lim_{R \to \infty} \int_a^R f(x)\,dx$$

**Example**

$$\int_0^\infty e^{-x}\,dx = \int_0^R e^{-x}\,dx = [-e^{-x}]_0^R = 1 - e^{-R}$$

$\to 1$ as $R \to \infty$

**Example**

$\int_1^\infty x^{-s}\,dx = \frac{1}{s-1}$ converges $s > 1$

diverges $s \leq 1$

$$\int_1^R x^{-s}\,dx = \left[-\frac{x^{1-s}}{s-1}\right]_1^R = \frac{1 - \frac{1}{R^{s-1}}}{s-1}$$

$s > 1$ $\to \infty$ converges to $\frac{1}{s-1}$

$s < 1$ $\to \infty$ diverges

$$\int_1^R x^{-1}\,dx = [\log x]_1^R = \log R - \log 1 \to \infty \text{ as } R \to \infty$$

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# MATH 102 L02

## 1.18 Integrals over infinite ranges -

$\int_0^\infty \cos x\,dx$ diverges a $[\sin x]_0^R = \sin R$

$\sin R$ doesn't converge to a fixed value as $R \to \infty$

## 1.19/1.20 Integration using partial fractions -

$$\int_2^\infty \frac{1}{x^2(2x+1)}\,dx$$

$$= \frac{Ax+B}{x^2} + \frac{C}{2x+1} \quad \therefore B = 1 \quad A = -2, C = 4$$

$$= \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1} = \frac{1}{x^2} - \frac{2}{x} + \frac{2}{x+1/2}$$

$$= \int_2^R \left(\frac{1}{x^2} - \frac{2}{x} + \frac{2}{x+1/2}\right)dx$$

$$= \left[2\log(x+1/2) - 2\log x - 1/x\right]_2^R = \left[2\log\left(\frac{x+1/2}{x}\right) - \frac{1}{x}\right]_2^R$$

$$= 2\log\left(\frac{R+1/2}{R}\right) - \frac{1}{R} - 2\log 5/4 + 1/2$$

$$\frac{R+1/2}{R} = 1 + \frac{1}{2R} \to 1 \text{ as } R \to \infty \quad \frac{1}{R} \to 0$$

converges to $2\log 4/5 + 1/2$

## 1.22 Integrals over the real line.

$$\int_{-\infty}^b f(x)\,dx = \lim_{A \to -\infty} \int_A^b f(x)\,dx$$

Improper Integral over real line.

$$\int_{-\infty}^\infty f(x)\,dx = \int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx$$

Example -
$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ Let $x = \tan t$

$dx/dt = \sec^2 t = x^2+1$
$\int_0^{\infty} \frac{dx}{1+x^2} = \int_0^{\pi/2} dt = \pi/2$

$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \pi/2$ similarly
Therefore
$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$ and converges

Example -
$\int_{-\infty}^{\infty} x\,dx$ $\int_0^R x\,dx = \frac{1}{2}R^2 \to \infty$ as $R \to \infty$
so diverges
$\int_{-R}^{R} x\,dx = 0$ HOWEVER if infinite areas above and below x-axis it diverges.

1.25 Integrals of unbounded functions -

Suppose: $f$ is continuous $[a, b]$ but discontinuous at $a$.

Variant of MATH101 chp 5

Let $\delta$ stand for a small positive number.
for $\delta > 0$ we can integrate over $(a+\delta, b)$ and form $\int_{a+\delta}^{b} f(x)dx$

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^{b} f(x)dx$$

This is defined where this limit exists otherwise it is then divergent

Example -

$\int_0^1 \log x\,dx$ is improper as diverges at $x=0$

$$\int_\delta^1 \log x\,dx = [x\log x - x]_\delta^1 = -1 - \delta\log\delta + \delta$$

$\delta\log\delta \to 0$ as $\delta \to 0+$

it converges to $-1$ as $\delta \to 0+$

1.25 cont Likewise if $f$ is discontinuous at $b$ we can integrate over $(a, b-\delta)$ and define

$$\int_a^b f(x)\,dx = \boxed{\lim_{\delta\to 0+}\int_a^{b-\delta} f(x)\,dx}$$

where this limit exists otherwise it diverges

Example -

$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$ $\frac{1}{\sqrt{1-x^2}} \to \infty$ as $x \to 1-$ so improper

Let $x = \sin t$ $\sin t \to 1-$ as $t \to (\pi/2)-$ so limits $0 < x < 1$ convert $0 < t < \pi/2$

$dx/dt = \cos t$ $\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cos t}{\cos t}\,dt = \int_0^{\pi/2} dt = \pi/2$$

1.30 Comparison test for integrals -

Let $f$ and $g$ be continuous real functions. If $|f(x)| \leq g(x)$ for all $x \in (a,b)$ finite or infinite

If $\int_a^b g(x)\,dx$ converges $\int_a^b f(x)\,dx$ converges.

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b g(x)\,dx$$

Example -

$\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges

$|\sin x| \le 1$ $\quad \frac{|\sin x|}{x^2} \le \frac{1}{x^2}$ $\quad \int_1^\infty \frac{dx}{x^2}$

$\int_1^R \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^R = 1 - \frac{1}{R} \to 1$ converges

## 1.32 Integral from theory of differentials

when $s > 0$ $\quad F(s) = \int_0^\infty \sin ax\, e^{-sx}\,dx = \frac{a}{s^2+a^2}$

$\int \sin ax\, e^{-sx}\,dx = -\frac{1}{s}\sin ax\, e^{-sx} + \frac{a}{s}\int \cos ax\, e^{-sx}\,dx$

By parts

$= -\frac{1}{s}\sin ax\, e^{-sx} - \frac{a}{s^2}\cos ax\, e^{-sx} - \frac{a^2}{s^2}\int \sin ax\, e^{-sx}\,dx$

$\left(1+\frac{a^2}{s^2}\right) = -\frac{s\sin ax + a\cos ax}{s^2} e^{-sx} + c$

$= -\frac{s\sin ax + a\cos ax}{a^2+s^2} e^{-sx} + \boxed{c'}$ $\quad \frac{c}{1+\frac{a^2}{s^2}}$

$\int_0^R \sin ax\, e^{-sx}\,dx = -\left[\frac{s\sin ax + a\cos ax}{a^2+s^2} e^{-sx}\right]_0^R$

$= \frac{a - e^{-Rs}(s\sin aR + a\cos aR)}{a^2+s^2}$

$|s\sin aR + a\cos aR|$

$e^{-Rs}(s\sin aR + a\cos aR) \to 0$

so converges to $\frac{a}{a^2+s^2}$ as $R \to \infty$

# 1.34 Laplace Transformations -

KEY TOPIC

$F(s) = \int_0^\infty f(x) e^{-sx} dx$ Denoted $\mathcal{L}[f](s)$

used to solve differentials

## Example

Let $f(x) = e^{ax}$ Laplace transformation defined for $s > a$ with

$$F(s) = \frac{1}{s-a}$$

If $a \neq s$, $\int_0^R e^{ax} e^{-sx} dx = \left[ \frac{e^{(a-s)x}}{(a-s)} \right]_0^R$

as $R \to \infty$ $= \frac{-1}{a-s}$

If $a < s$ $e^{(a-s)R} \to 0$ so $F(s)$ $s > a$ if $a > s$

$e^{(a-s)R} \to \infty$ so doesn't converge

If $a = s$ $\int_0^R e^{(a-s)x} dx = \int_0^R 1 \, dx = R$ so doesn't converge either

## Example 1.36 Laplace on Trig -

$f(x) = \sin ax$ $g(x) = \cos ax$

Laplace - $s > 0$

$F(s) = \frac{a}{s^2+a^2}$ $G(s) = \frac{s}{s^2+a^2}$ Proved in 1.32 - 1.33.

## Laplace transforms of Hyperbolics -

Laplace of $\sinh ax$ is $H(s) = \frac{a}{s^2-a^2}$ $(s > |a|)$

$\int_0^\infty e^{ax} e^{-sx} dx = \frac{1}{s-a}$ $\int_0^\infty e^{-ax} e^{-sx} dx = \frac{1}{s+a}$

$H(s) = \frac{1}{2} \left( \frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{1}{2} \frac{s+a-(s-a)}{(s-a)(s+a)}$

The Laplace transforms of sin, cos, sinh and cosh are all rational functions

b) For the circle sub $x = a\sin u$

$\int_0^t (a^2 - x^2)^{1/2}\,dx$

$x = t,\ u = \sin^{-1}(t/a)$

$\int_0^t (a^2 - x^2)^{1/2}\,dx = \int_0^{\sin^{-1}(t/a)} (a^2 - a^2\sin^2 u)^{1/2}.(a\cos u)\,du$

$= \int_0^{\sin^{-1}(t/a)} a^2\cos^2 u\,du$

$= a^2/2 \int_0^{\sin^{-1}(t/a)} (1+\cos 2u)\,du$

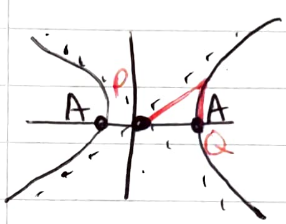
$= a^2/2 \left[u + \sin u\cos u\right]_0^{\sin^{-1}(t/a)}$

$= a^2/2 \sin^{-1}(t/a) + t/2\sqrt{a^2 - t^2}$


c) For the hyperbola sub $x = a\cosh u$

$\int_a^t (x^2 - a^2)^{1/2}\,dx$

<u>Geometric Meanings</u>. Hyperbola centre $(0,0)$ vertex $A = (a, 0)$ lines $y = \pm x$ asymptotes $(a\cosh u, a\sinh u)$ is a typical point.




Triangle OPQ $P = (a\cosh u, a\sinh u)$

$Q = (a\cosh u, 0)$

Area of triangle $= 1/2\,|OQ||QP|$

$= a^2/2 \sinh u\cosh u$ region APQ below hyperbola $=$

$\int_a^{a\cosh u} \sqrt{x^2 - a^2}\,dx$

Let $x = a\cosh u$ $dx/du = a\sinh u$

$\sqrt{x^2 - a^2} = a\sqrt{\cosh^2 u - 1} = a\sinh u$

$\int_a^{a\cosh u} \sqrt{x^2 - a^2}\,dx = \int_0^u a^2\sinh^2 u\,du = a^2/2 \int_0^u (\cosh 2u$

Area of AOP $=$ Area of OPQ $-$ Area APQ

$a^2/2 \int (\cosh 2u - 1)\,du = a^2/2 \left[1/2 \sinh 2u - u\right]_0^u$

$= a^2/2 \sinh u\cosh u - \dfrac{a^2 u}{2}$

$= \dfrac{a^2 u}{2}$

## 1.12 Standard substitutions for $\sqrt{\pm a^2 \pm x^2}$ -

| Integrand Factor | Substitution |
|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin u$ or $x = a\,\text{sech}\, u$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan u$ or $x = a\sinh u$ |
| $\sqrt{x^2 - a^2}$ | $x = a\cosh u$ or $x = a\sec u$ |

## 1.13 Substitutions -

| Factor | Substitution | Derivative | Simplification |
|---|---|---|---|
| $\sqrt{a^2-x^2}$ | $a\sin u$ | $dx/du = a\cos u$ | $a^2-x^2 = a^2\cos^2 u$ |
| $\sqrt{a^2+x^2}$ | $a\tan u$ | $dx/du = a\sec^2 u$ | $a^2+x^2 = a^2\sec^2 u$ |
| $\sqrt{x^2-a^2}$ | $a\cosh u$ | $dx/du = a\sinh u$ | $x^2-a^2 = a^2\sinh^2 u$ |

## 1.14 Useful Identities -

$\cosh^2 u - 1 = \sinh^2 u$ $\qquad$ $\sec^2 u = 1 + \tan^2 u$

$\cos u = \dfrac{\pm 1}{\sqrt{1+\tan^2 u}}$ $\qquad$ $\sin u = \dfrac{\pm \tan u}{\sqrt{1+\tan^2 u}}$

**Example** -

$\int_0^1 (1+x^2)^{-3/2}\,dx$

Let $x = \tan u$ $\quad dx/du = \sec^2 u$ $\quad 1+x^2 = 1+\tan^2 u$
$= \sec^2 u$

$$\int_0^{\pi/4} (\sec^2 u)^{-3/2} \sec^2 u\,du = \int_0^{\pi/4} \cos u\,du = \frac{1}{\sqrt{2}}$$

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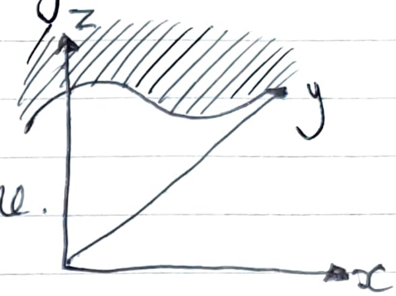
# MATH 102 L03 Chapter 2 Partial Differentiation

## 2.1 Functions of two variables -

Let $(x,y)$ be co-ordinates. $z = f(x,y)$ $f$ = function
Then $f$ is a function of the two variables $x$ and $y$.

e.g Let $(x,y)$ be the co-ordinates of a point in the plane and defining a point on a map. y-axis = north. x-axis = east. $f(x,y)$ is the height at the point $(x,y)$

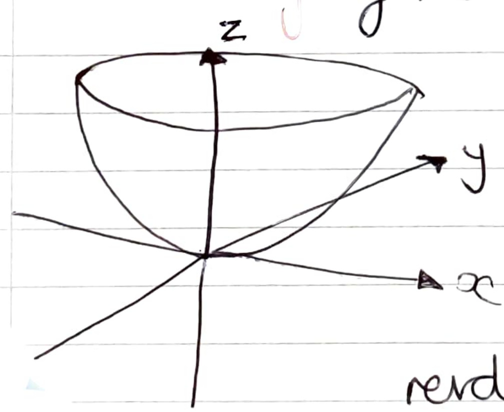



In 3d co-ordinate axis the domain of $f$ is a subset of $(x,y)$ plane. co-domain is in the z-axis

$f$ is continuous if $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$

## 2.2 Paraboloid -

$z = x^2 + y^2$ gives a paraboloid a bowl-shape.




This has exactly one axis of symmetry and no center of symmetry.

This is the surface of revolution of a parabola

more can be learnt on Wolfram Alpha

## 2.3 Partial Deriatives -

$\partial$ = lower case delta

Consider $f(x,y)$ as a function of $x$ and $y$ separately
Look at rates of change in each direction
Let $y=b$, consider $f(x,b)$

variant of first derivative

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

when this limit exists. $\partial$ is used for partial derivatives to distinguish from ordinary derivatives

Let $x=a$, consider $f$ with respect to $y$

" "

$$\frac{\partial f}{\partial x}(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

when this limit exists

**Example -**

find partial derivatives of $f(x,y)=x^2+xy$

Let $(x,y)=(a,b)$
consider
$f(a+h,b)=(a+h)^2+(a+h)b = a^2+2ah+h^2+ab+bh$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$= \lim_{h\to 0}(2a+b+h) = 2a+b$$

$f(a,b+k)-f(a,b) = a^2+a(b+k)-a^2-ab = ak$

$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k} = a$$

**General Rule**: To determine $\frac{\partial f}{\partial x}$ differentiate with respect to $x$, keeping $y$ constant.
To determine $\frac{\partial f}{\partial y}$ differentiate with respect to $y$, keeping $x$ constant.

**Example** -
Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f = x^3 + x^2y + y^2$

$\frac{\partial f}{\partial x} = 3x^2 + 2xy$ $\qquad$ $\frac{\partial f}{\partial y} = x^2 + 2y$

**Example** -
Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = e^{xy} + x + y$

Consider $e^{xy}$ as $e^{bx}$

$$\frac{\partial f}{\partial x} = be^{bx} + 1 = ye^{xy} + 1$$

$$\frac{\partial f}{\partial y} = xe^{xy} + 1$$

**Example** -

Using Product Rule

$f(x,y) = (x^2+xy)(\sin(y^2+xy))$. Let $y = b$

$\frac{\partial f}{\partial x}$ $\quad f(x,b) = (x^2+bx)\sin(bx+b^2)$

$$\frac{\partial f}{\partial x} = \begin{array}{l}(2x+b)\sin(bx+b^2) + b(x^2+bx)\cos(bx+b^2)\\ (2x+y)\sin(xy+y^2) + y(x^2+xy)\cos(xy+y^2)\end{array}$$

Set $x = a$

$$\frac{\partial f}{\partial y} = \begin{array}{l}(ay+a^2)'\sin(y^2+ay) + (ay+a^2)(\sin(y^2+ay))'\\ x\sin(y^2+xy) + (x+2y)(x^2+xy)\cos(y^2+xy)\end{array}$$

Three variables

**Example -**

Let $f(x,y,z) = x^2 + xyz^2$

$$\frac{\partial f}{\partial x} = 2x + yz^2 \qquad \frac{\partial f}{\partial y} = xz^2 \qquad \frac{\partial f}{\partial z} = 2xyz$$

## 2.9 Physical Applications of partial differentiation

**Example -** The equation of state of an ideal gas:

$$K = A\exp\left(\frac{TS - H}{RT}\right)$$

$K, R, A$ constants $\quad H, S, T$ variables.

Find $\frac{\partial H}{\partial T}$, rate of change of $H/T$ As $S$ is constant.

$(\div A) \quad K/A = \exp\left(\frac{TS-H}{RT}\right)$

$(c = \log K/A) \quad \exp(c) = K/A = \exp\left(\frac{TS-H}{RT}\right)$

$$c = \frac{TS - H}{RT} \qquad TS - H = cRT$$

$H = T(S - cR) \quad S, c, R$ constant

$$\frac{\partial H}{\partial T} = \frac{\partial}{\partial T}(\overset{(H)}{T(S-cR)}) = S - cR = H/T$$

**NOTATION:** we sometimes write partials as subscripts

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y} \qquad f_z = \frac{\partial f}{\partial z}$$

This makes it easier to see larger problems such as Higher - Order partial derivatives

## 2.12 Higher-order partial derivatives

These are formed by repeated differentiation

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \quad \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Order matters

$f_{xy}$ means $(f_x)_y$ meaning $x$ happens before $y$

Example -

Find partial derivatives of $f(x,y) = 2x^3 + 3xy^2 + y^4$

$f_x = 6x^2 + 3y^2 \quad f_y = 6xy + 4y^3$

$f_{xx} = (f_x)_x = 12x \quad f_{xy} = (f_x)_y = 6y$

$f_{yx} = (f_y)_x = 6y \quad f_{yy} = (f_y)_y = 6x + 12y^2$

Continue for all non-zero partial derivatives

$f_{xxx} = (f_{xx})_x = 12 \quad f_{xyy} = (f_{xy})_y = 6$

$f_{yxy} = (f_{yx})_y = 6 \quad f_{yyx} = (f_{yy})_x = 6$

$f_{yyy} = (f_{yy})_y = 24y$

$f_{yyyy} = (f_{yyy})_y = 24$

Key Theorem

## 2.14 Equality of mixed partials (Theorem)

If $f: \mathbb{R}^2 \to \mathbb{R}$ has the property $f_{xy}$ and $f_{yx}$ exist and are continuous $f_{xy} = f_{yx}$

## 2.15 Wave equation

verify $u(t,x) = 2\cosh(x+t) - 3\sinh(x-t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$u_x = 2\sinh(x+t) - 3\cosh(x-t)$

$u_t = 2\sinh(x+t) + 3\cosh(x-t)$

$u_{xx} = 2\cosh(x+t) - 3\sinh(x-t)$

$u_{tt} = 2\cosh(x+t) - 3\sinh(x-t)$

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# MATH 102 L04 Chap 3 Polar Co-ordinates / Curves

CP2 Chp5 3.1 Polar co-ordinates -

Let $r$ be radius and $\theta \in [0, 2\pi)$

$x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2$ (pythagoras) $\tan\theta = y/x$

$\theta = a \quad r = r_0$ (notation when comparing familiar curves)

## 3.2 Parametrising curves -

Let $t$ be $\mathbb{R}$ and thought of as time. Let $(x(t), y(t))$ be points that describe the curve $C$ as $t$ - varies.

Parametric form of $C$.

**Example -**

From sin and cos defs.

Circle $x^2 + y^2 = 1$

$$x = \frac{1-t^2}{1+t^2} \qquad y = \frac{2t}{1+t^2}$$

Let $t = \tan\theta/2 \quad x = \cos\theta \quad y = \sin\theta$

$$1 + t^2 = 1 + \tan^2\theta/2 = \sec^2\theta/2$$

$$\frac{1-t^2}{1+t^2} = \underbrace{\left(\frac{1-t^2}{1+t^2}\right)^2}_{x^2} + \underbrace{\left(\frac{2t}{1+t^2}\right)^2}_{y^2} = 1$$

Proof like in L01

$$\frac{1-t^2}{1+t^2} = \frac{1-\tan^2\theta/2}{\sec^2\theta/2} = \cos^2\theta/2 - \sin^2\theta/2 = \cos\theta$$

$$\frac{2t}{1+t^2} = \sin\theta$$

## 3.4 Parameters for circles -

Example Let C be the circle with diameter 1 and centre $(1/2, 0)$. y-axis = tangent to C
Find parametric form and rate of change

$r = 1/2$ centre $= (1/2, 0)$
co-ordinates $(2(x - 1/2), 2y)$

$$x = 1/2 + 1/2\left(\frac{1-t^2}{1+t^2}\right) = 1/2\left(\frac{1+t^2+1-t^2}{1+t^2}\right) = \frac{1}{1+t^2}$$

$$y = 1/2\left(\frac{2t}{1+t^2}\right) = \frac{t}{1+t^2}$$

## 3.4 Tangent Vector to a curve -

Suppose curve C is smooth (tangents at each point)
Tangent is in the direction of vector $\left(\frac{dx}{t}, \frac{dy}{t}\right)$ at t.
Tangent to C at $P = (x(t), y(t))$ has gradient

$$dy/dx = \frac{dy/dt}{dx/dt}$$

If the moving point flies off C it leaves in direction $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. Vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ = velocity vector.

$\frac{dx}{dt}$ = horizontal velocity

$\frac{dy}{dt}$ = vertical velocity.

The normal to C at P $= -dx/dy$.

## 3.5 Length of a curve -

Let $P$ and $Q$ be points on $C$ $(x(t), y(t))$

$P = (x(a), y(a))$ $Q = (x(b), y(b))$

Length $\overrightarrow{PQ}$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

similar to modulus of points.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$ for arc-length from $P$ to $(x(t), y(t))$.

This is a definition of the length of $C$.

## 3.6 Rectifiable Curves -

Rectification means working out arclength.

You can find the arclength $s(t)$ for the following.

circle $x^2 + y^2 = a^2$

parabola $y^2 = 4ax$

Neil's curve $y^2 = x^3$

Tsirnhausen's cubic $3y^2 = x^2(1-x)$

catenary $y = \cosh x$

logarithmic spirals $r = ae^{b\theta}$ (in polar co-ordinates)

astroid $|x|^{2/3} + |y|^{2/3} = 1$

## 3.7 Length of a graph -

Suppose $y = f(x)$ $a \le x \le b$

$$L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$ Length of Graph

Example:

for $y = x^{3/2}$ of Neil's Curve. $dy/dx = 3/2\sqrt{x}$

arc from $(0) \to (t, t^{3/2})$ =

$$\int_0^t \sqrt{1 + 9/4x}\, dx = 3/2 \int_0^t \sqrt{x + 4/9}\, dx = 3/2\left[2/3(x + 4/9)^{3/2}\right]_0^t$$

$$= (t + 4/9)^{3/2} - (4/9)^{3/2} = (t + 4/9)^{3/2} - 8/27$$

## 3.8 Arc length calculation

Example: arc length of logarithmic spiral $r = e^\theta$
$\theta = 0, r = 1$ to $\theta = \pi, r = e^\pi$

$x = r\cos\theta \quad y = r\sin\theta$

$x = e^\theta\cos\theta \quad y = e^\theta\sin\theta$

$\frac{dx}{d\theta} = e^\theta(\cos\theta - \sin\theta) \quad \frac{dy}{d\theta} = e^\theta(\cos\theta + \sin\theta)$

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 2e^{2\theta}$

$L = \int_0^\pi \sqrt{2}\, e^\theta d\theta = [\sqrt{2}e^\theta]_0^\pi = \sqrt{2}(e^\pi - 1)$

## 3.9 The ellipse E

ellipse = regular oval shape, sum of distances from two points is constant.

ellipse equation

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a, b > 0 \text{ constants}$$

centre of ellipse $= (0,0)$ axes are $y = 0 \; -a \leq x \leq a$ and $x = 0 \; -b \leq y \leq b$

parametrise by $x = a\cos t \; y = b\sin t \quad 0 \leq t \leq 2\pi$
If $a = b$ E is a circle $r = a$, centre $(0,0)$

## 3.10 Area of ellipse E

Area $= \pi ab$

Proof:

$A = 4\int_0^a y(x)dx = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\, dx \quad x = a\sin t$

$0 < x < a, \; 0 < t < \pi/2 \quad dx/dt = a\cos t \quad \sqrt{1 - \frac{x^2}{a^2}}$
$= \sqrt{1 - \sin^2 t} = \cos t$

$A = 4ab\int_0^{\pi/2}\cos^2 t\, dt = 2ab\int_0^{\pi/2}(1 + \cos 2t)dt = 2ab[t + \tfrac{1}{2}\sin 2t]_0^{\pi/2}$
$= \pi ab$

3.11 Perimeter of ellipse E -

Suppose $a \geq b$. Let $K = \sqrt{1-\frac{b^2}{a^2}}$ Perimeter =

$$a\int_0^{2\pi} \sqrt{1-k^2\sin^2 t}\, dt = 4a\int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\, dt$$

Proof: $x = a\sin t$ $y = b\cos t$

$dx/dt = a\cos t$ $dy/dt = -b\sin t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\cos^2 t + b^2\sin^2 t =$$

$$a^2 + (b^2 - a^2)\sin^2 t$$

$$= a^2\left(1 + \left(\frac{b^2}{a^2} - 1\right)\sin^2 t\right) = a^2(1 - k^2\sin^2 t)$$

Then use arc length from 3.5

3.12 Elliptic Integrals -

Complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta$$

$k$ is called the modulus. Prop 3.11. Perimeter is $4aE(k)$ $k = \sqrt{1-\frac{b^2}{a^2}}$ $k$ is known as the eccentricity of the ellipse.

These cannot be calculated precisely and we must approximate $E(k)$

3.13 Simpson's Rule

Let $f(x)$ be differentiable

$$\int_a^b f(x)\,dx = \frac{(b-a)}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\} + E$$

$$E = -\frac{(b-a)^5 f^{(4)}(c)}{2880}$$

In most cases this gives a good approximation for 3.12. When $\deg(x) < 4$: Error $= 0$.

Example:

Using Simpson's Rule to estimate a perimeter of
$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$a = 5,\ b = 3$

$$k = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - 9/25} = 4/5$$

By 3.11 perimeter $= 20E(4/5)$
Estimate $E(4/5)$

| $\theta$ | $0$ | $\pi/4$ | $\pi/2$ |
|---|---|---|---|
| $\sin\theta$ | $0$ | $1/\sqrt{2}$ | $1$ |
| $\sqrt{1 - 16/25 \sin^2\theta}$ | $1$ | $\sqrt{17}/5$ | $3/5$ |
| weight | 1 | 4 | 1 |

By simpson's $\int_0^{\pi/2} \sqrt{1 - 16/25 \sin^2\theta}\, d\theta = \pi/12\left(1 + 4/5\sqrt{17} + 3/5\right)$
$= 1.28$

Perimeter $= 20(1.28) = 25.6$.

30/11/2021

# MATH 102 L05

## 3.16 Change in a function along a curve -

We want to analyse how $f(x,y)$ changes as we move from $(x,y)$ to $(x+h, y+k)$ in the plane.

Change in $f(x,y)$ is:

$$f(x+h, y+k) - f(x,y) = \left(k\frac{\partial f}{\partial y} + h\frac{\partial f}{\partial x}\right)$$
$$(f(x+h, y+k) - f(x+h, y)) + (f(x+h,y) - f(x,y))$$

Therefore this will involve partial derivatives in the $x$ and $y$ directions.

When parametric, $f(x(t), y(t))$ is a function of $t$.

## 3.17 Chain rule for functions of two variables -

Suppose $C$ is a curve: $(x(t), y(t))$ and $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Because $(\partial x \times dx)$ $(\partial y \times dy)$ cancel

**Proof: (Rough)**

Let $x = x(t)$ and $y = y(t)$. Let $h = x(t+s) - x(t)$.
Let $k = y(t+s) - y(t)$. Then $\frac{h}{s} \to \frac{dx}{dt}$ and $\frac{k}{s} \to \frac{dy}{dt}$ as $s \to 0$.

$$\frac{f(x(t+s), y(t+s)) - f(x(t+s), y)}{s}$$

$$= \frac{f(x+h, y+k) - f(x+h, y)}{k} \cdot \frac{k}{s} \to \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

And the same for the other way. Summing both expressions we reach the req value. $\square$

Example. (Straight Line) -

Let $(x(t), y(t)) = (a+th, b+tk)$. Then $(x(t), y(t))$ is a straight-line segment. For any $f(x,y)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k.$$

$h = \frac{dx}{dt}$ $k = \frac{dy}{dt}$.

Example (Circle) -

Let $x(t) = \cos t$ and $y(t) = \sin t$. Find $\frac{df}{dt}$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t$$

$$\text{so } \frac{df}{dt} = -\sin t \frac{\partial f}{\partial x} + \cos t \frac{\partial f}{\partial y} = -y f_x + x f_y$$

## 3.20 Implicitly defined functions -

Graphs $\longleftrightarrow$ functions

Curves $\longleftrightarrow$ Implicit functions

The graph of a function has a property that each $x$ is associated with only one $y$ value. For curves they can correspond to multiple $y$ (implicitly defined functions).

Example (Circle)

$x^2+y^2-1=0$ defines a circle. This is the union (OR) of the semicircles $y_2 = \sqrt{1-x^2}$ and $y_1 = -\sqrt{1-x^2}$ where $y_1, y_2$ are functions of $x \in [-1, 1]$.

## 3.21 Contours -

A suitable function $f: \mathbb{R}^2 \to \mathbb{R}$ of the form $f(x,y)$ can give rise to a curve in the plane.

Needs Thinking About

**Example (Contour) -**

Let $f(x,y)$ be the height of land at point $(x,y)$. A walker walks at constant height $h$. His path is the curve.

$$C_h = \{(x,y) : f(x,y) = h\}$$

This follows the contours of his map. His direction of travel is a tangent to $C_h$ and parallel to $(dx, dy)$

**3.22 Implicit Functions -**

For differentiable $f(x,y)$, $f(x,y) = 0$ defines a curve $C$ in the plane. We can say $y$ is an implicit function of $x$.
$f(a,b) = 0$ and $f_y(a,b) \neq 0$ such that $f(x,y) = 0$ generally forms a graph of a function $y(x)$.
(This can sometimes be solved for $y$)

**Example -**

$\tan^{-1}\left(\frac{y}{x}\right) - \log\left(\sqrt{x^2+y^2}\right) = c$ gives $y$ as an implicit function of $x$. This example appears in differential eqns

**3.23 Gradient of the tangent to a curve -**
**(Find $dy/dx$ without solving for $y$)**

Let $f(x,y) = 0$ define a curve $C$. $y$ = implicit differentiable. Gradient at $(x,y)$ =

$$\left[\frac{dy}{dx} = -\frac{f_x}{f_y}\right]$$

(Can be several $dy/dx$ per $x$ for curves)

## Proof of gradient formula -

Set $x = t$, $y = y(t)$. $(t, y(t))$ describes C near $(a,b)$ as t varies.

$0 = f(t, y(t))$ for all t near a

From chain Rule -

$$0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$x = t$, $dx/dt = 1$ $dy/dt = dy/dx$

$$dy/dx = -\frac{f_x}{f_y}$$

which equals gradient of tangent. To differentiate implicitly means to write:

$f_x + f_y \, dy/dx = 0$ without attempting to find an explicit formula for y in terms of x.

## Examples of implicit differentiation -

$x^2 + 4y^2 = 1$ defines an ellipse. Find $dy/dx$.

$f(x,y) = x^2 + 4y^2 - 1$. $f(x,y) = 0$ gives ellipse. Now differentiate for x.

$$\frac{d}{dx} f(x,y) = 2x + 8y \, dy/dx = 0$$

$8y \, dy/dx = -2x$ $\quad dy/dx = -\frac{x}{4y}$

Parametrize the curve. $x = \cos t$, $y = \frac{1}{2}\sin t$.

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}\cos t}{-\sin t} = \frac{\cos t}{4 \cdot \frac{1}{2}\sin t} = -\frac{x}{4y}$$

as claimed.

# Chapter 4: Maxima and Minima in Two Variables

## 4.1 Maxima and Minima -

Local Maximum

Local Minimum

Local Maximum when $f(a,b) \geq f(x,y)$

Local Minimum when $f(a,b) \leq f(x,y)$

## 4.2 Stationary Points -

$P=(a,b)$ is a stationary point if

$$\left(\frac{\partial f}{\partial x}\right)_P = 0 = \left(\frac{\partial f}{\partial y}\right)_P$$

the subscript means we evaluate the partial derivatives at $P$

<u>Proposition</u>: Local Maxima and Minima occur at stationary points

<u>Proof</u>:

Suppose $(a,b)$ is a local max or min. Clearly the function $f(x,b)$ has a local max or min. The first-order partial derivative of $f$ to $x$ must be zero at $(a,b)$ and the same for $y$. □

<u>Example finding stationary points</u> -

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

$\frac{\partial f}{\partial x} = 3x^2 - 3 \quad \frac{\partial f}{\partial y} = 3y^2 - 12. \quad x^2 = 1 \quad y^2 = 4$

Four stationary points.

$(x,y) = (1,2), (1,-2), (-1,2), (-1,-2)$

Next we want to find out if these are min or max?

$f(1,2) = 2 \quad f(1,-2) = 34 \quad f(-1,2) = 6 \quad f(-1,-2) = 38.$

## 4.4 more stationary points -

$f(x,y) = xe^{-(x^2+y^2)}$

$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)} - 2x^2e^{-(x^2+y^2)}$ $\frac{\partial f}{\partial y} = -2xye^{-(x^2+y^2)}$

$f_x = f_y = 0$ if $1-2x^2 = -2xy = 0$.

Therefore $y=0$, $x = \pm\frac{1}{\sqrt{2}}$

Stationary points $(\pm\frac{1}{\sqrt{2}}, 0)$

$f(x,y) = \pm\frac{1}{\sqrt{2}}e^{-1/2}$

## 4.5 Taylor's Theorem (2 Variables) -

Key Theorem

Let $f(x,y)$ be differentiable. Then

**Theorem**:

Let $P=(a,b)$ be a point in the plane. Then for $h$ and $k$ sufficiently small.

$$f(a+h,b+k) = f(a,b) + \left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + *$$

* Higher order terms.

where partial derivatives are evaluated at P.

**Discussion**.

when $(a,b)$ isn't stationary it works as a good approximation.

when $(a,b)$ is stationary the term in braces is 0. and it approximates near to $(a,b)$. This is caused by a quadratic expression.

This is useful in determining max, min or stationary.

If $g(x,y)$ is a function and $G(t) = g(a+th, b+tk)$

$$G'(t) = h\frac{\partial g}{\partial x}(a+th, b+tk) + k\frac{\partial g}{\partial y}(a+th, b+tk).$$

$$G'(t) = hg_x + kg_y$$

## 4.7 Sketch Proof of Taylor's Theorem -

The straight line from $(a,b)$ to $(a+h, b+k)$ is given by $x = a+th$ and $y = b+tk$ $0 \le t \le 1$.

$F(t) = f(a+th, b+tk)$ $F(0) = f(a,b)$ $F(1) = f(a+h, b+k)$

Maclaurin series for $F$:

$$F(t) = F(0) + \frac{F'(0)}{1}t + \frac{F''(0)}{2}t^2$$ simplifies at $t=1$

$$F(1) = F(0) + \frac{F'(0)}{1} + \frac{F''(0)}{2} + \dots$$

$$F'(t) = h\frac{\partial f}{\partial x}(a+th, b+tk) + k\frac{\partial f}{\partial y}(a+th, b+tk)$$

$$F''(t) = \frac{d}{dt}\left(h\frac{\partial f}{\partial x}(a+th, b+tk) + k\frac{\partial f}{\partial y}(a+th, b+tk)\right) =$$

$$h\frac{d}{dt}\frac{\partial f}{\partial x} + k\frac{d}{dt}\frac{\partial f}{\partial y} = h\left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial y \partial x}\right) + k\left(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

Last step by Theorem 2.14

This is the formula for $f(a+h, b+k)$ □

HESSIAN DISCRIMINANT

## 4.9 Classification of stationary points via discriminant

Hessian Discriminant:

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$\Delta_P$ = Discriminant valued at $P$. $P$ is non-degenerate if $\Delta_P \neq 0$.

## 4.10 Examples of Hessian Discriminant -

Find stationary points of $f(x,y) = x^2 + xy - 2y^2 - 2$
Calculate Hessian discriminant at each stationary point.

$f_x = 2x + y \quad f_y = x - 4y \quad f_{xx} = 2 \quad f_{xy} = 1 \quad f_{yy} = -4$

Stationary: $2x + y = x - 4y = 0 \quad x = 4y \quad y = -2x$

$x = y = 0 \quad = (0,0)$

Hessian: $f_{xx}f_{yy} - f_{xy}^2 = 2(-4) - (1)^2 = -9$

## 4.11 Hessian Discriminant test -

If $(f_{xx})_P > 0$ and $\Delta_P > 0$ Minimum at $P$.
If $(f_{xx})_P < 0$ and $\Delta_P > 0$ Maximum at $P$
If $\Delta_P < 0$ Saddle

## 4.14 Cases in Hessian Discriminant test -

i) $\Delta > 0$ and $A > 0$ so $f(a+h, b+k) \geq f(a,b)$ and minimum

ii) $\Delta > 0$ and $A < 0$ so $f(a+h, b+k) \leq f(a,b)$ and maximum

iii) when $\Delta < 0$ $f$ increases and decreases so saddle.

?/12/2021

# MATH 102 LOG [STRIKE]

## Example of Hessian Discriminant test -

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3 \quad f_y = 3y^2 - 12 \quad f_{xx} = 6x \quad f_{yy} = 6y$

$f_{xy} = 0$

Hessian: $f_{xx}f_{yy} - f^2_{xy} = 36xy$

$\Delta_P$ is positive if $x$ and $y$ have the same sign and negative if opposite signs. $f_{xx}$ is positive iff $x>0$.

| Stationary Point | $\Delta$ | $f_{xx}$ | nature |
|---|---|---|---|
| (1, 2) | 72 | 6 | local min |
| (1, -2) | -72 | 6 | saddle |
| (-1, 2) | -72 | -6 | saddle |
| (-1, -2) | 72 | -6 | local max |

## 4.17 Geometric example of a minimum -

Open-topped rectangular box has sides $x, y, z$ and volume 32. Which choice of side lengths is surface area smallest?

$V = xyz \quad SA = xy + 2xz + 2yz$

$V = 32 \quad z = 32/xy$

$S(x,y) = xy + \frac{64}{y} + \frac{64}{x}$

(min)

$S_x = y - \frac{64}{x^2} \quad S_y = x - \frac{64}{y^2} \quad S_{xx} = \frac{128}{x^3} \quad S_{xy} = 1 \quad S_{yy} = \frac{128}{y^3}$

Stationary: $x = \frac{64}{y^2} \quad y = \frac{64}{x^2} \quad x = 64 \cdot \frac{x^4}{64^2} = \frac{x^4}{64}$

$x^3 = 64 \quad x = 4 \quad , y = 4 \quad P = (4, 4)$

$S_{xx} = 2 \quad S_{xy} = 1 \quad S_{yy} = 2 \quad \Delta = 3 \quad P =$ minimum

$= 48.$

# Chapter 5: Double Integrals.

## 5.1 Double Integrals -

Consider rectangular box $B$ in $Oxyz$ space
$a \le x \le b$ $c \le y \le d$ height $= z$.
Partially fill box with sand $z = f(x,y)$
volume of sand:

$$V = \iint_R f(x,y)\,dx\,dy$$

Interpret as: Split rectangle into $N^2$ smaller rectangles $R_{ij}$ of sides $h$ by $k$ $(x_i, y_j) = (a+hi, c+kj)$ for $1 < i$ $j \le N$ with $b = a+Nh$ and $d = c+Nk$

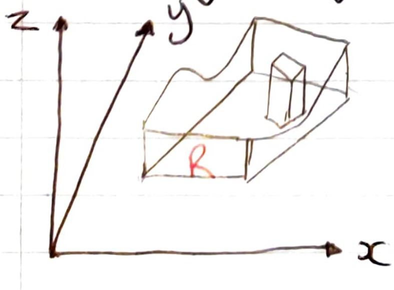
## 5.3 Double Integrals as limits -

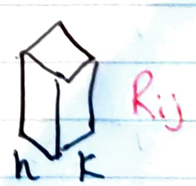
Over the small rectangle $R_{ij}$ total volume of the sand $= \sum_{i,j} f(x_i, y_j)hk$. as $h,k \to 0+$ this converges.

$$\iint_R f(x,y)\,dx\,dy = \lim_{h,k \to 0+} \sum f(x_i, y_j)hk$$ to be the double integral of $f$ over $R$.

Fundamental idea: double integral of $f$ over $R$ is the volume under the surface: $\{z = f(x,y) : (x,y) \in R\}$ above the region $R$

## 5.2 Diagram of Double Integral -



## 5.4 Repeated Integrals -

Given a continuous function $f(x,y)$ on R

For $y$:

$\int_a^b f(x,y)\,dx$ gives a function of $y$.

We can integrate to form the repeated integral

$$\int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy$$

For $x$:

$\int_c^d f(x,y)\,dy$ gives a function of $x$.

We can integrate for the other repeated integral

$$\int_a^b \left\{ \int_c^d f(x,y)\,dy \right\} dx$$

The term iterated integrals is commonly used.

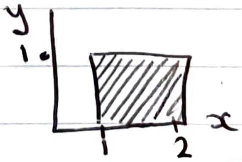
## 5.5 Areas of Sections -

$\int_a^b f(x,y)\,dx$ equals the area of the section that is cut by the plane through $(x,0,0)$ parallel to the co-ordinate plane $Oyz$.

Evaluate

$$\int_1^2 \left\{ \int_0^1 xy^2\,dy \right\} dx \text{ and } \int_0^1 \left\{ \int_1^2 xy^2\,dx \right\} dy$$

Sketch: Region




First

$$\int_0^1 xy^2\,dy = \left[\frac{xy^3}{3}\right]_0^1 = \frac{x}{3}(1^3 - 0^3) = \frac{x}{3}$$

$$\int_1^2 \frac{x}{3}\,dx = 1/2$$

Second

$$\int_1^2 xy^2\,dx = \frac{3y^2}{2} \quad \int_0^1 \frac{3y^2}{2}\,dy = 1/2.$$

## 5.7 Equality of Repeated Integrals -

It is no coincidence repeated integrals produce the same result!

**Fubini Theorem:**

Let $f$ be continuous on a bounded rectangle $R$. Then the repeated integrals are equal and their common value is the double integral.

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy = \int_a^b \left\{ \int_c^d f(x,y)\,dy \right\} dx$$

## 5.8 Notation for double integrals -

We can shorten the braces,

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy$$

## 5.9 Examples of Repeated Integrals

Let $R$ be the rectangle $0 \leq x \leq 2$ $0 \leq y \leq 1$

$\iint_R (x+y)\,dx\,dy$ ← Solve

$$\int_0^1 \int_0^2 (x+y)\,dx\,dy = \int_0^1 \left[ \frac{x^2}{2} + xy \right]_0^2 dy = 3$$

## 5.10 Integration over unbounded regions -

We can integrate over regions by splitting up $R$ into sets of small diameter and approximating.

$$\int_0^1 \int_0^\infty x \sin xy\, e^{-y}\,dy\,dx = 1 - \pi/4$$

$\int_0^\infty \dots =$ Laplace Transform of $\sin ax$ ($\sin ay$)

it is $\frac{x^2}{x^2+1}$

$$\int_0^1 \frac{x^2}{x^2+1}\,dx = 1 - \pi/4$$

## 5.11 Properties of Double Integrals –

For a region R, functions f and g and a constant C the following hold:

Common for integrals

i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$;

ii) $\iint_R Cf\,dx\,dy = C\iint_R f\,dx\,dy$

iii) For disjoint regions S and T with union $R = S \cup T$

OR

$$\iint_R f\,dx\,dy = \iint_S f\,dx\,dy + \iint_T f\,dx\,dy$$

iv) Area of region R $\quad A = \iint_R dx\,dy$

v) If $m \le f(x,y) \le M$ for all $(x,y)$ in R and R has area A then.

$$mA \le \iint_R f(x,y)\,dx\,dy \le MA.$$

$(V = 1 \cdot A = A)$

7/12/2021

# MATH 102 L07

## Chapter 6: Differential Equations

### 6.1 First-Order Differential Equations

$dy/dx = g(x,y)$ where $g(x,y)$ is a function of two variables

Begin by considering $dy/dx = f(x)$ In this form they are integrable.

**Example** -

Solve differential equation $dy/dx = \frac{1}{x-2}$

Integrate $\frac{1}{x-2} = \log|x-2| + c$

$y = \log|x-2| + c$

Initial / Boundary conditions can be used for finding values of constants

### 6.4 Terminology -

General Solution = answer without specifying c

Particular Solution = value of c.

Blows up = when model becomes undefined

**Example** -

$dx/dt = te^{2t} \quad x(0) = 3/4$

$\int te^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + c$

$(x(0)) = -1/4 + c = 3/4 \quad c = 1$

$x(t) = (\frac{1}{2}t - \frac{1}{4})e^{2t} + 1$

CP2 Chp 7

## 6.7 Separable First-Order Differentials

separable $dy/dx = f(y)g(x)$

$$dy/dx = y^2(x^2+1)$$

1) Make sides distinct
2) $\frac{1}{y^2}dy = (x^2+1)dx$ and integrate

## 6.11 Newton's Law of Cooling -

Rate of change of the temperature T is proportional to the difference between it's temperature and the temperature $\tau$ of the surroundings

$$\frac{dT}{dt} = -\lambda(T-\tau) \quad \lambda = \text{positive constant}$$

$$\int \frac{dT}{T-\tau} = -\int \lambda \, dt$$

## 6.13 The Logistic Equation -

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$ Used to model population growth

$K$ = theoretical max / carrying capacity

## 6.16 First order linear differential equations.

These are where we use the integrating factor

$dy/dx + p(x)y = q(x)$ multiply all by $e^{\int p(x)dx}$ and integrate

Homogeneous = when $q(x) = 0$
Non-homogeneous otherwise

We need the particular integral and complementary function

* see CP2 chp7.

General Solution -

$$y = CF + PI$$

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# MATH 102 L08

## Example -

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 6x^2 + 1$$

$m^2 + 5m + 6 \quad (m+2)(m+3)$

$y = Ae^{-2x} + Be^{-3x}$ (CF)

$y = \lambda x^2 + \mu x + \delta$

$y' = 2\lambda x + \mu$

$y'' = 2\lambda$

Then sub back in and solve,

$y = Ae^{-2x} + Be^{-3x} + x^2 - 5/3x + 11/9$.

In the circumstances where the PI has the same power as the CF we increase the power of e to solve. e.g $e^x$ and $e^x$ use $e^{2x}$ instead or $xe^x$

## Example -

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-2x}$$

$m^2 + m - 2 = (m+2)(m-1) \quad y = Ae^{-2x} + Be^{-x}$

$e^{-2x}$ = same

$y = Cxe^{-2x}$

$y' = Ce^{-2x} - 2Cxe^{-2x}$

$y'' = 4Cxe^{-2x} - 4Ce^{-2x}$

$y = Ae^{-2x} + Be^{-x} - 1/3xe^{-2x}$

# Further Laplace Transforms - Denoted $L(f)$

$$F(s) = \int_0^\infty f(x) e^{-sx} dx = \text{Laplace of } f(x)$$

**Theorem:**

Let $f(x)$ and $g(x)$ be functions whose Laplace exist for $s > \alpha$ and $c$ a constant. $L(f+g) = L(f) + L(g)$ and $L(cf) = cL(f)$

Example Laplace of $\cos ax$

Deduce from Laplace of $\sin ax$

$f(x) = \sin ax \quad f'(x) = a\cos ax$

$$L(a\cos ax) = -f(0) + s\frac{a}{s^2+a^2} = \frac{as}{s^2+a^2}$$

$$\int_0^\infty e^{-sx} \cos ax \, dx = \frac{s}{s^2+a^2} \text{ for } s > a.$$

**Theorem:**

If Laplace of $f(x)$ converges and equals $F(s)$ then Laplace of $e^{ax}f(x)$ converges for $s > s_0 + a$.

$$L(e^{ax} f(x))(s) = F(s-a)$$

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# MATH 102 L09

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**Example using Laplace** - Find inverse Laplace of $\frac{5}{(s+1)(s^2+4)}$

$$\frac{5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$5 = A(s^2+4) + (Bs+C)(s+1)$$

$$5 = As^2 + 4A + Bs^2 + Bs + Cs + C$$

$$A = -B \quad B = -C, \; A = C, \; A = 1, \; B = -1, \; C = 1$$

$$= \frac{1}{s+1} + \frac{-s+1}{s^2+4}$$

$$= \frac{1}{s+1} + \frac{1}{s^2+4} - \frac{s}{s^2+4}$$ by Theorem 6.47

$$= e^{-x} + \tfrac{1}{2}\sin 2x - \cos 2x$$

**Example** -

$dy/dx + y = \tfrac{5}{2}\sin 2x \quad y(0) = 0$

$$L\left(\frac{dy}{dx} + y\right) = \tfrac{5}{2}\left(\frac{2}{s^2+4}\right) = \frac{5}{s^2+4}$$

$$L\left(\frac{dy}{dx}\right) + L(y) = (-y(0) + sL(y)) + L(y) = (s+1)L(y)$$

$$L(y) = \frac{5}{(s+1)(s^2+4)}$$

$L^{-1} \quad y = e^{-x} + \tfrac{1}{2}\sin 2x - \cos 2x$

## Chapter 7: Recurrence Relations + Generating Functions

**Definition:**

A sequence $a_0, a_1 \ldots a_n$ is recursive if each member is expressed as a function of a previous member.

$a_n = f_n(a_0, a_1, \ldots a_{n-1})$ ← recurrence relation

e.g Factorial, Compound Interest

## Recurrence relations of order K -

$a_n = fn(a_{n-1}, a_{n-2}, \ldots a_{n-k})$ = order K

Compound Interest, Factorials = first order

Fibonacci = Second Order.

Linear recurrence relations

$a_n = c_0 a_0 + \ldots c_{n-k} a_{n-k}$

## 7.3 Generating functions -

Given a sequence $a_0, a_1, \cdots a_n$ it's generating function is the power series,

$A(z) = 1 + a_1 z + \ldots a_n z^n$

Large fibonacci example

## 7.7 Generating function of a linear recurrence -

Let $a_0, a_1, \ldots a_n$ be a sequence satisfying a linear recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$ where $c_i, 1 \le i \le k$ Then

$A(z) := \sum_{n=0}^{\infty} a_n z^n$

$A(z) = \frac{P(z)}{Q(z)}$ rational function

## 7.8 Auxiliary (characteristic) equation of a recurrence relation -

Let $a_0, a_1, \ldots a_n$ be a sequence satisfying linear recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$

Then $x^k = c_1 x^{k-1} + c_2 x^{k-2} + \ldots + c_k$

characteristic equation

Roots of characteristic equations ($r$) are solutions ($a_n = r^n$ is a solution.)

Recurrence relations can be homogeneous or non-homogeneous depending on coefficients and whether there is a g(n) coefficient.

General solution of inhomogeneous =
general solution of recurrence + particular solution

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# MATH 102 L10

## Recurrence Relation Example -

$a_n = 3a_{n-1} - 4a_{n-3}$

$a_0 = 0 \quad a_1 = 2 \quad a_2 = -1$

This is a recurrence order 3 with characteristic polynomial $x^3 - 3x^2 + 4$ $\quad x = -1$ is a root.

1 = simple root, 2 is a double root

General solution $= (-1)^n b_1 + b_2 2^n + b_3 n 2^n$

$0 = b_1 + b_2 \quad 2 = -b_1 + 2b_2 + 2b_3 \quad -1 = b_1 + 4b_2 + 8b_3$

$b_1 = -1 \quad b_2 = 1 \quad b_3 = -1/2$

$a_n = (-1)^{n+1} + 2^n - n2^{n-1}$

## Recurrence Relation inhomogenous Example -

$a_n = a_{n-1} + 2a_{n-2} + 3n$

$a_0 = 0 \quad a_1 = 3$

consider $a_n = a_{n-1} + 2a_{n-2}$ $\quad x^2 = x + 2,\ x = 2$ or $-1$

$a_n = (-1)^n b_1 + 2^n b_2$

Try: $a_n = d_0 + d_1 n$

$(d_0 + d_1 n) = d_0 + d_1(n-1) + 2(d_0 + d_1(n-2)) + 3n$

$0 = (2d_1 + 3)n + 2d_0 - 5d_1$

$d_0 = -15/4 \quad d_1 = -3/2$

$a_n = (-1)^n b_1 + 2^n b_2 - 3n/2 - 15/4$

Sub $a_0$ and $a_1$

$0 = b_1 + b_2 - 15/4$

$b_1 = -1/4 \quad b_2 = 4$

$a_n = (-1)^{n+1}/4 + 2^{n+1} - (3/2 n - 15/4)$

More examples L30.

## 7.19 Generating functions of inhomogeneous recurrence relations -

We want to reduce everything to the generating function $\frac{1}{1-z} = 1+z+\dots+z^n+\dots$ which is for $a_n = 1$.

**Example:**
Find in a closed form the generating function of $a_n = n$, $n = 1, 2, \dots$

$$A(z) = \sum_{n=1}^{\infty} n z^n$$

Set $B(z) = 1+z+z^2+\dots = \frac{1}{1-z}$

$B'(z) = 1+2z+\dots+nz^{n-1}+\dots$

$zB'(z) = z+2z^2+\dots = A(z)$

Therefore $A(z) = zB'(z) = z\left(\frac{1}{1-z}\right)' = \frac{z}{(1-z)^2}$

## 7.23 Recurrence Relations and differentials -

$\frac{d^2y}{dx^2} + y = 0$ Express $y$ as a power series,

$n$ increments

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$dy/dx = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$d^2y/dx^2 = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=0}^{\infty} c_n x^n = -\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = -\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$c_{n+2} = -\frac{1}{(n+2)(n+1)} c_n$$

Then simplify.

Example -

L31. $d^2y/dx^2 - 2x\, dy/dx + y = 0 \qquad y(0) = 1 \qquad dy/dx(0) = -1$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad dy/dx = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad d^2y/dx^2 = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 2 \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$(n+2)(n+1) c_{n+2} - (2n-1) c_n = 0$$

$$c_{n+2} = \frac{2n-1}{(n+2)(n+1)} c_n$$

$$c_{2n+2} = \frac{4n-1}{(2n+2)(2n+1)} c_{2n} = \frac{(4n-1)(4n-5)}{(2n+2)(2n+1)2n(2n-1)} c_{2n-2}$$

$$= \frac{(-1)\cdot 3 \cdot 7 \cdot 11 \cdot (4n-1)}{(2n+2)!} c_0$$

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1}$$

$$= -c_0 \sum_{n=0}^{\infty} \frac{3 \cdot 7 \cdot 11 \cdot (4n-5)}{(2n)!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

$y(0) = 1$ , $c_0 = -1$

$dy/dx(0) = -1$ , $c_1 = -1$

$$y = \sum_{n=0}^{\infty} \frac{3 \cdot 7 \cdot 11 \cdot (4n-5)}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$