8/4/2022

MATH 105 L01 + L02

Videos/Notes -

Matrices -

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$ Action of a matrix on a vector

$w = Mv$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Defines a transformation of the plane which sends $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

Bigger Matrices -

An $m \times n$ matrix has m rows, n columns.

An $m \times n$ matrix A gives us a map $\mathbb{R}^n \to \mathbb{R}^m$

Matrices only multiply if $(m \times n) \times (o \times p)$ if n and o are equal. e.g

$$A\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{pmatrix}$$ By turning each line left of matrix A.

e.g

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$ This is a rotation in the x-y plane. (around z-axis)

e.g

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$ Projection to x-y plane

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$ Inclusion of xy-plane in $\mathbb{R}^3$

e.g $$\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ y \end{pmatrix}$$

x-axis goes to $\begin{pmatrix} x \\ 2x \\ 0 \end{pmatrix}$ when $y=0$

y-axis goes to $\begin{pmatrix} y \\ 0 \\ y \end{pmatrix}$ when $x=0$

e.g a transformation
The line through O in the $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ direction is called the Kernel of our matrix for transformation :

$$\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x-z\\y-z\end{pmatrix} \quad \text{so} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

Matrix Multiplication -

$$A = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix} \quad B = \begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}$$ Both give us maps $\mathbb{R}^2 \to \mathbb{R}^2$

If we do B followed by A then we get AB :

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}B_{11}x + B_{12}y\\ B_{21}x + B_{22}y\end{pmatrix} =$$

$$= \begin{pmatrix}A_{11}(B_{11}x + B_{12}y) + A_{12}(B_{21}x + B_{22}y)\\ A_{21}(B_{11}x + B_{12}y) + A_{22}(B_{21}x + B_{22}y)\end{pmatrix}$$

$$= \begin{pmatrix}x(A_{11}B_{11} + A_{12}B_{21}) + y(A_{11}B_{12} + A_{12}B_{22})\\ x(A_{21}B_{11} + A_{22}B_{21}) + y(A_{21}B_{12} + A_{22}B_{22})\end{pmatrix}$$ RESULT of applying a matrix to a vector

The same as :

$$\begin{pmatrix}A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22}\\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$ Known as $AB\begin{pmatrix}x\\y\end{pmatrix}$

$$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$ Matrix Multiplication isn't commutative

Index Notation -

$$A = \begin{pmatrix}A_{11} & A_{12} & \dots & A_{1n}\\ A_{21} & & & \\ \dots & & & \\ A_{m1} & & & A_{mn}\end{pmatrix}$$ for $m \times n$ indices notation

$(AB)_{ij}$ = ij^{th} element of AB

$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{in}B_{nj}$ FORMULA

$$= \sum_{k=1}^{n} A_{ik}B_{kj}$$

Other Operations -

Matrix Exponentiation

$$\exp(A) = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

e.g $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

e.g $A = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} = -t^2 I$ $A^3 = -t^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$A^4 = -t^4$

$$\exp A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{t^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{t^3}{6}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= (\cos t) I + (\sin t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

6/5/2022

MATH 105 L03

Dot Products + Orthogonality -

Given $v, w \in \mathbb{R}^n$ what is the angle between them?

Definition: The dot product $v \cdot w = v_1w_1 + v_2w_2 + \dots v_nw_n$

when $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

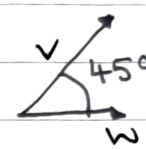
Theorem: $v \cdot w = |v||w|\cos\theta$ θ = angle between v and w

Example: $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $v \cdot w = 1 \times 0 + 0 \times 1 = 0$

$\theta = \pi/2$ rad $\cos \pi/2 = 0$

Orthogonal (at right angles)

Example: $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|v| = \sqrt{1+1} = \sqrt{2}$
$|w| = 1$



$= \pi/4$

$v \cdot w = 1 + 0 = 1$

$|v||w|\cos\theta = \sqrt{2} \times 1 \times \cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ so $\theta = \pi/4$

Dot Products and Transposition -

$v_1w_1 + \dots v_nw_n = (v_1 \ v_2 \dots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \xrightarrow{\text{transposition}} (v_1, v_2 \dots v_n) = v^T$ transpose of v

$v \cdot w = v^T w$

$(AB)^T = B^T A^T$

Dot Products and Orthogonal Matrices -

$n \times n$ matrix A is orthogonal if $A^TA = I$

$\Leftrightarrow$ all columns of A are orthogonal to one another and each has length 1 when considered as a vector

If $A^TA = I$ then geometric transformation A preserves dot products. In particular orthogonal matrices/transformations don't change length of vectors

Rotations -

Example -

rotating xy-plane

$$A = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

z axis fixed

Example -

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad Bu = u \text{ where } u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ -x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x = z$
$y = y$
$-x = z$
so $x = z = 0$
and y is free
$u = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ ie y axis

= rotation about y axis

Pick v orthogonal to $u = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$

e.g $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

angle$(v, Bv) = \theta$

$v \cdot Bv = |v||Bv| \cos\theta$

$Bv = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad v \cdot Bv = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 \quad \cos\theta = 0$

$\Rightarrow$ orthogonal

and rotation by $\pi/2$ or $3\pi/2$

Examples -

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad Cu = u \quad u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad Cu = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

$x = z \quad y = x \quad z = y \quad = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ direction

Pick v orthogonal

$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} v \quad Cv = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad |v| = \sqrt{2} \quad |Cv| = \sqrt{2}$

$v \cdot Cv = 2\cos\theta$

$\cos\theta = -1/2 \quad \theta = 2\pi/3$

5/5/2022

MATH 105 L04

Simultaneous Eqns and Row Operations -

$x - y = -1$
$x + y = 3$ is the same as $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 3 \end{array}\right]$ is called the augmented matrix and is shorthand notation for the above

Row Operations -

Type I: $R_i \rightarrow R_i + \lambda R_j$ $\quad \lambda \in \mathbb{R}$

e.g $x - y = -1$
$x + y = 3$ Replace Row 2 by Row 2 - Row 1

$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 3 \end{array}\right]$ $\quad R_2 \rightarrow R_2 + (-1)R_1$
$\quad R_2 \rightarrow R_2 - R_1$

$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 4 \end{array}\right]$

Type II $R_i \rightarrow \lambda R_i$ $\quad \lambda \in \mathbb{R}$ e.g multiplying/dividing a row

Type III $R_i \leftrightarrow R_j$

Swapping Rows
(reordering eqns)

Aim of using row operations is to try to put augmented matrix into the form $[I \mid b]$ ← vector of constants

It is not always possible to do this

Echelon Form -

$x - y = -1$
$x + y = 3$ $= \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 3 \end{array}\right]$

$x = 1$
$y = 2$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ SOLVED

Definition: Given a matrix and a row, the leading entry in this row is the left most non-zero entry left of the bar.

e.g $\left[\begin{array}{cccc|c} 0 & 5 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 7 & 1 & 3 & 4 & -1 \end{array}\right]$ NOT in echelon form.

Definition: A matrix is in echelon form if all zero rows are at the bottom and the leading entries move strictly to the right as you go down the rows. e.g $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e.g $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form

called echelle.

Reduced Echelon Form -

It's in echelon form, all leading entries are 1 and :

* in a column containing a leading entry there are no other non-zero entries

= column.

Example -

$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & -1 & 13 \end{array}\right]$

$w + y + z = 5$

$x + 2y - z = 13$

2 eqns 4 unknowns So use eqns to express dependent variables in terms of others (free variables)

$w = 5 - y - z$

$x = 13 - 2y + z$

w, x dependent

y, z free

w only appears in 1. x only in 2

Example -

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$w + 2x + z = 1$

$y + 8z = 2$

w, y dependent x, z free

$w = 1 - 2x - z$

$y = 2 - 8z$

Echelon Examples

$x + 2y + z = 5$
$-x + y + 2z = 1$
$x \quad - z = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 3 & 3 & 6 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 3 & 3 & 6 \\ 0 & -2 & -2 & -4 \end{array}\right]$$

$R_2 \rightarrow \frac{1}{3}R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -4 \end{array}\right]$$

$R_3 \rightarrow -\frac{1}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Echelon Form

Need to get reduced echelon and get rid of 2

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $R_1 \rightarrow R_1 - 2R_2$

Reduced Echelon Form

$x - z = 1 \qquad y + z = 2 \qquad 0 = 0$

$x = 1 + z \qquad y = 2 - z \qquad z$ free x, y dependent

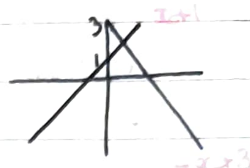
15/2022

MATH 105 L05

Geometric Viewpoint Simultaneous Eqns -

$x - y = -1$ $(x=1, y=2)$ $y = x+1$

$x + y = 3$ $y = -x+3$



$L = \{a_1x + a_2y = b\} \longrightarrow$ Line in $\mathbb{R}^2$

slope $= -a_1/a_2$ $y = \frac{-a_1}{a_2}x + \frac{b}{a_2}$

intercept $= b/a_2$

$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is orthogonal to L.

$x - y + z = 1$ $\xrightarrow{\text{each gives}}$ a plane in $\mathbb{R}^3$

$x - z = 0$

$x = z$ x free

$2x - y = 1 \Rightarrow y = 2x - 1$

line of intersection is $\begin{pmatrix} x \\ 2x-1 \\ x \end{pmatrix}$

$a_1x + a_2y + a_3z = b \longrightarrow$ Plane in $\mathbb{R}^3$

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is orthogonal to this plane $z = b/a_3$ is z-intercept

For 4 variables each equation $a_1w + a_2x + a_3y + a_4z = b$ defines a hyperplane in $\mathbb{R}^4$

If $Av = b$ has k free variables in it's general solution then the subspace of solutions is k-dimensional.

Subspaces -

Definition: A subset $V \in \mathbb{R}^n$ is a linear subspace if

- $\forall v, w \in V$, $v + w \in V$ "closed under addition"
- $\forall v \in V, \lambda \in \mathbb{R}$, $\lambda v \in V$ "closed under rescaling"

Non-empty Linear subspaces always contain $0 \in \mathbb{R}^n$ (rescale any v by $\lambda = 0$)

$x - y = -1$
$x + y = 3$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a solution $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is.

Affine Subspaces -

$V \subseteq \mathbb{R}^n$ is an affine subspace if $\exists w \in \mathbb{R}^n$ and a linear subspace, $U \in \mathbb{R}^n$ such that $V = \{w + u : u \in U\}$ $= w + U$.

Lemma: Given a system of simultaneous linear equations $Av = b$ The set of solutions is an affine subspace of $\mathbb{R}^n$. It is a linear subspace iff $b = 0$.

Lemma: If $V, W \subseteq \mathbb{R}^n$ linear subspaces then so is $V \cap W$.

Lemma: Affine subspace is linear $\Leftrightarrow$ it contains the origin

Inverses -

Theorem: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2 by 2 matrix with $ad - bc \neq 0$ then the matrix $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is an inverse.

Definition: Let A be an $n \times n$ (square) matrix. We say that A is invertible if $\exists B$ such that $AB = BA = I$. If such a B exists then it's unique and we call it A^{-1}. Suppose B and C are both inverses for A.

$B = BAC = C$

Lemma: If A and B are invertible matrices with inverses A^{-1}, B^{-1} then AB is invertible with inverse $B^{-1}A^{-1}$

__Theorem__ : Given an $n \times n$ matrix A, form the augmented matrix $(A \mid I)$. Use row operations to put the left hand side into reduced echelon form $\rightarrow (B \mid C)$.

If $B = I$ then A is invertible with inverse C

If $B \neq I$ then A is not invertible

$(A \mid I) \rightarrow (I \mid A^{-1})$

__Example__ :

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \quad R2 \longmapsto R2 - R1$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \quad R2 \longmapsto \tfrac{1}{2} R2$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\tfrac{1}{2} & \tfrac{1}{2} \end{array}\right] \quad R1 \longmapsto R1 + R2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & 1 & -\tfrac{1}{2} & \tfrac{1}{2} \end{array}\right] \qquad A^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

A^{-1}

__Example__ :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \quad R3 \longmapsto R3 - R1$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \quad R3 \longmapsto R3 - 3R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -15 & -1 & -3 & 1 \end{array}\right] \quad R3 \longmapsto -\tfrac{1}{15} R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & \tfrac{1}{15} & \tfrac{1}{5} & -\tfrac{1}{15} \end{array}\right] \quad R2 \longmapsto R2 - 4R3 \text{ and } R1 \longmapsto R1 - 3R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \tfrac{4}{5} & -\tfrac{3}{5} & \tfrac{1}{5} \\ 0 & 1 & 0 & -\tfrac{4}{15} & \tfrac{1}{5} & \tfrac{4}{15} \\ 0 & 0 & 1 & \tfrac{1}{15} & \tfrac{1}{5} & -\tfrac{1}{15} \end{array}\right]$$

Further operations ....

until $A^{-1} = \begin{bmatrix} \tfrac{4}{3} & -1 & -\tfrac{1}{3} \\ -\tfrac{4}{15} & \tfrac{1}{5} & \tfrac{4}{15} \\ \tfrac{1}{15} & \tfrac{1}{5} & -\tfrac{1}{15} \end{bmatrix}$

1/5/2022

MATH 105 L06

Elementary Matrices -

Definition: Let i, j be numbers between 1 and n ($i \neq j$).

Let $\lambda \in \mathbb{R}$

$$E_{ij}(\lambda) = \begin{array}{cc} & \text{column } j \\ & \begin{bmatrix} 1 & 0 & 0 \\ & 1 \quad \lambda & \\ & & \ddots \end{bmatrix} \end{array} \text{row } i$$

e.g. $E_{32}(-5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$

Lemma: Let A and $E_{ij}(\lambda)$ be $n \times n$ matrices. Then $E_{ij}(\lambda)A$ is the matrix obtained from A by doing

$R_i \longmapsto R_i + \lambda R_j$

Definition: $E_i(\lambda) = \begin{array}{cc} & i \\ i & \begin{bmatrix} 1 & & \\ & \lambda & \\ & & \ddots \end{bmatrix} \end{array}$ Elementary Matrix of Type II

Lemma: $E_i(\lambda)A$ is obtained from A by $R_i \longmapsto \lambda R_i$

Lemma: $E_{ij}(\lambda)$ is invertible with inverse $E_{ij}(-\lambda)$

$E_i(\lambda)$ is invertible with inverse $E_i(1/\lambda)$

Theorem: If A is a matrix and you put $(A \mid I)$ into reduced echelon form $(B \mid C)$ then A is invertible iff $B = I$. If $B = I$ then $C = A^{-1}$

Corollary: A product of elementary matrices is invertible and conversely any invertible matrix is a product of elementary matrices.

Determinants -

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse iff $ad - bc \neq 0$.

$n!$ choices in general. At a permutation of $1, 2, \dots n$.
Sign of choice is $+1$ if permutation is even, -1 otherwise
For $n \times n$:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

$\operatorname{sgn} = \pm 1$

Theorem:

Any permutation can be written as either an odd number of transpositions or an even number but not both

Properties of Determinants

Lemma: If two rows coincide then $\det(A) = 0$

e.g. $\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix} = 0$

Lemma: If A' is obtained from A by $R_i \mapsto R_i + \lambda R_j$ then $\det A' = \det A$

Lemma: If A' is obtained from A by switching two rows $R_i \longleftrightarrow R_j$ then $\det A' = -\det A$

Lemma: If A' is obtained from A $R_i \mapsto \lambda R_i$ then $\det A' = \lambda \det A$

Determinant Examples -

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MATH 105 L07

Further Properties of Determinants -

Theorem: An invertible $n \times n$ matrix $\iff \det A \neq 0$.

Theorem: $\det(AB) = \det A \det B$

Cofactor Expansion -

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\left.\begin{matrix} A_{11}A_{22}A_{33} \\ -A_{11}A_{23}A_{32} \end{matrix}\right\} A_{11}(A_{22}A_{33} - A_{23}A_{32})$$

$$\left.\begin{matrix} -A_{12}A_{21}A_{33} \\ +A_{12}A_{23}A_{31} \end{matrix}\right\} A_{12}(A_{23}A_{31} - A_{21}A_{33})$$

$$\left.\begin{matrix} +A_{13}A_{21}A_{32} \\ -A_{13}A_{22}A_{31} \end{matrix}\right\} A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$\det A = A_{11}\det C_{11} - A_{12}\det C_{12} + A_{13}\det C_{13} \dots \pm A_{1n}\det C_{1n}$

C_{ij} = submatrix obtained by removing row i and column j.

Geometry of Determinants -

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Let S be the unit square in $\mathbb{R}^2$ and $A(S)$ be the image of S under A.

$|\det A| = \text{area}(A(S))$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A(S)$ has area 1 so $\det A = 1$

Theorem: If A is an $n\times n$ matrix and S is the unit cube in $\mathbb{R}^n$ then $|\det A| = \text{volume}(A(S))$

$A(S)$ is parallelepiped

Theorem: Let $a_1, \dots a_n$ be vectors in $\mathbb{R}^n$. Consider the simplex with vertices at $0, a_1, \dots a_n$.

Then volume of this simplex $= \frac{1}{n!}\det A$.

Remark: $|\det A|$ is scale factor for volumes under A.

19/5/2022

MATH 105 L08

Eigenvectors and Eigenvalues -

Definition: A $n \times n$ matrix $v \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$

$Av = \lambda v$ (for some vector v) If $Av = \lambda v$ Then we say v is an eigenvector of A with eigenvalue λ.

$v \neq 0$ (evec) (eval)

$Av = \lambda v$ is a family of equations one for each λ.

The λs for which $Av = \lambda v$ has a solution $v \neq 0$ are called the eigenvalues of A.

Example: $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and told $\lambda = 1$ is an eigenvalue of A

$$Av = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x \end{bmatrix} = \lambda v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$2x - y = x$
$x = y$ $\Rightarrow$ $x = y$ so solutions (eigenvectors) are all vectors of the form $\begin{bmatrix} x \\ x \end{bmatrix}$.

For fixed λ, $Av = \lambda v$ is a system of simultaneous equations equivalent to $(A - \lambda I)v = 0$

Finding Eigenvalues -

$Av = \lambda v$

Theorem: $Av = \lambda v$ has a solution $v \neq 0$ iff λ is a root of the characteristic polynomial of A: $\det(A - tI) = 0$

variable

polynomial in t of degree n

size of A.

Eigenexamples -

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(A - tI) = \det\begin{bmatrix} 2-t & 1 \\ 1 & 1-t \end{bmatrix}$

$= (2-t)(1-t) - 1 = t^2 - 3$ roots: $\frac{3 \pm \sqrt{9-4}}{2}$

$= \frac{3 \pm \sqrt{5}}{2}$ so to find eigenvectors: solve:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + y \\ x + y \end{bmatrix} = \begin{bmatrix} \frac{3 \pm \sqrt{5}}{2} x \\ \frac{3 \pm \sqrt{5}}{2} y \end{bmatrix} \Rightarrow \begin{matrix} y = \frac{-1 \pm \sqrt{5}}{2} x \\ x = \frac{1 \pm \sqrt{5}}{2} y \end{matrix}$$

eigenvalues: $\begin{bmatrix} x \\ \frac{-1 \pm \sqrt{5}}{2} x \end{bmatrix}$ for $\lambda = \frac{3 \pm \sqrt{5}}{2}$

Eigenspaces -

If v is an evec of A with eval λ (λ-evec) then so is kv for any $k \in \mathbb{C}$.

$$A(kv) = kAv = k\lambda v = \lambda(kv)$$

Can get away with writing "the evec for eval λ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$"

e.g. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(I - \lambda I) = (1-\lambda)^2 \Rightarrow \lambda = 1$ is the eval.

$Iv = v$ for any v so the general evec is $\begin{bmatrix} x \\ y \end{bmatrix}$ and therefore an eigenplane

Theorem: The set of evecs with eval λ form a complex subspace of $\mathbb{C}^n$

2/5/2022

MATH 105 L09

Eigenapplications: Differential Equations -

$x(t)$, $y(t)$

$\dot{x} = ax + by$ $\quad$ a, b, c, d constants $\quad$ $\dot{x} = dx/dt$

$\dot{y} = cx + dy$

$v(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ $\quad$ $\dot{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ $\quad$ $\dot{v} = Av$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let λ_1, λ_2 be evals of A and let u_1, u_2 be evecs.

Write v as $\alpha u_1 + \beta u_2$

e.g $v = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad$ α, β components of v in the eigendirections

$\dot{v} = \dot{\alpha} u_1 + \dot{\beta} u_2 = Av = \alpha A u_1 + \beta A u_2$

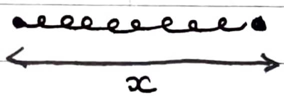
$= \alpha \lambda_1 u_1 + \beta \lambda_2 u_2$

<u>Components</u>: $\left.\begin{aligned} u_1 &: \dot{\alpha} = \alpha \lambda_1 \\ u_2 &: \dot{\beta} = \beta \lambda_2 \end{aligned}\right\}$ decoupled

Example -

$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}\right\}$ $\quad$ $\ddot{x} = \dot{y} = -x$ $\quad$ $\ddot{x} = -x$ "Simple harmonic motion"

$(y = \dot{x})$ $\quad$ $\dot{y} = \ddot{x} = \dots$



$m\ddot{x} = -kx$

$(m = k = 1)$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\det\begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix} = t^2 + 1 \Rightarrow \lambda_1 = i \quad \lambda_2 = -i$

evecs: $\lambda_1 = i$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ix \\ iy \end{bmatrix}$ $\quad$ $\begin{aligned} y &= ix \\ -x &= iy \end{aligned}$ $\quad$ $u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda_2 = -i$, $\quad u_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha u_1 + \beta u_2 = \alpha\begin{bmatrix} 1 \\ i \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ \alpha i - \beta i \end{bmatrix}$

$\alpha + \beta = x \quad i(\alpha - \beta) = y$

$\Rightarrow \alpha = \dfrac{x - iy}{2} \qquad \beta = \dfrac{x + iy}{2}$

$\dot{\alpha} = \lambda_1 \alpha$ $\Rightarrow$ $\dot{\alpha} = i\alpha$ $\qquad \dfrac{\dot{\alpha}}{\alpha} = \dfrac{d}{dt}(\log \alpha) = i$

$\dot{\beta} = \lambda_2 \beta$ $\qquad\qquad \dot{\beta} = -i\beta$

$\Rightarrow \log \alpha = it + \text{constant}$

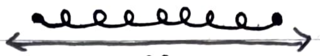
$\Rightarrow \alpha = C_1 e^{it}$

and $\beta = C_2 e^{-it}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow \begin{aligned} x &= C_1 e^{it} + C_2 e^{-it} \\ y &= iC_1 e^{it} - iC_2 e^{-it} \end{aligned}$$

$\dot{x} = y$

$\dot{y} = -x$



Pick $x(0) = 1 \quad = C_1 + C_2$

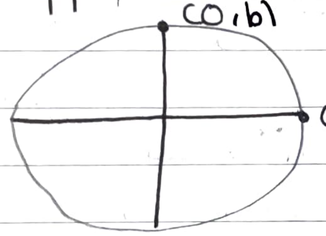
$y(0) = 0 \quad = iC_1 - iC_2 \quad \Rightarrow C_1 = C_2$ and $C_1 = C_2 = 1/2$

$\Rightarrow x(t) = 1/2\,(e^{it} + e^{-it}) = \cos(t)$

$y(t) = i/2\,(e^{it} - e^{-it}) = -\sin(t)$

directed inwards (origin →)

Eigenapplications : Ellipsoids -



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The longest and shortest diameter are of lengths 2a or 2b respectively

a = semimajor axis $\qquad$ b = semiminor axis $(a > b)$

General Equation of an ellipse (centre of mass at O)

$$Ax^2 + Bxy + Cy^2 = 1$$

Definition:

$Ax^2 + Bxy + Cy^2$ is called positive definite if it is always positive for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

Ellipse : $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : Ax^2 + Bxy + Cy^2 = 1 \right\}$ where A, B, C give a positive definite quadratic form.

Theorem: If you pick co-ordinates so that the new x and y axis point along the eigenvectors of $\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$ ~~then~~ and so that:

u_1 has length 1 and sits at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in new co-ords and u_2 sits at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then the equation of the ellipse becomes:

$$\lambda_1 x^2 + \lambda_2 y^2 = 1$$ where λ_1, λ_2 are the eigenvalues

$v = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad M = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$ symmetric matrix

$$v^T M v = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} Ax + By/2 \\ Bx/2 + Cy \end{bmatrix} = Ax^2 + \frac{Bxy}{2} + \frac{Bxy}{2} + Cy^2$$

$$= Ax^2 + Bxy + Cy^2$$

Lemma: If λ_1, λ_2 are distinct eigenvalues of M then u_1, u_2 (evecs) are orthogonal. ($M^T = M$)

Example: $\frac{3}{2}(x^2 + y^2) - xy = 1$ $\quad M = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$

$$\det(M - tI) = \det \begin{bmatrix} 3/2 - t & -1/2 \\ -1/2 & 3/2 - t \end{bmatrix} = (3/2 - t)^2 - 1/4$$
$$= t^2 - 3t + 2$$

roots $\frac{3 \pm \sqrt{9-8}}{2} = 1$ or 2

evecs: $\lambda = 1$ $\quad \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \frac{3}{2}x - \frac{1}{2}y = x \Rightarrow x = y$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$ $\quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This tells us semimajor/minor axis are $\underset{\text{major}}{1}$ and $\underset{\text{minor}}{\frac{1}{\sqrt{2}}}$

u_1 points in direction of semimajor axis $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 45° rotation clockwise

so

same for higher dimension ellipsoids in dim 3

Eigenapplications : Dynamics -

$v \longrightarrow Av \longrightarrow A^2v \longrightarrow A^3v \; \dots\dots \; A^n v$

what happens as $n \to \infty$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} \qquad \begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_2 &= 2 \\ F_3 &= 3 \end{aligned}$$

$$A \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix}$$

Theorem :

$$\lim_{n \to \infty} \left[\frac{F_{n+1}}{F_n} \right] = \frac{1+\sqrt{5}}{2}$$

Proof :

Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $\alpha u_1 + \beta u_2$

where u_1, u_2 are λ_1-, λ_2- eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^n(\alpha u_1 + \beta u_2) = \alpha A^n u_1 + \beta A^n u_2$$

$Au_1 = \lambda_1 u_1 \quad A^2 u_1 = \lambda_1 A u_1 = \lambda_1^2 u_1 \quad A^n u_1 = \lambda_1^n u_1$

$A^n u_2 = \lambda_2^n u_2$

$$\Rightarrow A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha \lambda_1^n u_1 + \beta \lambda_2^n u_2$$

$$= \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

Claim : $\lambda_1 = 1.618\dots = \frac{1+\sqrt{5}}{2} \qquad \lambda_1^n \to \infty$ as $n \to \infty$

$\lambda_2 = -0.618\dots = \frac{1-\sqrt{5}}{2} \qquad \lambda_2^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} (\text{slope of } \alpha \lambda_1^n u_1 + \beta \underbrace{\lambda_2^n u_2}_{\to 0})$$

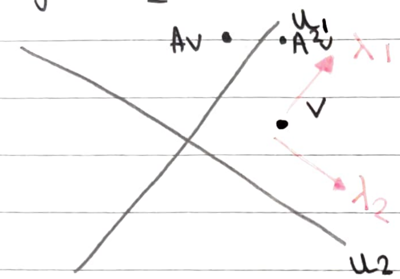
$$= \lim_{n\to\infty} \text{slope of } u_1 = \text{slope of } u_1$$

Eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$?

$\det \begin{bmatrix} -t & 1 \\ 1 & 1-t \end{bmatrix} = t^2 - t + 1$ roots: $\dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Eigenvectors: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1 \pm \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x+y \end{bmatrix}$

$y = \dfrac{1 \pm \sqrt{5}}{2} x$ so slope $= \dfrac{1 \pm \sqrt{5}}{2}$ = golden ratio when we take + sign $\square$

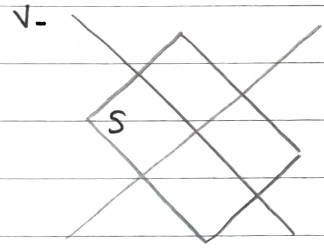


$v = \alpha u_1 + \beta u_2$

$Av = \alpha \lambda_1 u_1 + \beta \lambda_2 u_2$

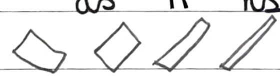
as $A^n v$ as n grows $A^n v$ gets more parallel to u_1 but nses like ${}^{\circ}/_{\circ}$ etc.

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\lambda \pm$ evals: $= \dfrac{3 \pm \sqrt{5}}{2}$ $\lambda_+ > 1$, $0 < \lambda_- < 1$



square called S

as A^n nses e.g



and gets closer to v+

v+ stretches
v- shrinks

Called Arnold Cat Map

where using a grid when starting at some ^ at a square grid chosen point the original image will be recreated as it once was. Phenomenon: Poincaré recurrence

a phenomenon in dynamical systems.

26/5/2022

MATH 105 L10

Linear Maps -

A map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ is linear if $\exists$ an $m \times n$ matrix A such that $f(v) = Av \quad \forall v \in \mathbb{R}^n$.

Example :

maps back to itself, and is differentiable

$\{\text{polynomials of degree} \le n\} \circlearrowleft \frac{d}{dx}$ linear map

$P(x) = a_n x^n + \dots + a_1 x + a_0 \longrightarrow v_p = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\frac{dP}{dx} = n a_n x^{n-1} + \dots 2a_2 x + a_1 \longrightarrow \begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ na_n \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ 0 & 0 & 0 & 3 \dots & 0 \\ & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 \dots n \end{bmatrix}$$

Matrix D

$Dv_p = v_{\frac{dP}{dx}}$ so differentiation can be thought of as a linear map

A map is linear if $f(v+w) = f(v) + f(w)$ and $\forall v, w$

$f(\lambda v) = \lambda f(v)$ $\quad \forall v, \lambda \in \mathbb{R}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{R}^n$

$$\left.\begin{aligned} \text{so } \frac{d}{dx}(P+Q) &= \frac{dP}{dx} + \frac{dQ}{dx} \\ \frac{d}{dx}(\lambda P) &= \lambda \frac{dP}{dx} \end{aligned}\right\} \frac{d}{dx} \text{ is linear}$$

Lemma / theorem: f is linear $\left(\begin{aligned} f(v+w) &= f(v) + f(w) \\ f(\lambda v) &= \lambda f(v) \end{aligned} \right) \iff \exists$ matrix A such that $f(v) = Av$

Kernels -

f is a linear map if	V is a linear subspace if
1. $f(v+w) = f(v) + f(w)$	$v, w \in V \Rightarrow v+w \in V$
2. $f(\lambda v) = \lambda f(v)$	$v \in V \Rightarrow \lambda v \in V \quad (\forall \lambda \in \mathbb{R})$

Given a linear map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ we will associate to it two subspaces.

$\ker(f) \subseteq \mathbb{R}^n$ and $\operatorname{im}(f) \subseteq \mathbb{R}^m$

Kernel $Ker(f) := \{ v \in \mathbb{R}^n : f(v) = 0 \}$

$f(v) = Av$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ vertical projection to xy-plane

$Ker(f) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ Kernel = Light ray through 0 $= \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$

Lemma: $Ker(f)$ is a subspace

Remarks:

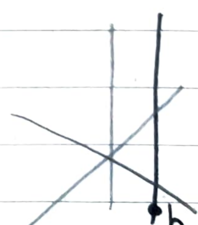
1. $0 \in Ker(f)$ because $f(0) = 0$
2. If f is invertible then $Ker f = \{0\}$
   $f(v) = 0 \Rightarrow v = f^{-1}(0) = 0$
3. The "Kernel" in a nut is the little bit in the middle that's left when you strip away the husk.
   If $f(v) = Av$ then we can think of $Ker(f)$ as the space of solutions to the simultaneous equations $Av = 0$ which is the intersection of the hyperplanes
   $$A_{11}v_1 + \dots + A_{1n}v_n = 0, \dots \quad A_{m1}v_1 + \dots + A_{mn}v_n = 0$$
   The bit left over after intersecting all the hyperplanes.

Lemma:

Consider the simultaneous equations $\overset{m \times n}{A}v = b \in \mathbb{R}^m$ and let $f(v) = Av$. The space of solutions if non-empty is a translate of $Ker(f)$

Example:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $Av = b$ $b \in xy$ plane



(Vid 18) Proof: We saw if v_0 is a solution then the set of all solutions is the affine subspace $v_0 + U$ $U = \{v : Av = 0\}$
$= \ker(f) = \ker(A)$ $\Box$.

In particular we see if $Av = b$ has a solution then it has a k-dimensional space of solutions where k is the dimension of $\ker(f)$ $\dim \ker(f)$.

Here k = number of free variables of the matrix when put into reduced echelon form

This is The Nullity of the Matrix (Dim of kernel)

Images -

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$\ker(f) \subseteq \mathbb{R}^n$, $\operatorname{im}(f) \subseteq \mathbb{R}^m$

Image of f $\operatorname{im}(f) = \{b \in \mathbb{R}^m : b = f(v) \text{ for some } v \in \mathbb{R}^n\}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \operatorname{im}(f) = xy\text{-plane}$$

$$\operatorname{im}(f) = \left\{ b : b = f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \right\}$$

Example:

$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$ defines a linear map $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x \\ y \end{bmatrix}$$

Remarks:

1) $0 \in \operatorname{im}(f)$ because $f(0) = 0$
2) If f is invertible then $\operatorname{im}(f) = \mathbb{R}^m$ because for any $b \in \mathbb{R}^m$ $f^{-1}(b)$ satisfies $f(f^{-1}(b)) = b$

Lemma: $\operatorname{Im}(f)$ is a subspace

Lemma: $Av = b$ has a solution iff $b \in \operatorname{im}(f)$
$(f(v) = Av)$

Definition: The rank of A is defined as the dimension of the image (A) similar to nullity

Theorem: A is an $m \times n$ matrix $(f: \mathbb{R}^n \to \mathbb{R}^m)$ then rank A + nullity A = n (number of columns)

Example:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 0 (image is 0-dimensional)
nullity = 3 (kernel is 3-dimensional)
$0 + 3 = 3$. $(n = 3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 1 (image is 1-dimensional (x))
nullity = 2 (free variables)
$1 + 2 = 3$

nullity = number of free variables in reduced echelon form
rank = number of leading entries in reduced echelon form

Vector Spaces – (Taste of things to come in Linear Algebra)

A $\boxed{\text{vector space}}$ is a set V with operations

add : $V \times V \to V$ $\quad (v, w) \mapsto v + w$

rescaling : $\mathbb{R} \times V \to V$ $\quad (\lambda, v) \mapsto \lambda v$

and an element $0 \in V$ such that

$u + (v + w) = (u + v) + w \qquad v + w = w + v$

$v = 0 + v = v + 0 \qquad v + (-v) = 0$

$\lambda(\mu v) = (\lambda\mu)v \qquad (\lambda + \mu)v = \lambda v + \mu v$

Example:

$V = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$

$(f+g)(x) = f(x) + g(x) \quad (\lambda f)(x) = \lambda f(x)$

Subspace $\mathcal{C}^0(\mathbb{R}) \subseteq V$ continuous functions

$\mathcal{C}$ = subspace $\qquad \mathcal{C}^1(\mathbb{R}) \subseteq C^0(\mathbb{R})$ once continuously differentiable

etc, nested subspaces $\mathcal{C}^3(\mathbb{R}) \subseteq \mathcal{C}^2(\mathbb{R}) \subseteq \mathcal{C}^1(\mathbb{R})$

$\frac{d}{dx} : \mathcal{C}(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ is a linear map

$f \longmapsto \frac{df}{dx}$

What is the Kernel of $\frac{d}{dx}$?

$\frac{df}{dx} = 0 \Rightarrow f$ is constant

$Ker(\frac{d}{dx}) \subseteq \mathcal{C}^1(\mathbb{R})$

one dimensional subspace of constant functions

$\frac{df}{dx} = \lambda f$ if f is a λ eigenfunction

$f(x) = Ce^{\lambda x}$ C = constant

$\frac{d^2 f}{dx^2} = \lambda f$ $\quad A\cos(x\sqrt{-\lambda}) + B\sin(x\sqrt{-\lambda})$

2 dimensional eigenspace

Example :

$x \in \mathbb{C}$ is algebraic if $\exists$ polynomial P with rational coefficients such that $P(x) = 0$

e.g $\sqrt{2}$ is algebraic $x^2 = 2$

i is algebraic $x^2 = -1$

The set of all algebraic numbers is written $\bar{\mathbb{Q}}$

Claim : $\bar{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space

$x, y \in \bar{\mathbb{Q}} \Rightarrow x + y \in \bar{\mathbb{Q}}$

$x \in \bar{\mathbb{Q}}, \lambda \in \mathbb{Q} \Rightarrow \lambda x \in \bar{\mathbb{Q}}$

If $x \in \bar{\mathbb{Q}}$ then $\exists P(z) = a_n z^n + \ldots + a_0$ with $a_k \in \mathbb{Q}$ such that $P(x) = 0$

$\lambda \in \mathbb{Q}$ $\quad \frac{a_n}{\lambda^n} z^n + \frac{a_{n-1}}{\lambda^{n-1}} z^{n-1} + \ldots + a_0$ is a polynomial with rational coefficients

and $R(\lambda x) = P(x) = 0$

In fact $x, y \in \bar{\mathbb{Q}} \Rightarrow xy \in \bar{\mathbb{Q}}$

$Gal(\bar{\mathbb{Q}} : \mathbb{Q}) = \{$ invertible linear maps $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}} \}$

Group = Gal

Galoire Theory