Lecture 19

Matrix Inverse and Transpose

- **Matrix Inverse and Transpose**

- The *inverse* of a square, nonsingular matrix [A] is that matrix which, when multiplied by [A] yields the identity matrix. The inverse of matrix [A] is denoted by $[A]^{-1}$ in mathematics. In MATLAB, the inverse of [A] is calculated by inv(A))

  $$[A]\,[A]^{-1} = [A]^{-1}[A] = [I] \quad \text{(in mathematics)}$$

  (A matrix is singular if its determinant is 0. A singular matrix does not have the inverse.)

- The transpose of a matrix involves transforming its rows into columns and its columns into rows,

  $$[A]^T \Rightarrow (a_{ij})^T = a_{ji} \quad \text{(in mathematics)}$$

  A' (in MATLAB)

  Example:
  ```
  >> vec = [1, 2, 3];
  >> vecnew = vec'
  >> vecnew = [1, 2, 3]'
  ```

Simultaneous Linear Equations

➢ Simultaneous Linear Equations

- A system of simultaneous linear equations can be written in matrix notation as:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\quad\Rightarrow\quad
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}
=
\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}
$$

a_{ij}

$$[A]\{x\} = \{b\}$$

- [A] => coefficient matrix
- {b} => solution vector
- {x} => vector of unknowns

Simultaneous Linear Equations

- **Solving with MATLAB**

- MATLAB provides two direct ways to solve systems of linear algebraic equations [A]{x} = {b}
  - ✓ 1st way: Left-division

    x = A\b (left division, **NOT** right division)

Example: Find solution to the following system of equations

$0.3\,x_1 + 0.52\,x_2 + x_3 = -0.01$

$0.5x_1 + x_2 + 1.9x_3 = 0.67$

$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$

$$A = \begin{pmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}, b = \begin{pmatrix} -0.01 \\ 0.67 \\ -0.44 \end{pmatrix}$$

```
>> A = [0.3  0.52  1;  0.5  1  1.9;  0.1  0.3  0.5];
>> b = [-0.01;  0.67;  -0.44];
>> x = A\b
```

Solving with MATLAB

- **Solving with MATLAB**

- MATLAB provides two direct ways to solve systems of linear algebraic equations $[A]\{x\} = \{b\}$
  - ✓ 2nd way: Matrix inverse

$$[A]\{x\} = \{b\}$$

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{b\} \qquad ([A]^{-1}[A] = [I])$$

$$[I]\{x\} = [A]^{-1}\{b\} \qquad ([I]\{x\} = \{x\})$$

$$\{x\} = [A]^{-1}\{b\}$$

In MATLAB, We use: `x = inv(A)*b`

`inv` is a built-in function of MATLAB. It calculates the inverse of a matrix.

Example: `>> x=inv(A)*b`

Solving Small Numbers of Equations

- **Solving Small Numbers of Equations**

- There are many ways to solve a system of linear equations:
  - ✓ Cramer’s rule
  - ✓ Method of elimination
  - ✓ Other computer methods

Cramer's Rule

➢ **Cramer's Rule**

- Cramer's Rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D (which is the determinant of the coefficient matrix) and with the numerator obtained from D by replacing the column of coefficients of the unknown in the equation by the constants $b_1, b_2, \ldots, b_n$
- Application of method

Find solution to the following system of equations

$$0.3\,x_1 + 0.52\,x_2 + x_3 = -0.01$$

$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$

$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Cramer's Rule

- Find the coefficient matrix, and solution vector:

$$A = \begin{pmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}, b = \begin{pmatrix} -0.01 \\ 0.67 \\ -0.44 \end{pmatrix}$$

- Find the determinant D of the coefficient matrix

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = 0.3 \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.52 \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1 \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = -0.0022$$

The determinant is calculated by det(A) in MATLAB.

- To solve for x_2, find determinant D_2 by replacing D's second column with the vector b

$$D_2 = \begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix} = 0.3 \begin{vmatrix} 0.67 & 1.9 \\ -0.44 & 0.5 \end{vmatrix} - 0.01 \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1 \begin{vmatrix} 0.5 & 0.67 \\ 0.1 & -0.44 \end{vmatrix} = 0.0649$$

- Divide

$$x_2 = \frac{D_2}{D} = \frac{0.0649}{-0.0022} = -29.5$$

Cramer’s Rule

```
clc
clear

A = [0.3 0.52 1; 0.5 1 1.9; 0.1 0.3 0.5];
b = [-0.01  0.67  -0.44]’;
;
dD = det(A);          % calculate the denominator D

% to find x1
C = A;                % create a new coefficient matrix
C(:,1) = b;           % replace 1st column with  b
nD = det(C);          % calculate the numerator
x1 = nD/dD;           % calculate the 1st root

% to find x2
C = A;                % create a new coefficient matrix
C(:,2) = b;           % replace 2nd column with b
nD = det(C);          % calculate the numerator
x2 = nD/dD;           % calculate the 2nd root

% to find x3
C = A;                % create a new coefficient matrix
C(:,3) = b;           % replace the 3rd column with b
nD = det(C);          % calculate the numerator
x3 = nD/dD;           % calculate the 3rd root

[x1 x2 x3]
```

- **Cramer’s Rule**

- Cramer’s Rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D (which is the determinant of the coefficient matrix) and with the numerator obtained from D by replacing the column of coefficients of the unknown in equation by the constants b_1, b_2, …, b_n

Cramer's Rule (Any Number of Equations)

```
clc
clear

A = [0.3 0.52 1; 0.5 1 1.9; 0.1 0.3 0.5];
b = [-0.01  0.67  -0.44]';

n = length(b);

dD = det(A);

fprintf('\nThe solution to the %d*%d
system of equations is \n\n',n,n)

for i = 1:n
   C = A;
   C(:,i) = b;
   nD = det(C);
   x(i) = nD/dD;
   fprintf('x(%d) = %.3f\n',i,x(i))
end
```

➢ **Cramer's Rule**

- Cramer's Rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D (which is the determinant of the coefficient matrix) and with the numerator obtained from D by replacing the column of coefficients of the unknown in equation by the constants b_1, b_2, …, b_n

- **Homework on Canvas**