Lecture 16

Open Methods

- Open methods are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root.
- Fixed point iteration:
  - ✓ Given an equation

    $f(x)=0$
  - ✓ Rearrange this equation so that x is on the left side of the equation:

    $x = g(x)$
  - ✓ Make an initial guess for x value,

    $x = x_0$
  - ✓ use the iterative scheme with the recursive relation to estimate the new x,

    $x_{i+1} = g(x_i), \quad i = 0, 1, 2, \ldots$

Example:

$f(x) = x^2 - x - 2 = 0$

$-x = -x^2 + 2$

$x = x^2 - 2$

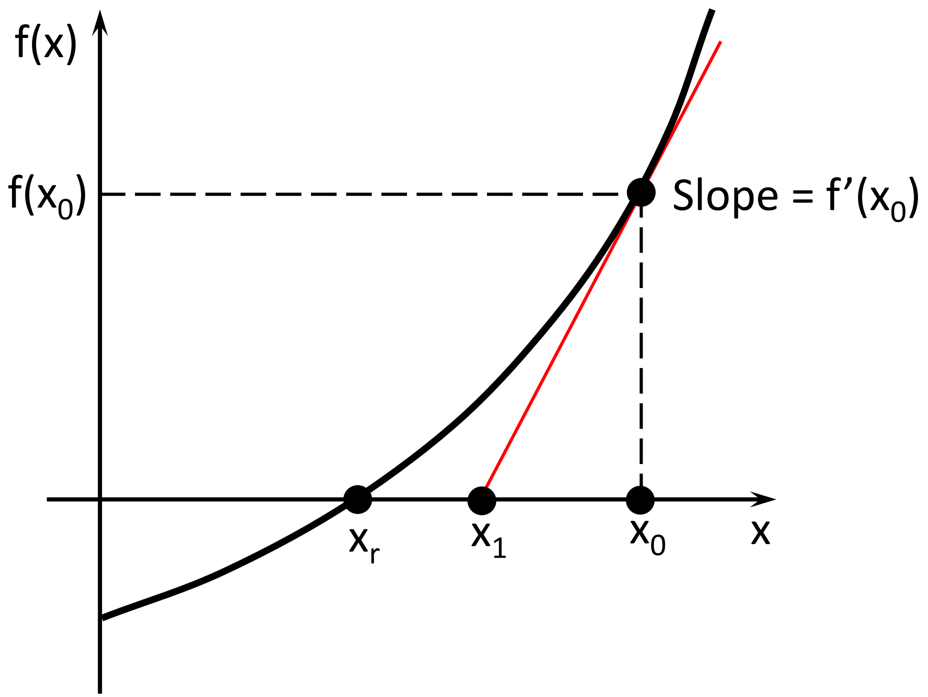
$g(x) = x^2 - 2$

$x = x_0$

$x_{i+1} = g(x_i) = x_i^2 - 2$

Newton-Raphson Method

- Newton-Raphson method: An open method to find the root of f(x) = 0.
- New-Raphson method is the most widely used method.
- The basic idea is that a continuous and differentiable function can be approximated by a straight line tangent to it.



Curve slope at x_0 is,

$$f'(x_0) = \frac{f(x_0) - 0}{(x_0 - x_1)}$$

Rearrange the expression,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton-Raphson Method

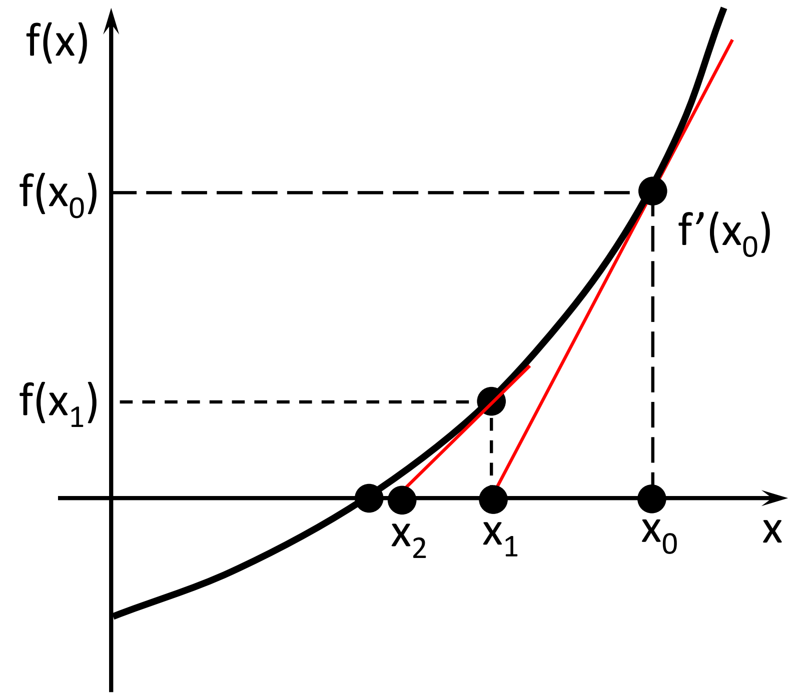
- Suppose we want to find the root of a continuous, differentiable function f(x), and we know the root we are looking for is near the point $x = x_0$. The Newton-Raphson method tells us that a better approximation for the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- This process may be repeated as many times as necessary to get the desired accuracy. In general, for any x value x_i, the next value is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- This is the formula for the Newton-Raphson method.
- If $|\varepsilon_a| < \varepsilon_S$, the loop stops. Otherwise calculate new value of *x*.



For fix-point iteration method,
$x_{i+1} = g(x_i)$, $i = 0, 1, 2, \ldots$

Two function handles are defined in NR method

Newton-Raphson Method

- Step 1: Given an equation f(x) = 0, make an initial guess for x value,

  $x = x_0$

- Step 2: Determine x_{i+1} using the equation:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Step 3: Determine ε_a using the equation $\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$
- If $|\varepsilon_a| < \varepsilon_S$, iteration stops. Otherwise, go to Step 2, and calculate new value of x

(A while loop will be used)

Two function handles are defined in NR method: f(x) and f'(x)

Newton-Raphson Method

- Example: solve the equation: $f(x) = x^2 - x - 2 = 0$ using Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$f(x) = x^2 - x - 2$
$f'(x) = 2x - 1$

```
>> f = @(x) x.^2 - x - 2;
>> x = linspace(-10,10,40);
>> plot(x,f(x))
```

Fixed-Point Iteration

```
clc
clear

xo = input('Enter a guess for the root of f(x):  ');
g = @(x) x.^2 – 2;
n = 4;
es = 0.5*10^(2-n);

[xr,cnt] = fixed_point(es,xo,g);

fprintf('\nThe root of f(x) is %.3f\n', xr)
fprintf('\nIt took %d iterations to converge \n\n', cnt)
```

```
function [xr, cnt] = fixed_point(es, xo, g)
ea = 1;
cnt = 0;
while ea > es
    xn = g(xo);
    ea = 100*abs((xn-xo)/xn);
    xo = xn;
    cnt = cnt +1;
end
xr = xn;
```

Example: $f(x) = x^2 - x - 2 = 0$

$-x = -x^2 + 2$

$x = x^2 - 2$

$g(x) = x^2 - 2$

- Step 1: Given an equation f(x) = 0, move *x* to the left hand side, the right hand side is denoted by *g(x),*

  x = g(x)

- Step 2: Make an initial guess for x value,

  $x = x_0$

- Step 3: Determine x_{i+1} using the equation:

  $x_{i+1} = g(x_i)$

- Step 4: Determine ε_a using the equation $\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$
- If $|\varepsilon_a| < \varepsilon_s$ stop. Otherwise, go to Step 3, and calculate new value of x

Newton-Raphson Method

```
clc
clear

xo = input('Enter a guess for the root of f(x):  ');
fx = @(x) x.^2 – x – 2;
dfx = @(x) 2*x -1;
n = 4;
es = 0.5*10^(2-n);

[xr,cnt] = new_rap(es,xo,fx,dfx);

fprintf('\nThe root of f(x) is %.3f\n', xr)
fprintf('\nIt took %d iterations to converge \n\n', cnt)
```

Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Example:

$f(x) = x^2 - x - 2 = 0$

$f'(x) = 2*x-1$

```
function [xr, cnt] = new_rap(es, xo, fx, dfx)
ea = 1;
cnt = 0;
while ea > es
    xn = xo – fx(xo)/dfx(xo);
    ea = 100*abs((xn-xo)/xn);
    xo = xn;
    cnt = cnt +1;
end
xr = xn;
```

Newton-Raphson Method

- Newton-Raphson method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- The Newton-Raphson method is a convenient method for functions whose derivatives can be evaluated analytically. It may not be convenient for functions whose derivatives cannot be evaluated analytically.
- What should we do for functions whose derivatives cannot be evaluated analytically?

Secant Method

- A slight variation of Newton's method for functions whose derivatives are difficult to evaluate. For these cases the derivatives can be approximated by a backward finite difference

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Two starting (initial) values are required for Secant method. Try to make two values close to each other

- Substitute this expression into $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, we have,

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

- This is the formula for the Secant method.
- Requires two initial values of x to estimate f'(x), but they are not required to 'bracket' the root, so it is not classified as a "bracketing" method.
- The secant method has the same properties as Newton's method. Convergence is not guaranteed.
- If $|\varepsilon_a| < \varepsilon_s$, the loop stops. Otherwise calculate new value of *x*.

Secant Method

```
clc
clear

x1 = input('Enter a guess for the root of f(x):  ');
x2 = input('Enter another guess for the root of f(x):  ');

fx = @(x) x^2 – x – 2;


n = 4;
es = 0.5*10^(2-n);

[xr,cnt] = secant_method(es,x1,x2,fx);

fprintf('\nThe root of f(x) is %.3f\n', xr)
fprintf('\nIt took %d iterations to converge \n\n', cnt)
```

```
function [xr, cnt] = secant_method(es, x1,x2, fx)
ea = 1;
cnt = 0;
while ea > es
    xn = x2 – fx(x2)*(x2-x1)/(fx(x2)-fx(x1));
    ea = 100*abs((xn-x2)/xn);
    x1 = x2;
    x2 = xn;
    cnt = cnt +1;
end
xr = xn;
```

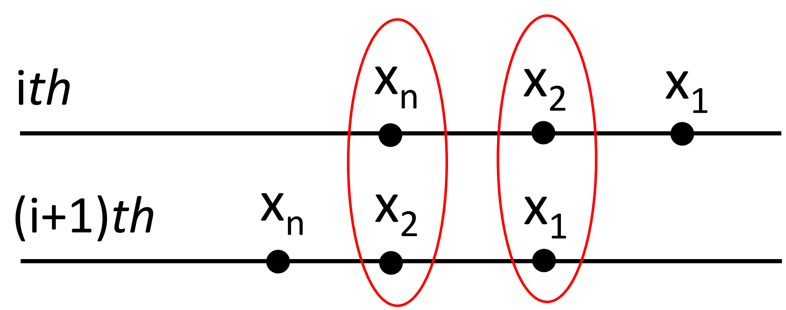
Secant method

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

Example:

$f(x) = x^2 - x - 2 = 0$

In the program, we use x_1 instead of x_{i-1}, and use x_2 instead of x_i.



Modified Secant Method

- Here a fractional perturbation is used to estimate the derivative

$$f'(x_i) \cong \frac{f(x_i+\delta x_i)-f(x_i)}{\delta x_i}$$

- Substitute this expression into $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, we have,

$$x_{i+1} = x_i - f(x_i)\frac{\delta x_i}{f(x_i+\delta x_i)-f(x_i)}$$

- This is the formula for the modified secant method.
- In this method we have the term δx_i . This term is actually δ multiplied by x_i .
- Requires an initial estimate x_i and a value for δ.
- If $|\varepsilon_a| < \varepsilon_S$, the loop stops. Otherwise calculate new value of *x*.

Newton-Raphson Method

- **Homework on Canvas**