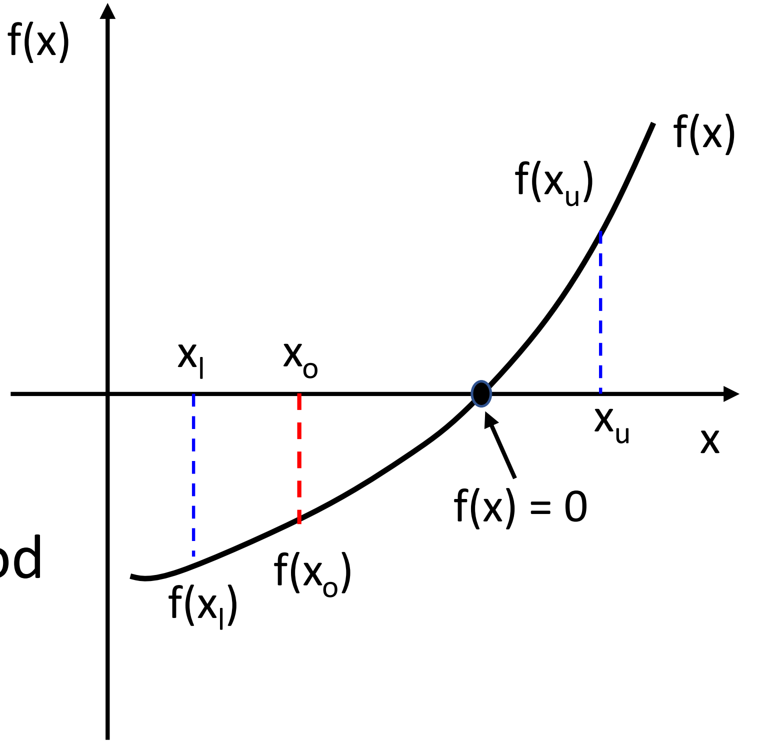
Lecture 15

Methods for Finding Root of Equation f(x) = 0

- There are basically two types of methods for find the root of equation f(x) = 0

- Bracketing techniques
  - ✓ Bisection method
  - ✓ False position method

- Open methods
  - ✓ Fixed-point iteration method
  - ✓ Newton-Raphson method



Open Methods

- Open methods are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root.
- Open methods may sometimes "diverge" or move away from the true root as the computation progresses.
- Bracketing methods are "convergent" because they move closer to the true root as computation progresses
- Convergence of open methods depends on the starting point (initial guess) and how the function behaves
- We should have some kind of idea of what our root should be. Generally we should plot the function.

Fixed-Point Iteration Method

- Fixed point: A point is called a fixed point if it satisfies the equation

  $x = g(x)$

- Fixed point iteration:
  - ✓ Given an equation

    $f(x)=0$

  - ✓ Rearrange this equation so that x is on the left side of the equation, and the right side is defined as g(x):

    $x = g(x)$

  - ✓ Make an initial guess for x value,

    $x = x_0$

  - ✓ use the iterative scheme with the recursive relation,

    $x_{i+1} = g(x_i), \quad i = 0, 1, 2, \ldots$

Fixed-Point Iteration Method

✓ The recursive relation can be expanded as,

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$x_3 = g(x_2)$

$x_4 = g(x_3)$

...

$x_{i+1} = g(x_i)$

...

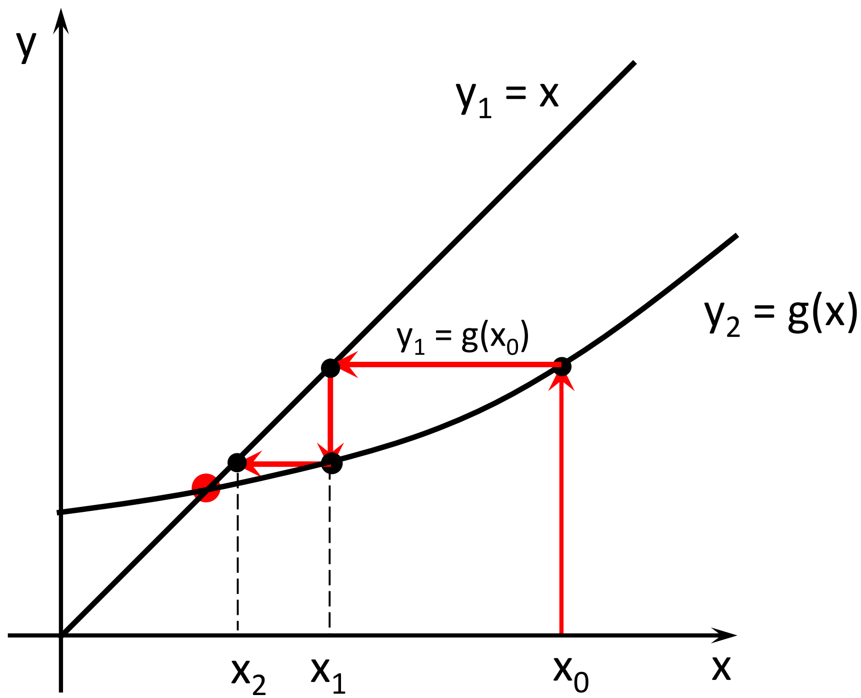
until ea<es

Original equation n: x = g(x)

$$\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$$

Fixed-Point Iteration Method

- $x = g(x)$ can be expressed as a pair of functions:
  $y_1(x) = x$ and $y_2(x) = g(x)$
- Plot two functions separately
- The root of $x = g(x)$ is given by the intersection of $y_1(x)$ and $y_2(x)$



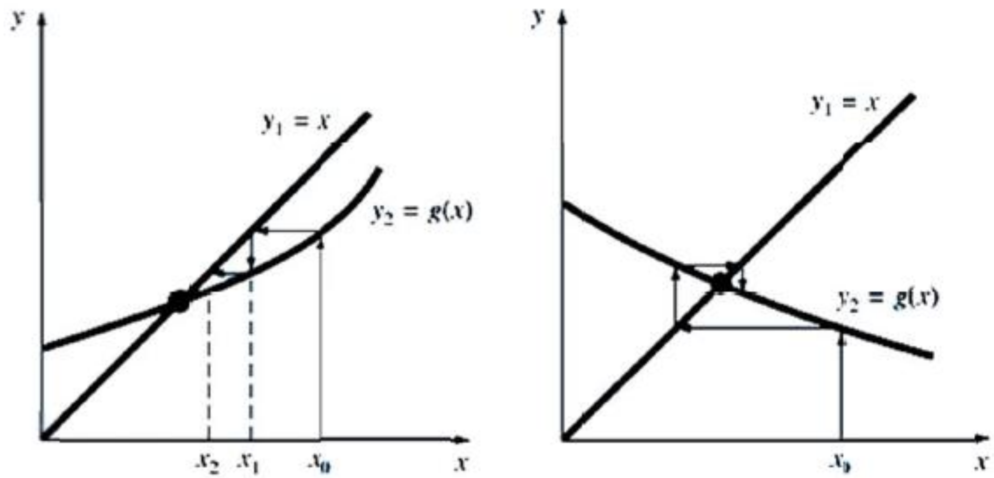
$x_1 = g(x_0)$

$x_2 = g(x_1)$

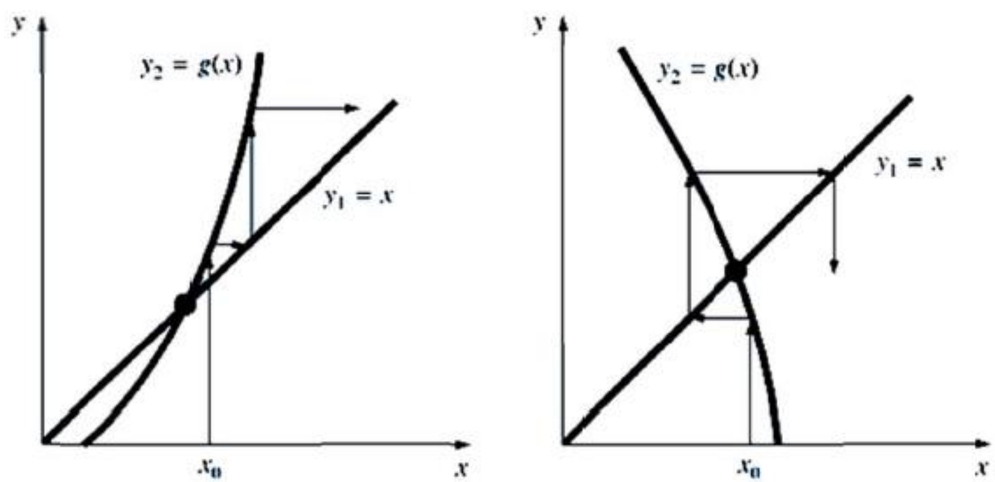
$x_3 = g(x_2)$

Fixed-Point Iteration Method

converging



diverging



- Fixed-point iteration converges if

$$|g'(x)|<1$$

Fixed-Point Iteration Method

- Rearrange the equation so that x is on the left side of the equation:

$$f(x) = 0$$

$$x = g(x)$$

- Example: $f(x) = x^2 - x - 2 = 0$

Method 1:

$f(x) = x^2 - x - 2 = 0$

$-x = -x^2 + 2$

$x = x^2 - 2$

$g(x) = x^2 - 2$

Method 2:

$f(x) = x^2 - x - 2 = 0$

$x^2 = x + 2$

$x = \sqrt{x + 2}$

$g(x) = \sqrt{x + 2}$

Method 3:

$f(x) = x^2 - x - 2 = 0$

$x^2 = x + 2$

$x = \dfrac{x + 2}{x}$

$g(x) = 1 + \dfrac{2}{x}$

- **Either method can be used to find the root**
- **The solution of each method may converge and may diverge**
- **A lot of times, we have to try different g(x)**

Fixed-Point Iteration Method

- Step 1: Given an equation f(x) = 0, move *x* to the left hand side, and move everything other than x to the right hand side. The right hand side is denoted by *g(x),*

  $x = g(x)$

- Step 2: Make an initial guess for x value,

  $x = x_0$

- Step 3: Determine x_{i+1} using the equation:

  $x_{i+1} = g(x_i)$

- Step 4: Determine ε_a using the equation $\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$

- If $|\varepsilon_a| < \varepsilon_S$, iteration stops. Otherwise, go to Step 3, and calculate new value of x

  (A while loop will be used)

Fixed-Point Iteration Method

- Example: solve the equation: $f(x) = x^2 - x - 2 = 0$

  Rearrange the equation as,

  $-x = -x^2 + 2$

  $x = x^2 - 2$

  $g(x) = x^2 - 2$

```
>> g = @(x) x.^2 – 2;
>> x = linspace(-10,10,20);
>> plot(x,g(x))
>> hold on
>> plot(x,x)
```

Fix-Point Iteration

We always have the same structure with the iterative techniques

```
clc
clear

xo = input('Enter a guess for the root of f(x):  ');
g = @(x) x.^2 – 2;   % define the function handle of g(x)
n = 4;
es = 0.5*10^(2-n);

[xr,cnt] = fixed_point(es,xo,g);

fprintf('\nThe root of f(x) is %.3f\n', xr)
fprintf('\nIt took %d iterations to converge \n\n', cnt)
```

```
function [xr, cnt] = fixed_point(es, xo, g)
ea = 1;
cnt = 0;
while ea > es
    xn = g(xo);
    ea = 100*abs((xn-xo)/xn);
    xo = xn;
    cnt = cnt +1;
end
xr = xn;
```

Example: $f(x) = x^2 - x - 2 = 0$

$-x = -x^2 + 2$

$x = x^2 - 2$

$g(x) = x^2 - 2$

- Step 1: Given an equation f(x) = 0, move *x* to the left hand side, the right hand side is denoted by *g(x),*

  x = g(x)

- Step 2: Make an initial guess for x value,

  $x = x_0$

- Step 3: Determine x_{i+1} using the equation:

  $x_{i+1} = g(x_i)$

- Step 4: Determine ε_a using the equation $\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$
- If $|\varepsilon_a| < \varepsilon_s$ stop. Otherwise, go to Step 3, and calculate new value of x

Fixed-Point Iteration Method

- **Summary**
- Open methods – fixed point iteration method

- **Homework on Canvas**