Lecture 20

Naïve Gauss Elimination

- **Naïve Gauss Elimination**

- For larger systems, Cramer's Rule can become unwieldy.
- Instead, a sequential process of removing unknowns from equations using forward elimination followed by back substitution may be used – this is Naïve Gauss elimination

Naïve Gauss Elimination

➢ **Forward Elimination (transform coefficient matrix to upper triangular)**

- Starting with the first row, add or subtract multiples of that row to eliminate the first coefficient from the second row and beyond.

  (row_i)' = (row_i) – (row_1)(a_{i1}/a_{11})

  i = 2, 3 … n

- Continue this process with the second row to remove the second coefficient from the third row and beyond.

  (row_i)'' = (row_i)' – (row_2)'(a'_{i2}/a'_{22})

  i = 3, 4 … n

- Stop when an upper triangular matrix remains.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & \boxed{a_{32}} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{bmatrix}$$

$$\begin{array}{lccccc} & a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ - & (a_{11} & a_{12} & a_{13} & a_{14} & b_1)\,\frac{a_{21}}{a_{11}} \\ \hline & \mathbf{0} & a_{22} - a_{12}\frac{a_{21}}{a_{11}} & a_{23} - a_{13}\frac{a_{21}}{a_{11}} & a_{24} - a_{14}\frac{a_{21}}{a_{11}} & b_2 - b_1\frac{a_{21}}{a_{11}} \\ & \mathbf{0} & a'_{22} & a'_{23} & a'_{24} & b'_2 \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{bmatrix}$$

$$\begin{array}{rccccc} & a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ -(& a_{11} & a_{12} & a_{13} & a_{14} & b_1)\dfrac{a_{31}}{a_{11}} \\ \hline & \mathbf{0} & a'_{32} & a'_{33} & a'_{34} & b'_3 \end{array}$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ \boxed{a_{41}} & \boxed{a_{42}} & \boxed{a_{43}} & \boxed{a_{44}} & \boxed{b_4} \end{array}\right]$$

$$\begin{array}{rccccc} & a_{41} & a_{42} & a_{43} & a_{44} & b_4 \\ -(& a_{11} & a_{12} & a_{13} & a_{14} & b_1)\dfrac{a_{41}}{a_{11}} \\ \hline & \mathbf{0} & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 & \\ -(0 & a'_{22} & a'_{23} & a'_{24} & b'_2) & \dfrac{a'_{32}}{a'_{22}} \\ \hline 0 & \mathbf{0} & a''_{33} & a''_{34} & b''_3 & \end{array}$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Naïve Gauss Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

➢ **Back Substitution**

- Starting with the last row, solve for the unknown, then substitute that value into the next highest row.
- Because of the upper-triangular nature of the matrix, each row will contain only one more unknown.

$$a_{11}*x_1 + a_{12}*x_2 + a_{13}*x_3 = b_1$$
$$a'_{22}*x_2 + a'_{23}*x_3 = b'_2$$
$$a''_{33} * x_3 = b''_3$$

$$x_3 = b''_3/a''_{33}$$

$$x_2 = (b'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (b_1 - a_{13}x_3 - a_{12}x_2)/a_{11}$$

Naïve Gauss Elimination

- **Example**
- Solve the linear equations
- $$\begin{cases} 0.3x_1 + 0.52x_2 + x_3 = -0.01 \\ 0.5x_1 + x_2 + 1.9x_3 = 0.67 \\ 0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44 \end{cases}$$

- The coefficients are stored in a data file. (use ‘load’ function to read in data from an external data file. the load function will read from the file filename.ext and create a matrix with the same name as the file,)

Naïve Gauss Elimination

```
clc
clear

load matin.dat    % load data from external file

C = matin;        % create a new matrix

%Forward Elimination
C(2,:) = C(2,:) - C(1,:)*(C(2,1)/C(1,1));    % zero out C21
C(3,:) = C(3,:) - C(1,:)*(C(3,1)/C(1,1));    % zero out C31

C(3,:) = C(3,:) - C(2,:)*(C(3,2)/C(2,2));    % zero out C'32

%Back Substitution
x(3) = C(3,4)/C(3,3);
x(2) = (C(2,4)-C(2,3)*x(3))/C(2,2);
x(1) = (C(1,4)-C(1,3)*x(3)-C(1,2)*x(2))/C(1,1);

disp(C)
disp(x)
```

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$(row\_i)' = (row\_i) - (row\_1)(a_{i1}/a_{11})$

$i = 2, 3 \ldots n$

$(row\_i)'' = (row\_i)' - (row\_2)'(a'_{i2}/a'_{22})$

$i = 3, 4 \ldots n$

$$x_3 = b''_3/a''_{33}$$

$$x_2 = (b'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (b_1 - a_{13}x_3 - a_{12}x_2)/a_{11}$$

Naïve Gauss Elimination

```
clc
clear

load matin.dat
C = matin

[nr,nc] = size(C);

%Forward Elimination
 for i = 1:nr-1              % i: columns to zero out
    for j = i+1:nr           % j: rows to work on
       C(j,:) = C(j,:) - C(i,:)*C(j,i)/C(i,i);
    end
 end

%Back Substitution
 for i = nr:-1:1
    sm = 0
    for j = nr:-1:i+1
       sm = sm + C(i,j)*x(j);
    end
    x(i) = (C(i,nr+1) - sm)/C(i,i);
 end

 disp(C)
 disp(x)
```

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{bmatrix}$$

$$C_{i,i}*x_i + C_{i,i+1}*x_{i+1} + \ldots + C_{i,nr}*x_{nr} = b_i$$

$$sm = C_{i,i+1}*x_{i+1} + \ldots + C_{i,nr}*x_{nr}$$

$$x_i = (b_i - sm)/C_{i,i}$$

- **Homework on Canvas**