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# Reinforcement Learning: function approximation

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# Markov Decision Process

- The MDP gives us a precise formulation of the environment, given a state st we select an action at and observe $s_{t+1}$ and $r_t$ according to the transition probabilities P:
  - $S$ - set of possible states
  - $s_t \in S$ - state at step $t$
  - $A$ - set of possible actions
  - $a_t \in A$ - selected action at step $t$
  - $R$ - Reward function. The reward at step $t$ is given by $r_{t+1} = R(s_t, a_t, s_{t+1})$
  - $P$ - transition probabilities such that $s_{t+1}P(s|s_t, a_t)$, i.e.
  - $\rho$ - Initial state distribution such that $s_0\rho(s)$
- Agent is defined with a policy function $\pi(a|s)$, mapping from states to actions and can be either deterministic or non-deterministic

# Markov Decision Process

- Given an MDP and a policy, an episode can be produced by repeating of:
  - $a_t \pi(a|s_t)$
  - $s_{t+1} P(s|s_t, a_t)$
  - $r_{t+1} = r(s_t, a_t, s_{t+1})$
- which produce:

$$\text{episode} := s_0, a_0, r_1, s_1, a_1, r_1, \ldots, s_{\tau 1}, a_{\tau 1}, r_{\tau 1}, s_\tau$$

- Optimal Solution gives:

$$\max_{\pi} E[\sum_{t=1}^{\tau} r_t] \tag{1}$$

# Markov Decision Process

- Value function defined as :

$$V^{\pi}(s) = E_{\pi}[\sum_{t=1}^{\tau} r_t | s_0 = s] \tag{2}$$

$$V^{*}(s) = \max_{\pi} E_{\pi}[\sum_{t=1}^{\tau} r_t | s_0 = s] \tag{3}$$

- Bellman equation: A recursive relation for value function:

$$V^{*}(s) = \max_{a \in A} E[r_{t+1} + V^{*}_{s_{t+1}} | s_t = s, a_t = t] \tag{4}$$

- $(TV)(s)$ is the Bellman operator and we can recursively calculate it and update $V(s)$ (value iteration) to reach optimal (Monotonicity and Contraction mapping)

$$(TV)(s) = \max_{a \in A} E[r_{t+1} + V^{*}_{s_{t+1}} | s_t = s, a_t = t] \tag{5}$$

$$V_{k+1} = TV_k \tag{6}$$

- policy iteration use the same idea, but instead of updating value function, it update Policy $\pi$

# Markov Decision Process

- Another approach: State-Value Function: define a quantity $Q : S \times A \rightarrow \mathbb{R}$:

$$Q^{\pi}(s,a) = \bar{R}(s,a) + \sum_{s' \in S}^{T} P_{s,a}(s') V^{\pi}(s') \tag{7}$$

- Recursively Calculate optimal by using Bellman Operator:
  - $FQ(s,a) = \bar{R}(s,a) + \sum_{s' \in S}^{T} P_{s,a}(s') \max_{a' \in A} Q(s', a')$
  - $Q(s,a) = FQ(s,a)$

- Greedy action selection is simple:

$$\pi(s) = \arg\max_{a' \in A} Q(s_{t+1}, a') \tag{8}$$

# MDP vs RL

- Difference between Markov Decision Process and Reinforcement learning:

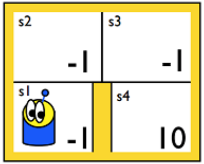
  - MDP: the transition matrix $P(s'|s, a)$ is known $\rightarrow$ used to find the optimal agent
  - RL: $P(s'|s, a)$ unknown and need to be learned :
    i. Iteracting with environment
    ii. Requiring explicit knowledge of $P$

- But the same idea of state-action value function can be used for Reinforcement learning

# Q-learning and SARSA

- Q-learning:

  i. Build a Q-table which stores $Q(s, a)$ for each $s$ and $a$ (randomly initialized). i.e.




$\alpha = .7$

| | ⇧ | ⇩ | ⇦ | ⇨ |
|---|---|---|---|---|
| $s_1$ | 0 | 0 | 0 | 0 |
| $s_2$ | 0 | 0 | 0 | 0 |
| $s_3$ | 0 | 0 | 0 | 0 |
| $s_4$ | 0 | 0 | 0 | 0 |

Q-Table

# Q-learning and SARSA

- Q-learning:

  ii. update $Q(s,a)$ with:

$$Q_{k+1} := (1-\gamma_k)Q_k + \gamma_k(r + \max_{a' \in A} Q_k(s',a')) \tag{9}$$

  where $\gamma_k$ is the learning rate, with $\sum_{k=1}^{\infty} \gamma_k = \infty$ and $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$ :

$$Q_{k+1} = Q_k + \gamma_k(r + \max_{a' \in A} Q_k(s',a') - Q_k(s,a)) \tag{10}$$

- Q-learning Demo: google

# Q-learning and SARSA

- Q-learning result:

# Q-learning and SARSA

- Exploration: $\epsilon - greedy$

$$a_t = \begin{cases} \arg\max_{a \in A} Q_{s_t,a}, & \text{w.p. } 1-\epsilon. \\ unif(A), & \text{w.p. } \epsilon. \end{cases} \tag{11}$$

- SARSA:
  - update based on the current play $(s, a, r, s', a')$

$$Q_{k+1} = Q_k + \gamma_k(r + Q_k(s', a') - Q_k(s, a)) \tag{12}$$

  - Similar to Q-learning but is On-policy

# Deep Q-Networks

- Drawback of Q-learning and SARSA:
  - Q-table can be too big if environment is complicate i.e. $1^6 \times 1^3$ maze
- Alternative Algorithm: DQN:
  - Use a function approximator to estimate action-value function with Q-Network
- Steps:

  $i$ store transition$(s_t, s_t, r_{t+1}, s_{t+1})$ in memory
  $ii$ sample mini-batch of transitions, optimise MSE between Q-network and Q-learning targets:

$$\text{minimize } L_w = E_{s,a,r,s'}[(r + \gamma \max_{a'} Q(s', a'; w^-) - Q(s, a; w))^2] \tag{13}$$

  $iii$

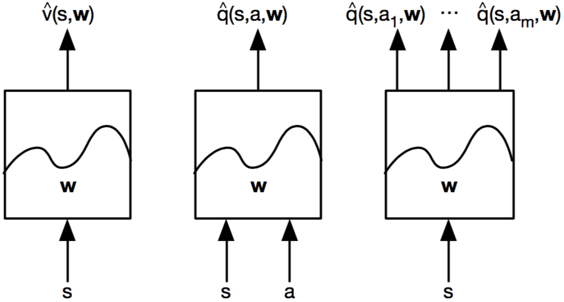
# Value Iteration: Problem

- Recall: greedy action selection

$$\pi(s) = \arg\max_{a' \in A} Q(s_{t+1}, a') \tag{14}$$

- Problem: deterministic, strategy fixed, not practical in Partially-Observed environment

# Value Function Approximation




- Tablular methods: impossible to record all states for real word problems
- Function approximation: generalize from seen states to unseen states

# Value Function Approximation

- Goal: find parameter vector $w$ minimising mean-squared error between approximate value function $\hat{v}(S, w)$ and true value function $v_\pi(S)$

$$J(w) = ||v_\pi(S) - \hat{v}(S, w)||_2^2 \tag{15}$$

- Stochastic gradient descent

$$\Delta w = \alpha(v_\pi(s) - \hat{v}(S, w))\nabla_w \hat{v}(S, w) \tag{16}$$

- In reality we don't have the true value function $v_\pi(S)$
  - For Monte-Carlo, use discounted return $G_t$
  - For TD, use $R_{t+1} + \lambda\hat{v}(S_{t+1}, w)$

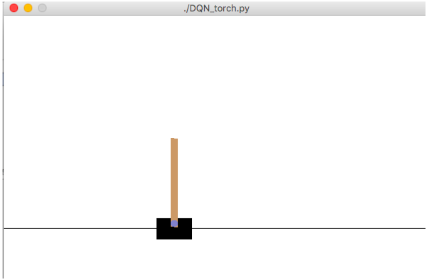
# Deep Q-Networks (DQN)

- Take action $a_t$ according to $\epsilon$-greedy policy
- Store transition $(s_t, a_t, r_{t+1}, s_{t+1})$ in memory $D$
- Sample random mini-batch of transitions $(s, a, r, s')$ from $D$
- Compute Q-learning targets w.r.t. old, fixed parameters $w^-$
- Optimise MSE between Q-network and Q-learning targets

$$L(w) = \mathrm{E}_{s,a,s,r' \sim D_i}[(r + \gamma \max_{a'} Q(s', a'; w^-) - Q(s, a; w))^2] \tag{17}$$

- Two important tricks in ensuring convergence: experience replay and fixed target

# Deep Q-Networks(DQN): play games in OpenAI gym




- States are represented by 4-element tuples (position,cart velocity,angle,tip velocity)
- Actions can be either moving left or right
- Function approximator is a feed foward neural network
- 1 hidden layer with 10 neurons, 2 output neurons representing value estimation for two actions
- Implemented using torch and tensorflow, can stay alive for 1 minute

# Policy based method

Value based method (previous sildes):

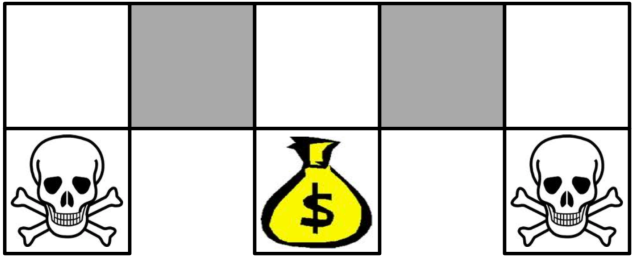
- Main focus is on state-action value evaluation
- Policy improvement is based on greedy or $\epsilon$-greedy strategy w.r.t state-action values
- Return deterministic policy

Policy based method (slides after this page):

- Policy is a function of observations
- Policy improvement is based on gradient w.r.t some objective function
- State-action value not neccessary for policy updates
- Return stochastic policy

# Policy based method

What's wrong with value based methods?



- Main problem: deterministic policy
- No good for partially observable environment
- The agent cannot differentiate the grey states
- An optimal deterministic will either go left or right

# Policy Gradient: problem formulation

- Policy is a function of observation: $\pi_\theta(\cdot)$
- Trajectory $\tau$: $\{s_0, a_0, r_0, s_1, a_1, r_1, ...\}$ is treated as random variable
- Distribution of $\tau$ is determined by policy $\pi_\theta$
- For each trajectory, total reward is defined as $R(\tau)$
- Ultimate goal: optimize expectation $E_{\pi_\theta}[R(\tau)]$ w.r.t $\theta$

# Policy Gradient: approximate the gradient

- What does the gradient look like?

$$
\begin{aligned}
\nabla_\theta E_{\pi_\theta}[R(\tau)] &= \nabla_\theta \sum_\tau P_\theta(\tau) R(\tau) \\
&= \sum_\tau \nabla_\theta P_\theta(\tau) R(\tau) \\
&= \sum_\tau P_\theta(\tau) \frac{\nabla_\theta P_\theta(\tau)}{P_\theta(\tau)} R(\tau) = E_{\pi_\theta}[\nabla_\theta \ln P_\theta(\tau) R(\tau)]
\end{aligned} \tag{18}
$$

- Equation 18 tell us: gradient can be represented as an expectation
- Why it is important: expectation can be approximated by sampling

# Policy Gradient: approximate the gradient

- Why the gradient even exists?

$$P(\tau) = P(s_0) \prod_{i=0}^{\infty} \pi_\theta(a_i, s_i) P(s_{i+1} | s_i, a_i) \tag{19}$$

- Assumption: there is an underlying MDP specifying $P(s_{i+1}|s_i, a_i)$ and $P(s_0)$

$$\begin{aligned}
\nabla_\theta \ln P(\tau) &= \nabla_\theta \ln [P(s_0) \prod_{i=0}^{\infty} \pi_\theta(a_i, s_i) P(s_{i+1} | s_i, a_i)] \\
&= \nabla_\theta \ln P(s_0) + \nabla_\theta \sum_{i=0}^{\infty} [\ln \pi_\theta(a_i, s_i) + \ln P(s_{i+1} | s_i, a_i)] \qquad (20) \\
&= \nabla_\theta \sum_{i=0}^{\infty} \ln \pi_\theta(a_i, s_i)
\end{aligned}$$

# Policy Gradient: understanding the formula

Combine all equation in previous slides, one important formula:

$$\nabla_\theta E_{\pi_\theta}[R(\tau)] = E_{\pi_\theta}[R(\tau)\nabla_\theta \sum_{s_i,a_i \in \tau} \ln \pi_\theta(a_i, s_i)] \tag{21}$$

Intuition from equation 21, adjustment magnitude of policy on $\pi_\theta(a, s)$:

- In proportion to the total reward gained from trajectories containing $(a, s)$
  - Rationale: good actions lead to good trajectories, while bad actions lead to bad ones
  - What about good actions in trajectories with bad overall performance? work on it later
- In inverse proportion to the probability of performing action $a$ on state $s$
  - consider actions sampled frequently but with small positive effect
  - mitigate case where 'not-so-good' actions are rewarded frequently

# Vanilla Policy Gradient: **REINFORCE**

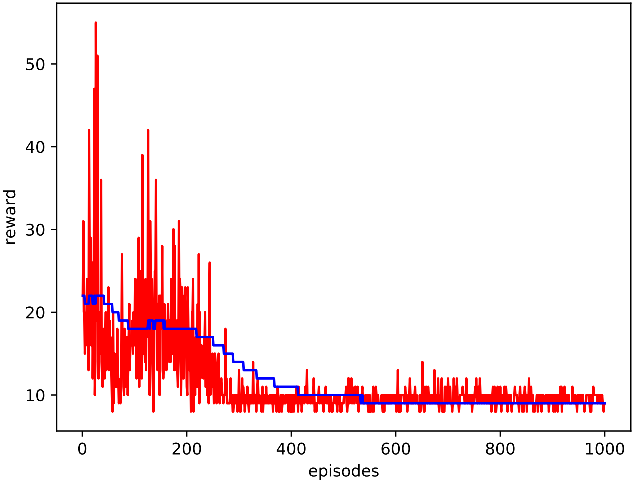
So far, we obtain the first policy gradient algotithm called **REINFORCE** [Williams, R. J.]

**Algorithm 1** Generic Policy Gradient

1: Initialize policy parameters $\theta$, learning rate $\alpha$
2: **for** each iteration **do**
3: Collect trajectories $\{\tau_1, \tau_2, \tau_3, \ldots \tau_k\}$ using policy $\pi_\theta$
4: Estimate gradient $\hat{grad} = \frac{1}{k}\sum_{i=1}^{k}[R(\tau_i)\nabla_\theta \sum \ln \pi_\theta(a, s)]$
5: Update policy $\theta \leftarrow \theta + \alpha \cdot \hat{grad}$
6: **end for**
7: **Return** policy $\pi_\theta$

# Vanilla Policy Gradient: **REINFORCE**

Experiment on CartPole game:
Bad performance, worse than random play after 1000 episodes of training



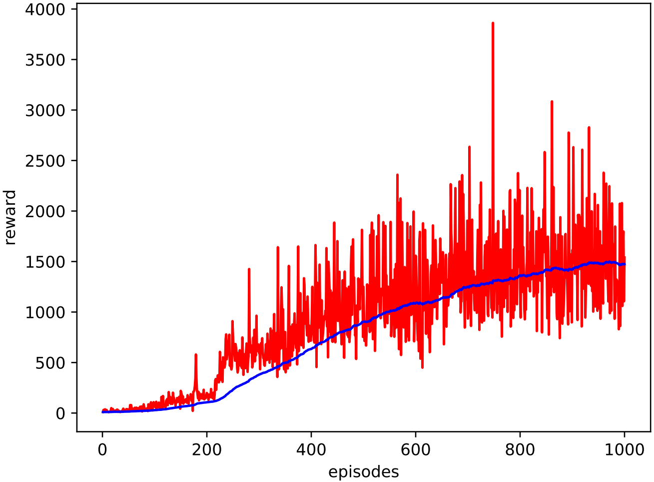
# Improvement for **REINFORCE**: baseline

- One natural quetion: what if reward is always positive?
- Do we have to always increase $\pi_\theta(a, s)$ because $R(\tau)$ is positive (as in formula 21)?
- Actually, we only cares about the relative performance of trajectories
- Observation from formula 22: we can remove any constant term $A$ from the expectation with out introducing bias.
- $A$ can be the average performance for all trajectories, it is referenced as a baseline

$$
\begin{aligned}
E_{\pi_\theta}[\sum_a A \cdot \nabla \ln \pi(a, S)] &= \sum_a \pi_\theta(a, S) A \frac{\nabla_\theta \pi_\theta(a, S)}{\pi_\theta(a, S)} \\
&= A \sum_a \nabla_\theta \pi(a, S) \\
&= A \cdot \nabla_\theta \sum_a \pi_\theta(a, S) = A \nabla_\theta(1) = 0
\end{aligned} \tag{22}
$$

# Improvement for **REINFORCE**: baseline

Experiment on CartPole game:
Better than before: an upgoing trend of rewards



# Improvement for **REINFORCE**: advantage function

- Why we award/punish an action (s,a) based on the entire trajectory reward?
- Markov property: the action $a_t$ only affects rewards after time $t$.

$$E_{\pi_\theta}[R_{0:i-1}(\tau)\nabla_\theta \ln \pi_\theta(a_i, s_i)] = 0 \tag{23}$$

- Actually, we can exploit the markov property to refine formula 21

$$\begin{aligned}\nabla_\theta E_{\pi_\theta}[R(\tau)] &= E_{\pi_\theta}[\sum(\nabla_\theta \ln \pi_\theta(a_i, s_i)(R_{0:i-1}(\tau) + R_{i:\infty}(\tau)))] \\ &= E_{\pi_\theta}[\sum(\nabla_\theta \ln \pi_\theta(a_i, s_i)R_{i:\infty}(\tau))]\end{aligned} \tag{24}$$

- Recall that subtraction of baseline doesn't change the expectation

$$\nabla_\theta E_{\pi_\theta}[R(\tau)] = E_{\pi_\theta}[\sum \nabla_\theta \ln \pi_\theta(a_i, s_i)(R_{i:\infty}(\tau) - V_{\pi_\theta}(s_i))] \tag{25}$$

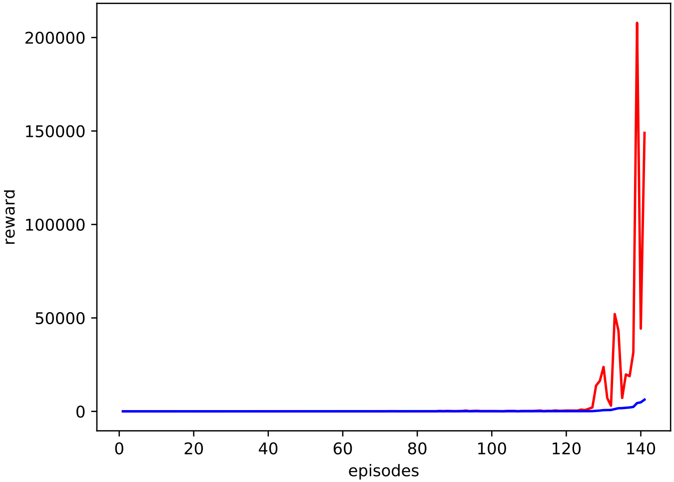
# Improvement for **REINFORCE**: advantage function

- As shown in equation 25, $R_{i:\infty}(\tau) - V_{\pi_\theta}(s_i)$ is the actual term that determines the magnitude we adjust probability $\pi_\theta(a_i, s_i)$
- $R_{i:\infty}(\tau) - V_{\pi_\theta}(s_i)$ is also known as the advantage function
- Rationale: extra reward gained when performing certain action $a_i$ on state $s_i$ compared to average reward from that state under policy $\pi_\theta$

# Improvement for **REINFORCE**: advantage function

Experiment on CartPole game:

Only after 141 episodes of training, surviving time boosted to 20k!



# Actor-Critic: a combination

So far, we mainly focused on pure value-based and pure policy-based methods ...

- Value-based: problem of deterministic policy in partially observed environments
- Policy-based: credit assignment problem (delay between action and reward)
- Why not combine them?
- Still use policy function
- Also adopt an estimator for state values to approximate advantage function
- Policy updates without delay!

# Actor-Critic: algorithm

Now there are two function to learn: policy function $\pi_\theta$ is known as actor and value estimator $\hat{V}_\varphi$ is known as critic, hence the model named **Actor-Critic**.

**Algorithm 2** Actor-Critic

Initialize actor parameters $\theta$, critic parameters $\varphi$, learning rate $\alpha$
**for** each episode **do**
    Interact with environment for some time, get trajectory $\tau : \{s_0, a_0, r_0, s_1, a_0, r_0, ..., s_k\}$
    **If** $s_k$ is Terminal, $R_{k:\infty} \leftarrow 0$ **else** $R_{k:\infty} \leftarrow V_\varphi\hat{(}s_k)$
    **for** each step $i$ in $k-1:0$ **do**
        Estimate return $R_{i:\infty} \leftarrow r_i + R_{i+1:\infty}$
        Estimate advantage $A(a_i, s_i) \leftarrow R_{i:\infty} - V_\varphi\hat{(}s_i)$
    **end for**
    Estimate policy gradient $gr\hat{a}d = \frac{1}{k-1}\sum_{i=1}^{k-1}\sum[A(a_i, s_i)\nabla_\theta \ln \pi_\theta(a_i, s_i)]$
    Update actor $\theta \leftarrow \theta + \alpha \cdot gr\hat{a}d$
    Calculate loss for critic: $l(\varphi) \leftarrow \frac{1}{k-1}\sum_{i=0}^{k-1} A^2(a_i, s_i)$
    Update critic $\varphi \leftarrow \varphi + \alpha \cdot \nabla_\varphi l(\varphi)$
**end for**
**Return** policy $\pi_\theta$

# Further improvement: clipped objective function

- In previous section, policy gradient method works by computing gradient estimator in form

$$\hat{g} = \hat{E}_t[\nabla_\theta \ln \pi_\theta(a_t, s_t)\hat{A}_t] \tag{26}$$

- The estimator $\hat{g}$ can be obtained by differentiating the objective

$$L^{PG}(\theta) = \hat{E}_t[\ln \pi_\theta(a_t, s_t)\hat{A}_t] \tag{27}$$

- Multiple steps to optimize $L^{PG}$ on same trajectory: destructively large policy updates [Schulman, John et al.].
- To improve sample efficiency, they adopt strategy of clipping the surrogate objective function in form $L^{CPI}(\theta)$:

# Further improvement: clipped objective function

$$L^{CPI}(\theta) = \hat{E}_t[\frac{\pi_\theta(a_t, s_t)}{\pi_{\theta_{old}}(a_t, s_t)}\hat{A}_t] \tag{28}$$

- $\pi_{\theta_{old}}$: fixed term generated by old policy
- $\pi_\theta$: current policy being optimized.
- The ratio $\frac{\pi_\theta(a_t,s_t)}{\pi_{\theta_{old}}(a_t,s_t)}$ is denoted as $r_t(\theta)$
- $r_t(\theta)$ measures the difference between current policy and old policy
- we don't want too big a update step, hence some constraint based on $r_t(\theta)$
- In practise we use the gradient of following objective function

$$L^{CLIP}(\theta) = \hat{E}_t[\min(r_t(\theta)\hat{A}_t, \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t] \tag{29}$$

# Proximal Policy Optimization(PPO)

The algorithm is known as Proximal Policy Optimization [Schulman, John et al.]

**Algorithm 3** Proximal Policy Optimization

Initialize actor parameters $\theta$, critic parameters $\varphi$, learning rate $\alpha$, clip coefficient $\epsilon$
Initialize old policy $\pi_{\theta_{old}} \leftarrow \pi_\theta$
**for** each episode **do**
  **for** each time period in episode **do**
    denote current state as $s_0$, continue interacting with environment for some time
    get trajectory $\tau : \{s_0, a_0, r_0, s_1, a_0, r_0, ..., s_k\}$
    **If** $s_k$ is Terminal, $R_{k:\infty} \leftarrow 0$ **else** $R_{k:\infty} \leftarrow V_\varphi\hat{(}s_k)$
    **for** each step $i$ in $k-1:0$ **do**
      Estimate return $R_{i:\infty} \leftarrow r_i + R_{i+1:\infty}$
      Estimate advantage $A(a_i, s_i) \leftarrow R_{i:\infty} - V_\varphi\hat{(}s_i)$
    **end for**
    calculate loss for actor $l(\theta) \leftarrow \frac{1}{k}\sum_{t=0}^{k-1} \min(r_t(\theta)\hat{A}_t, \mathrm{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t$
    apply multiple steps gradient ascent on $l(\theta)$ to update $\theta$
    calculate loss for critic: $l(\varphi) \leftarrow \frac{1}{k-1}\sum_{i=0}^{k-1} A^2(a_i, s_i)$
    apply multiple steps gradient descent on $l(\varphi)$ to update $\varphi$
    renew old policy $\pi_{\theta_{old}} \leftarrow \pi_\theta$
  **end for**
**end for**
**Return** policy $\pi_\theta$

# Proximal Policy Optimization(PPO): some demo

Test on OpenAI gym Agents implemented and trained using Pytorch

For detailed information about task environment, check this list

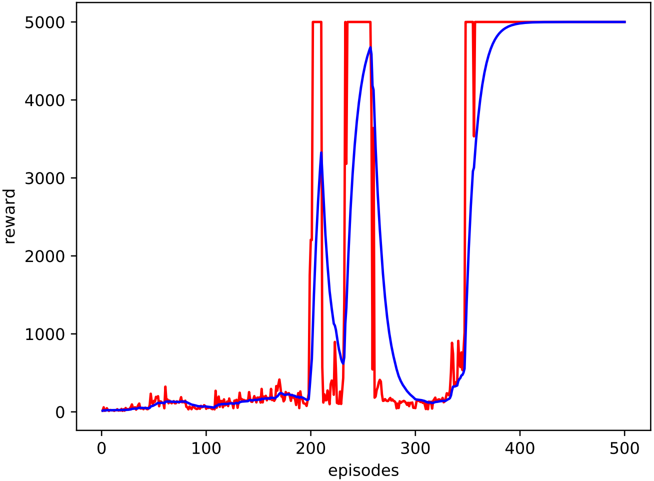
- CartPole-v0: no training and trained
- MountainCar-v0: no training and trained
- LunarLander-v2: no training and trained
- Pendulum-v0: no training and trained

Some strategyies in our training:

- For continous action space (like Pendulum-v0): discretize it
- Set a maximum number (5000) of steps for each episode during training
- Use a large batch size (512) to perform gradient descent
- Adopt different step size for Actor and Critic updates
- Have a look at our code on github

# Further improvement: high score buffer replay

- The learning curve is like:



# Further improvement: high score buffer replay

- A Typical training curve in Reinforcement Learning
- Not stable: immature policy, more frequent explorational moves
- Another problem: cases where positive signals are extremely rare
- Idea comes naturally: store those trajectories with high score in a buffer
- Use importance sampling to learn from high score buffer from time to time