Quaternion Extended Kalman Filter

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State Definition

$$\boldsymbol{q}_k = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} \tag{1}$$

$$\boldsymbol{\omega}_k = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2}$$

State Transition Equation

$$\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{q}_k \\ \boldsymbol{\omega}_k \end{bmatrix} \tag{3}$$

Quaternion rate of change

$$\dot{q}_w = \frac{1}{2}\left(-q_x\omega_x - q_y\omega_y - q_z\omega_z\right) \tag{4}$$

$$\dot{q}_x = \frac{1}{2}\left(q_w\omega_x + q_y\omega_z - q_z\omega_y\right) \tag{5}$$

$$\dot{q}_y = \frac{1}{2}\left(q_w\omega_y - q_x\omega_z + q_z\omega_x\right) \tag{6}$$

$$\dot{q}_z = \frac{1}{2}\left(q_w\omega_z + q_x\omega_y - q_y\omega_x\right) \tag{7}$$

Defining the omega operator as

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{8}$$

Defining the Xi operator as

$$\Xi(\boldsymbol{q}) = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z \\ q_w & -q_z & q_y \\ q_z & q_w & -q_x \\ -q_y & q_x & q_w \end{bmatrix} \tag{9}$$

State differential equation

Here $w(t)$ is white noise - this is a constant angular velocity model.

$$\dot{\boldsymbol{q}} = \boldsymbol{\Omega}(\boldsymbol{\omega})\boldsymbol{q} = \Xi(\boldsymbol{q})\boldsymbol{\omega} \tag{10}$$

$$\dot{\boldsymbol{\omega}} = w(t) \tag{11}$$

$$\frac{\partial}{\partial t}\boldsymbol{x} = \begin{bmatrix} 0 & \Xi(\boldsymbol{q}) \\ 0 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ I \end{bmatrix} w(t) \tag{12}$$

Deriving Kalman Filter Predict Equation

If we take Laplace transforms of 12, we obtain:

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \begin{bmatrix} 0 & \Xi(\boldsymbol{q}) \\ 0 & 0 \end{bmatrix} \boldsymbol{X}(s) + \begin{bmatrix} 0 \\ I \end{bmatrix} W(t) \tag{13}$$

Rearranging:

$$\boldsymbol{X}(s) = \begin{bmatrix} sI & -\Xi(\boldsymbol{q}) \\ 0 & sI \end{bmatrix}^{-1} \boldsymbol{x}(0) + \begin{bmatrix} sI & -\Xi(\boldsymbol{q}) \\ 0 & sI \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} W(t) \tag{14}$$

And taking inverse Laplace transforms:

$$\boldsymbol{x}(\Delta t) = \begin{bmatrix} I & \Delta t \Xi(\boldsymbol{q}) \\ 0 & I \end{bmatrix} \boldsymbol{x}(0) + \int_0^{\Delta t} \begin{bmatrix} (\Delta t - \tau)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(\tau) d\tau \tag{15}$$

Finding the Process Noise[1]

$$\boldsymbol{w} = \int_0^{\Delta t} \begin{bmatrix} (\Delta t - \tau)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(\tau) d\tau \tag{16}$$

The expected value is zero:

$$E\left[\boldsymbol{w}\right] = 0 \tag{17}$$

The expected process noise is therefore:

$$\begin{aligned}
Q =& E\left[\boldsymbol{w}\boldsymbol{w}^\top\right] && (18)\\
=& E\left[\left(\int_0^{\Delta t} \begin{bmatrix} (\Delta t - u)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(u) du\right) \left(\int_0^{\Delta t} \begin{bmatrix} (\Delta t - v)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(v) dv\right)^\top\right] && (19)\\
=& \int_0^{\Delta t}\int_0^{\Delta t} \begin{bmatrix} (\Delta t - u)(\Delta t - v)\Xi(\boldsymbol{q})\Xi(\boldsymbol{q})^\top & (\Delta t - u)\Xi(\boldsymbol{q}) \\ (\Delta t - v)\Xi(\boldsymbol{q})^\top & I \end{bmatrix} E[w(u)w(v)^\top] du dv && (20)\\
=& q \begin{bmatrix} \frac{\Delta t^3}{3}\Xi(\boldsymbol{q})\Xi(\boldsymbol{q})^\top & \frac{\Delta t^2}{2}\Xi(\boldsymbol{q}) \\ \frac{\Delta t^2}{2}\Xi(\boldsymbol{q})^\top & \Delta t \end{bmatrix} && (21)
\end{aligned}$$

where $E[w(u)w(v)^\top] = q$.

Kalman Filter Prediction Equation

$$\boldsymbol{x}_{k+1|k} = f(\boldsymbol{x}_{k|k}) = \begin{bmatrix} I & \Delta t \Xi(\boldsymbol{q}_{\boldsymbol{k}|\boldsymbol{k}}) \\ 0 & I \end{bmatrix} \boldsymbol{x}_{k|k} \tag{22}$$

$$\boldsymbol{\Sigma}_{k+1|k} = F_k \boldsymbol{\Sigma}_{k|k} F_k^\top + Q_k \tag{23}$$

$$F_k = \frac{\partial f(\boldsymbol{x}_{k|k})}{\partial \boldsymbol{x}_{k|k}} = \begin{bmatrix} I + \Delta t \boldsymbol{\Omega}(\boldsymbol{\omega}_{\boldsymbol{k}|\boldsymbol{k}}) & \Delta t \Xi(\boldsymbol{q}_{\boldsymbol{k}|\boldsymbol{k}}) \\ 0 & I \end{bmatrix} \tag{24}$$

[1]Similar derivations can be found in Estimation II.

Measurement Equations

Rotation Matrix from state quaternion

$$R(\boldsymbol{x}) = \begin{bmatrix} 2(q_w^2 + q_x^2) - 1 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & 2(q_w^2 + q_y^2) - 1 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & 2(q_w^2 + q_z^2) - 1 \end{bmatrix} \tag{25}$$

Accelerometer

$$h(\boldsymbol{x}) = R(\boldsymbol{q})^\top \boldsymbol{g} \tag{26}$$

where $\boldsymbol{g} = \begin{bmatrix} 0 & 0 & g \end{bmatrix}^\top$ is the expected gravity field vector.

Magnetometer

$$h(\boldsymbol{x}) = R(\boldsymbol{q})^\top \boldsymbol{m} \tag{27}$$

where $\boldsymbol{m} = \begin{bmatrix} m_x & m_y & m_z \end{bmatrix}^\top$ is the expected magnetic field vector at the location of the sensor.

Kalman Filter Update Equations

Innovation

$$\boldsymbol{y}_k = \boldsymbol{z}_k - h(\boldsymbol{x}_{k+1|k}) \tag{28}$$

$$S_k = H_k \boldsymbol{\Sigma}_{k+1|k} H_k^\top + R_k \tag{29}$$

$$H_k = \frac{\partial h(\boldsymbol{x}_{k+1|k})}{\partial \boldsymbol{x}_{k+1|k}} \tag{30}$$

Kalman Gain

$$K_k = \boldsymbol{\Sigma}_{k+1|k} H_k^\top S_k^{-1} \tag{31}$$

Updates

$$\boldsymbol{x}_{k+1|k+1} = \boldsymbol{x}_{k+1|k} + K_k \boldsymbol{y}_k \tag{32}$$

$$\boldsymbol{\Sigma}_{k+1|k+1} = (I - K_k H_k)\, \boldsymbol{\Sigma}_{k+1|k} \tag{33}$$

Results

Experiment

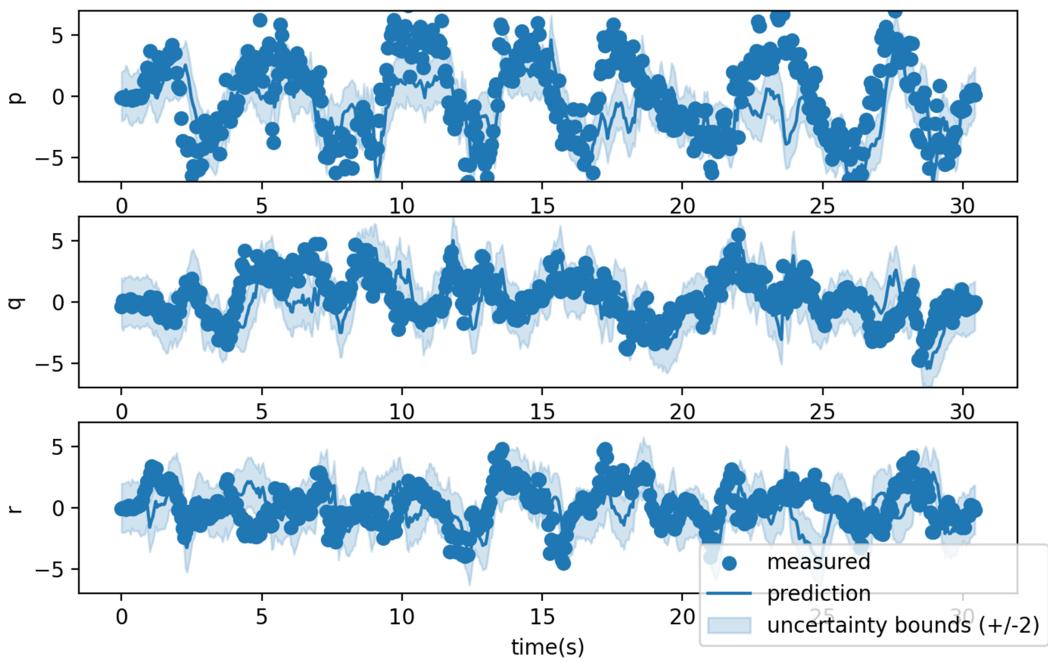
We collect measurements from three sensors, an accelerometer, gyroscope and magnetometer.

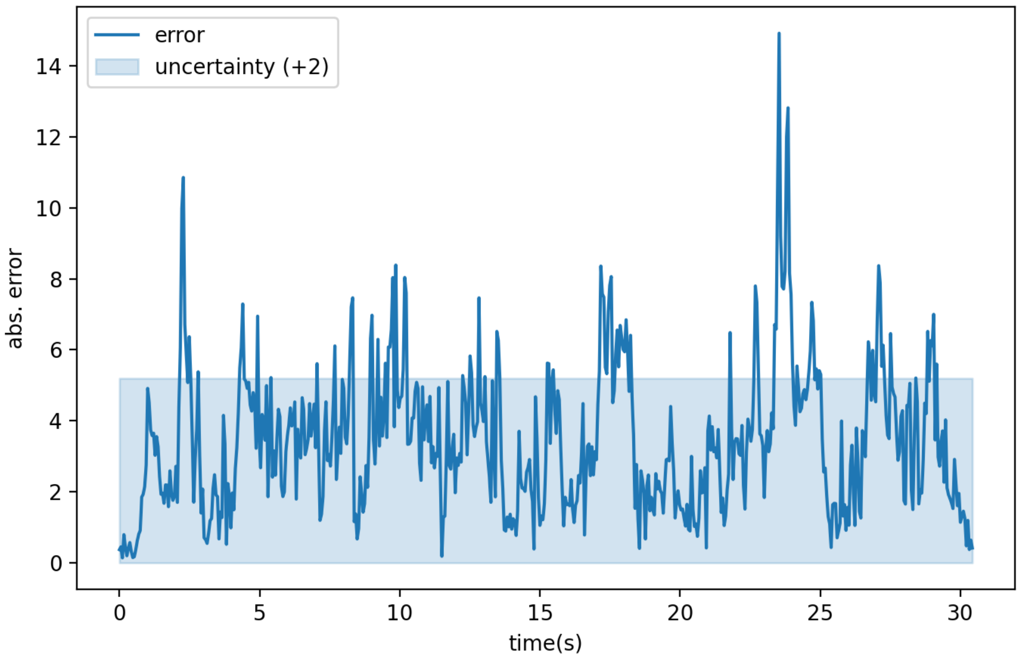
We run the Kalman filter using the accelerometer measurements aswell as the magnetometer measurements.

We then compare our predicted body rates to the measured gyroscope measurements. The results are shown in the image below.

Results

Here $[p \quad q \quad r] \triangleq [\omega_x \quad \omega_y \quad \omega_z]$ for the three plots below.




error
uncertainty (+2)
abs. error
time(s)
0
2
4
6
8
10
12
14
0
5
10
15
20
25
30