Quaternion Extended Kalman Filter

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Contents

1 State Definition

$$\boldsymbol{q}_k = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} \tag{1}$$

$$\boldsymbol{\omega}_k = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2}$$

2 State Transition Equation

$$\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{q}_k \\ \boldsymbol{\omega}_k \end{bmatrix} \tag{3}$$

2.1 Quaternion rate of change

$$\dot{q}_w = \frac{1}{2}\left(-q_x\omega_x - q_y\omega_y - q_z\omega_z\right) \tag{4}$$

$$\dot{q}_x = \frac{1}{2}\left(q_w\omega_x + q_y\omega_z - q_z\omega_y\right) \tag{5}$$

$$\dot{q}_y = \frac{1}{2}\left(q_w\omega_y - q_x\omega_z + q_z\omega_x\right) \tag{6}$$

$$\dot{q}_z = \frac{1}{2}\left(q_w\omega_z + q_x\omega_y - q_y\omega_x\right) \tag{7}$$

2.2 Defining the omega operator as

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{8}$$

2.3 Defining the Xi operator as

$$\Xi(\boldsymbol{q}) = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z \\ q_w & -q_z & q_y \\ q_z & q_w & -q_x \\ -q_y & q_x & q_w \end{bmatrix} \tag{9}$$

2.4 State differential equation

Here $w(t)$ is white noise - this is a constant angular velocity model.

$$\dot{\boldsymbol{q}} = \boldsymbol{\Omega}(\boldsymbol{\omega})\boldsymbol{q} = \Xi(\boldsymbol{q})\boldsymbol{\omega} \tag{10}$$

$$\dot{\boldsymbol{\omega}} = w(t) \tag{11}$$

$$\frac{\partial}{\partial t}\boldsymbol{x} = \begin{bmatrix} 0 & \Xi(\boldsymbol{q}) \\ 0 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ I \end{bmatrix} w(t) \tag{12}$$

2.4.1 Deriving Kalman Filter Predict Equation

If we take Laplace transforms of 12, we obtain:

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \begin{bmatrix} 0 & \Xi(\boldsymbol{q}) \\ 0 & 0 \end{bmatrix} \boldsymbol{X}(s) + \begin{bmatrix} 0 \\ I \end{bmatrix} W(t) \tag{13}$$

Rearranging:

$$\boldsymbol{X}(s) = \begin{bmatrix} sI & -\Xi(\boldsymbol{q}) \\ 0 & sI \end{bmatrix}^{-1} \boldsymbol{x}(0) + \begin{bmatrix} sI & -\Xi(\boldsymbol{q}) \\ 0 & sI \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} W(t) \tag{14}$$

And taking inverse Laplace transforms:

$$\boldsymbol{x}(\Delta t) = \begin{bmatrix} I & \Delta t \Xi(\boldsymbol{q}) \\ 0 & I \end{bmatrix} \boldsymbol{x}(0) + \int_0^{\Delta t} \begin{bmatrix} (\Delta t - \tau)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(\tau) d\tau \tag{15}$$

2.4.2 Finding the Process Noise

Similar derivations can be found in Estimation II.

$$\boldsymbol{w} = \int_0^{\Delta t} \begin{bmatrix} (\Delta t - \tau)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(\tau) d\tau \tag{16}$$

The expected value is zero:

$$E\left[\boldsymbol{w}\right] = 0 \tag{17}$$

The expected process noise is therefore:

$$\begin{aligned}
Q =& E\left[\boldsymbol{w}\boldsymbol{w}^\top\right] && (18)\\
=& E\left[\left(\int_0^{\Delta t} \begin{bmatrix} (\Delta t - u)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(u) du\right)\left(\int_0^{\Delta t} \begin{bmatrix} (\Delta t - v)\Xi(\boldsymbol{q}) \\ I \end{bmatrix} w(v) dv\right)^\top\right] && (19)\\
=& \int_0^{\Delta t}\int_0^{\Delta t} \begin{bmatrix} (\Delta t - u)(\Delta t - v)\Xi(\boldsymbol{q})\Xi(\boldsymbol{q})^\top & (\Delta t - u)\Xi(\boldsymbol{q}) \\ (\Delta t - v)\Xi(\boldsymbol{q})^\top & I \end{bmatrix} E[w(u)w(v)^\top] du dv && (20)\\
=& q \begin{bmatrix} \frac{\Delta t^3}{3}\Xi(\boldsymbol{q})\Xi(\boldsymbol{q})^\top & \frac{\Delta t^2}{2}\Xi(\boldsymbol{q}) \\ \frac{\Delta t^2}{2}\Xi(\boldsymbol{q})^\top & \Delta t \end{bmatrix} && (21)
\end{aligned}$$

where $E[w(u)w(v)^\top] = q$.

3 Extended Kalman Filter

3.1 Kalman Filter Prediction Equation

$$\boldsymbol{x}_{k+1|k} = f(\boldsymbol{x}_{k|k}) = \begin{bmatrix} I & \Delta t \Xi(\boldsymbol{q}_{k|k}) \\ 0 & I \end{bmatrix} \boldsymbol{x}_{k|k} \tag{22}$$

$$\boldsymbol{\Sigma}_{k+1|k} = F_k \boldsymbol{\Sigma}_{k|k} F_k^\top + Q_k \tag{23}$$

$$F_k = \frac{\partial f(\boldsymbol{x}_{k|k})}{\partial \boldsymbol{x}_{k|k}} = \begin{bmatrix} I + \Delta t \boldsymbol{\Omega}(\boldsymbol{\omega}_{k|k}) & \Delta t \Xi(\boldsymbol{q}_{k|k}) \\ 0 & I \end{bmatrix} \tag{24}$$

3.2 Measurement Equations

3.2.1 Rotation Matrix from state quaternion

$$R(\boldsymbol{x}) = \begin{bmatrix} 2(q_w^2 + q_x^2) - 1 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & 2(q_w^2 + q_y^2) - 1 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & 2(q_w^2 + q_z^2) - 1 \end{bmatrix} \tag{25}$$

3.2.2 Accelerometer

$$h(\boldsymbol{x}) = R(\boldsymbol{q})^\top \boldsymbol{g} \tag{26}$$

where $\boldsymbol{g} = \begin{bmatrix} 0 & 0 & g \end{bmatrix}^\top$ is the expected gravity field vector.

3.2.3 Magnetometer

$$h(\boldsymbol{x}) = R(\boldsymbol{q})^\top \boldsymbol{m} \tag{27}$$

where $\boldsymbol{m} = \begin{bmatrix} m_x & m_y & m_z \end{bmatrix}^\top$ is the expected magnetic field vector at the location of the sensor.

3.3 Kalman Filter Update Equations

3.3.1 Innovation

$$\boldsymbol{y}_k = \boldsymbol{z}_k - h(\boldsymbol{x}_{k+1|k}) \tag{28}$$

$$S_k = H_k \boldsymbol{\Sigma}_{k+1|k} H_k^\top + R_k \tag{29}$$

$$H_k = \frac{\partial h(\boldsymbol{x}_{k+1|k})}{\partial \boldsymbol{x}_{k+1|k}} \tag{30}$$

3.3.2 Kalman Gain

$$K_k = \boldsymbol{\Sigma}_{k+1|k} H_k^\top S_k^{-1} \tag{31}$$

3.3.3 Updates

$$\boldsymbol{x}_{k+1|k+1} = \boldsymbol{x}_{k+1|k} + K_k \boldsymbol{y}_k \tag{32}$$

$$\boldsymbol{\Sigma}_{k+1|k+1} = (I - K_k H_k) \boldsymbol{\Sigma}_{k+1|k} \tag{33}$$

3.4 Results

3.4.1 Experiment

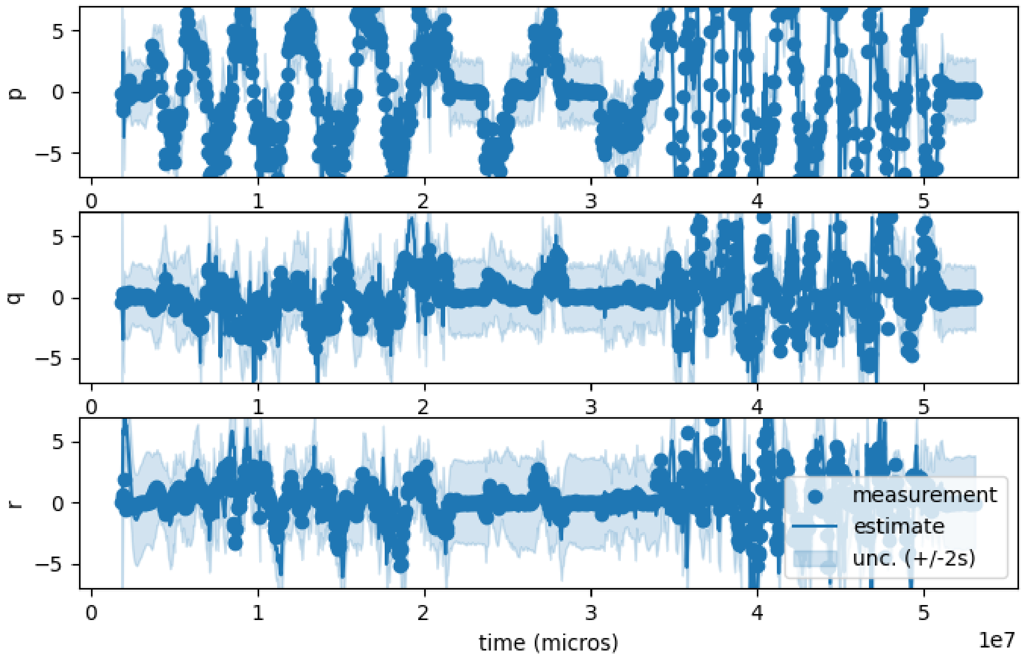
We collect measurements from three sensors, an accelerometer, gyroscope and magnetometer.

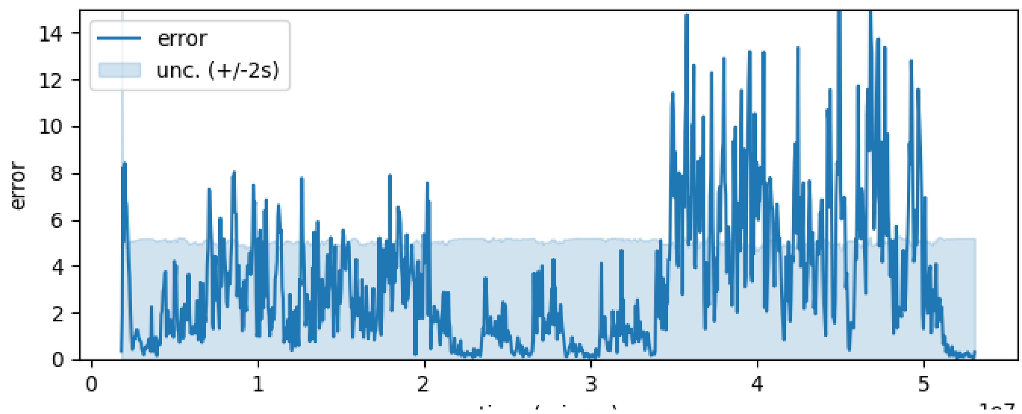
We run the Kalman filter using the accelerometer measurements aswell as the magnetometer measurements.

We then compare our predicted body rates to the measured gyroscope measurements. The results are shown in the image below.

3.4.2 Results

Here $[p \quad q \quad r] \triangleq [\omega_x \quad \omega_y \quad \omega_z]$ for the three plots below.





4 Unscented Kalman Filter

4.1 Weighted Mean and Covariance

If $\boldsymbol{x} \in \mathcal{X}, \boldsymbol{y} \in \mathcal{Y}$ are two sets of points with the same number of points, the weighted mean can be expressed as, (for some set of weights w):

$$\mu_{\mathcal{X}} = \frac{1}{n} \sum_{i=1}^{n} w_i \boldsymbol{x}_i \tag{34}$$

The weighted covairance:

$$\Sigma_{\mathcal{X},\mathcal{Y}} = \frac{1}{n} \sum_{i=1}^{n} w_i \left(\boldsymbol{x}_i - \mu_{\mathcal{X}}\right) \left(\boldsymbol{y}_i - \mu_{\mathcal{Y}}\right)^{\top} \tag{35}$$

4.2 Square root of a matrix

There are numerous methods to find the square root of a matrix, we present one of them.

Using singular value decomposition:

$$\Sigma = UWV^{\top} \implies \sqrt{\Sigma} = V\sqrt{W}V^{\top} \tag{36}$$

$$W = \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & \cdots & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & w_n \end{bmatrix} \implies \sqrt{W} = \begin{bmatrix} \sqrt{w_0} & 0 & \cdots & 0 \\ 0 & \sqrt{w_1} & \cdots & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix} \tag{37}$$

4.3 Kalman Filter Prediction Equations

We begin by calculating the sigma points:

$$\mathcal{X}_{k|k}^{0} = \boldsymbol{x}_{k|k} \tag{38}$$

$$\mathcal{X}_{k|k}^{i} = \boldsymbol{x}_{k|k} + \sqrt{(N+\lambda)\Sigma_{k|k}} \qquad i = 1, \ldots, N \tag{39}$$

$$\mathcal{X}_{k|k}^{i} = \boldsymbol{x}_{k|k} - \sqrt{(N+\lambda)\Sigma_{k|k}} \qquad i = N, \ldots, 2N \tag{40}$$

We update the sigma points:

$$\mathcal{X}_{k+1|k}^{i} = f(\mathcal{X}_{k|k}^{i}) \qquad i = 0, \ldots, 2N \tag{41}$$

We now update the state and covariance using equations 34 and 35:

$$\boldsymbol{x}_{k+1|k} = \mu_{\mathcal{X}_{k+1|k}} \tag{42}$$

$$\boldsymbol{\Sigma}_{k+1|k} = \Sigma_{\mathcal{X}_{k+1|k}, \mathcal{X}_{k+1|k}} \tag{43}$$

4.4 Measurement Update Equations

We begin by calculating the sigma points:

$$\mathcal{Z}_{k}^{i} = h(\mathcal{X}_{k+1|k}^{i}) \qquad i = 0, \ldots, 2N \tag{44}$$

Using the measurement sigma points, we find the innovation covariance, and the state measurement covariance:

$$\boldsymbol{\Sigma}_{\boldsymbol{z},\boldsymbol{z}} = \Sigma_{\mathcal{Z}_{k+1|k}, \mathcal{Z}_{k+1|k}} \tag{45}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{x},\boldsymbol{z}} = \Sigma_{\mathcal{X}_{k+1|k}, \mathcal{Z}_{k+1|k}} \tag{46}$$

4.4.1 Kalman Gain

$$K_k = \boldsymbol{\Sigma}_{x,z}\boldsymbol{\Sigma}_{z,z}^{-1} \tag{47}$$

4.4.2 Updates

$$\boldsymbol{x}_{k+1|k+1} = \boldsymbol{x}_{k+1|k} + K_k\left(\boldsymbol{z}_k - \mu_{\mathcal{Z}}\right) \tag{48}$$

$$\boldsymbol{\Sigma}_{k+1|k+1} = \boldsymbol{\Sigma}_{k+1|k} - K\Sigma_{z,z}K^{\top} \tag{49}$$

4.5 Results

Using the same experiment as above we run the UKF on the data.

Here $[p \quad q \quad r] \triangleq [\omega_x \quad \omega_y \quad \omega_z]$ for the three plots below.