7.8 Groups of Order 12, Math 102 **Name:** *James Wang*
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Important Facts

Proposition 2.11.4: Let H and K be subgroups of a group G, and let $f : H \times K \to G$ be the multiplication map, defined by $f(h,k) = hk$. Its image is the set $HK = \{hk \mid h \in H, k \in K\}$.
a) f is injective if and only if $H \cap K = \{1\}$.
b) f is a homomorphism from the product group $H \times K$ to G if and only if elements of K commute with elements of $H : hk = kh$.
c) If H is a normal subgroup of G, then HK is a subgroup of G.
d) f is an isomorphism from the product group $H \times K$ to G if and only if $H \cap K = \{1\}$, $HK = G$, and also H and K are normal subgroups of G.

Lemma 7.5.5 a): For every $n \geq 3$, the alternating group A_n is generated by 3-cycles.

Definition 7.7.1: Let G be a group of order n, and let p be a prime integer that divides n. Let p^e denote the largest power of p that divides n, so that $n = p^e m$, where m is an integer not divisible by p. Subgroups H of G of order p^e are Sylow p-subgroups of G.

Theorem 7.7.4 Second Sylow Theorem a): Let G be a finite group whose order is divisible by a prime p. The Sylow p-subgroups of G are conjugate subgroups.

Lemma 7.7.5: A group G has just one Sylow p-subgroup H if and only if that subgroup is normal.

Theorem 7.7.6 Third Sylow Theorem: Let G be a finite group whose order n is divisible by a prime p. Say that $n = p^e m$, where p does not divide m, and let s denote the number of Sylow p-subgroups. Then s divides m and s is congruent to 1 modulo p.

Exercise 7.4 b) Sketch:

We want to show no simple group has order p^2q, where p, q are prime.
Suppose $|G| = p^2q$, where p, q are prime. Let s_p be the number of Sylow p-subgroups, and let s_q be the number of Sylow q-subgroups. Third Sylow Theorem $\implies$ $s_q = 1, p$, or p^2.

Case 1: Suppose $s_q = 1$. Then there is only one Sylow q-subgroup of G.

Case 2: Suppose $s_q = p$. Third Sylow Theorem $\implies$ $p \equiv 1 \bmod q$, so $p > q$.
Third Sylow Theorem $\implies$ $s_p \mid q$ and $s_p \equiv 1 \bmod p$.

Case 3: Suppose $s_q = p^2$. Since q is prime, the intersection of all of the Sylow q-subgroups is trivial. Moreover, each of them is cyclic, and each of them consists of the identity element and $q - 1$ elements of order q.

Groups of Order 12 Sketch:

$12 = 2^2 \cdot 3$. Let s_2 be the number of Sylow 2-subgroups, and let s_3 be the number of Sylow 3-subgroups. Third Sylow Theorem $\implies$ $s_2 = 1$ or 3, and $s_3 = 1$ or 4.

Let H be a Sylow 2-subgroup and K be a Sylow 3-subgroup. Then $|H| = 4$ and $|K| = 3$.

Claim: Either a Sylow 2-subgroup or Sylow 3-subgroup must be normal, so $s_2 = 3$ and $s_3 = 4$ cannot happen at the same time.

Case 1: H and K are both normal, so $s_2 = s_3 = 1$.

$$G \cong C_4 \times C_3 \text{ or } G = C_2 \times C_2 \times C_3$$

Case 2: H is normal but K is not, so $s_2 = 1$ and $s_3 = 4$.

$$G \cong A_4$$

Case 3: K is normal but H is not, so $s_2 = 3$ and $s_3 = 1$.

We obtain the relation $xy = y^2x$.

Case 3a): K is normal but H is not and $H = C_4$.

$$G \cong \{x, y \mid x^4 = 1, y^3 = 1, xy = y^x\}$$

Case 3b): K is normal but H is not and $H = C_2 \times C_2$.

$$G \cong D_6$$