Week 5 Final, Math 102 **Name:** *James Wang*
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(16.5.1)

(a)

Let H be the group of automorphisms generated by σ. Since $\sigma^2(t) = (t^{-1})^{-1} = t$, $H = \{e, \sigma\}$ and $H \cong C_2$. Observe that t is a root of the polynomial $f(x) = (x - t)(x - t^{-1}) = x^2 - (t + t^{-1})x + 1$, which means $[\mathbb{C}(t) : \mathbb{C}(t + t^{-1})] \leq 2$. Clearly, $\mathbb{C}(t + t^{-1})$ is a subfield of the fixed field $\mathbb{C}(t)^H$. Finally, by the Fixed Field Theorem, $[\mathbb{C}(t) : \mathbb{C}(t)^H] = |H| = 2$. Hence it must be $[\mathbb{C}(t)^H : \mathbb{C}(t + t^{-1})] = 1$ and $\mathbb{C}(t)^H = \mathbb{C}(t + t^{-1})$.

(b)

Let H be the group of automorphisms generated by σ. Since $\sigma(t) = it$, $\sigma^2(t) = -t$, $\sigma^3(t) = -it$, and $\sigma^4(t) = t$, we know $H = \{e, \sigma, \sigma^2, \sigma^3\}$ and $H \cong C_4$. Observe that t is a root of the polynomial $f(x) = (x - it)(x + t)(x + it)(x - t) = (x^2 + t^2)(x^2 - t^2) = x^4 - t^4$, which means $[\mathbb{C}(t) : \mathbb{C}(t^4)] \leq 4$. Clearly, $\mathbb{C}(t^4)$ is a subfield of the fixed field $\mathbb{C}(t)^H$. Finally, by the Fixed Field Theorem, $[\mathbb{C}(t) : \mathbb{C}(t)^H] = |H| = 4$. Hence it must be $[\mathbb{C}(t)^H : \mathbb{C}(t^4)] = 1$ and $\mathbb{C}(t)^H = \mathbb{C}(t^4)$.

(c)

Let H be the group of automorphisms generated by σ. Since $\sigma^2(t) = -(-t) = t$ and $\tau^2(t) = (t^{-1})^{-1} = t$, we know $H = \{e, \sigma, \tau, \tau\sigma\}$, and $H \cong C_2 \times C_2$. Observe that t is a root of the polynomial $f(x) = (x-t)(x+t)(x-t^{-1})(x+t^{-1}) = (x^2-t^2)(x^2-t^{-2}) = x^4-(t^2+t^{-2})x^2+1$, which means $[\mathbb{C}(t) : \mathbb{C}(t^2 + t^{-2})] \leq 4$. Clearly, $\mathbb{C}(t^2 + t^{-2})$ is a subfield of the fixed field $\mathbb{C}(t)^H$. Finally, by the Fixed Field Theorem, $[\mathbb{C}(t) : \mathbb{C}(t)^H] = |H| = 4$. Hence it must be $[\mathbb{C}(t)^H : \mathbb{C}(t^2 + t^{-2})] = 1$ and $\mathbb{C}(t)^H = \mathbb{C}(t^2 + t^{-2})$.

(d)

Let H be the group of automorphisms generated by σ. Since $\sigma(t) = \omega t$, $\sigma^2(t) = \omega^2 t$, $\sigma^3(t) = t$, $\tau(t) = t^{-1}$, and $\tau^2(t) = t$, we know $H = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$, and $H \cong S_3$. Observe that t is a root of the polynomial $f(x) = (x-t)(x-t^{-1})(x-\omega t)(x-\omega t^{-1})(x-\omega^2 t)(x-\omega^2 t^{-1}) = x^6 - (t^3 + t^{-3})x^3 + 1$, which means $[\mathbb{C}(t) : \mathbb{C}(t^3 + t^{-3})] \leq 6$. Clearly, $\mathbb{C}(t^3 + t^{-3})$ is a subfield of the fixed field $\mathbb{C}(t)^H$. Finally, by the Fixed Field Theorem, $[\mathbb{C}(t) : \mathbb{C}(t)^H] = |H| = 6$. Hence it must be $[\mathbb{C}(t)^H : \mathbb{C}(t^3 + t^{-3})] = 1$ and $\mathbb{C}(t)^H = \mathbb{C}(t^3 + t^{-3})$.

(16.5.3)

Recall that given a field extension K/F, an element $\alpha \in K$ is either transcendental or algebraic. We use proof by contrapositive: Suppose that $\alpha \in \mathbb{C}(t)$ is algebraic over $\mathbb{C}$, we want to show $\alpha \in \mathbb{C}$. By the definition of algebraic, we know α is the root of some polynomial $f(x) \in \mathbb{C}[x]$. Furthermore, the Fundamental Theorem of Algebra allows us to factor $f(x) = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_n)$, where $\beta_i \in \mathbb{C}$ and $n = \deg(f)$. Since $f(\alpha) = 0$, it must be $\alpha = \beta_i$ for some $1 \leq i \leq n$. Hence $\alpha \in \mathbb{C}$.

QED

(16.6.1)

Since $x^3 + x + 1$ has no root in $\mathbb{Q}$, it is irreducible over $\mathbb{Q}$. So if α is a complex root, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. By observation, moreover, $\sqrt{-31}$ is a root of the polynomial $x^2 + 31$, so $[\mathbb{Q}(\sqrt{-31}) : \mathbb{Q}] = 2$. Since 2 does not divide 3, $\mathbb{Q}(\sqrt{-31}) \nsubseteq \mathbb{Q}(\alpha)$ and $\sqrt{-31} \notin \mathbb{Q}(\alpha)$.

By inspection, suppose u_1, u_2, and u_3 are roots of $x^3 + x + 1$, then using the discriminant formula $-4p^3 - 27q^2$ Artin provides for $f(x) = x^3 + px + q$ on page 482, we get

$$D(u) = (u_1 - u_2)^2(u_1 - u_3)^2(u_2 - u_3)^2 = -4(1)^3 - 27(1)^2 = -31.$$

It follows

$$(u_1 - u_2)(u_1 - u_3)(u_2 - u_3) = \sqrt{-31}.$$

Since K is a splitting field of the polynomial, $u_1, u_2, u_3 \in K$.

Hence $(u_1 - u_2)(u_1 - u_3)(u_2 - u_3) = \sqrt{-31} \in K$.

(Exercise 3) Let K/F be a Galois extension with $[K:F]=10$ with Abelian Galois group.

(1) How many intermediate fields L are there?

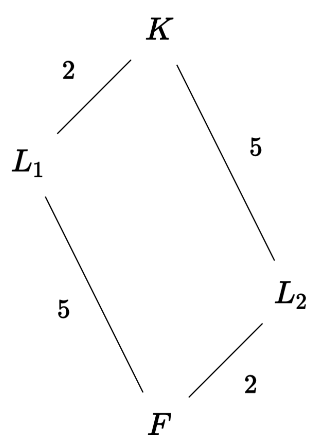
Since K/F is a Galois extension and $[K:F]=10$, $|G(K/F)|=10$. We also know $G(K/F)$ is abelian. By the Structure Theorem for Finite Abelian Groups, we can write $G(K/F)$ as a product of cyclic groups. The only options are $C_1 \times C_{10}$ and $C_2 \times C_5$. Nevertheless, notice that $\gcd(2,5)=1$, so $C_1 \times C_{10} \cong C_2 \times C_5$. They are both cyclic groups of order 10.

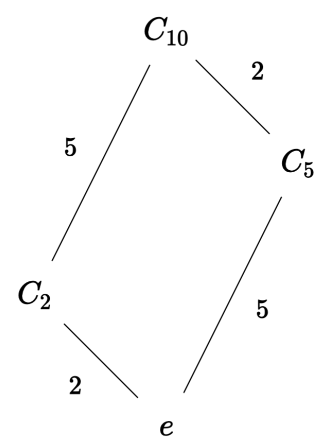
Furthermore, we know every subgroup of a cyclic group is cyclic, and by Lagrange's Theorem, the order of the subgroup has to divide the order of the group. Thus, the subgroups of C_{10} are e, C_2, C_5, and C_{10}, and the only proper subgroups are C_2 and C_5.

By the Main Theorem of Galois Theory, we know there is a bijective correspondence between subgroups of $G(K/F)$ and intermediate fields of K/L, so there are 2 intermediate fields L.

(2) What can you say about the degrees $[L:F]$ and $[K:L]$ for each of these intermediate fields?

Let H be subgroups of $G(K/L)$. If L corresponds to H, then $[K:L]=|H|$ and $[L:F]=[G:H]$.





As shown in the illustrations, $[K:L_1]=|C_2|$ and $[L_1:F]=[C_{10}:C_2]$, so L_1 corresponds to C_2. $[K:L_2]=|C_5|$ and $[L_2:F]=[C_{10}:C_5]$, so L_2 corresponds to C_5.