Week 11 Final, Math 102 **Name:** *James Wang*
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(6.3.1)

We want to verify the rules in 6.3.3.

1) Let $v' = \rho_\theta(v)$. Since ρ_θ is an orthogonal linear operator and t_a is a translation, similar to the proof in 6.2.8, we have

$$\rho_\theta t_v(x) = \rho_\theta(x+v) = \rho_\theta(x) + \rho_\theta(v) = \rho_\theta(x) + v' = t_{v'}\rho_\theta(x)$$

Hence we have verified $\rho_\theta t_v = t_{v'}\rho_\theta$.

2) Let $v' = r(v)$. By observation, the reflection matrix is orthogonal, because

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

meaning r is also an orthogonal linear operator. By a similar reasoning as part 1), we can verify $rt_v = t_{v'}r$.

3) We perform the following matrix multiplications:

$$r\rho_\theta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

$$\rho_{-\theta}r = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

Hence we have verified $r\rho_\theta = \rho_{-\theta}r$.

4) The book provides the statement $t_v t_w = t_{v+w}$ in 6.2.8.

5) We perform the following matrix multiplication:

$$\begin{aligned}
\rho_\theta \rho_\eta &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\eta - \sin\theta\sin\eta & -(\sin\theta\cos\eta + \sin\eta\cos\theta) \\ \sin\theta\cos\eta + \sin\eta\cos\theta & \cos\theta\cos\eta - \sin\theta\sin\eta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\eta) & -\sin(\theta+\eta) \\ \sin(\theta+\eta) & \cos(\theta+\eta) \end{bmatrix} \\
&= \rho_{\theta+\eta}
\end{aligned} \tag{1}$$

Hence we have verified $\rho_\theta \rho_\eta = \rho_{\theta+\eta}$.

6) Clearly,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence we have verified $rr = 1$.

$$QED$$

(6.5.1)

Suppose l_1, l_2 are lines through the origin in $\mathbb{R}^2$ that intersect in an angle π/n, and let r_i be the reflection about l_i. We want to show that r_1, r_2 generate a dihedral group D_n.

Since we are already given two reflections r_1, r_2, to show that they generate D_n, it suffices to show that they generate a rotation $\rho_{2\pi/n}$.

By observation, we note that doing the reflection r_2 is the same as first doing a rotation $\rho_{\pi/n}$, then the reflection r_1, and finally the rotation $\rho_{-\pi/n}$, so we have the following:

$$r_2 = \rho_{-\pi/n} r_1 \rho_{\pi/n}$$

In using the third rule in 6.3.3, we can simplify to

$$\begin{aligned} r_2 &= \rho_{-\pi/n} r_1 \rho_{\pi/n} \\ &= r_1 \rho_{\pi/n} \rho_{\pi/n} \\ &= r_1 \rho_{2\pi/n} \end{aligned} \tag{2}$$

Thus we see that $r_1^{-1} r_2 = \rho_{2\pi/n}$, and a rotation by $\rho_{2\pi/n}$ can indeed be generated by r_1, r_2. Hence r_1, r_2 generate a dihedral group D_n.

$$QED$$

(6.3.5)

We want to write formulas for the isometries in terms of a complex variable $z = x + iy$.

1) translation t_a by a vector $a = a_1 + ia_2$:

$$\begin{aligned} t_a(z) &= z + a \\ &= (x + iy) + (a_1 + ia_2) \\ &= (x + a_1) + i(y + a_2) \end{aligned} \tag{3}$$

2) rotation ρ_θ by an angle θ about the origin:

$$\begin{aligned} \rho_\theta(z) &= e^{i\theta}(x + iy) \\ &= (\cos\theta + i\sin\theta)(x + iy) \\ &= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta) \end{aligned} \tag{4}$$

3) reflection r about the e_1-axis:

$$\begin{aligned} r(z) &= r(x + iy) \\ &= x - iy \\ &= \overline{z} \end{aligned} \tag{5}$$