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Sept 6, 2023

(12.3.1)

a):

We see whether we can use double inclusion to show that the ideal generated by an element of $\mathbb{Z}[x]$ and $ker(\phi)$ are subsets of each other.

Suppose $f(x) \in ker(\phi)$, then $\phi(f(x)) = f(1+\sqrt{2}) = 0$. By observation, although $(1+\sqrt{2})$ is not an integer root, it is promising that we can manipulate it into something we can use, because it resembles the numerator of the quadratic formula of finding roots of polynomials in the form $ax^2 + bx + c$, where $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

For the sake of keeping computations simple, we assume $a = 1$, then it follows $b = -2$ and $c = -1$, giving us x^2-2x-1. It is easy to check that $\phi(x^2-2x-1) = 0$, so $x^2-2x-1 \in ker(\phi)$. More importantly, it is also true that any polynomial $f(x) = g(x) \cdot (x^2 - 2x - 1)$, or the ideal $(x^2 - 2x - 1)$ is in $ker(\phi)$.

We now already have one side of the inclusion: ideal $(x^2 - 2x - 1) \subseteq ker(\phi)$. We then want to see whether the other inclusion is true.

Suppose $p(x) \in ker(\phi)$, it may also be true that $p(x) = q(x)(x^2 - 2x - 1) + r(x)$, and $\phi(r(x)) = 0$. By the definition of Euclidean Algorithm, it must be true $r(x) = mx + n$. We see that $\phi(r(x)) = 0$ if and only if $m(1 + \sqrt{2}) + n = 0$, yet there are no integer solutions for this other than $m = n = 0$.

Therefore, any $p(x) \in \mathbb{Z}[x]$ that is also in $ker(\phi)$ must have a remainder of 0 when divided by $x^2 - 2x - 1$, which gives us the inclusion $ker(\phi) \subseteq$ ideal $(x^2 - 2x - 1)$.

The kernel of ϕ is a principal ideal. A generator is $x^2 - 2x - 1$.

Note: part b) of this problem is similar

(12.4.3)

We notice that since $\mathbb{Q}$ is a field, $\mathbb{Q}[x]$ must be an Euclidean Domain and therefore also a Principal Ideal Domain, because every Euclidean Domain is a PID.

We then note a corollary about PIDs: Let R be a PID, then the maximal ideals of R are the principal ideals generated by the irreducible elements. So the real question is whether $x^4 + 6x^3 + 9x + 3$ is irreducible in $\mathbb{Q}[x]$.

We use the Eisenstein Criterion to determine this. By observation, 3 is a prime integer that makes the polynomial's coefficients meet the requirements:

1) 3 does not divide 1.

2) 3 divides all other coefficients: $6, 9, 3$

3) $3^2 = 9$ does not divide 3.

Therefore, the polynomial $x^4 + 6x^3 + 9x + 3$ is irreducible in $\mathbb{Q}[x]$, and this polynomial does generate a maximal ideal in $\mathbb{Q}$.

$$QED$$