Week 10 Final, Math 102 **Name:** *James Wang*
Nov 14, 2023

(7.2.1)

a) We want to find the centralizer and the order of the conjugacy class of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{F}_3)$.

We first find the centralizer. By definition it is the set of invertible 2×2 matrices such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

We see it must be that $a = d$ and $c = 0$, and to make the matrix invertible, entries of the first column cannot be all 0, so we have

$$Z\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \in \{1,2\}, b \in \{0,1,2\} \right\}$$

which is of size 6.

We then find the size of the conjugacy class. Using the formula in Exercise 3.5.4, we know

$$|GL_2(\mathbb{F}_3)| = 3(3+1)(3-1)^2 = 48$$

so by the counting formula we have

$$\left| C\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \right| = \frac{|GL_2(\mathbb{F}_3)|}{\left| Z\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \right|} = \frac{48}{6} = 8.$$

b) We want to find the centralizer and the order of the conjugacy class of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in GL_2(\mathbb{F}_5)$.

We first find the centralizer. By definition it is the set of invertible 2×2 matrices such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

We see it must be that $b = c = 0$, and to make the matrix invertible, entries of neither column can be all 0, so we have

$$Z\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \in \{1, 2, 3, 4\}\right\}$$

which is of size 16.

We then find the size of the conjugacy class. Using the formula in Exercise 3.5.4, we know

$$|GL_2(\mathbb{F}_5)| = 5(5+1)(5-1)^2 = 480$$

so by the counting formula we have

$$\left|C\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\right| = \frac{|GL_2(\mathbb{F}_5)|}{\left|Z\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\right|} = \frac{480}{16} = 30.$$

(7.3.2)

Let Z be the center of a group G. Suppose G/Z is a cyclic group, we want to show that G is abelian and therefore $G = Z$.

Since G/Z is cyclic and the elements of a quotient group are cosets, we know there exists some xZ where $x \in G$ such that $< xZ > = G/Z$. Thus for all $gZ \in G/Z$, there exists some $n \in \mathbb{N}$ such that $gZ = (xZ)^n$. By the operation of cosets and the coset recognition lemma, we have

$$gZ = (xZ)^n$$
$$gZ = x^n Z$$
$$(x^n)^{-1} g \in Z$$
$$(x^n)^{-1} g = z \text{ where } z \in Z$$
$$g = x^n z$$

This means we cam express all $g \in G$ in the form $g = x^n z$ with $n \in \mathbb{N}$ and $z \in Z$.

Now take arbitrary $g_1, g_2 \in G$, we know $g_1 = x^{n_1} z_1, g_2 = x^{n_2} z_2$. We want to consider $g_1 g_2$. Since $z_1, z_2 \in Z$ and group operations are associative, we can rearrange terms on the right hand side to get the desired result:

$$\begin{aligned} g_1 g_2 &= x^{n_1} z_1 x^{n_2} z_2 \\ &= x^{n_1} x^{n_2} z_1 z_2 \\ &= x^{n_1+n_2} z_1 z_2 \\ &= x^{n_2} x^{n_1} z_1 z_2 \\ &= x^{n_2} x^{n_1} z_2 z_1 \\ &= x^{n_2} z_2 x^{n_1} z_1 \\ &= g_2 g_1 \end{aligned} \tag{1}$$

$$QED$$

(7.5.2)

We want to find the centralizer of (1 2) in S_5, which is the set of elements in S_5 that is fixed when conjugate by (1 2).

Since (1 2) is a transposition, when we use it to conjugate some $\tau \in S_5$, we are essentially swapping the places of 1 and 2 in the cycle notation of τ. If in τ either 1 or 2 is sent to an element that is not 1 or 2, then τ after conjugation must be different. Hence the centralizer of (1 2) in S_5 are the elements that either fix 1 and 2 or swap 1 and 2, while the permutation of $3, 4$, and 5 does not really matter. Thus the centralizer of (1 2) in S_5 is of the structure $S_2 \times S_3$.

(7.2.3)

Suppose a group G of order 12 contains a conjugacy class $C(x)$ of order 4. We want to show that $Z(G)$ is trivial.

Since under the conjugation action, the conjugacy class $C(x)$ is the orbit of x and the centralizer $Z(x)$ is the stabilizer of x, by the counting formula we have

$$|Z(x)| = \frac{|G|}{|C(x)|} = \frac{12}{4} = 3$$

Recall further that we always have $Z(G) \leqslant Z(x)$, so by Lagrange's Theorem, we have either $|Z(G)| = 1$ or $|Z(G) = 3|$. Now suppose for a contradiction that $|Z(G) = 3|$, then it must be $Z(x) = Z(G)$. Since we always have $x \in Z(x)$, it follows that $x \in Z(G)$.

Yet, if $x \in Z(G)$, then $Z(x) = G$ by definition, so $|C(x)| = 1$ by the counting formula.

We have reached a contradiction with our initial assumption of $|C(x)| = 4$. Hence it must be that $|Z(G)| = 1$.

$$QED$$