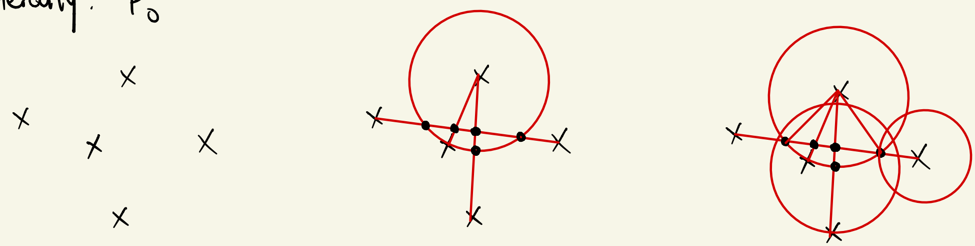
Key Ideas:

* What does it mean for a point/number to be constructible?
* The set of constructible numbers is an extension field of the rationals.

constructible $\Leftrightarrow$ lies in that extension field

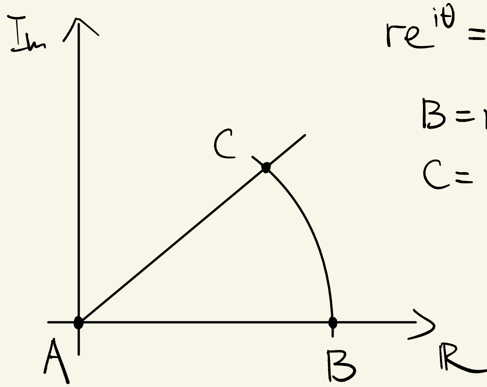
Generally: P_0



* A point is constructible if it is constructible from the set

$$P_0 = \{(0,0), (1,0)\} \subseteq \mathbb{R}^2$$

* $z = a + bi \in \mathbb{C}$ is constructible if $(a,b) \in \mathbb{R}^2$ is constructible



$re^{i\theta} = r(\cos\theta + i\sin\theta)$

$B = r$

$C = re^{i\theta}$

bisecting an angle $\Rightarrow$ constructing $re^{i\theta/2}$ from $re^{i\theta}$

* Let $K \subseteq \mathbb{C}$ be the set of constructible numbers,

K is a subfield of $\mathbb{C}$, so $K/\mathbb{Q}$ and $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$

(*) Let $F \subseteq K$ be a field generated by r & c construction, and suppose α is constructible from F in 1 step:

what about $[F(\alpha):F]$?

3 ways new points are constructed:

① 2 lines intersecting $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ $[F(\alpha):F] = 1$

② 1 line intersecting 1 circle $\begin{cases} ax+by=c \\ (x-d)^2+(y-e)^2 = r^2 \end{cases}$ $[F(\alpha):F] \leq 2$

③ 2 circles intersecting $\begin{cases} (x-d_1)^2+(y-e_1)^2 = r_1^2 \\ (x-d_2)^2+(y-e_2)^2 = r_2^2 \end{cases}$ $[F(\alpha):F] \leq 2$

reduced to case ②

Theorem: $[F(\alpha):F] = 2$

can ignore times when $[F(\alpha):F] = 1$, because α is in the original field, α does not generate a field extension

Corollary: if α is constructible, $[\mathbb{Q}(\alpha):\mathbb{Q}] = 2^n$, $n \in \mathbb{N}$

$\mathbb{Q}$ is the smallest field containing 1, 0