Week 4 Final, Math 102 **Name:** *James Wang*
Sept 27, 2023

(16.1.1)

(a)

The polynomial is not fixed under the transposition $(u_1\ u_2)$, so it is not symmetric. Since the polynomial is stabilized by e and the two 3-cycles, the orbit has size 2.

Orbits:

$$u_1{}^2u_2 + u_2{}^2u_3 + u_3{}^2u_1$$
$$u_2{}^2u_1 + u_1{}^2u_3 + u_3{}^2u_2.$$

(b)

The polynomial is symmetric, so the orbit is only itself.

By the systematic method, we first set $u_3 = 0$, and get the polynomial $g^\circ = (u_1 + u_2)u_2u_1$. When using the elementary symmetric functions $s_1^\circ = u_1 + u_2$ and $s_2^\circ = u_1u_2$, we see that $g^\circ = s_1^\circ s_2^\circ$. We know the product s_1s_2 differs from the given polynomial by some hs_3.

By inspection, the product

$$s_1s_2 = (u_1 + u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)$$

degree 3 and the given polynomial is also degree 3, so the h must be a constant. Furthermore, the product contains $3u_1u_2u_3$, and the given polynomial only contains $2u_1u_2u_3$.

Hence $(u_1 + u_2)(u_2 + u_3)(u_1 + u_3) = s_1s_2 - s_3$.

(c)

The polynomial is not fixed under the transposition $(u_1\ u_2)$, because only one term becomes negative, so it is not symmetric. Since the polynomial is stabilized by e and the two 3-cycles (two terms become negative), the orbit has size 2.

Orbits:

$$(u_1 - u_2)(u_2 - u_3)(u_1 - u_3)$$
$$(u_2 - u_3)(u_1 - u_3)(u_2 - u_1).$$

(d)

The polynomial is not fixed under the transposition $(u_1\ u_2)$ as it becomes the given polynomial's additive inverse, so it is not symmetric. Since the polynomial is stabilized by e and the two 3-cycles, the orbit has size 2.

Orbits:

$$u_1^3 u_2 + u_2^3 u_3 + u_3^3 u_1 - u_1 u_2^3 - u_2 u_3^3 - u_3 u_1^3$$
$$u_2^3 u_1 + u_1^3 u_3 + u_3^3 u_2 - u_2 u_1^3 - u_1 u_3^3 - u_3 u_2^3.$$

(e)

The polynomial is symmetric, so the orbit is only itself.

We use special values of variables to determine the coefficients. Since the given polynomial is degree 3, it can be written as a linear combination of $c_1 s_1^3 + c_2 s_1 s_2 + c_3 s_3$, because s_3, $s_1 s_2$, and s_1^3 are the only degree 3 monomials of the elementary symmetric polynomials.

First let $u = (1, 0, \ldots, 0)$, we get $s_1^3 = u_1^3$, so $c_1 = 1$.

Then let $c_1 = 1$ and $u = (1, 1, 0, \ldots, 0)$, we get $(u_1 + u_2)^3 = u_1^3 + 3u_1^2 u_2 + 3u_1 u_2^2 + u_2^3$. To get back to $u_1^3 + u_2^3$, we should subtract by $3(u_1^2 u_2 + u_1 u_2^2) = 3s_1 s_2$, so $c_2 = -3$.

Finally let $c_1 = 1$, $c_2 = -3$, and $u = (1, 1, 1, 0, \ldots, 0)$, we get $(u_1 + u_2 + u_3)^3 - 3(u_1 + u_2 + u_3)(u_1 u_2 + u_1 u_3 + u_2 u_3) = u_1^3 + u_2^3 + u_3^3 - 3u_1 u_2 u_3$. To get back to $u_1^3 + u_2^3 + u_3^3$, we should add by $3u_1 u_2 u_3 = 3s_3$, so $c_3 = 3$.

Putting everything together, we have $u_1^3 + u_2^3 + \cdots + u_n^3 = s_1^3 - 3s_1 s_2 + 3s_3$.

(16.2.1)

By definition, the discriminant is

$$D(u) = (u_1 - u_2)^2(u_1 - u_3)^2 \cdots (u_{n-1} - u_n)^2 = \prod_{i<j}(u_i - u_j)^2.$$

Since every permutation can be written as a product of transpositions, to prove the discriminant is a symmetric function, we only need to show that it is fixed under every transposition.

Suppose we are swapping u_l and u_m. Factors of the discriminant that do not contain u_l and u_m remain the same after the swap. Factors that contain both u_l and u_m would change from $(u_l - u_m)^2$ to $(u_m - u_l)^2$, which is the same thing. For factors that contain one of u_l and u_m, say u_l, there must exist a term $(u_l - u_k)^2$ or $(u_k - u_l)^2$ in the discriminant where k is arbitrary and $k \neq l$. Similarly, there must also exist a term $(u_m - u_k)^2$ or $(u_k - u_m)$ in the discriminant where $k \neq m$. So they still form the same polynomial after the swap.

Hence the discriminant is a symmetric function because it is fixed by every transposition.

$$QED$$

(16.3.2)

(a)

Since the given polynomial is degree 3, by the Fundamental Theorem of Algebra, it has 3 roots in the complex field. By inspection, the polynomial $x^3 - 2$ is irreducible over $\mathbb{Q}$, and $\sqrt[3]{2}$ is a root for $x^3 - 2$, so $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

Since $\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R} \subset \mathbb{C}$, it still does not contain the two complex roots. Moreover, we recall that if $\alpha \in \mathbb{C}$ is a root of a real polynomial f, then the complex conjugate $\bar{\alpha}$ is also a root of f. This means the two other roots are conjugate pairs and they lie in the same field extension.

We now have the factorization $x^3 - 2 = (x - \sqrt[3]{2})(x - \alpha)(x - \bar{\alpha})$, and $(x - \alpha)(x - \bar{\alpha})$ is a degree 2 irreducible polynomial over $\mathbb{Q}(\sqrt[3]{2})$, so $[\mathbb{Q}(\sqrt[3]{2}, \alpha) : \mathbb{Q}(\sqrt[3]{2})] = 2$.

Putting everything together, we know the degree of the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $[\mathbb{Q}(\sqrt[3]{2}, \alpha) : \mathbb{Q}] = 6$.

(b)

By inspection, we can factor the polynomial as:

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x + i)(x - i)(x + 1)(x - 1)$$

Although $x^2 + 1$ is irreducible over $\mathbb{Q}$, it does split in $\mathbb{Q}(i)$, so $\mathbb{Q}(i)$ is a splitting field for $x^4 - 1$.

We know $[\mathbb{Q}(i) : \mathbb{Q}] = 2$.

(c) (I'm not sure about the exact argument but I have the following)

We first solve for the roots:

$$x^4 + 1 = 0 \implies x^4 = -1 \implies x^2 = \pm i \implies x = \pm\sqrt{\pm i}$$

Two solutions for x is $\pm\frac{\sqrt{2}}{2}(1 + i)$. By inspection, we only need to add $\sqrt{2}$ and i into the field $\mathbb{Q}$ to get a splitting field of $x^4 + 1$, and we know $[\mathbb{Q}(\sqrt{2}, i) : Q] = 2$.