Week 12 Final, Math 102 **Name:** *James Wang*
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<u>Exercise 5.1.3:</u>

We want to determine whether O_n is isomorphic to the product group $SO_n \times \{\pm I\}$.

Recall Proposition 2.11.4d states that let H and K be subgroups of G and $f : H \times K \to G$ be the multiplication map, f is an isomorphism if and only if $H \cap K = \{1\}$, $HK = G$, and H and K are normal subgroups of G.

Since $[O_n : SO_n] = 2$ and $\{\pm I\} = Z(O_n)$, we know SO_n and $\{\pm I\}$ are normal subgroups in O_n. Moreover, given $A \in O_n$, if $\det(A) = 1$, we can associate it with $(A, I) \in SO_n \times \{\pm I\}$, and if $\det(A) = -1$, we can associate it with $(-A, -I) \in SO_n \times \{\pm I\}$, so $SO_n\{\pm I\} = O_n$.

It remains to see whether $SO_n \cap \{\pm I\} = \{I\}$, and we claim this answer depends on whether n is odd or even. When n is odd, $\det(-I) = -1$, so $SO_n \cap \{\pm I\} = \{I\}$, yet when n is even, $\det(-I) = 1$, so $SO_n \cap \{\pm I\} = \{\pm I\}$ which is nontrivial.

Hence O_n is isomorphic to $SO_n \times \{\pm I\}$ only when n is odd.

QED

Exercise 9.3.4:

We want to determine the centralizer of j in SU_2.

Recall $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and let $\begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix}$ be an arbitrary element of SU_2.

Note that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, and consider the following matrix multiplication:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix} = \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -x_2 + x_3 i & x_0 - x_1 i \\ -x_0 - x_1 i & -x_2 - x_3 i \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 i & x_0 + x_1 i \\ -x_0 + x_1 i & -x_2 + x_3 i \end{bmatrix}$$

For the above equality to be true, it must be that $x_1 = 0, x_3 = 0$. Hence $x_0^2 + x_2^2 = 1$, and

$$Z(j) = \left\{ \begin{bmatrix} x_0 & x_2 \\ -x_2 & x_0 \end{bmatrix} : x_0^2 + x_2^2 = 1 \right\} \in SL_2(\mathbb{R}).$$

Note that $x_0^2 + x_2^2 = 1$ means the determinant of every matrix in $Z(j)$ must be 1.

Let $P \in Z(j)$, note further that

$$PP^T = \begin{bmatrix} x_0 & x_2 \\ -x_2 & x_0 \end{bmatrix} \begin{bmatrix} x_0 & -x_2 \\ x_2 & x_0 \end{bmatrix} = \begin{bmatrix} x_0^2 + x_2^2 & -x_2 x_0 + x_2 x_0 \\ -x_2 x_0 + x_0 x_2 & x_2^2 + x_0^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which means every matrix in $Z(j)$ must also be orthogonal.

Hence $Z(j) = SO_2$.

$$QED$$

Exercise 5.1.2:

Given a matrix A that represents a rotation of $\mathbb{R}^3$ through the angle θ about a pole u, we want to find its complex eigenvalues. By the Fundamental Theorem of Algebra, we know that A has 3 complex eigenvalues.

By Euler's Theorem, the 3×3 rotation matrices are elements of SO_3, so $\det(A) = 1$.

Moreover, by Lemma 5.1.29, $\det(A) = 1$ means A has an eigenvalue equal to 1.

Finally, by Corollary 5.1.28, since $A \in SO_3$ and it represents the rotation $\rho_{(u,\theta)}$ with spin (u, θ), we know $tr(A) = 1 + 2\cos(\theta)$.

Let λ_1, λ_2 be the two complex eigenvalues that are not 1, and putting the above information together, we have

$$\lambda_1 \cdot \lambda_2 = 1 \quad \text{and} \quad 1 + \lambda_1 + \lambda_2 = 1 + 2\cos(\theta).$$

Substituting $\lambda_2 = \dfrac{1}{\lambda_1}$ into the second equation, we get

$$\begin{aligned} \lambda_1 + \frac{1}{\lambda_1} &= 2\cos\theta, \\ \lambda_1^2 + 1 &= 2\lambda_1 \cos\theta, \\ \lambda_1^2 - 2\lambda_1 \cos\theta + 1 &= 0. \end{aligned}$$

Using the quadratic formula, we obtain

$$\begin{aligned} \lambda_1 &= \cos\theta + \sqrt{\cos^2\theta - 1} = \cos\theta + \sqrt{-\sin^2\theta}, \\ \lambda_2 &= \cos\theta - \sqrt{\cos^2\theta - 1} = \cos\theta - \sqrt{-\sin^2\theta}. \end{aligned}$$

Hence the complex eigenvalues of A are 1 and $\cos\theta \pm \sqrt{-\sin^2\theta}$.

QED