Week 9 Final, Math 102 **Name:** *James Wang*
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(14.5.1)

math.stackexchange.com/questions/4013135/understanding-the-solution-of-14-5-1-in-artin

Note: I couldn't convince myself to solve this problem by multiplying 2 to the fractions, so I got another approach from the link cited above.

Let $R = \mathbb{Z}[\delta]$, where $\delta = \sqrt{-5}$. We want to find a presentation matrix as R-module for the ideal $(2, 1+\delta)$.

We first choose $(2, 1+\delta)$ as the set of generators for V. We then consider the surjective homomorphism $\varphi : R^2 \to V$ defined by $(x, y) \mapsto 2x + (1+\delta)y$. Clearly, elements of $\ker(\varphi)$ are the relation vectors, and by definition $\ker(\varphi) = \{(x, y) \in R^2 : 2x + (1+\delta)y = 0\}$.

We first solve for x and get $x = \frac{-(1+\delta)y}{2}$. If $y = 2$, then $x = -1-\delta$. Since $2, -1-\delta \in \mathbb{Z}[\delta]$, we have obtained a relation of $(2)(-1-\delta) + (1+\delta)(2) = 0$.

We then solve for y and get $y = \frac{-2x}{1+\delta}$, or equivalently $y(1+\delta) = -2x$. With trial and error and some wishful thinking, we see that if $y = 1-\delta$, then $x = -3$. Since $1-\delta, -3 \in \mathbb{Z}[\delta]$, we have obtained another relation of $(2)(-3) + (1+\delta)(1-\delta) = 0$, which is independent from our first relation.

Finally, we list the coordinate vectors as columns of the presentation matrix and get

$$A = \begin{bmatrix} -1-\delta & -3 \\ 2 & 1-\delta \end{bmatrix}$$

Since φ is surjective, $V \cong R^2/\ker(\varphi) = R^2/AR^2$, so matrix A presents the module V, which is the ideal $(2, 1+\delta)$.

QED

(14.5.2)

We want to identify the abelian group presented by the matrix $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 6 \end{bmatrix}$.

By the operations allowed in integer matrices and the deletion rule, we obtain

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 3 & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{\text{deleting } R_1, C_1} \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix} \xrightarrow{\text{deleting } R_1, C_1} \begin{bmatrix} 7 \end{bmatrix}.$$

Hence the matrix presents an abelian group of one generator (x) and one relation $7x = 0$.

The abelian group is therefore $\mathbb{Z}/7\mathbb{Z}$.

QED

(14.7.2)

We want to write the abelian group generated by x and y with the relation $3x + 4y = 0$ as a direct sum of cyclic groups. Given the relation, we can write the presentation matrix

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and by the operations allowed in integer matrices, we obtain

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence the matrix presents an abelian group of two generators (x, y) and one relation $x = 0$. As a direct sum of cyclic groups, this is $\mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z} = \{0\} \oplus \mathbb{Z} = \mathbb{Z}$.

QED

(14.7.4e)

We want to identify the abelian group with the presentation matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

By the operations allowed in integer matrices and the deletion rule, we obtain

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{deleting } R_2, C_1} \begin{bmatrix} -1 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 \end{bmatrix}.$$

Hence the matrix presents an abelian group of one generator (x) and one relation $x = 0$.

The abelian group is therefore $\mathbb{Z}/\mathbb{Z} = \{0\}$.

$$QED$$

(14.7.5)

We want to determine the number of isomorphism classes of abelian groups of order 400. By the alternate form of the Structure Theorem for Abelian Groups, we know every finite abelian group is a direct sum of cyclic groups of prime power orders, and the orders of the cyclic subgroups are uniquely determined by the group.

Since the prime factorization of $400 = 16 \cdot 25 = 2^4 \cdot 5^2$, there are 5 ways to partition 2^4: $2^4, 2^3 \cdot 2, 2^2 \cdot 2^2, 2^2 \cdot 2 \cdot 2$, and $2 \cdot 2 \cdot 2 \cdot 2$, and there are 2 ways to partition 5^2: 5^2, and $5 \cdot 5$, meaning there are 10 ways to partition the prime factorization $2^4 \cdot 5^2$ into prime powers.

Hence there are 10 isomorphism classes of abelian groups of order 400, which are

$$
\begin{gathered}
C_{16} \oplus C_{25}, \\
C_8 \oplus C_2 \oplus C_{25}, \\
C_4 \oplus C_4 \oplus C_{25}, \\
C_4 \oplus C_2 \oplus C_2 \oplus C_{25}, \\
C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_{25}, \\
C_{16} \oplus C_5 \oplus C_5, \\
C_8 \oplus C_2 \oplus C_5 \oplus C_5, \\
C_4 \oplus C_4 \oplus C_5 \oplus C_5, \\
C_4 \oplus C_2 \oplus C_2 \oplus C_5 \oplus C_5, \\
C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_5 \oplus C_5.
\end{gathered}
$$

$$QED$$