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**Exercise 6.4:**

Let $H$ be a normal subgroup of prime order $p$ in a finite group $G$. Suppose that $p$ is the smallest prime that divides the order of $G$. We want to show that $H$ is in the center $Z(G)$. Let $Z(x)$ be the centralizer of $x$ and $C(x)$ be the conjugacy class of $x$. Recall that for all $x \in G$, $x \in Z(G)$ if and only if $|Z(x)| = |G|$ if and only if $|C(x)| = 1$. Recall further that since $H$ is a normal subgroup of $G$, $H$ is a union of the conjugacy classes. Therefore, to prove that $H$ is in the center $Z(G)$, it suffices to show that $|H| = \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}}$.

Suppose for a contradiction that $|H|$ has a nontrivial conjugacy class $C(y)$, then $|C(y)| \Big| |G|$. Since the identity element is always in a conjugacy class by itself, we know $|C(y)| \leq p - 1$. However, $p$ is the smallest prime that divides $|G|$ by assumption, so $|C(y)|$ cannot divide $|G|$ and we have reached a contradiction. Hence $H$ must be a union of trivial conjugacy classes, and $H$ is in the center $Z(G)$.

$QED$

**Exercise 7.4 a):**

We want to show no simple group has order $pq$, where $p, q$ are prime.

Suppose $|G| = pq$, where $p, q$ are prime. Let $s_p$ be the number of Sylow $p$-subgroups. Without loss of generality, also suppose $p > q$. By the Third Sylow Theorem, we know $s_p \mid q$ and $s_p \equiv 1 \mod p$. Since $q$ is prime, $s_p = 1$ or $q$. Given the assumption $p > q$, for $s_p \equiv 1 \mod p$ to be true, it must be $s_p = 1$. Hence there is only one Sylow $p$-subgroup, which makes it normal, so $G$ is not simple.

$$QED$$

**Exercise 8.1:**

We want to determine the group on the following list that $S_3 \times C_2$ is isomorphic to:

$$C_4 \times C_3$$

$$C_2 \times C_2 \times C_3$$

$$A_4$$

$$D_6$$

$$G = \{x, y \mid x^4 = 1, y^3 = 1, xy = y^2x\}$$

Since $S_3$ is not abelian, we can eliminate the two abelian groups $C_4 \times C_3$ and $C_2 \times C_2 \times C_3$.

Since elements of $S_3$ have order $1, 2,$ or 3 and elements of $C_2$ have order 1 or 2, $S_3 \times C_2$ does not have an order 4 element (4 is not a least common multiple of any two of the orders). But $G$ has an order 4 element, so we can also eliminate $G$.

Since $S_3 \times C_2$ has three order 3 elements but we know $A_4$ has eight order 3 elements from the proof in the classification of groups of order 12, we can finally eliminate $A_4$.

Hence $S_3 \times C_2$ can only be isomorphic to $D_6$.

**Exercise 7.4 b):**

We want to show no simple group has order $p^2q$, where $p, q$ are prime.

Suppose $|G| = p^2q$, where $p, q$ are prime. Let $s_p$ be the number of Sylow $p$-subgroups, and let $s_q$ be the number of Sylow $q$-subgroups. By the Third Sylow Theorem, $s_q = 1, p$, or $p^2$.

Case 1: Suppose $s_q = 1$. Then there is only one Sylow $q$-subgroup of $G$, which has to be normal. Hence $G$ is not simple.

Case 2: Suppose $s_q = p$. Then there are $p$ Sylow $q$-subgroups of $G$. By the Third Sylow Theorem, we know $p \equiv 1 \bmod q$, so $p > q$. By the Third Sylow Theorem again, we know $s_p \mid q$ and $s_p \equiv 1 \bmod p$. Putting the restrictions together, it must be $s_p = 1$, so there is only one Sylow $p$-subgroup of $G$, which has to be normal. Hence $G$ is not simple.

Case 3: Suppose $s_q = p^2$. Then there are $p^2$ Sylow $q$-subgroups of $G$. Since $q$ is prime, each of the $p^2$ Sylow $q$-subgroups is cylic, and the intersection of all of them is trivial, so each of them consists of the identity element and $q - 1$ elements of order $q$. Since each Sylow $p$-subgroup has order $p^2$, the identity element and the rest $p^2q - (p^2(q-1)+1) = p^2 - 1$ elements that is not in any Sylow $q$-subgroup form the only Sylow $p$-subgroup of $G$, which has to be normal. Hence $G$ is not simple.

$QED$