Week 8 Final, Math 102 **Name:** *James Wang*
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(14.1.1)

Let R be a ring, and let V denote the R-module R, we want to determine all homomorphisms $\varphi : V \to V$. Let $\varphi(1_R) = x \in R$, we claim that all homomorphisms are of the form $\varphi(v) = vx$. If φ is a homomorphism of R-modules, then by definition, it must be that

$$\varphi(v) = \varphi(v \cdot 1_R) = v\varphi(1_R) \text{ for all } v \in V,$$

so the image is uniquely determined by where 1_R gets sent to.

It remains to show that all φ of this form are indeed homomorphisms, and we just check the definitions using axioms (distributivity and associativity) of an R-module V:

Indeed,

$$\varphi(v_1 + v_2) = (v_1 + v_2)x = v_1x + v_2x = \varphi(v_1) + \varphi(v_2)$$
$$\varphi(rv_1) = (rv_1)x = r(v_1x) = r\varphi(v_1)$$

for all $v_1, v_2 \in V$ and $r \in R$:

Hence all homomorphisms $\varphi : V \to V$ are of the form $\varphi(v) = vx$.

QED

<u>(14.2.4)</u>

a)

Let I be an ideal of a ring R. We claim if I is a free R-module, then $I = \{0\}$ or $I = (\alpha)$ where $\alpha \in R$ is not a zero divisor.

$I = \{0\}$ is always a free R-module because we can use the basis $\{\emptyset\}$ by convention.

Now suppose $I \neq \{0\}$, we want to show $I = (\alpha)$ and $\alpha \in R$ is not a zero divisor. Since I is a free R-module by assumption, it must have a basis by definition. Suppose for a contradiction that there are two or more elements in each basis of I, and we pick x, y from an arbitrary basis. Since $x \cdot y + (-x) \cdot y = 0$, we have obtained a linear dependence relationship, which contradicts the assumption that x, y are in a basis. Thus, a basis of I can only contain exactly one element of R, which we denote by α, and $I = (\alpha)$.

Recall that $I \neq \{0\}$, so $\alpha \neq 0$. Notice moreover that for α to be linearly independent, it must be that for all $r \in R$, if $r\alpha = 0$, then $r = 0$. This is equivalent to showing α cannot be a zero divisor in R.

Hence I is a free R-module when $I = \{0\}$ or $I = (\alpha)$ and $\alpha \in R$ is not a zero divisor.

$$QED$$

b)

Let I be an ideal of a ring R. We claim if R/I is a free R-module, then $I = \{0\}$ or $I = R$.

When $I = \{0\}$, $R/I = R/(0) = R$ is always a free R-module because for all $r \in R$, we have $r = r \cdot 1_R$, so $\{1_R\}$ is a basis.

When $I = R$, $R/I = R/(1_R) = 0$, and similar to the argument from last part, this is always a free R-module because we can use the basis $\{\emptyset\}$.

It remains to show that if I is proper, then R/I is not a free R-module. Suppose for a contradiction that $I = (a)$ where $a \neq 0_R$ and $a \neq 1_R$, and R/I is a free R-module. Let $x \neq 0$ be an element of a basis of R/I, then $x = r + I$ where $r \in R$. It follows

$$a \cdot x = a \cdot (r + I) = (ar + I).$$

By the definition of ideals, $ar \in I$, so $ar + I \in I$. This means $a \cdot x = 0 \in R/I$, and we have obtained a linear dependence relationship, which contradicts the assumption that x is an element of a basis.

Hence R/I is a free R-module when $I = \{0\}$ or $I = R$.

$$QED$$

(14.2.1)

Let $R = \mathbb{C}[x, y]$ and let M be the ideal of R generated by the two elements x, y. We claim M is not a free R-module. Using the result from 14.2.4 part a), it suffices to show that $\mathbb{C}[x, y]$ is not a Principal Ideal Domain.

Let $I = (x, y)$ and suppose for a contradiction that I is principal, then there exists some $k \in \mathbb{C}[x, y]$ such that $I = (k)$, or equivalently $k|x$ and $k|y$. But since x, y are relatively prime, k has to be a unit in R, meaning $I = \mathbb{C}[x, y]$. Thus, there exist $a, b \in \mathbb{C}[x, y]$ such that $ax + by = 1$. By observation, however, for all $a, b \in \mathbb{C}[x, y]$, the constant term of $ax + by$ is always 0. We have reached a contradiction.

Hence $\mathbb{C}[x, y]$ is not a Principal Ideal Domain, so M is not a free R-module.

$$QED$$