

---

### Important Facts

- The symmetric polynomials is a subring of  $R[u]$ .
- Every permutation in  $S_n$  can be written as a product of transpositions.
- The orbit sums and monomials of the elementary symmetric functions are two bases of the symmetric polynomials.

### Theorem 16.1.6 Symmetric Functions Theorem

Every symmetric polynomial  $g(u_1, \dots, u_n)$  with coefficients in a ring  $R$  can be written in a unique way as a polynomial in the elementary symmetric functions  $s_1, \dots, s_n$ .

### Lemma 16.1.11

Let  $g$  be a symmetric polynomial of degree  $d$  in the variables  $u_1, \dots, u_n$ , and suppose that  $g^\circ = Q(s_1^\circ, \dots, s_{n-1}^\circ)$ . Then  $g = Q(s_1, \dots, s_{n-1}) + s_n h$ , where  $h$  is a symmetric polynomial in  $u_1, \dots, u_n$  of degree  $d - n$ .

### Corollary 16.1.12

Suppose that a polynomial  $f(x) = x^n - a_1 x^{n-1} + \dots \pm a_n$  has coefficients in a field  $F$ , and that it splits completely in an extension field  $K$ , with roots  $\alpha_1, \dots, \alpha_n$ . Let  $g(u_1, \dots, u_n)$  be a symmetric polynomial in  $u_1, \dots, u_n$  with coefficients in  $F$ . Then  $g(\alpha_1, \dots, \alpha_n)$  is an element of  $F$ .

### Proposition 16.1.14

Let  $p_1 = p_1(u_1, \dots, u_n)$  be a polynomial, let  $\{p_1, \dots, p_k\}$  be its orbit for the operation of the symmetric group on the variables, and let  $w = w_1, \dots, w_k$  be another set of variables, where  $k$  is the number of polynomials in the orbit of  $p_1$ . If  $h(w_1, \dots, w_k)$  is a symmetric polynomial in  $w$ , then  $h(p_1, \dots, p_k)$  is a symmetric polynomial in  $u$ .

### Confusion

- What does the second induction on the degree of a symmetric polynomial look like?