Week 6 Final, Math 102 **Name:** *James Wang*
Oct 11, 2023

(1.1) Find the Galois group and the correspondence of the polynomial $x^6 - 4x^3 + 4$.

Let K be the splitting field of the polynomial $x^6 - 4x^3 + 4$, the Galois group is $G(K/\mathbb{Q})$.

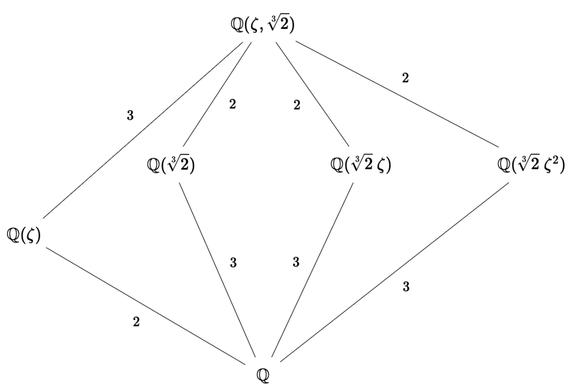
By observation, $x^6 - 4x^3 + 4 = (x^3 - 2)^2$. Notice that although this polynomial has 6 roots, there are only 3 distinct roots: $\{\sqrt[3]{2}, \sqrt[3]{2}\,\zeta, \sqrt[3]{2}\,\zeta^2\}$, where $\zeta = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$.

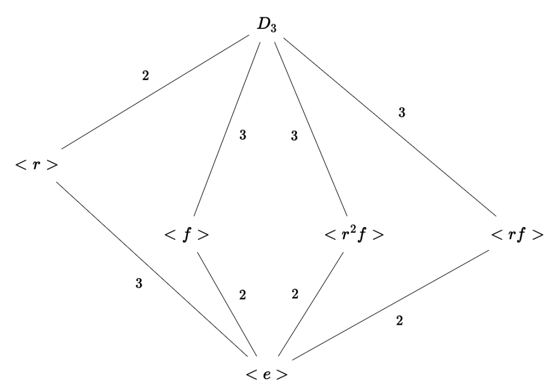
This means the Galois group of the polynomial $x^6 - 4x^3 + 4$ is a subgroup of S_3, and $|S_3| = 6$.

By Exercise 16.3.2, the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta, \sqrt[3]{2})$, and $[\mathbb{Q}(\zeta, \sqrt[3]{2}) : \mathbb{Q}] = 6$.

Since we have obtained a splitting field, it must be $[\mathbb{Q}(\zeta, \sqrt[3]{2}) : \mathbb{Q}] = |G(\mathbb{Q}(\zeta, \sqrt[3]{2})/\mathbb{Q})| = 6$.

Hence the Galois group is the entire S_3. Since $S_3 \cong D_3$, we proceed to draw lattices of D_3, where r is $120°$ rotation and f is flip.





Since every element in each subfield is fixed when acted by elements in the corresponding subgroup, we define

$$r : \sqrt[3]{2} \mapsto \sqrt[3]{2}\,\zeta \quad \text{and} \quad \zeta \mapsto \zeta$$
$$f : \sqrt[3]{2} \mapsto \sqrt[3]{2} \quad \text{and} \quad \zeta \mapsto \zeta^2,$$

and this gives exactly what we want.

Note that

$$r^2 f(\sqrt[3]{2}\,\zeta) = r^2(\sqrt[3]{2}\,\zeta^2) = \sqrt[3]{2}\,\zeta^4 = \sqrt[3]{2}\,\zeta$$
$$rf(\sqrt[3]{2}\,\zeta^2) = r(\sqrt[3]{2}\,\zeta^4) = r(\sqrt[3]{2}\,\zeta) = \sqrt[3]{2}\,\zeta^2$$

(1.2) Find the Galois group and the correspondence of the polynomial $x^4 - 5x^2 + 6$.

Let K be the splitting field of the polynomial $x^4 - 5x^2 + 6$, the Galois group is $G(K/\mathbb{Q})$.

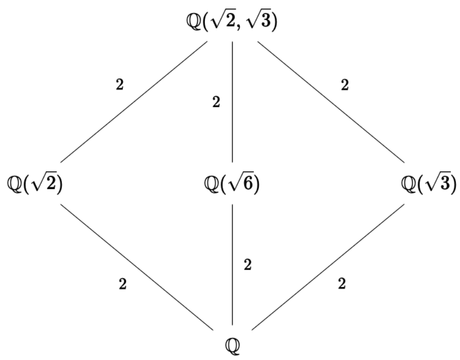
By observation, $x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$, and the roots are $\{\pm\sqrt{2}, \pm\sqrt{3}\}$.

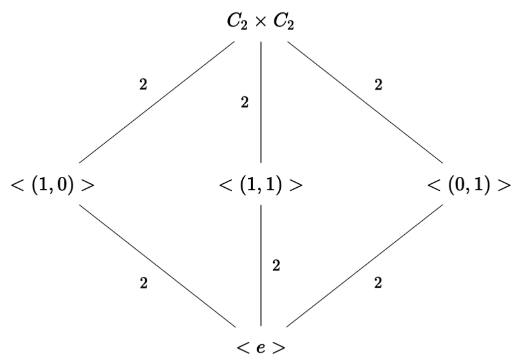
Thus, the splitting field of $x^4 - 5x^2 + 6$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Since we have a splitting field, we know it must be $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = |G(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})| = 4$.

There are two groups of order 4: C_4 and $C_2 \times C_2$. Since C_4 only has one proper subgroup, whereas we know $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has 3 proper subfields, the Galois group can only be $C_2 \times C_2$.

We proceed to draw the lattices.





Since every element in each subfield is fixed when acted by elements in the corresponding subgroup, we define

$$(1,0) = \sigma_1 : \sqrt{2} \mapsto -\sqrt{2} \;\text{ and }\; \sqrt{3} \mapsto \sqrt{3}$$
$$(0,1) = \sigma_2 : \sqrt{2} \mapsto \sqrt{2} \;\text{ and }\; \sqrt{3} \mapsto -\sqrt{3}$$

and this gives exactly what we want.

Note that

$$(1,1) = \sigma_2\sigma_1 : \sqrt{2} \mapsto -\sqrt{2} \;\text{ and }\; \sqrt{3} \mapsto -\sqrt{3}$$
$$\sigma_2\sigma_1(\sqrt{6}) = \sigma_2\sigma_1(\sqrt{2}\sqrt{3}) = \sigma_2(-\sqrt{2}\sqrt{3}) = (-\sqrt{2})(-\sqrt{3}) = \sqrt{6}$$

(16.12.1)

Suppose we have an arbitrary Galois extension K/F of degree 10, then by definition, the corresponding Galois group $G(K/F)$ must be order 10. Using the classification of groups, we know there exist only two groups of order 10 up to isomorphism, which are C_{10} and D_5. In both cases, there exist index 2 normal subgroups: C_5 and the dihedral subgroup generated by a 72° rotation. By the Main Theorem of Galois Theory, there must exist an intermediate field L such that $[K : L] = 5$ and $[L : F] = 2$. Since 2 is prime, this means there is a chain of subfields $F \subseteq L \subseteq K$ of $\mathbb{C}$ such that L is a Galois extension of F of prime degree. This satisfies Artin's second definition of solvable in Proposition 16.12.2. Hence every Galois extension of degree 10 is solvable.

$$QED$$