Week 7 Final, Math 104 **Name:** *James Wang*
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Exercise 60.2:

Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. We want to show that X is homeomorphic to the projective plane P^2.

We proceed with the idea in Corollary 22.3, which is illustrated by the diagram below, but the difference is that the surjective continuous map we build is a composition using S^2 as an intermediate stage.

$$\begin{tikzcd}
B^2 \arrow[dd, "p"'] \arrow[dr, "g"] & & \\
& S^2 \arrow[dr, "h"] & \\
X^* \arrow[rr, "f"'] & & P^2
\end{tikzcd}$$

Define $g : B^2 \to S^2$ as $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$, which maps B^2 to the upper hemisphere of S^2. We can convince ourselves that g is continuous since the outputs get close as the inputs get close (Professor said in class that we need not prove this here). We then define $h : S^2 \to P^2$ by identifying each point in S^2 with its antipodal point, which is the canonical quotient map. Since quotient maps must be continuous, we have that $h \circ g$ is continuous. We claim that $h \circ g$ is also surjective. Take any $y \in P^2$, then $h^{-1}(y)$ is a pair of antipodal points. Since they are antipodal, one of them (we denote by s) must be on the upper hemisphere of S^2, so h is surjective. Furthermore, by taking the x, y coordinates of s, we find some $b \in B^2$ such that $g(b) = s$, so $h(g(b)) = y$ and $g \circ h$ is surjective.

We proceed to define $X^* = \{(h \circ g)^{-1}(\{y\}) \mid y \in P^2\}$, and give it the quotient topology. Although we now have everything to invoke the Corollary, for it to prove what we want, we have to show X, the quotient space obtained from B^2 by identifying antipodal points of S^1, partition B^2 the same way as X^* does. We first look at X. If $b \in B^2$ and $b \in S^1$, then b is the representative of the equivalence class $\{b, -b\}$ in X. If $b \in B^2$ but $b \notin S^1$, then b is by itself an equivalence class in X. We then look at X^*. If $b \in B^2$ and $b \in S^1$, then it means the z-coordinate of $g(b)$ is 0, that $g(b)$ and its antipodal point are both on the equator of S^2 (both belong to the upper hemisphere), so $\{b, -b\}$ are an equivalence class in X^*. If $b \in B^2$ and $b \notin S^1$, then the antipodal point of $g(b)$ is on the lower hemisphere, so b is by itself an equivalence class in X^*. Thus, $X^* = X$.

We now invoke the Corollary, which says f is a homeomorphism if and only if $h \circ g$ is a quotient map. By construction, h is already a quotient map. We finally claim that g is an open map, which is obvious because g carries every open neighborhood of B^2 to an open neighborhood in S^2. Since being an open map is a stronger condition than being a quotient map, g is also a quotient map. By a result in the book, we know the composition of quotient maps is a quotient map, so $h \circ g$ is a quotient map.

Putting everything together, we have that X^* is homeomorphic to P^2, which means X is homeomorphic to P^2.

QED

Exercise 63.1:

Let C_1 and C_2 be disjoint simple closed curves in S^2.

(a) We want to show that $S^2 - C_1 - C_2$ has precisely three components.

Since a simply closed curve separates S^2, we can let W_1, V_1 denote the separations by C_1, and W_2, V_2 denote the separations by C_2. Then, $S^2 - C_1 = W_1 \cup V_1$ and $S^2 - C_2 = W_2 \cup V_2$. Without loss of generality, suppose W_1 and W_2 are disjoint, so V_1 and V_2 cannot be disjoint. Now by some set theory facts:

$$\begin{aligned} S^2 - C_1 - C_2 &= (W_1 \cup V_1) \cap (W_2 \cup V_2) \\ &= (W_1 \cap W_2) \cup (W_1 \cap V_2) \cup (V_1 \cap W_2) \cup (V_1 \cap V_2) \\ &= \emptyset \cup W_1 \cup W_2 \cup (V_1 \cap V_2). \end{aligned} \tag{1}$$

This shows $S^2 - C_1 - C_2$ has at least three components. We then show it has at most three components by showing $V_1 \cap V_2$ is connected. Consider $\overline{W_1} = W_1 \cup C_1$ and $\overline{W_2} = W_2 \cup C_2$. By Theorem 63.3, since $\overline{W_1}$ and $\overline{W_2}$ are closed in S^2, $S^2 - \overline{W_1} \cap \overline{W_2} = S^2 - \emptyset = S^2$, which is simply connected, and neither $\overline{W_1}$ nor $\overline{W_2}$ separates S^2, then $\overline{W_1} \cup \overline{W_2}$ does not separate S^2. By definition, this means

$$\begin{aligned} S^2 - \overline{W_1} \cup \overline{W_2} &= S^2 - (W_1 \cup C_1) - (W_2 \cup C_2) \\ &= (S^2 - W_1 \cup C_1) \cap (S^2 - W_2 \cup C_2) \\ &= V_1 \cap V_2 \end{aligned} \tag{2}$$

is connected.

Putting everything together, we have proved that $S^2 - C_1 - C_2$ has precisely three components.

QED

(b) We want to show that these three components have boundaries C_1 and C_2 and $C_1 \cup C_2$, respectively.

By the Jordan Curve Theorem, the boundaries of W_1 and W_2 are C_1 and C_2, respectively, because $\overline{W_1} - W_1 = C_1$ and $\overline{W_2} - W_2 = C_2$ by construction. It remains to show $\overline{V_1 \cap V_2} - V_1 \cap V_2 = C_1 \cup C_2$, which we again use set theory. We have shown in part (a) that $V_1 \cap V_2 = S^2 - \overline{W_1} \cup \overline{W_2}$, and we can convince ourselves that $\overline{V_1 \cap V_2} = S^2 - W_1 \cup W_2$. Putting everything together:

$$
\begin{aligned}
\overline{V_1 \cap V_2} - V_1 \cap V_2 &= (S^2 - W_1 \cup W_2) - (S^2 - \overline{W_1} \cup \overline{W_2}) \\
&= (\overline{W_1} - W_1) \cup (\overline{W_2} - W_2) \\
&= C_1 \cup C_2.
\end{aligned} \tag{3}
$$

We have proved what we want.

QED