Week 10 Final, Math 104 **Name:** *James Wang*
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Exercise 74.7:

If $m > 1$, we want to show the fundamental group of the m-fold projective plane P_m is not abelian.

By Theorem 74.4, we know $\pi_1(P_m) = (\alpha_1, \alpha_2 \ldots, \alpha_m \mid \alpha_1^2\alpha_2^2 \cdots \alpha_m^2)$, which is the quotient of the free group on m generators (we denote by F_m) by one relation. We also note that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = (\beta_1, \beta_2 \mid \beta_1^2, \beta_2^2)$, which is the quotient of the free group on two generators by two relations.

We first define a function $h : F_m \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ by

$$\alpha_i = \begin{cases} \beta_1 & \text{if } i = 1, \\ \beta_2 & \text{if } i = 2, \\ 1 & \text{if } i > 2. \end{cases}$$

By Lemma 69.1, since F_m is a free group with system of free generators $\{\alpha_i\}$, we know given the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and its elements $\{\beta_i, 1\}$, there is a unique homomorphism $h_* : F_m \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ such that our definition of h is satisfied.

Now let N be the least normal subgroup containing $\alpha_1^2\alpha_2^2 \cdots \alpha_m^2$. Since the relations of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ give

$$h_*(\alpha_1^2\alpha_2^2 \cdots \alpha_m^2) = h_*(\alpha_1^2)h_*(\alpha_2^2) \cdots h_*(\alpha_m^2) = \beta_1^2\beta_2^2 \cdots 11 = 11 \cdots 11 = 1,$$

we know $N \subseteq \ker(h_*)$. By the universal property of quotients, there exists a unique homomorphism $\varphi : F_m/N \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ such that $\varphi(gN) = h_*(g)$, where $F_m/N \cong \pi_1(P_m)$.

Since h_* maps generators of $\pi_1(P_m)$ to generators of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, h_* and φ are both surjective. Moreover, note that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is not abelian. We can finally conclude that $\pi_1(P_m)$ is not abelian because there does not exist any surjective homomorphism from an abelian group to a non-abelian group. Otherwise, take $k_1, k_2 \in \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, there exist $g_1, g_2 \in \pi_1(P_m)$ such that $\varphi(g_1) = k_1$ and $\varphi(g_2) = k_2$. By properties of homomorphisms, we have

$$k_1 k_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = \varphi(g_2 g_1) = \varphi(g_2)\varphi(g_1) = k_2 k_1,$$

which is a contradiction.

$$QED$$

Exercise 78.2:

Let H^2 be the subspace of $\mathbb{R}^2$ consisting of all points (x_1, x_2) with $x_2 \geq 0$.

(a) We want to show that no point of H^2 of the form $(x_1, 0)$ has a neighborhood (in H^2) that is homeomorphic to an open set of $\mathbb{R}^2$.

On one hand, using the idea illustrated in Figure 78.3, we note that every point of H^2 of the form $(x_1, 0)$ has arbitrarily small neighborhoods W for which $W - (x_1, 0)$ is simply connected, as it is contractible to a point. On the other hand, using the idea illustrated in Figure 78.4, take any point $(y_1, y_2) \in \mathbb{R}^2$ and consider the open ball $B_r(y_1, y_2)$. Let V be any neighborhood contained in $B_r(y_1, y_2)$, and let $B_\varepsilon(y_1, y_2)$ be a smaller ball contained in V, the inclusion map $i : B_\varepsilon(y_1, y_2) - (y_1, y_2) \to B_r(y_1, y_2) - (y_1, y_2)$ induces an isomorphism of fundamental groups, so the inclusion map $k : V - (y_1, y_2) \to B_r(y_1, y_2) - (y_1, y_2)$ induces a surjection. This shows that (y_1, y_2) does not have arbitrarily small neighborhoods V for which $V - (y_1, y_2)$ is simply connected. Hence they cannot be homeomorphic.

QED

(b) We want to show that $x \in \partial X$ if and only if there is a homeomorphism h mapping a neighborhood of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times 0$.

$(\Longrightarrow)$: Suppose $x \in \partial X$, then by definition x has no neighborhood homeomorphic with an open set of $\mathbb{R}^2$. Since X is a 2-manifold with boundary, by definition, for every $x \in X$, we can let U be the neighborhood of x and V be the open set in H^2 such that $p : U \to V$ is the homeomorphism. Suppose for a contradiction that $p(x) \notin \mathbb{R} \times 0$, that $p(x) = (x_1, x_2)$ with $x_2 > 0$, then we can define $W := B_{x_2/2}(x_1, x_2) \cap V$, which is a neighborhood of $p(x)$ contained in V that is disjoint from $\mathbb{R} \times 0$. Since W is an open set of $\mathbb{R}^2$ and p restricted to the pre-image $p^{-1}(W)$, which we define as h, is a homeomorphism from a neighborhood of x to W, we have reached a contradiction with the definition that x has no neighborhood homeomorphic with an open set of $\mathbb{R}^2$. Hence it must be that $p(x) \in \mathbb{R} \times 0$.

$(\Longleftarrow)$: Take $x \in X$, suppose there is a homeomorphism $h : U \to V$ where U is a neighborhood of x and V is an open set of H^2 such that $h(x) \in \mathbb{R} \times 0$. Then by part (a), we know $h(x)$ has no neighborhood homeomorphic to an open set of $\mathbb{R}^2$. Since h is a homeomorphism, which preserves properties expressed in terms of the open sets, we know x has no neighborhood homeomorphic to an open set of $\mathbb{R}^2$. By definition, this means $x \in \partial X$.

$$QED$$

74.2:

Consider the space X obtained from a seven-sided polygonal region by means of the labelling scheme $abaaab^{-1}a^{-1}$. We want to show that the fundamental group of X is the free product of two cyclic groups.

By choosing a different point to start the labelling scheme, we can get

$$abaaab^{-1}a^{-1} = b^{-1}a^{-1}abaaa = a^3.$$

Hence it is clear that $\pi_1(X) = (a, b \mid a^3)$. By Theorem 68.7, we can let $G_1 = G_2 = \mathbb{Z}$, then we see that $N_1 = 3\mathbb{Z}$ and $N_2 = 1$. If N is the least normal subgroup of G that contains N_1 and N_2, then $G/N \cong \pi_1(X) \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}$.

$$QED$$

74.2:

If $n > 1$, we want to show the fundamental group of the n-fold torus T_n is not abelian.

By Theorem 74.3, we know $\pi_1(T_n) = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n])$, which is the quotient of the free group on $2n$ generators (we denote by F_{2n}) by one relation. We are also given the hint to consider the group (γ, δ), which is the free group F_2 on 2 generators.

We first define a function $h : F_{2n} \to F_2$ by

$$\alpha_i = \begin{cases} \gamma & \text{if } i = 1, \\ \delta & \text{if } i \neq 1. \end{cases}$$

$$\beta_i = \begin{cases} \gamma & \text{if } i = 1, \\ \delta & \text{if } i \neq 1. \end{cases}$$

By Lemma 69.1, since F_{2n} is a free group with system of free generators $\{\alpha_i, \beta_i\}$, we know given the group F_2 and its elements $\{\gamma, \delta\}$, there is a unique homomorphism $h_* : F_{2n} \to F_2$ such that our definition of h is satisfied.

Now let N be the least normal subgroup containing $[\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$. We have

$$h_*([\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]) = h_*([\alpha_1, \beta_1]) \cdots h_*([\alpha_n, \beta_n]) = \gamma\gamma\gamma^{-1}\gamma^{-1} \cdots \delta\delta\delta^{-1}\delta^{-1} = 1,$$

which means $N \subseteq \ker(h_*)$. By the universal property of quotients, there exists a unique homomorphism $\varphi : F_{2n}/N \to F_2$ such that $\varphi(gN) = h_*(g)$, where $F_{2n}/N \cong \pi_1(T_n)$.

Since h_* maps generators of $\pi_1(T_n)$ to generators of F_2, h_* and φ are both surjective. Moreover, note that F_2 is not abelian. We can finally conclude that $\pi_1(T_n)$ is not abelian because there does not exist any surjective homomorphism from an abelian group to a non-abelian group. Otherwise, take $k_1, k_2 \in F_2$, there exist $g_1, g_2 \in \pi_1(T_n)$ such that $\varphi(g_1) = k_1$ and $\varphi(g_2) = k_2$. By properties of homomorphisms, we have

$$k_1 k_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = \varphi(g_2 g_1) = \varphi(g_2)\varphi(g_1) = k_2 k_1,$$

which is a contradiction.

$$QED$$