Week 2 Final, Math 104 **Name:** *James Wang*
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Exercise 16.5 (Rephrased):

Let X and Y be two sets, each with two topologies. So we have 4 topological spaces: $(X, \mathcal{T}), (X, \mathcal{T}'), (Y, \mathcal{U}), (Y, \mathcal{U}')$. Let $\mathcal{S}$ be the product topology on $X \times Y$ induced by $\mathcal{T}$ and $\mathcal{U}$, and let $\mathcal{S}'$ be the product topology on $X \times Y$ induced by $\mathcal{T}'$ and $\mathcal{U}'$. (So a basis for $\mathcal{S}$ consists of sets of the form $V \times W$ for $V \in \mathcal{T}$ and $W \in \mathcal{U}$, but S itself does not just equal $\mathcal{T} \times \mathcal{U}$.)

a)

Suppose that $(X, \mathcal{T})$ is coarser than $(X, \mathcal{T}')$ and $(Y, \mathcal{U})$ is coarser than $(Y, \mathcal{U}')$. We want to show that $(X \times Y, \mathcal{S})$ is coarser than $(X \times Y, \mathcal{S}')$.

By definition, a basis $\mathcal{B}$ for $(X \times Y, \mathcal{S})$ is all sets of the form $V \times W$ where $V \in \mathcal{T}$ and $W \in \mathcal{U}$. Take arbitrary V and W, since $(X, \mathcal{T})$ is coarser than $(X, \mathcal{T}')$ and $(Y, \mathcal{U})$ is coarser than $(Y, \mathcal{U}')$, we have $V \in \mathcal{T} \subseteq \mathcal{T}'$ and $W \in \mathcal{U} \subseteq \mathcal{U}'$. This means $V \times W$ is also an element of the basis $\mathcal{B}'$ of the product topology on $X \times Y$ induced by $\mathcal{T}'$ and $\mathcal{U}'$, which is $(X \times Y, \mathcal{S}')$. Thus, by Lemma 13.1, every open set of $(X \times Y, \mathcal{S})$ is a union of elements of $\mathcal{B}$, which is also a union of elements of $\mathcal{B}'$, so $(X \times Y, \mathcal{S})$ is coarser than $(X \times Y, \mathcal{S}')$.

$$QED$$

b)

Suppose that $(X \times Y, \mathcal{S})$ is coarser than $(X \times Y, \mathcal{S}')$. Does it necessarily follow that $(X, \mathcal{T})$ is coarser than $(X, \mathcal{T}')$ and that $(Y, \mathcal{U})$ is coarser than $(Y, \mathcal{U}')$?

We claim the statement is true. Take arbitrary $V \in \mathcal{T}$ and $W \in \mathcal{U}$, so that $V \times W$ is a basis element of $(X \times Y, \mathcal{S})$, which also means that $(V \times W) \in (X \times Y, \mathcal{S})$. Since $(X \times Y, \mathcal{S})$ is coarser than $(X \times Y, \mathcal{S}')$ by assumption, we have $(V \times W) \in (X \times Y, \mathcal{S}) \subseteq (X \times Y, \mathcal{S}')$. Thus, $V \in \mathcal{T}'$ and $W \in \mathcal{U}'$, and $(X, \mathcal{T})$ is coarser than $(X, \mathcal{T}')$ and $(Y, \mathcal{U})$ is coarser than $(Y, \mathcal{U}')$.

$$QED$$

__**Exercise 17.13:**__

Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

$(\Longrightarrow)$:

Suppose X is Hausdorff. We want to show Δ is closed. Take arbitrary $(x_1, x_2) \in (X \times X) - \Delta$, so that $x_1 \neq x_2$. By the definition of X being Hausdorff, there exist neighborhoods U_1 and U_2 containing x_1 and x_2 respectively such that $U_1 \cap U_2 = \emptyset$. Hence $U_1 \times U_2$ is a neighborhood containing $(x_1, x_2) \in X \times X$. Moreover, we claim $(U_1 \times U_2) \cap \Delta = \emptyset$, because if there were some element in their intersection, it must be of the form $(x_3, x_3) \in U_1 \times U_2$, which means $U_1 \cap U_2 \neq \emptyset$, contradicting the definition of X being Hausdorff. Thus, $(x_1, x_2) \in (U_1 \times U_2) \subseteq (X \times X) - \Delta$, so $(X \times X) - \Delta$ is open and Δ is closed in $X \times X$.

$(\Longleftarrow)$:

Suppose Δ is closed in $X \times X$. We want to show that X is Hausdorff. By definition, Δ being closed means $(X \times X) - \Delta$ is open. Take arbitrary $x_1, x_2 \in X$ such that $x_1 \neq x_2$, meaning $(x_1, x_2) \in (X \times X) - \Delta$. Since $(X \times X) - \Delta$ is open, by definition there exist open sets $U_1, U_2 \in X$ and $(U_1 \times U_2) \subseteq (X \times X) - \Delta$ such that $(x_1, x_2) \in (U_1 \times U_2) \subseteq (X \times X) - \Delta$. It must be that U_1 and U_2 are neighborhoods containing x_1 and x_2 respectively. Moreover, we claim $U_1 \cap U_2 = \emptyset$, because if there were some x_3 in their intersection, then it means $x_3 \in U_1$ and $x_3 \in U_2$, so $(x_3, x_3) \in (U_1 \times U_2) \subseteq (X \times X) - \Delta$, contradicting the fact that we have subtracted the set $\Delta = \{x \times x \mid x \in X\}$. Hence the definition of X being Hausdorff is satisfied.

QED

Exercise 13.6:

We want to show that the topologies $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable. Since each basis element is itself an element of the topology, it suffices to show that the bases $\mathcal{B}_l = \{[a, b)\}$ and $\mathcal{B}_k = \{(a, b) - K\}$ where $K = \{1/n \mid n \in \mathbb{Z}_+\}$ of the topologies $\mathbb{R}_l$ and $\mathbb{R}_k$, respectively, are not subsets of each other.

$(\mathcal{B}_l \nsubseteq \mathcal{B}_k)$:

Clearly, $[1, 3) \in \mathcal{B}_l$ but $[1, 3) \notin \mathcal{B}_k$, because $1 \in K$.

$(\mathcal{B}_k) \nsubseteq \mathcal{B}_l)$:

Clearly, $((-1, 1) - K) \in \mathcal{B}_k$. Note that $0 \in ((-1, 1) - K)$ because $0 \neq 1/n$ for all $n \in \mathbb{Z}_+$. Also note that for a half interval in the form $[a, b)$ to contain 0, it must be that $a \leq 0$ and $b > 0$. By a fact from real analysis, however, there exists some $1/n$ where $n \in \mathbb{Z}_+$ such that $0 < 1/n < b$. This means that for this particular $1/n$, $1/n \in [a, b)$ but $1/n \notin ((-1, 1) - K)$.

QED

Exercise 17.10:

We want to show that every order topology is Hausdorff. Let $\mathcal{T}$ denote the order topology on a simply ordered set X and choose $x_1, x_2 \in X$ such that $x_1 \neq x_2$. We construct the open sets depending on how x_1 and x_2 are ordered.

Without loss of generality, if x_2 is the immediate successor of x_1 such that $x_1 < x_2$, then let $U_1 = (-\infty, x_2)$ and $U_2 = (x_1, \infty)$. Clearly, U_1 and U_2 are both open. Moreover, $x_1 \in U_1$ and $x_2 \notin U_1$, and $x_2 \in U_2$ and $x_1 \notin U_2$, so that $U_1 \cap U_2 = \emptyset$.

Without loss of generality, if x_2 is a successor of x_1 but not the immediate one, such that $x_1 < k < x_2$ for some $k \in X$, then let $U_1 = (-\infty, k)$ and $U_2 = (k, \infty)$. Clearly, U_1 and U_2 are both open. Moreover, k is in neither set and separates the two sets, so that $U_1 \cap U_2 = \emptyset$. Hence the definition of X being Hausdorff is satisfied.

$$QED$$