22 The Quotient Topology, Math 104 **Name:** *Zoe Markman, James Wang*
February 6, 2024

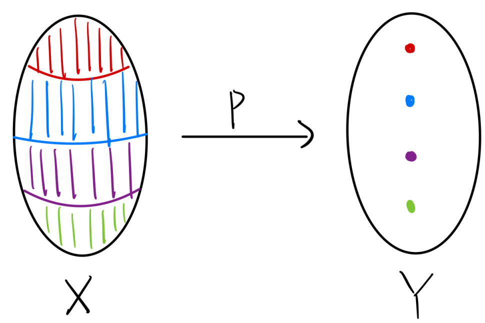
Definition: Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** if

$$U \subseteq Y \text{ is open } \iff p^{-1}(U) \subseteq X \text{ is open.}$$

Definition: If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is the **quotient topology** induced by p.

Definition: Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called the **quotient space** of X^*.

Remark:
- The quotient topology is the finest topology on Y such that p is still continuous.
- A quotient map needs not to be an open/closed map.
- There is a correspondence between surjective functions and partitions of X.



Motivation: Pasting by squishing points to the same equivalence class.

Example: $[0, 1] \subseteq \mathbb{R}$ with the equivalence relation $0 \sim 1$ is homeomorphic to a circle.

Example 4: Let X be the closed unit ball $\{x \times y | x^2 + y^2 \leq 1\}$. Let X^* be the partition of X consisting of:

- All one point sets $\{x \times y\}$ where $x^2 + y^2 < 1$.
- The set of all points on the unit circle

We construct a homeomorphism to the unit sphere.

Example 5: Let X be the rectangle $[0, 1] \times [0, 1]$. Define a partition X^* of X which contains:

- All one point sets $\{x \times y\}$ where $0 < x < 1$ and $0 < y < 1$.
- $\{x \times 0, x \times 1\}$ where $0 < x < 1$
- $\{0 \times y, 1 \times y\}$ where $0 < y < 1$
- $\{0 \times 0, 0 \times 1, 1 \times 0, 1 \times 1\}$

Recall: For a group G, and a subgroup N, you can create a quotient group G/N. We classify elements by the equivalence relation $\sim$ of belonging to the same coset, such that we can consider G/N as equivalent to $G/\sim$.

Our quotient topology is similarly constructed from an equivalence relation. We partition the set X, to create a set X^*, and construct a topology from the equivalence relation of two elements being in the same equivalence class.

Theorem: Let H be a normal subgroup of G, and let H be closed in the topology of G. Define $xH = \{x \cdot h | h \in H\}$ (i.e. the left coset of H in G). Give G/H the quotient topology. Then G/H is a topological group.