Week 11 Final, Math 104 **Name:** *James Wang*
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Exercise 79.5(b):

Let $T = S^1 \times S^1$ be the torus; let $x_0 = b_0 \times b_0 \in T$. If E is a covering space of T, we want to show E is homeomorphic either to $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T.

Consider the covering map $p : E \to T$, and choose some $e_0 \in p^{-1}(x_0)$, we can then define $p_*(\pi_1(E, e_0))$, which is a subgroup of $\pi_1(T, x_0)$. Since $\pi_1(T, x_0)$ is a free abelian group of rank 2, let $\{(1, 0), (0, 1)\}$ be the standard ordered basis, we know there are three possibilities of $p_*(\pi_1(E, e_0))$ by the hint:

1) $p_*(\pi_1(E, e_0)) = 0$,

2) $\{(m, 0)\}$ is a basis for $p_*(\pi_1(E, e_0))$, $m \in \mathbb{Z}^+$,

3) $\{(m, 0), (0, n)\}$ is a basis for $p_*(\pi_1(E, e_0))$, $m, n \in \mathbb{Z}^+$.

Using the result from part (a), the homeomorphism $h : (T, x_0) \to (T, x_0)$ induces a group isomorphism $h_* : \pi_1(T, x_0) \to \pi_1(T, x_0)$, which we can see as a change of basis map, and define $h_*((1, 0)) = [\alpha_1]$ and $h_*((0, 1)) = [\alpha_2]$. Clearly, $h \circ p : E \to P$ is also a covering map, and our three possibilities become:

1) $(h \circ p)_*(\pi_1(E, e_0)) = 0$,

2) $\{m\alpha_1\}$ is a basis for $(h \circ p)_*(\pi_1(E, e_0))$, $m \in \mathbb{Z}^+$,

3) $\{m\alpha_1, n\alpha_2\}$ is a basis for $(h \circ p)_*(\pi_1(E, e_0))$, $m, n \in \mathbb{Z}^+$.

We then define the following canonical covering maps $p' : E' \to T$ with different choices of E' by choosing some $e'_0 \in p'^{-1}(x_0)$, and observe $p'_*(\pi_1(E, e'_0))$:

1') $p' : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ by $(x, y) \mapsto (e^{x(2\pi i)}, e^{y(2\pi i)})$, then $p'_*(\pi_1(E, e'_0)) = 0$,

2') $p' : S^1 \times \mathbb{R} \to S^1 \times S^1$ by $(x, y) \mapsto (x^m, e^{y(2\pi i)})$, then $p'_*(\pi_1(E, e'_0))$ is a $\mathbb{Z}$-linear combination of the basis $\{m\alpha_1\}$, where $m \in \mathbb{Z}^+$,

3') $p' : S^1 \times S^1 \to S^1 \times S^1$ by $(x, y) \mapsto (x^m, y^n)$, then $p'_*(\pi_1(E, e'_0))$ is a $\mathbb{Z}$-linear combination of the basis $\{m\alpha_1, n\alpha_2\}$, where $m, n \in \mathbb{Z}^+$.

Since $(h \circ p)_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_0))$ for each of the three possibilities, by Theorem

79.2, there must be an equivalence $h : E \to E'$ for each possibility. Hence E must be homeomorphic to either $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T.

The following is a diagram for case 2, and the diagrams for cases 1 and 3 are similar:

$$\begin{tikzcd}
(E, e_0) \arrow[d, "p"'] & (S^1 \times \mathbb{R}, e_0') \arrow[d, "p'"] \\
(T, x_0) \arrow[r, "h"'] & (T, x_0)
\end{tikzcd}$$

QED

__**Exercise 81.4:**__

Let G be a group of homeomorphisms of X. The action of G on X is said to be fixed-point free if no element of G other than the identity e has a fixed point. Suppose X is Hausdorff and G is a finite group of homeomorphisms of X whose action is fixed-point free, we want to show that the action of G is properly discontinuous.

Take arbitrary $x \in X$ and $g_1, g_2 \in G$. We first claim that the action of G on X is fixed-point free means that if $g_1 \neq g_2$, then $g_1(x) \neq g_2(x)$. This is because suppose otherwise that $g_1 \neq g_2$ but $g_1(x) = g_2(x)$, we then have $g_2 g_1^{-1} \neq e$ and $g_2 g_1^{-1}(x) = x$, which contradicts with the fixed-point free definition.

Thus, given X is Hausdorff and $g_1(x) \neq g_2(x)$, we can construct disjoint neighborhoods U_1 and U_2 of $g_1(x)$ and $g_2(x)$, respectively. We then define $W := \bigcap\limits_{g_i \in G} g_i^{-1}(U_i)$, which is a neighborhood of x because $x \in g_i^{-1}(U_i)$ for each g_i, each g_i is continuous so $g_i^{-1}(U_i)$ is open, and G is finite so that a finite intersection of open sets is open. Using this construction, we see that $g_i(W) \subseteq U_i$ for each U_i. Furthermore, note that $W = e(W) \subseteq U_e$, where e is treated as the identity homeomorphism on X. By our previous construction, we know all neighborhoods U_i are pairwise disjoint, so $U_i \cap U_e = \emptyset$. It follows that $g_i(W) \cap W = \emptyset$ whenever $g_i \neq e$, which is precisely the definition of properly discontinuous.

$$QED$$

79.2:

(a) We want to show that every continuous map $f : P^2 \to S^1$ is nullhomotopic.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ P^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{0} \arrow[d, "p_*"] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \mathbb{Z} \end{tikzcd}$$

Let f be an arbitrary continuous map $f : P^2 \to S^1$, and let $\mathbb{R}$ be the covering space of S^1. We proceed by the General Lifting Lemma. We know $\pi_1(S^1) = \mathbb{Z}$, and since $\pi_1(P^2)$ is a group of order 2, we denote it by $\mathbb{Z}/2\mathbb{Z}$. We claim the only homomorphism $f_* : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the trivial homomorphism. This is because in order for $f_*(1+1) = f_*(0) = 0 = f(1) + f(1)$ to make sense, it must be that $f_*(1) = 0$. Hence we have $f_*(\pi_1(P^2)) \subseteq p_*(\pi_1(\mathbb{R}))$, and we can now invoke the General Lifting Lemma and lift f to $\tilde{f} : P^2 \to \mathbb{R}$. Recall by Exercise 51.3 that $\mathbb{R}$ is contractible, so the set of continuous functions $[P^2, \mathbb{R}]$ up to homotopy has a single element. Since $\tilde{f}$ is continuous and every constant function is continuous, we can conclude that $\tilde{f}$ is nullhomotopic. We then get a homotopy $F : P^2 \times I \to \mathbb{R}$ such that $F(x, 0) = \tilde{f}(x)$ and $F(x, 1) = e_c$, which is a constant map. We can then compose p to F and get a homotopy $p \circ F : P^2 \times I \to S^1$, in which we have $p \circ F(x, 0) = p \circ \tilde{f}(x) = f(x)$ and $p \circ F(x, 1) = p \circ e_c = e_{p(c)}$, which is another constant map. This shows that f is nullhomotopic.

$$QED$$

b) We want to find a continuous map of the torus into S^1 that is not nullhomotopic.
We claim $f : S^1 \times S^1 \to S^1$ defined by $(x, y) \mapsto x$ is such a map. f is continuous because it is a projection onto the first coordinate. Also note that f is surjective. Now using the contrapositive of Corollary 58.6, since $f_* : \pi_1(S^1 \times S^1) \to \pi_1(S^1)$ is surjective, and $\pi_1(S^1) = \mathbb{Z}$, f_* is clearly non-trivial. Hence f is not nullhomotopic.

$$QED$$