72 Adjoining a Two-cell, Math 104 **Name:** *Phil Rehwinkel, James Wang*
March 28, 2024

Theorem 70.1 & 70.2:

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i_*"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
& \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

If ϕ_1 and ϕ_2 are homomorphisms compatible on $U \cap V$ such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then we have a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$.

Let j be the extension of j_1 and j_2, then $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective, and

$$\pi_1(U, x_0) * \pi_1(V, x_0) \Big/ < i_1(g)^{-1} i_2(g) >$$

for $g \in \pi_1(U \cap V, x_0)$.

Theme of Seifert-van Kampen: Assuming the hypothesis of Seifert-van Kampen, the fundamental group $\pi_1(X, x_0)$ is equivalent to the free product of $\pi_1(U, x_0) * \pi_1(V, x_0)$, together with the relation $i_1(x) = i_2(x)$ for all $x \in U \cap V$.

Theorem 72.1

Let A be a closed, path-connected subspace of a Hausdorff space X. Let $h : B^2 \to X$ be a continuous map that maps the interior of the closed disc into $X - A$ and maps the boundary of the closed disc, S^1, into A. Let p be a point on the boundary of the disc; then $a = h(p)$, and $k : (S^1, p) \to (A, a)$ be the map obtained by restricting the domain of h to the boundary of the disc.

Then, the homomorphism of fundamental groups

$$i_* : \pi_1(A, a) \to \pi_1(X, a)$$

induced by the inclusion i is surjective, and has as its kernel the least normal subgroup of $\pi_1(A, a)$ containing the image of k_*.

A way of thinking about Theorem 72.1, from Professor Miller: Given the continuous map h, we know the fundamental group $\pi_1(X, x_0)$ is equivalent to the fundamental group $\pi_1(A)$ together with a relation determined by the image on the boundary (S^1) of the two-cell, without any new generator.

Example (Fundamental Group of the Torus):