Week 5 Final, Math 104 **Name:** *James Wang*
Feb 22, 2024

The given problem: Suppose that X is a contractible space. Prove that there is a bijection between $[X, Y]$ and the set of path components of Y.

Suppose X is contractible, by definition we know the identity map is nullhomotopic, meaning it is homotopic to a constant map. Translating this, we get a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = p$, where p is an element of X. Let $C(Y)$ denote the set of path components of Y, so that $y_1, y_2 \in [Y_i] \in C(Y)$ if and only if there is a path between them. Now consider the map $\varphi : [X, Y] \to C(Y)$ defined by $\varphi([f]) = [Y_i]$, where $f(p) \in [Y_i]$. We first show that φ is well-defined. Take arbitrary continuous $f_1, f_2 : X \to Y$ such that $f_1 \simeq f_2$, then by definition there exists $H : X \times I \to Y$ such that $H(x, 0) = f_1(x)$ and $H(x, 1) = f_2(x)$ for all $x \in X$. Recall that X is contractible, so we can compose a constant path in Y to H and get a homotopy between $f_1(p)$ and $f_2(p)$. This means there is a path between $f_1(p)$ and $f_2(p)$, so we have shown $\varphi(f_1)$ and $\varphi(f_2)$ are in the same path component. We next show that φ is injective. Take arbitrary continuous $f_1, f_2 : X \to Y$ and suppose $\varphi(f_1) = \varphi(f_2)$, which means $f_1(p)$ and $f_2(p)$ are in the same path component. Since X is contractible, it follows quickly that $f_1(p) \simeq f_2(p)$ and $f_1 \simeq f_2$. We finally show that φ is surjective. Take any $[Y_i] \in C(Y)$, since X is contractible, recall we have defined a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = p$. We can also define some $g : X \to Y$ so that $g(x) \in [Y_i]$. Together we finally define a homotopy $K : X \times I \to Y$ by $K(x, t) = g(F(x, t))$, through which we see $K(x, 0) = g(x)$ and $K(x, 1) = g(p)$. Since $g(x) \in [Y_i]$ by construction, it must also be that $g(p) \in [Y_i]$, and we have found some continuous $g : X \to Y$ such that $\varphi([g]) = [Y_i]$.

QED

Exercise 53.6b):

Let $p : E \to B$ be a covering map. We want to prove that if B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Let $\{U_\alpha\}$ be an open covering of E. Take arbitrary $x \in E$, since $p(x) \in B$ and $p^{-1}(b)$ is finite for each $b \in B$, we can denote $p^{-1}(p(\{x\})) = \{x, x_1, \dots, x_n\}$. Take an open set V' in B that contains $p(x)$, by the definition of a covering map, $p^{-1}(V')$ is divided into disjoint open sets $T_x, T_{x_1}, \dots, T_{x_n}$, with each being homeomorphic to V' through the restriction on p. Since $\{U_\alpha\}$ covers E, take $U_{\alpha_x}, U_{\alpha_{x_1}}, \dots, U_{\alpha_{x_n}} \in \{U_\alpha\}$ such that $x \in U_{\alpha_x}, x_1 \in U_{\alpha_{x_1}}, \dots, x_n \in U_{\alpha_{x_n}}$. Consider $V_x = p(U_{\alpha_x} \cap T_x) \cap p(U_{\alpha_{x_1}} \cap T_{x_1}) \cap \cdots \cap p(U_{\alpha_{x_n}} \cap T_{x_n})$. We know V_x is open in B because p is an open map, and $p^{-1}(V_x)$ is divided into $n+1$ disjoint open sets in E, each is homeomorphic to V_x. We then repeat the above process for every $x \in E$, then the set $\{V_x\}_{x \in X}$ is an open cover of B. Since B is compact, let $\{V_{x_1}, V_{x_2}, \dots, V_{x_n}\}$ be a finite subcover of B. Denote $p^{-1}(p\{x_1\}) = \{x_1, e_{1,1}, e_{1,2}, \dots, e_{1,n_1}\}$, so we know $p^{-1}(V_{x_1}) \subseteq U_{\alpha_{x_{1,1}}} \cup U_{\alpha_{e_{1,1}}} \cup \cdots \cup U_{\alpha_{e_{1,n_1}}}$. Similarly, $p^{-1}(V_{x_2}) \subseteq U_{\alpha_{x_{2,1}}} \cup U_{\alpha_{e_{2,1}}} \cup \cdots \cup U_{\alpha_{e_{2,n_2}}}$, and so on. This means each of $p^{-1}(V_{x_1}), p^{-1}(V_{x_2}), \dots, p^{-1}(V_{x_k})$ is covered by a finite number of elements in $\{U_\alpha\}$. Hence any open cover of $E = p^{-1}(V_{x_1}) \cup p^{-1}(V_{x_2}) \cup \cdots \cup p^{-1}(V_{x_k})$ can be reduced to a finite subcover with elements in $\{U_\alpha\}$, so E is compact.

$$QED$$

Exercise 52.5:

Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. We want to show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism.

Let $f : \mathbb{R}^n \to Y$ be the extended continuous map of h so that $f|_A = h$, and let $j : A \to \mathbb{R}^n$ be the inclusion map such that $j(a) = a$ for all $a \in A$. Note that $f \circ j = h$. Since both j and f are continuous, $j_* : \pi_1(A, a_0) \to \pi_1(\mathbb{R}^n, a_0)$ and $f_* : \pi_1(\mathbb{R}^n, a_0) \to \pi_1(Y, y_0)$ exist. By the functorial property, we have $f_* \circ j_* = (f \circ j)_* = h_*$. However, $\pi_1(\mathbb{R}^n, a_0)$ must be trivial, so for the maps to be defined, j_* and f_* must be the trivial homomorphism, and h_* is the trivial homomorphism.

QED

Exercise 53.1:

Let Y have the discrete topology. We want to show that if $p : X \times Y \to X$ is projection on the first coordinate, then p is a covering map.

We need to show p is continuous and surjective, and every $x \in X$ has a neighborhood U such that $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in X where for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

Since $p : X \to Y$ is a projection map, p is clearly continuous and surjective.

Take arbitrary $x \in X$ and consider the entire space X as the neighborhood of x. It follows that $p^{-1}(X) = \{X \times \{y\}\}$ for all $y \in Y$. Clearly, for each $y \in Y$, $X \times \{y\}$ is open by the product topology because X is open and $\{y\}$ is open (Y has the discrete topology). Moreover, take $y_1, y_2 \in Y$, we see that $(X \times \{y_1\}) \cap (X \times \{y_2\}) = \emptyset$, so the open sets are also disjoint. Finally, we claim that in fixing $y \in Y$, $X \times \{y\}$ is homeomorphic to X. This is because $f : X \times \{y\} \to X$ defined by $(x, y) \mapsto x$ is bijective and continuous, and $f^{-1} : X \to X \times \{y\}$ defined by $x \mapsto (x, y)$ is also continuous.

QED