Week 3 Final, Math 104 **Name:** *James Wang*
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Exercise 22.4 a):

Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ if } x_0 + y_0^2 = x_1 + y_1^2.$$

Let X^* be the corresponding quotient space. What familiar space is it homeomorphic to?

We claim X^* is homeomorphic to $\mathbb{R}$ by using Corollary 22.3.

Using the hint, we define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x \times y) = x + y^2$. Since the sum of continuous functions is continuous, we know g is continuous. Moreover, g is also surjective because for all $x \in \mathbb{R}$, $g(x \times 0) = x$. Finally, $X^* = \{g^{-1}(\{z\}) \mid z \in \mathbb{R}\}$. Thus, we have everything we need to invoke Corollary 22.3, which tells us that g induces a bijective continuous map $f : X^* \to \mathbb{R}$, which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "p"'] \arrow[dr, "g"] & \\
X^* \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

Thus, to show X^* is homeomorphic to $\mathbb{R}$, it remains to show that g is a quotient map.

We claim $\varphi : \mathbb{R} \to \mathbb{R}^2$ defined by $\varphi(x) = (x \times 0)$ is a continuous right inverse of g. Since each coordinate function of φ is continuous, φ is continuous by Theorem 18.4. Since $g(\varphi(x)) = g(x \times 0) = x$, we know $g \circ \varphi$ is the identity map of $\mathbb{R}$. By the result of Exercise 22.2a), we know g is a quotient map.

Together, we have proved that X^* is homeomorphic to $\mathbb{R}$.

QED

Note: I proved Exercise 22.2 later in this submission.

Exercise 26.5:

Let A and B be disjoint compact subspaces of the Hausdorff space X. We want to show that there exist disjoint open sets U and V containing A and B, respectively.

Suppose A and B are disjoint compact subspaces of the Hausdorff space X. For each $a \in A$, we know by Lemma 26.4 that there exist disjoint open sets U_i and V_i of X containing a and B, respectively. Note that the set $\bigcup U_i$ is an open covering of A because it is a union of open sets and contains each $a \in A$. Recall that A by assumption is also compact, so $\bigcup U_i$ can be reduced to a finite subcollection $U_1 \cup U_2 \cup \cdots \cup U_n$ that still covers A, which is also open.

Now take the corresponding V_i of each U_i, and consider their intersection $V_1 \cap V_2 \cap \cdots \cap V_n$. Note that it is open because it is the intersection of finitely many open sets. Moreover, since $B \subseteq V_i$ for each V_i by construction, $B \subseteq (V_1 \cap V_2 \cap \cdots \cap V_n)$. Finally, since it must be that $U_i \cap V_i = \emptyset$ for each pair of U_i and V_i, we know for sure that whatever is in U_i cannot be in $V_1 \cap V_2 \cap \cdots \cap V_n$, so $(U_1 \cup U_2 \cup \cdots \cup U_n) \cap (V_1 \cap V_2 \cap \cdots \cap V_n) = \emptyset$.

Hence we have found disjoint open sets $U = U_1 \cup U_2 \cup \cdots \cup U_n$ and $V = V_1 \cap V_2 \cap \cdots \cap V_n$ containing A and B respectively.

QED

Exercise 22.2:

a)

Let $p : X \to Y$ be a continuous map. We want to show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

Recall that for $p : X \to Y$ to be a quotient map, it must be surjective, and a subset $U \subseteq Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

We first check surjectivity:

Since by assumption there exists $f : Y \to X$ such that $p \circ f = \mathrm{id}_Y$, we know f is a right inverse of p, and by a result from Exercise 2.5, this means p is surjective.

We then check the if and only if statement:

$(\Longrightarrow)$: Suppose U is an open subset of Y. Recall by assumption that p is continuous, so that $p^{-1}(U)$ is an open subset of X by the definition of continuity.

$(\Longleftarrow)$: Suppose $p^{-1}(U)$ is an open subset of X. Recall by assumption that $f : Y \to X$ is also continuous, so that $f^{-1}(p^{-1}(U))$ is an open subset of Y by the definition of continuity. Recall further that $p \circ f = \mathrm{id}_Y$, so it must also be that $(p \circ f)^{-1} = f^{-1} \circ p^{-1} = \mathrm{id}_Y$, and $f^{-1}(p^{-1}(U)) = U$. Hence U must be an open subset of Y.

We have thus proved that $p : X \to Y$ is a quotient map.

$$QED$$

b) If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. We want to show that a retraction is a quotient map.

Let $\varphi : A \to X$ be the inclusion function defined by $\varphi(a) = a$ for all $a \in A$. By Theorem 18.2b), we know φ is continuous. Note that r is continuous, and note that $r(\varphi(a)) = a$ for all $a \in A$, meaning that $r \circ \varphi$ equals the identity map of A. By the result of part a), we can conclude that r is a quotient map.

$$QED$$

Exercise 24.2:

Let $f : S^1 \to \mathbb{R}$ be a continuous map. We want to show there exists a point c of S^1 such that $f(c) = f(-c)$.

Let $g : S^1 \to \mathbb{R}$ be a new function defined by $g(x) = f(x) - f(-x)$ for all $x \in S^1$. Thus, to show that there exists some $c \in S^1$ such that $f(c) = f(-c)$, it suffices to show that there exists some $c \in S^1$ such that $g(c) = 0$.

Note that $g(-x) = f(-x) - f(x) = -g(x)$ for all $x \in S^1$, which means 0 has to be between $g(-x)$ and $g(x)$. Note further that since g is a difference of continuous functions, it must also be continuous. Thus, we have everything we need to invoke the Intermediate Value Theorem, which tells us that there exists some $c \in [-x, x]$ such that $g(c) = 0$.

$$QED$$