Week 4 Final, Math 104 **Name:** *James Wang*
Feb 16, 2024

<u>Exercise 31.7:</u>

Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a perfect map.)

a)

We want to show that if X is Hausdorff, then so is Y.

Take distinct points $y_1, y_2 \in Y$ and consider $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$. Since a function only has one output for each input, we know that $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are disjoint. Moreover, by the definition of a perfect map, we also know that $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are compact. Now recall that we have proved in Exercise 26.5 that disjoint compact subspaces of a Hausdorff space have disjoint open sets containing each of them, so there exist disjoint open subsets U_1 and U_2 of X containing $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ respectively.

We claim that $Y \setminus p(X \setminus U_1)$ and $Y \setminus p(X \setminus U_2)$ are disjoint neighborhoods of y_1 and y_2. Let's just focus on $Y \setminus p(X \setminus U_1)$ for now. We first show that it is open in Y. Since U_1 is open and p is a closed map, $p(X \setminus U_1)$ is closed and $Y \setminus p(X \setminus U_1)$ is open. We then show that it is disjoint from $Y \setminus p(X \setminus U_2)$. Suppose for a contradiction that there exists some $z \in (Y \setminus p(X \setminus U_1))$ and $z \in (Y \setminus p(X \setminus U_2))$. Then this means $z \notin p(X \setminus U_1)$ and $z \notin p(X \setminus U_2)$, or equivalently $p^{-1}(\{z\}) \subseteq U_1$ and $p^{-1}(\{z\}) \subseteq U_2$. This, however, contradicts with our initial construction of U_1 and U_2 being disjoint. We finally show that $y_1 \in (Y \setminus p(X \setminus U_1))$. Suppose for a contradiction that $y_1 \notin (Y \setminus p(X \setminus U_1))$, then it must be $y_1 \in p(X \setminus U_1)$, or equivalently $p^{-1}(\{y_1\}) \nsubseteq U_1$. This, however, contradicts with our initial construction of $p^{-1}(\{y_1\}) \subseteq U_1$. The proof for $Y \setminus p(X \setminus U_2)$ is the same.

We have thus proved that for two distinct points $y_1, y_2 \in Y$, there exist neighborhoods of them respectively that are disjoint, which is the definition of Y being Hausdorff.

QED

Exercise 35.4:

Let Z be a topological space. If Y is a subspace of Z, we say that Y is a retract of Z if there is a continuous map $r : Z \to Y$ such that $r(y) = y$ for each $y \in Y$.

a)

We want to show that if Z is Hausdorff and Y is a retract of Z, then Y is closed in Z.

This is equivalent to showing $Z \setminus Y$ is open. Take arbitrary $z \in Z \setminus Y$. Since by definition $r : Z \to Y$, we have that $r(z) = y$ for some $y \in Y$. Clearly, $z \neq y$ because $z \notin Y$. Recall that Z is also Hausdorff by assumption, so there exist neighborhoods U_z and U_y of z and y, respectively, that are disjoint.

We claim that $(r^{-1}(U_y \cap Y)) \cap U_z$ is a neighborhood of z that is disjoint from Y. We first show that this set does contain z. Clearly, $z \in U_z$ by construction, and since $r(z) = y \in U_y \cap Y$, it follows quickly that $z \in r^{-1}(U_y \cap Y)$, so $z \in (r^{-1}(U_y \cap Y)) \cap U_z$. We then show that this set is open in Z. Since Y is a subspace of Z and U_y is open in Z, by the construction of the subspace topology, $U_y \cap Y$ is open in Y. Recall that a retract is by definition continuous, so $r^{-1}(U_y \cap Y)$ must also be open in Z. Furthermore, U_z is open in Z, so $(r^{-1}(U_y \cap Y)) \cap U_z$ is open in Z. We finally show that this set is disjoint from Y. Suppose for a contradiction that there exists $x \in ((r^{-1}(U_y \cap Y)) \cap U_z) \bigcap Y$. This means $x \in U_z$, and $r(x) \in (U_y \cap Y) \subseteq U_y$. However, it is also true that $x \in Y$, so together we have $r(x) = x \in U_y$. Hence U_z and U_y are not disjoint, and we have reached a contradiction with the definition of Z being Hausdorff. We have thus found a neighborhood of z in $Z \setminus Y$. Hence $Z \setminus Y$ is open and Y is closed in Z.

$$QED$$

b)

Let A be a two-point set in $\mathbb{R}^2$. We want to show that A is not a retract of $\mathbb{R}^2$.

Suppose for a contradiction that there exists a retract $r : \mathbb{R}^2 \to \{x_1, x_2\} \subseteq \mathbb{R}^2$ such that $r(x_1) = x_1$ and $r(x_2) = x_2$. Since $\mathbb{R}^2$ is connected and r is continuous, $r(\mathbb{R}^2)$ must be connected by Theorem 23.5. However, $r(\mathbb{R}^2) = \{x_1, x_2\}$ is not connected, because we can let $r = d(x_1, x_2)/2$. Then $B_r(x_1) \cap \{x_1, x_2\}$ and $B_r(x_2) \cap \{x_1, x_2\}$ are open in the subspace topology on $\{x_1, x_2\}$ and form a separation of $\{x_1, x_2\}$.

QED

Exercise 30.4:

We want to show that every compact metrizable space X has a countable basis.

For each $n \in \mathbb{N}$, since X is metrizable, the set $\{B_{1/n}(x) \mid x \in X\}$ is an open covering of X. Since X is compact, there exists a finite subset $\mathcal{A}_n \subseteq \{B_{1/n}(x) \mid x \in X\}$ that still covers X. We claim that $\mathbb{B} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is a countable basis for the topology on X.

Clearly, $\mathbb{B}$ is countable because each $\mathcal{A}_n$ is finite and the union is countable. We next show that $\mathbb{B}$ is a basis for the topology on X. First, since each $B_{1/n}(x)$ is open in X, every element of $\mathbb{B}$ is open in X. Next, we use Lemma 13.2. Let $V \subseteq X$ be open, then for each $v \in V$, there exists $r > 0$ such that $B_r(v) \subseteq V$. Moreover, since $1/n \to 0$, by definition, there exists $k \in \mathbb{N}$ such that $d(0, 1/k) < r/2$, or equivalently $1/k < r/2$. Recall that $\mathcal{A}_k$ by construction is a covering of X, so there exists $x \in X$ such that $v \in B_{1/k}(x) \in \mathcal{A}_n$.

We claim that $B_{1/k}(x) \subseteq B_r(v)$, because let $y \in B_{1/k}(x)$, by the triangle inequality we have

$$d(y, v) \leq d(y, x) + d(x, v) < 1/k + 1/k < r/2 + r/2 = r.$$

Note that $d(y, x) < 1/k$ because $y \in B_{1/k}(x)$ by assumption, and $d(x, v) < 1/k$ because $v \in B_{1/k}(x)$ by construction. Thus, we have obtained the following

$$v \in B_{1/k}(x) \subseteq B_r(v) \subseteq V.$$

By Lemma 13.2, since $v \in V$ is arbitrary and $B_{1/k}(x)$ is open, $v \in B_{1/k}(x) \subseteq V$ implies that $\mathbb{B}$ is a basis for the topolgy on X.

$$QED$$

Note: I only know how to do this problem because of my presentation on $\mathbb{R}^n$ being second-countable.