16 The Subspace Topology, Math 104 **Name:** *James Wang, Jeffrey Zhang*
February 1, 2024

Important Facts

Definition: Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a subspace of X.

An open set in a subspace is the intersection of any open set in X with the subset Y

Section 16 Example 1

Looking at the subset $Y = [0, 1]$ of the set $\mathbb{R}$, we can determine all possible open sets in Y by considering the intersection of any open set (a, b) of $\mathbb{R}$ with Y. This leads to four different types of open sets in the subspace (a, b), with $a, b \in Y$, $[0, b)$, with $b \in Y$, $(a, 1]$, with $a \in Y$, and $[0, 1]$ or $\emptyset$.

Section 16 Example 3

Now consider the subset $I = [0, 1]$. In the subspace $I \times I$ of $\mathbb{R} \times \mathbb{R}$ under dictionary order, we can see that the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open since it is the intersection of the open set $(\frac{1}{2} \times \frac{1}{2}, \frac{1}{2} \times \frac{3}{2})$ with $I \times I$. But, in the order topology, the set is clearly not open.

We can see that a set that is open in the subspace topology may not be open in the order topology.

Definition: Given an ordered set X, a subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y.

Question: Given an ordered set X and take $Y \subseteq X$, what is the relationship between the order topology on Y and the subspace topology Y inherits from X?

Theorem 16.3 with Proof Sketch:

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Let U be open in X and V be open in Y, we observe that

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B),$$

so the bases for the two topologies are the same.

Theorem 16.4 with Proof Sketch:

Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

We use double inclusion to show the order topology = the subspace topology:
$(\supseteq)$: $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y
$(\subseteq)$: any open ray of Y is the intersection of an open ray of X with Y

Claim:

Given an ordered set X and take $Y \subseteq X$, the subspace topology Y inherits from X is always finer than the order topology on Y.

The basis elements of the order topology are

$$B_1 = \{y \in Y \mid a < y < b\},\ B_2 = \{y \in Y \mid a_0 \leq y < b\},\ B_3 = \{y \in Y \mid a < x \leq b_0\}.$$

Note that

$$B_1 = (a, b) \cap Y,\ B_2 = [a_0, b) \cap Y,\ B_3 = (a, b_0] \cap Y.$$

The basis elements of the order topology is a subset of the basis of the subspace topology.