55 Retractions and Fixed Points, Math 104 **Name:** *Hillary Kim, James Wang*
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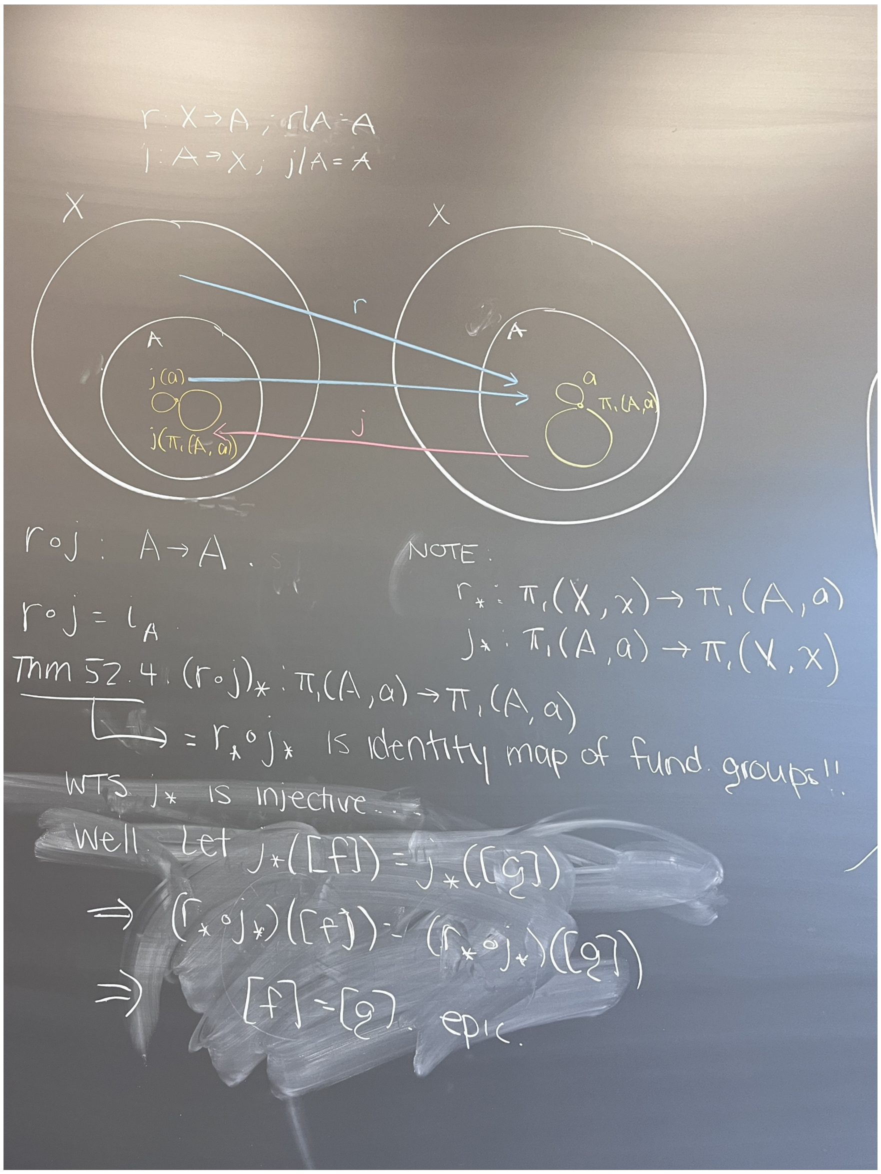
Theorem 1. *If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.*



Theorem 2. ***No retraction Theorem***
There is no retraction of B^2 onto S^1, aka the unit disk onto the unit circle.

This is a silly little definition-based proof. Let's look at the disk; it is star-convex (recall Chapter 52??). This means it must be simply connected, and thus its fundamental group is trivial (page 333).
Now let's look at S^1 and its fundamental group. By Theorem 54.5, we know the fundamental group of S^1 is isomorphic to the additive set of integers. Great. It is nontrivial.
Okay, now that those observations are out of the way, let's suppose towards a contradiction that there is a retraction of B^2 onto S^1. By Lemma 55.1 (above), this means that inclusion $j : S^1 \to B^2$ must induce an injective j_*. But guys. Horrible news ahead. j_* maps the fundamental group of S_1, which is nontrivial, to the fundamental group of B^2, which is trivial. So it can't be injective. Oh my god gasp a contradiction. Crazy time! I wanted to also briefly mention a generalization about this; the source I will link to Piazza.

Theorem 3. *Let X be a topological space homeomorphic to the underlying space of a 2-dimensional simplicial complex. Then there exists no retraction $r : X \to \partial X$*

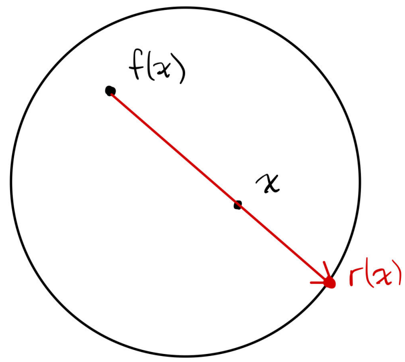
Several comments on this. Given my brief and hasty research on what a "simplical complex" means, I am very unsure. But, given our knowledge of the proof, I assume we can make the less stronger statement that

Theorem 4. *Let X be a topological space homeomorphic to a simply connected space. Then there exists no retraction $r : X \to \partial X$*

Again, I am very unsure. But this is cool!

Theorem 5. *(Brouwer fixed-point theorem for disc) If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.*

Proof Sketch (Hatcher):
Suppose for a contradiction...



We can define $r : B^2 \to S^1$, which is a retraction!
Note that $w_0 \simeq w_1$ in B^2, but $w_0 \not\simeq w_1$ in S^1. Yet we can force a composition that reaches a contradiction.

Corollary: Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue.

How is B homeomorphic to B^2 in the textbook proof?