**51 Homotopy of Paths, Math 104** **Name:** *Adin Aberbach, James Wang*
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**Definition:** Two paths $f$ and $f'$, mapping the interval $I = [0, 1]$ into $X$, are said to be **path homotopic** if they have the same initial point $x_0$ and the same final point $x_1$, and if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(s,0) = f(s) \quad &\text{and} \quad F(s,1) = f'(s), \\ F(0,t) = x_0 \quad &\text{and} \quad F(1,t) = x_1. \end{aligned}$$

**Definition:** If $f$ is a path in $X$ from $x_0$ to $x_1$, and if $g$ is a path in $X$ from $x_1$ to $x_2$, we define the **product** $f * g$ to be the path $h$ given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, 1/2], \\ g(2s-1), & \text{for } s \in [1/2, 1]. \end{cases}$$

The product on paths induces a well-defined operation on path-homotopy classes,

$$[f] * [g] = [f * g].$$

To verify this, let $F$ be a path homotopy between $f$ and $f'$ and let $G$ be a path homotopy between $g$ and $g'$. Define the new path homotopy as

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \in [0, 1/2], \\ G(2s-1,t), & \text{for } s \in [1/2, 1]. \end{cases}$$

**Theorem:** The operation $*$ has the following properties:
(1) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.
(2) (Right and left identities) Given $x \in X$, let $e_x$ denote the constant path $e_x : I \to X$ carrying all of $I$ to the point $x$. If $f$ is a path in $X$ from $x_0$ to $x_1$, then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(3) (Inverse) Given the path $f$ in $X$ from $x_0$ to $x_1$, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the **reverse** of $f$. Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$