Week 14 Final, Math 104 **Name:** *James Wang*
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Honor Problem 2022.11: Let X be the one point union of a torus and a 2-sphere.

a) Compute $\pi_1(X)$.

We know X is the one point union of a torus and a 2-sphere. We use the Seifert-Van Kampen Theorem to compute $\pi_1(X)$. Let U be the torus T with a little bit of S^2 such that it is open. Let V be the 2-sphere S^2 with a little bit of T such that it is open. By observation, $U \cap V$ deformation retracts to the wedge point of T and S^2, which means it is simply connected. Also note that U and V are path connected, because T and S^2 are path connected. We now have everything we need to use the Seifert-Van Kampen Theorem, and by a special case of it, we know $\pi_1(X) \cong \pi_1(U) * \pi_1(V) \cong (\mathbb{Z} \times \mathbb{Z}) * \{1\} \cong \mathbb{Z} \times \mathbb{Z}$.

QED

b) Describe the universal cover of X.

We know the universal cover of T is $\mathbb{R} \times \mathbb{R}$, and by the classification of covering spaces, we know the only connected covering space of S^2 is just S^2 itself. We claim the universal cover of X is $\mathbb{R} \times \mathbb{R}$, with a S^2 attached at every $x \times y$ where $x, y \in \mathbb{Z}$. This is a covering space because by letting every $x \times y$ map to the wedge point, every $x \in X$ would have a evenly covered neighborhood W. Furthermore, this covering space is simply connected, because given an arbitrary loop, we can decompose it into small pieces lying entirely within each S^2 or $\mathbb{R} \times \mathbb{R}$. Since S^2 is simply connected, every piece in each S^2 is path homotopic to the constant function at that particular $x \times y$. Now, this means the loop is path homotopic to a loop that lies entirely in $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ is also simply connected, we can conclude that this loop must be path homotopic to a point in $\mathbb{R} \times \mathbb{R}$. Hence we have shown that our construction is a universal cover.

QED

Honor Problem 2019.2: Let X be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. For each $t \in \mathbb{R}$, let X_t be the subset of all $f \in X$ such that $\sup_{x \in \mathbb{R}} f(x) < t$. Let τ be the coarsest topology on X that contains X_t for each $t \in \mathbb{R}$.

a) Show that (X, τ) is not Hausdorff.

We first make sense of the open sets in (X, τ). Since τ is the coarsest topology on X that contains each X_t, τ must also contain $\bigcup X_{t_i}$ and $\bigcap_{i=1}^{n} X_{t_i}$. By observation, nevertheless, these are still of the form X_t, since $\bigcup X_{t_i} = X_t$ where $t = \max\{t_i\}$, and $\bigcap_{i=1}^{n} X_{t_i} = X_t$ where $t = \min\{t_i\}$. (X, τ), of course, also contains $\emptyset$ and X. Now consider $f_1, f_2 \in X$ such that $\sup(f_1(x)) = 0$ and $\sup(f_2(x)) = \infty$. By our previous observation, the only neighborhood V of f_2 would have to be the entire X. This means any neighborhood U of f_1 would have to intersect V. Hence (X, τ) cannot be Hausdorff.

QED

b) Show that every subset $K \subseteq (X, \tau)$ that contains the identity function is compact.

Let $K \subseteq (X, \tau)$ contains the identity function $\mathrm{Id}_{\mathbb{R}}$, and note that $\sup(\mathrm{Id}_{\mathbb{R}}) = \infty$. Using similar reasoning as part a), we see that if $\mathcal{A}$ is an open covering of K, then $\mathcal{A}$ must also cover $\mathrm{Id}_{\mathbb{R}}$, yet the only open set in X that can cover $\mathrm{Id}_{\mathbb{R}}$ is just X itself, so $X \in \mathcal{A}$. Nevertheless, X also clearly covers K. This means one single element in $\mathcal{A}$ already covers K. Hence K must be compact.

QED

c) Let $Y \subseteq (X, \tau)$ be the subset of all constant functions. Is Y compact? Is it connected? Is it path connected?

Let $Y \subseteq (X, \tau)$ be the subset of all constant functions. We claim Y is not compact. Let $\mathcal{A} = \{X_{t_i}\}$ be an open covering of Y, and consider any finite subcollection $\mathcal{A}' = \{X_{t_1}, \ldots, X_{t_n}\} \subseteq \mathcal{A}$. Clearly, we can always set t to be larger than $\max\{t_1, \ldots, t_n\}$, then the constant function mapping all of $\mathbb{R}$ to t would not be contained in $\mathcal{A}'$. We claim Y is path connected. Let

f_a and f_b be two arbitrary constant functions mapping to a and b, respectively. We claim $\gamma : I \to Y$ defined by $s \mapsto f_{(1-s)a+sb}$ is a path from f_a to f_b. Clearly, $\gamma(0) = f_a$ and $\gamma(1) = f_b$. More importantly, γ is continuous, as it is constructed from the straight-line homotopy that moves $f_a(x)$ to $f_b(x)$ along every constant function f_c where c is between a and b. Finally, since Y is already shown to be path connected, it must also be connected.

$$QED$$