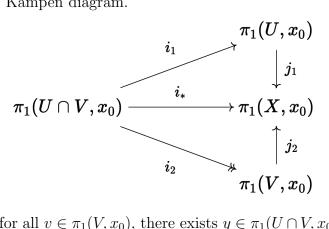
## Exercise 70.2:

Suppose that  $i_2$  is surjective.

(a) We want to show that  $j_1$  induces an epimorphism  $h: \pi_1(U, x_0)/M \to \pi_1(X, x_0)$  where M is the least normal subgroup of  $\pi_1(U, x_0)$  containing  $i_1(\ker(i_2))$ .

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We first use the hint and show  $j_1$  is surjective, and for now we only focus on the left half of the classic Seifert-van Kampen diagram.



Since  $i_2$  is surjective, for all  $v \in \pi_1(V, x_0)$ , there exists  $y \in \pi_1(U \cap V, x_0)$  such that  $i_2(y) = v$ . By commutativity, for all  $v \in \pi_1(V, x_0)$ , we have

$$j_2(v) = j_2 \circ i_2(y) = i_*(y) = j_1 \circ i_1(y)$$
 for some  $y \in \pi_1(U \cap V, x_0)$ .

Hence  $\operatorname{im}(j_2) \subseteq \operatorname{im}(j_1)$ . The Seifert-van Kampen Theorem tells us that the homomorphism  $j: \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$  of the free product that extends  $j_1$  and  $j_2$  is surjective, which means images of  $j_1$  and  $j_2$  generate  $\pi_1(X, x_0)$ . We can then conclude that  $\operatorname{im}(j_1)$  alone generates  $\pi_1(X, x_0)$ . This proves that  $j_1: \pi_1(U, x_0) \to \pi_1(X, x_0)$  is surjective.

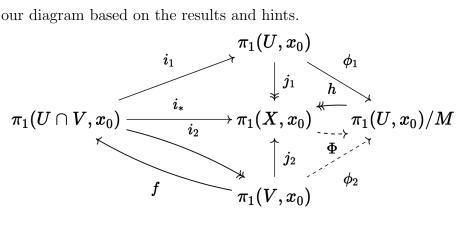
Recall that  $M = \langle i_1(\ker(i_2)) \rangle$ . Now to show  $h : \pi_1(U, x_0)/M \to \pi_1(X, x_0)$  is surjective, it suffices to show  $M \subseteq \ker(j_1)$ . Take arbitrary  $g \in \ker(i_2)$ . This result follows quickly from commutativity:

$$j_1 \circ i_1(g) = i_*(g) = j_2 \circ i_2(g) = j_2(e) = e.$$

$$QED$$

(b) We want to show that h is an isomorphism.

We update our diagram based on the results and hints.



Let  $\phi_1: \pi_1(U, x_0) \to \pi_1(U, x_0)/M$  be the projection. We first define the map  $\phi_2: \pi_1(V, x_0) \to \pi_1(U, x_0)$  $\pi_1(U,x_0)/M$ . Since  $i_2$  is surjective, we know for every  $v \in \pi_1(V,x_0)$ , there exists at least one  $y \in \pi_1(U \cap V, x_0)$  such that  $i_2(y) = v$ . We define the map  $f : \pi_1(V, x_0) \to \pi_1(U \cap V, x_0)$  by  $v \mapsto y$ . Note that even if there are multiple elements in  $\pi_1(U \cap V, x_0)$  that get mapped to v, we just choose one of them and make it y.

We define  $\phi_2: \pi_1(V, x_0) \to \pi_1(U, x_0)/M$  by  $v \mapsto \phi_1 \circ i_1 \circ f(v)$ . We then have to show that  $\phi_2$ is well-defined. Suppose for some  $v \in \pi_1(V, x_0)$ , there are two elements  $y_1, y_2 \in \pi_1(U \cap V, x_0)$ such that  $i_2(y_1) = i_2(y_2) = v$ , it suffices to show that defining  $f(v) = y_1$  or  $f(v) = y_2$ does not affect the output  $\phi_2(v)$ , that  $(i_1(y_1))M = (i_1(y_2))M$ . Recall by group theory we know  $i_2(y_1) = i_2(y_2)$  implies  $y_1y_2^{-1} \in \ker(i_2)$ . Chasing through the map  $i_1$ , we have the output  $i_1(y_1y_2^{-1})$ . Since by definition  $M = \langle i_1(\ker(i_2)) \rangle$ , we know  $i_1(y_1y_2^{-1}) \in M$ , which is equivalent to  $i_1(y_1)i_1(y_2^{-1}) \in M$  because  $i_1$  is a homomorphism. By the Coset Recognition Lemma, we have  $(i_1(y_1))M = (i_1(y_2))M$ . This shows the map  $\phi_2$  is well-defined.

We also need to verify that  $\phi_2$  is indeed a homomorphism. Since for any  $v \in \pi_1(V, x_0)$ , we can choose any  $y \in i_2^{-1}(v)$  and define f(v) = y, if  $f(v_1) = y_1$  and  $f(v_2) = y_2$ , we can define  $f(v_1v_2) = y_1y_2$ , because  $i_2$  is a homomorphism and  $i_2(y_1y_2) = i_2(y_1)i_2(y_2) = v_1v_2$ . Then

$$\phi_{2}(v_{1}v_{2}) = \phi_{1} \circ i_{1} \circ f(v_{1}v_{2}) 
= \phi_{1} \circ i_{1}(y_{1}y_{2}) 
= \phi_{1}(i_{1}(y_{1})i_{1}(y_{2})) 
= \phi_{1}(i_{1}(y_{1}))\phi_{1}(i_{1}(y_{2})) 
= (\phi_{1} \circ i_{1} \circ f(v_{1}))(\phi_{1} \circ i_{1} \circ f(v_{2})) 
= \phi_{2}(v_{1})\phi_{2}(v_{2}),$$
(1)

which shows that  $\phi_2$  is a homomorphism.

Since we have shown  $\phi_2$  is well-defined and we use  $\phi_1 \circ i_1$  in our definition of  $\phi_2$ , for all  $y \in \pi_1(U \cap V, x_0)$ , we can define  $f \circ i_2(y) = y$ . This way,

$$\phi_2 \circ i_2(y) = \phi_1 \circ i_1 \circ f \circ i_2(y) = \phi_1 \circ i_1(y)$$
 for all  $y \in \pi_1(U \cap V, x_0)$ ,

so we have everything to invoke Theorem 70.1. We then get a unique homomorphism  $\Phi$ :  $\pi_1(X, x_0) \to \pi_1(U, x_0)/M$ . By commutativity, we have

$$\Phi \circ h \circ \phi_1 = \Phi \circ j_1 = \phi_1$$

which shows that  $\Phi \circ h$  is the identity map on  $\pi_1(U, x_0)/M$ . Hence  $h : \pi_1(U, x_0)/M \to \pi_1(X, x_0)$  has a left inverse, so it must be injective. Recall we have proved h is also surjective in part (a). Together we have h is an isomorphism.

QED

## Hatcher:

Let X be the union of n lines through the origin in  $\mathbb{R}^3$ . We want to compute the fundamental group of  $\mathbb{R}^3 - X$ .

Consider the deformation retraction of the space  $\mathbb{R}^3 - X$  to  $S^2 \cap (\mathbb{R}^3 - X)$ , which can be interpreted as a 2-sphere with 2n holes (intersections of the surface of  $S^2$  and 2 ends of each line). Since a 2-sphere with one hole is homotopic equivalent to the interior of the unit disk  $B^2$ , a 2-sphere with 2n holes must be homotopic equivalent to a unit disk  $B^2$  with 2n-1 holes. Hence  $\pi_1(\mathbb{R}^3 - X)$  is the free group on 2n-1 generators.

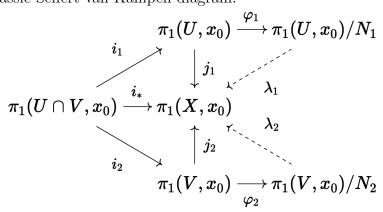
QED

## Exercise 70.1:

Suppose that the homomorphism  $i_*$  induced by inclusion  $i:U\cap V\to X$  is trivial.

(a) Show that  $j_1$  and  $j_2$  induce an epimorphism  $h: (\pi_1(U, x_0)/N_1) * (\pi_1(V, x_0)/N_2) \to \pi_1(X, x_0)$  where  $N_1$  is the least normal subgroup of  $\pi_1(U, x_0)$  containing image  $i_1$ , and  $N_2$  is the least normal subgroup of  $\pi_1(V, x_0)$  containing image  $i_2$ .

We modify the classic Seifert-van Kampen diagram.



We first define the projection maps

$$\varphi_1: \pi_1(U, x_0) \to \pi_1(U, x_0)/N_1$$
 and  $\varphi_2: \pi_1(V, x_0) \to \pi_1(V, x_0)/N_2$ .

By commutativity, we know  $j_1 \circ i_1 = j_2 \circ i_2 = i_*$ . Since  $i_*$  is trivial, we have  $\operatorname{im}(i_1) \subseteq \ker(j_1)$  and  $\operatorname{im}(i_2) \subseteq \ker(j_2)$ . Together with our definition of the least normal subgroups, this means  $N_1 = \langle \operatorname{im}(i_1) \rangle \subseteq \ker(j_1)$  and  $N_2 = \langle \operatorname{im}(i_2) \rangle \subseteq \ker(j_2)$ . By the Very Useful Lemma from Algebra, we get unique induced homomorphisms

$$\lambda_1: \pi_1(U, x_0)/N_1 \to \pi_1(X, x_0)$$
 satisfying  $\lambda_1(uN_1) = j_1(u)$ 

$$\lambda_2: \pi_1(V, x_0)/N_2 \to \pi_1(X, x_0)$$
 satisfying  $\lambda_2(uN_2) = j_2(u)$ 

This means  $\operatorname{im}(j_1) = \operatorname{im}(\lambda_1)$  and  $\operatorname{im}(j_2) = \operatorname{im}(\lambda_2)$ .

Since  $\operatorname{im}(j_1)$  and  $\operatorname{im}(j_2)$  generate  $\pi_1(X, x_0)$ , it must be that  $\operatorname{im}(\lambda_1)$  and  $\operatorname{im}(\lambda_2)$  also generate  $\pi_1(X, x_0)$ . Note that the map  $h: (\pi_1(U, x_0)/N_1) * (\pi_1(V, x_0)/N_2) \to \pi_1(X, x_0)$  is a homomorphism of the free product that extends the homomorphisms  $\lambda_1$  and  $\lambda_2$ . Hence h must be surjective.

QED