Week 9 Final, Math 104 **Name:** *James Wang*
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Exercise 70.2:

Suppose that i_2 is surjective.

(a) We want to show that j_1 induces an epimorphism $h : \pi_1(U, x_0)/M \to \pi_1(X, x_0)$ where M is the least normal subgroup of $\pi_1(U, x_0)$ containing $i_1(\ker(i_2))$.

We first use the hint and show j_1 is surjective, and for now we only focus on the left half of the classic Seifert-van Kampen diagram.

$$\begin{tikzcd} & \pi_1(U, x_0) \arrow[d, "j_1"] \\ \pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i_*"] \arrow[dr, two heads, "i_2"'] & \pi_1(X, x_0) \\ & \pi_1(V, x_0) \arrow[u, "j_2"'] \end{tikzcd}$$

Since i_2 is surjective, for all $v \in \pi_1(V, x_0)$, there exists $y \in \pi_1(U \cap V, x_0)$ such that $i_2(y) = v$. By commutativity, for all $v \in \pi_1(V, x_0)$, we have

$$j_2(v) = j_2 \circ i_2(y) = i_*(y) = j_1 \circ i_1(y) \text{ for some } y \in \pi_1(U \cap V, x_0).$$

Hence $\mathrm{im}(j_2) \subseteq \mathrm{im}(j_1)$. The Seifert-van Kampen Theorem tells us that the homomorphism $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ of the free product that extends j_1 and j_2 is surjective, which means images of j_1 and j_2 generate $\pi_1(X, x_0)$. We can then conclude that $\mathrm{im}(j_1)$ alone generates $\pi_1(X, x_0)$. This proves that $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ is surjective.

Recall that $M = < i_1(\ker(i_2)) >$. Now to show $h : \pi_1(U, x_0)/M \to \pi_1(X, x_0)$ is surjective, it suffices to show $M \subseteq \ker(j_1)$. Take arbitrary $g \in \ker(i_2)$. This result follows quickly from commutativity:

$$j_1 \circ i_1(g) = i_*(g) = j_2 \circ i_2(g) = j_2(e) = e.$$

QED

(b) We want to show that h is an isomorphism.

We update our diagram based on the results and hints.

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, two heads, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i_*"] \arrow[dr, bend left, two heads, "i_2"] & \pi_1(X, x_0) \arrow[r, dashed, bend right, "\Phi"'] & \pi_1(U, x_0)/M \arrow[l, bend right, two heads, "h"'] \\
 & \pi_1(V, x_0) \arrow[u, "j_2"] \arrow[ur, dashed, "\phi_2"'] \arrow[ul, bend left, "f"] &
\end{tikzcd}$$

Let $\phi_1 : \pi_1(U, x_0) \to \pi_1(U, x_0)/M$ be the projection. We first define the map $\phi_2 : \pi_1(V, x_0) \to \pi_1(U, x_0)/M$. Since i_2 is surjective, we know for every $v \in \pi_1(V, x_0)$, there exists at least one $y \in \pi_1(U \cap V, x_0)$ such that $i_2(y) = v$. We define the map $f : \pi_1(V, x_0) \to \pi_1(U \cap V, x_0)$ by $v \mapsto y$. Note that even if there are multiple elements in $\pi_1(U \cap V, x_0)$ that get mapped to v, we just choose one of them and make it y.

We define $\phi_2 : \pi_1(V, x_0) \to \pi_1(U, x_0)/M$ by $v \mapsto \phi_1 \circ i_1 \circ f(v)$. We then have to show that ϕ_2 is well-defined. Suppose for some $v \in \pi_1(V, x_0)$, there are two elements $y_1, y_2 \in \pi_1(U \cap V, x_0)$ such that $i_2(y_1) = i_2(y_2) = v$, it suffices to show that defining $f(v) = y_1$ or $f(v) = y_2$ does not affect the output $\phi_2(v)$, that $(i_1(y_1))M = (i_1(y_2))M$. Recall by group theory we know $i_2(y_1) = i_2(y_2)$ implies $y_1 y_2^{-1} \in \ker(i_2)$. Chasing through the map i_1, we have the output $i_1(y_1 y_2^{-1})$. Since by definition $M = < i_1(\ker(i_2)) >$, we know $i_1(y_1 y_2^{-1}) \in M$, which is equivalent to $i_1(y_1) i_1(y_2^{-1}) \in M$ because i_1 is a homomorphism. By the Coset Recognition Lemma, we have $(i_1(y_1))M = (i_1(y_2))M$. This shows the map ϕ_2 is well-defined.

We also need to verify that ϕ_2 is indeed a homomorphism. Since for any $v \in \pi_1(V, x_0)$, we can choose any $y \in i_2^{-1}(v)$ and define $f(v) = y$, if $f(v_1) = y_1$ and $f(v_2) = y_2$, we can define $f(v_1 v_2) = y_1 y_2$, because i_2 is a homomorphism and $i_2(y_1 y_2) = i_2(y_1) i_2(y_2) = v_1 v_2$. Then

$$\begin{aligned}\phi_2(v_1v_2) &= \phi_1 \circ i_1 \circ f(v_1v_2) \\ &= \phi_1 \circ i_1(y_1y_2) \\ &= \phi_1(i_1(y_1)i_1(y_2)) \\ &= \phi_1(i_1(y_1))\phi_1(i_1(y_2)) \\ &= (\phi_1 \circ i_1 \circ f(v_1))(\phi_1 \circ i_1 \circ f(v_2)) \\ &= \phi_2(v_1)\phi_2(v_2),\end{aligned} \tag{1}$$

which shows that ϕ_2 is a homomorphism.

Since we have shown ϕ_2 is well-defined and we use $\phi_1 \circ i_1$ in our definition of ϕ_2, for all $y \in \pi_1(U \cap V, x_0)$, we can define $f \circ i_2(y) = y$. This way,

$$\phi_2 \circ i_2(y) = \phi_1 \circ i_1 \circ f \circ i_2(y) = \phi_1 \circ i_1(y) \text{ for all } y \in \pi_1(U \cap V, x_0),$$

so we have everything to invoke Theorem 70.1. We then get a unique homomorphism $\Phi : \pi_1(X, x_0) \to \pi_1(U, x_0)/M$. By commutativity, we have

$$\Phi \circ h \circ \phi_1 = \Phi \circ j_1 = \phi_1,$$

which shows that $\Phi \circ h$ is the identity map on $\pi_1(U, x_0)/M$. Hence $h : \pi_1(U, x_0)/M \to \pi_1(X, x_0)$ has a left inverse, so it must be injective. Recall we have proved h is also surjective in part (a). Together we have h is an isomorphism.

$$QED$$

Hatcher:

Let X be the union of n lines through the origin in $\mathbb{R}^3$. We want to compute the fundamental group of $\mathbb{R}^3 - X$.

Consider the deformation retraction of the space $\mathbb{R}^3 - X$ to $S^2 \cap (\mathbb{R}^3 - X)$, which can be interpreted as a 2-sphere with $2n$ holes (intersections of the surface of S^2 and 2 ends of each line). Since a 2-sphere with one hole is homotopic equivalent to the interior of the unit disk B^2, a 2-sphere with $2n$ holes must be homotopic equivalent to a unit disk B^2 with $2n - 1$ holes. Hence $\pi_1(\mathbb{R}^3 - X)$ is the free group on $2n - 1$ generators.

$$QED$$

Exercise 70.1:

Suppose that the homomorphism i_* induced by inclusion $i : U \cap V \to X$ is trivial.

(a) Show that j_1 and j_2 induce an epimorphism $h : (\pi_1(U, x_0)/N_1) * (\pi_1(V, x_0)/N_2) \to \pi_1(X, x_0)$ where N_1 is the least normal subgroup of $\pi_1(U, x_0)$ containing image i_1, and N_2 is the least normal subgroup of $\pi_1(V, x_0)$ containing image i_2.

We modify the classic Seifert-van Kampen diagram.

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[r, "\varphi_1"] \arrow[d, "j_1"] & \pi_1(U, x_0)/N_1 \arrow[dl, dashed, "\lambda_1"] \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i_*"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) & \\
 & \pi_1(V, x_0) \arrow[r, "\varphi_2"'] \arrow[u, "j_2"'] & \pi_1(V, x_0)/N_2 \arrow[ul, dashed, "\lambda_2"']
\end{tikzcd}$$

We first define the projection maps

$$\varphi_1 : \pi_1(U, x_0) \to \pi_1(U, x_0)/N_1 \quad \text{and} \quad \varphi_2 : \pi_1(V, x_0) \to \pi_1(V, x_0)/N_2.$$

By commutativity, we know $j_1 \circ i_1 = j_2 \circ i_2 = i_*$. Since i_* is trivial, we have $\mathrm{im}(i_1) \subseteq \ker(j_1)$ and $\mathrm{im}(i_2) \subseteq \ker(j_2)$. Together with our definition of the least normal subgroups, this means $N_1 = < \mathrm{im}(i_1) > \subseteq \ker(j_1)$ and $N_2 = < \mathrm{im}(i_2) > \subseteq \ker(j_2)$. By the Very Useful Lemma from Algebra, we get unique induced homomorphisms

$$\lambda_1 : \pi_1(U, x_0)/N_1 \to \pi_1(X, x_0) \text{ satisfying } \lambda_1(uN_1) = j_1(u)$$
$$\lambda_2 : \pi_1(V, x_0)/N_2 \to \pi_1(X, x_0) \text{ satisfying } \lambda_2(uN_2) = j_2(u)$$

This means $\mathrm{im}(j_1) = \mathrm{im}(\lambda_1)$ and $\mathrm{im}(j_2) = \mathrm{im}(\lambda_2)$.

Since $\mathrm{im}(j_1)$ and $\mathrm{im}(j_2)$ generate $\pi_1(X, x_0)$, it must be that $\mathrm{im}(\lambda_1)$ and $\mathrm{im}(\lambda_2)$ also generate $\pi_1(X, x_0)$. Note that the map $h : (\pi_1(U, x_0)/N_1) * (\pi_1(V, x_0)/N_2) \to \pi_1(X, x_0)$ is a homomorphism of the free product that extends the homomorphisms λ_1 and λ_2. Hence h must be surjective.

$$QED$$