Week 8 Final, Math 104 **Name:** *James Wang*
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Exercise 67.4:

(b) If G is free abelian, we want to show it has no elements of finite order.

Suppose $\{a_\alpha\}$ is a basis of G, then take arbitrary $x \in G$, we know x can be expressed as

$$x = \sum c_{\alpha_i} a_{\alpha_i}, \text{ where } c_{\alpha_i} \in \mathbb{Z}.$$

Note that each $c_{\alpha_i} a_{\alpha_i}$ is an element of the subgroup G_{α_i}, which is generated by a_{α_i}.

Now suppose x has finite order, then there exists $m \in \mathbb{Z}^+$ such that $mx = 0$, which gives us

$$mx = m\left(\sum c_{\alpha_i} a_{\alpha_i}\right) = \sum mc_{\alpha_i} a_{\alpha_i} = 0,$$

meaning $mc_{\alpha_i} a_{\alpha_i} = 0$ for all α_i. This is because such elements, say $mc_{\alpha_1} a_{\alpha_1}$ and $mc_{\alpha_2} a_{\alpha_2}$, cannot be inverses of each other, for they must be in different subgroups G_1 and G_2.

Since G is free abelian, by definition, we know each subgroup generated by a_{α_i} is infinite cyclic (a_{α_i} is infinite order), so it must be that $mc_{\alpha_i} = 0$ for all α_i.

Finally, since $m, c_{\alpha_i} \in \mathbb{Z}$, $\mathbb{Z}$ is an integral domain, but $m \neq 0$, we have that $c_{\alpha_i} = 0$ for all α_i, implying $x = 0$.

Hence we have shown that if x is finite order, then it must be the identity 0, and G has no element of finite order.

QED

(c) We want to show the additive group of rationals has no elements of finite order, but it is not free abelian.

Take arbitrary $p, q \in \mathbb{Z}$ provided $q \neq 0$ and consider $\frac{p}{q}$, we see there does not exsit $m \in \mathbb{Z}^+$ such that $\frac{mp}{q} = 0$. Hence the additive group of rationals has no elements of finite order.

Now suppose for a contradiction that this group is free abelian with the basis $\{a_\alpha\}$, and consider $\frac{1}{2}a_{\alpha_j} \in (\mathbb{Q}, +)$ where a_{α_j} is a basis element, we know it can be expressed as

$$\tfrac{1}{2}a_{\alpha_j} = \sum c_{\alpha_i} a_{\alpha_i}, \text{ where } c_{\alpha_i} \in \mathbb{Z}.$$

Multiplying both sides by 2 gives us $a_{\alpha_j} = \sum 2c_{\alpha_i} a_{\alpha_i}$. Since a_{α_j} is a basis element and the expression of a_{α_j} as a direct sum must be unique, it follows from the equation that it must be $a_{\alpha_j} = 2c_{\alpha_j} a_{\alpha_j}$. This means $c_{\alpha_j} = \frac{1}{2}$, which is a contradiction with our knowledge that $c_{\alpha_j} \in \mathbb{Z}$.

QED

Exercise 68.4 (Proof of Theorem 68.4):

Let $\{G_\alpha\}_{\alpha\in J}$ be a family of groups. Suppose G and G' are groups and $i_\alpha : G_\alpha \to G$ and $i'_\alpha : G_\alpha \to G'$ are families of monomorphisms, such that the families $\{i_\alpha(G_\alpha)\}$ and $\{i'_\alpha(G_\alpha)\}$ generate G and G', respectively. If both G and G' have the extension property stated in Lemma 68.3, we want to show there is a unique isomorphism $\phi : G \to G'$ such that $\phi\circ i_\alpha = i'_\alpha$ for all α.

Since both G and G' have the extension property, we know there exist a unique homomorphism $\phi : G \to G'$ such that $\phi \circ i_\alpha = i'_\alpha$, and a unique homomorphism $\psi : G' \to G$ such that $\psi \circ i'_\alpha = i_\alpha$.

We now consider the homomorphism $\psi \circ \phi : G \to G$. Composing i_α to it, then

$$\psi \circ \phi \circ i_\alpha = \psi \circ i'_\alpha = i_\alpha \text{ for all } \alpha \in J.$$

Since it is also true that in $\mathrm{Id}_G : G \to G$ we have $\mathrm{Id}_G \circ i_\alpha = i_\alpha$ for all $\alpha \in J$, and $\psi \circ \phi$ is unique, it must be that $\psi \circ \phi = \mathrm{Id}_G$.

Similarly, we consider the homomorphism $\phi \circ \psi : G' \to G'$. Composing i'_α to it, then

$$\phi \circ \psi \circ i'_\alpha = \phi \circ i_\alpha = i'_\alpha \text{ for all } \alpha \in J.$$

Since it is also true that in $\mathrm{Id}_{G'} : G' \to G'$ we have $\mathrm{Id}_{G'} \circ i'_\alpha = i'_\alpha$ for all $\alpha \in J$, and $\phi \circ \psi$ is unique, it must be that $\phi \circ \psi = \mathrm{Id}_{G'}$.

Hence there is a unique isomorphism $\phi : G \to G'$ such that $\phi \circ i_\alpha = i'_\alpha$ for all α, because ϕ is a homomorphism with the inverse ψ.

$$QED$$

Exercise 68.3:

Let $G = G_1 * G_2$. Given $c \in G$, let cG_1c^{-1} denote the set of all elements of the form cxc^{-1}, for $x \in G_1$. It is a subgroup of G; we want to show its intersection with G_2 consists of the identity alone.

Suppose for a contradiction that there is a nontrivial element in the intersection $cG_1c^{-1} \cap G_2$, then we can see it from two different perspectives. Seeing it as an element of cG_1c^{-1}, we can express it as cxc^{-1}, where $x \in G_1$ and $x \neq 1$. Seeing it as an element of G_2, we can express it as a length one word $y \in G_2$, where $y \neq 1$. We note that $cxc^{-1} = y$.

Given $c \in G$, we now split into cases.

We first suppose the expression of c as a reduced word contains no element of G_2, that c is itself an element of G_1. Since $x \in G_1$ by construction, we have $cxc^{-1} \in G_1$, so we can express cxc^{-1} as a length one word $x' \in G_1$. We have thus found a nontrivial element $x' = y$ in the intersection $G_1 \cap G_2$, which is a contradiction because it must be that $G_1 \cap G_2 = \{1\}$.

We then suppose the expression of c as a reduced word contains some nontrivial element of G_2. Since $x \in G_1$, the expression of cxc^{-1} as a reduced word is at least length three. Recall we can also express the same element as just y, which is length one. We have thus found two different reduced words of an element, which is also a contradiction.

Hence it must be that $cG_1c^{-1} \cap G_2 = \{1\}$.

QED