Week 6 Final, Math 104 **Name:** *James Wang*
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Exercise 54.7:

We want to generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Using Theorem 53.1, we see that the map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map, so $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ with the same definition of p is also a covering map. Let $(e_{0_1}, e_{0_2}) = (0, 0)$, and let $b_0 = (p \times p)((e_{0_1}, e_{0_2}))$. Then $(p \times p)^{-1}(b_0)$ is the set $\mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{R} \times \mathbb{R}$ is convex, it must be simply connected, so the lifting correspondence

$$\phi : \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}$$

is bijective. We show that ϕ is a homomorphism, then the problem would be proved.

Given $[f]$ and $[g]$ in $\pi_1(S^1 \times S^1, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R} \times \mathbb{R}$ beginning at $(0, 0)$. Let $(n_1, n_2) = \tilde{f}(1)$ and $(m_1, m_2) = \tilde{g}(1)$; then $\phi([f]) = (n_1, n_2)$ and $\phi([g]) = (m_1, m_2)$, by definition. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}} = (n_1, n_2) + \tilde{g}(s)$$

on $\mathbb{R}$. Because $(p \times p)((n_1, n_2) + (x_1, x_2)) = (p \times p)((x_1, x_2))$ for all $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, the path $\tilde{\tilde{g}}$ is a lifting of g; it begins at (n_1, n_2). Then the product $f * \tilde{\tilde{g}}$ is defined, and it is the lifting of $f * g$ that begins at $(0, 0)$, which we check:

$$\begin{aligned}(p \times p) \circ (f * \tilde{\tilde{g}})(s) &= \begin{cases} (p \times p)(\tilde{f}(s)), & s \in [0, \frac{1}{2}] \\ (p \times p)(\tilde{\tilde{g}}(s)), & s \in [\frac{1}{2}, 1] \end{cases} \\ &= \begin{cases} f(s), & s \in [0, \frac{1}{2}] \\ g(s), & s \in [\frac{1}{2}, 1]. \end{cases}\end{aligned} \tag{1}$$

The end point of the path $f * \tilde{\tilde{g}}$ is $\tilde{\tilde{g}}(1) = (n_1, n_2) + (m_1, m_2)$. Then by definition,

$$\phi([f] * [g]) = (n_1, n_2) + (m_1, m_2) = \phi([f]) + \phi([g]).$$

$$QED$$

Exercise 58.5:

Recall that a space X is said to be contractible if the identity map of X to itself is nullhomotopic. We want to show that X is contractible if and only if X has the homotopy type of a one-point space.

$(\Longrightarrow)$:

Suppose X is contractible, so there exists some $m \in X$ such that $\text{Id}_X \simeq c_m$. Specifically, $c_m : X \to X$ is defined by $c_m(x) = m$ for all $x \in X$. Note that by restricting the codomain, we can obtain ${c_m}' : X \to \{m\}$. Now let $h : \{m\} \to X$ defined by $h(m) = m$. Together, first consider ${c_m}' \circ h : \{m\} \to \{m\}$, by which we have ${c_m}'(h(m)) = {c_m}'(m) = m$, meaning it is the identity map on $\{m\}$, so it must be ${c_m}' \circ h \simeq \text{Id}_m$. We then consider $h \circ {c_m}' : X \to X$, by which we have $h({c_m}'(x)) = h(m) = m$ for all $x \in X$, meaning it is a constant map to m, so it must be $h \circ {c_m}' \simeq c_m \simeq \text{Id}_X$. Note that ${c_m}' \circ h$ and $h \circ {c_m}'$ are both continuous because ${c_m}'$ is a constant function and h is the inclusion function. By definition, X has the homotopy type of the one-point space $\{m\}$.

$(\Longleftarrow)$:

Suppose X has the homotopy type of a one-point space, say $Y = \{y\}$. Then there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \text{Id}_X$ and $f \circ g \simeq \text{Id}_Y$. Note that since $Y = \{y\}$, it must be that $f(x) = y$ for all $x \in X$ and $g(y) = k$ for some $k \in X$. Take an arbitrary $x \in X$, then we have $g(f(x)) = g(y) = k$, meaning $g \circ f$ has to be a constant map. We have thus proved Id_X is homotopic to a constant map, so Id_X is nullhomotopic and X is contractible.

QED

Exercise 55.1:

We want to show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

Suppose r is a retraction of B^2 onto A:

$$r : B^2 \to A \text{ defined by } r(a) = a \text{ for all } a \in A.$$

We then consider an arbitrary continuous $f : A \to A$, and define $g : B^2 \to B^2$ as $g(x) = f \circ r(x)$ for all $x \in B^2$. Since r and f are both continuous, g is continuous, and the Brouwer fixed point theorem tells us that there exists $y \in B^2$ such that $g(y) = y = f(r(y))$.

Since $\mathrm{im}(f) \subseteq A$, $f(r(y)) = y \in A$. Thus $r(y) = y$ by definition, so $f(r(y)) = f(y) = y$. Hence y is a fixed point of f.

$$QED$$

Exercise 57.2:

We want to show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective. (Hint: If $p \in S^2$, then $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$.)

Suppose for a contradiction that g is not surjective, and take some $p \in S^2$ such that $p \notin \mathrm{im}(g)$. By the given hint, we know $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$, so let $\varphi : S^2 - \{p\} \to \mathbb{R}^2$ be the homeomorphism. We know φ must be continuous from the definition.

Now consider $\varphi \circ g : S^2 \to \mathbb{R}^2$, which must also be continuous. Note that this composition only makes sense because of our assumption that g is not surjective. By the Borsuk-Ulam Theorem, we know there exists $y \in S^2$ such that $\varphi(g(y)) = \varphi(g(-y))$, or equivalently $g(y) = g(-y)$.

We have reached a contradiction with the initial assumption that $g(x) \neq g(-x)$ for all $x \in S^1$, so g must be surjective.

$$QED$$