Week 1 Final, Math 104 **Name:** *James Wang*
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Exercise 1.2.5:

In general, let us denote the identity function for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that a function $g : B \to A$ is a left inverse for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a right inverse for f if $f \circ h = i_B$.

a)

We first show that if f has a left inverse, then f is injective.

Let $f : A \to B$ be a function with a left inverse $g : B \to A$, and suppose that for some $a_1, a_2 \in A$, $f(a_1) = f(a_2)$. Applying g to both sides yields $g(f(a_1)) = g(f(a_2))$, which after using the definition of left inverse is equivalent to $a_1 = a_2$. Hence if f has a left inverse, then f is injective.

We then show that if f has a right inverse, then f is surjective.

Let $f : A \to B$ be a function with a right inverse $h : B \to A$. Take an arbitrary $b \in B$, we want to find some $a \in A$ such that $f(a) = b$. We can let $a = h(b)$, then $f(h(b)) = b$ using the definition of right inverse. Hence if f has a right inverse, then f is surjective.

$$QED$$

b)

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ has a left inverse $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ but no right inverse. This is because f is injective but not surjective:

$$\sqrt{x_1} = \sqrt{x_2} \implies (\sqrt{x_1})^2 = (\sqrt{x_2}^2) \implies x_1 = x_2$$

The interval $(-\infty, 0)$ in the codomain is not being hit by f

c)

$f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$ has a right inverse $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \sqrt{x}$ but no left inverse. This is because f is surjective but not injective:

$$\text{For any } y \in [0, \infty), \text{ since } y > 0, \text{ we can let } x = \sqrt{y} \in \mathbb{R}, \text{ so that } f(\sqrt{y}) = (\sqrt{y})^2 = y$$

$$\text{Consider } x_1 = 3 \text{ and } x_2 = -3, \text{ that } f(3) = f(-3) = 9$$

d)

Yes, there exist functions with more than one left inverse and more than one right inverse.

Consider the function $f_1 : \{1, 2\} \to \{1, 2, 3\}$ defined by $f_1(1) = 1$ and $f_1(2) = 2$.

Let the left inverses be

$$g_1(1) = 1, g_1(2) = 2, g_1(3) = 2$$
$$g_2(1) = 1, g_2(2) = 2, g_2(3) = 1$$

We can check that

$$g_1(f_1(1)) = 1, g_1(f_1(2)) = 2$$
$$g_2(f_1(1)) = 1, g_2(f_1(2)) = 2$$

Consider the function $f_2 : \{1, 2, 3\} \to \{1, 2\}$ defined by $f_2(1) = 1$, $f_2(2) = 2$, and $f_2(3) = 1$.

Let the right inverses be

$$h_1(1) = 1, h_1(2) = 2$$
$$h_2(1) = 3, h_2(2) = 2$$

We can check that

$$f_2(h_1(1)) = 1, f_2(h_1(2)) = 2$$
$$f_2(h_2(1)) = 1, f_2(h_2(2)) = 2$$

e)

We want to show that if $f : A \to B$ has both a left inverse $g : B \to A$ and a right inverse $h : B \to A$, then f is bijective and $g = h = f^{-1}$.

Since f has both a left inverse and a right inverse, by the result of part a), f is both injective and surjective, hence bijective and $f^{-1} : B \to A$ exists.

Take an arbitrary $x \in B$, and consider the term $g(f(f^{-1}(x)))$, we see that

$$i_A(f^{-1}(x)) = g(f(f^{-1}(x))) = g(i_B(x)), \text{ so } f^{-1}(x) = g(x) \text{ and } f^{-1} = g.$$

Take an arbitrary $x \in B$, and consider the term $f^{-1}(f(h(x)))$, we see that

$$i_A(h(x)) = f^{-1}(f(h(x))) = f^{-1}(i_B(x)), \text{ so } h(x) = f^{-1}(x) \text{ and } h = f^{-1}.$$

Together, we can conclude that $g = h = f^{-1}$.

QED

Exercise 1.3.13:

We want to prove that if an ordered set A has the least upper bound property, then it has the greatest lower bound property.

Let A be an arbitrary ordered set with the least upper bound property, and take an arbitrary nonempty bounded subset $B \subseteq A$ (we can only discuss the LUB and GLB property when the chosen subset is bounded). Now consider the set of lower bounds of B, which we denote by X. By our construction, $B \neq \emptyset$ and $X \neq \emptyset$.

Since $X \subseteq A$ and it is bounded above by every $b \in B$, we can invoke the least upper bound property on X, by which we know there exists some $L \in A$ such that $L = \sup(X)$. We claim that $L = \inf(B)$. Note that every $b \in B$ is an upper bound of X and L being the least upper bound of X suggest that we have $b \geq L$ for every $b \in B$, so L is indeed a lower bound of B. It remains to show that L is the greatest lower bound, which is quick because L by construction is the supremum of X, which is the set of lower bounds of B.

Thus, given the least upper bound property on an arbitrary ordered set, we can find a greatest lower bound for any of its nonempty bounded subset.

$$QED$$

Exercise 1.10.7:

Let J be a well-ordered set. A subset J_0 of J is said to be inductive if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Longrightarrow \alpha \in J_0.$$

We want to show that if J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.

Suppose for a contradiction that $J_0 \neq J$, and consider $J \setminus J_0$. Since $J \setminus J_0 \subseteq J$ and $J \setminus J_0 \neq \emptyset$, J being well-ordered tells us that $J \setminus J_0$ has a smallest element, which we denote by x.

This means that everything in J that is smaller than x must be in J_0, so $S_x \subset J_0$. Now using the given definition of inductive, we know this implies $x \in J_0$. By construction, however, $x \in J \setminus J_0$, so $x \notin J_0$, and we have reached a contradiction.

We have thus proved that if J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$. Note that the proof still works if $S_x = \emptyset$.

$$QED$$