Week 13 Final, Math 104 **Name:** *James Wang*
April 27, 2024

Exercise 26.13:

Let G be a topological group.

a) Let A and B be subspaces of G. If A is closed and B is compact, we want to show that $A \cdot B$ is closed.

Since G is a topological group, we know the map $\phi : G \times G \to G$ defined by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is continuous. To show $A \cdot B$ is closed, we show that $G - (A \cdot B)$ is open. Take $c \in G - (A \cdot B)$, then it must be $\phi(c, b) = c \cdot b^{-1} \notin A$ for any $b \in B$, otherwise multiplying b to both sides gives $c = ab$ for some $a \in A$. Note that A is also closed by assumption, which means $G - A$ is open, and $\phi^{-1}(G - A)$ is open. Together, we have that $\phi(c, b) \in G - A$, and $(c, b) \in \phi^{-1}(G - A)$. Hence for each $b \in B$, we can find neighborhoods $c \in U_b$ and $b \in V_b$ such that $\phi(U_b \times V_b) \subseteq G - A$.

Now we take the union of all V_b to get an open covering $\bigcup\limits_{b \in B} V_b$ of B. Since B is compact, we can reduce this open covering to a finite subcollection $\bigcup\limits_{b \in B,\ i=1}^{n} V_{b,i}$ that still covers B. We claim that $W := \bigcap\limits_{b \in B,\ i=1}^{n} U_{b,i}$, which is the intersection of all those $U_{b,i}$ that correspond to those $V_{b,i}$ in the finite covering, is our desired neighborhood of c. Clearly, it contains c because each $U_{b,i}$ contains c, and it is open because it is a finite intersection of open sets. Finally, we show $W \subseteq G - (A \cdot B)$. Suppose otherwise, then there exists $w \in W$ such that $w = a \cdot b$ for some $a \in A$ and $b \in B$. Since $b \in B$, we can find some $V_{b,i}$ from the finite covering such that $b \in V_{b,i}$. Moreover, note that since $a \cdot b \in W$, we have that $a \cdot b \in U_{b,i}$ for all i. We can then choose the $U_{b,i}$ that corresponds to the $V_{b,i}$ containing b, and consider $\phi(a \cdot b, b) = a \cdot b \cdot b^{-1} = a$. This contradicts with our knowledge that $\phi(U_{b,i} \times V_{b,i}) \subseteq G - A$.

QED

b) Let H be a subgroup of G; let $p : G \to G/H$ be the quotient map. If H is compact, we want to show p is a closed map.

Let A be a closed set in G, then we see that

$$p(A) = \{aH \mid a \in A\},$$

which are all the left cosets obtained when left multiply H by $a \in A$. Since H is also a subgroup, take any $h \in H$, the left coset hH is just H itself. Hence the left cosets $(ah)H$ where $a \in A$ and $h \in H$ are also just aH. Together, we can conclude that

$$p^{-1}(p(A)) = \{ah \mid a \in A, h \in H\} = A \cdot H.$$

Since A is closed and H is compact, we can use the result from part a), which gives us that $A \cdot H$ is also closed. Recall that p is a quotient map, which by definition means a subset $U \subseteq G/H$ is closed in G/H if and only if $p^{-1}(U)$ is closed in G. Since $p^{-1}(p(A)) = A \cdot H$ is closed in G, $p(A)$ must be closed in G/H, and it follows immediately that p is a closed map.

$$QED$$

Exercise 52.7:

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

c)

We want to show the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same.

Take arbitrary $[f], [g] \in \pi_1(G, x_0)$, by observation and the well-defined group operation $\otimes$ on $\pi_1(G, x_0)$ verified in part b), we see that

$$[f] \otimes [g] = [f * e_{x_0}] \otimes [e_{x_0} * g] = [(f * e_{x_0}) \otimes (e_{x_0} * g)],$$

and we use the definitions to expand $((f * e_{x_0}) \otimes (e_{x_0} * g))(s)$ as

$$(f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ e_{x_0} & \text{if } s \in [\frac{1}{2}, 1] \end{cases} \cdot \begin{cases} e_{x_0} & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

This, after simplification, is just

$$\begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

which is precisely the definition of the product $(f * g)(s)$.

Hence $[(f * e_{x_0}) \otimes (e_{x_0} * g)] = [f * g]$, and it follows that $[f] \otimes [g] = [f] * [g]$.

QED

d) We want to show that $\pi_1(G, x_0)$ is abelian.

Take arbitrary $[f], [g] \in \pi_1(G, x_0)$, and consider $[f] * [g]$. By our result from part c), we know $[f] * [g] = [f] \otimes [g]$, and using similar ideas of the expansion procedure in part c), we get

$$[f] * [g] = [f] \otimes [g] = [e_{x_0} * f] \otimes [g * e_{x_0}] = [e_{x_0} * f] \cdot [g * e_{x_0}] = [e_{x_0} \cdot g] * [f \cdot e_{x_0}] = [g] * [f].$$

This proves $\pi_1(G, x_0)$ is abelian.

$$QED$$