65 The Winding Number, Math 104 **Name:** *Denny Tan, James Wang*
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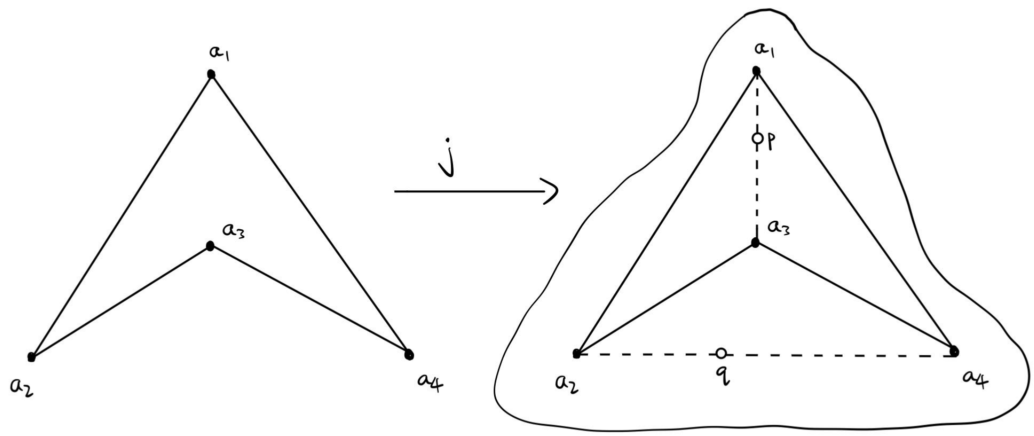
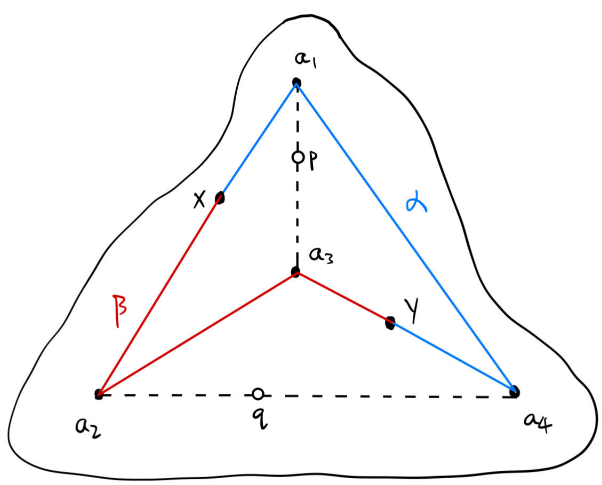
Motivation: Given a homeomorphism from S^1 to a simple closed curve C in $\mathbb{R}^2 - 0$, what can we say about the number of times S^1 gets wrapped around 0?

Lemma 65.1: Let G be a subspace of S^2 that is a complete graph on four vertices $a_1, \ldots, a_4$. Let C be the subgraph $a_1a_2a_3a_4a_1$, which is a simple closed curve. Let p and q be interior points of the edges a_1a_3 and a_2a_4, respectively. Then:
(a) The points p and q lie in different components of $S^2 - C$.
(b) The inclusion $j : C \to S^2 - p - q$ induces an isomorphism of fundamental groups.

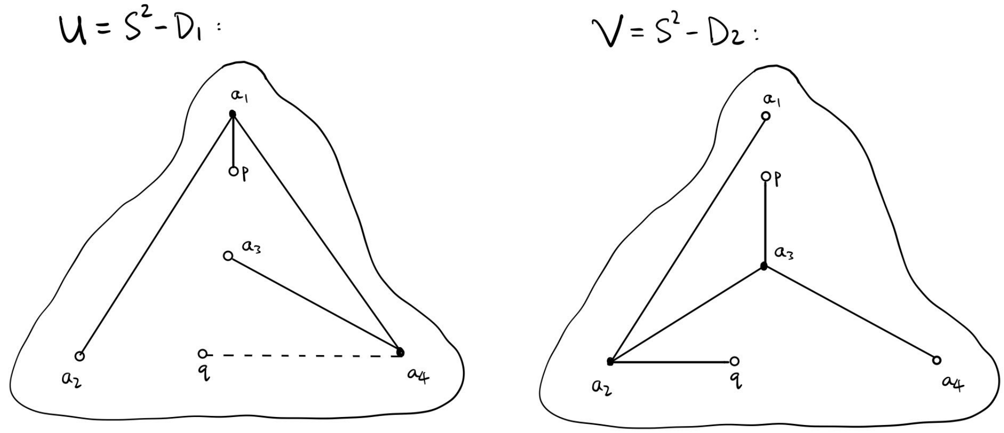
(b) Proof Sketch:





Claim: $\alpha * \beta = xa_1a_4ya_3a_2x$ generates the fundamental group $\pi_1(S^2 - p - q, x)$.

Let $D_1 = pa_3a_2q$ and $D_2 = qa_4a_1p$, then



Note that $U \cup V = S^2 - p - q$ and $U \cap V = S^2 - D$.
Since D is a simple closed curve in S^2, by the Jordan Curve Theorem, $U \cap V$ has 2 components. Moreover, by part (a) of this lemma, we know x and y are in different components of $S^2 - D$. Finally, $\alpha \in U$ is a path from x to y and $\beta \in V$ is a path from y to x.

We have everything to invoke Theorem 63.1(b). Since $\pi_1(S^2 - p - q, x)$ is infinitely cyclic, it is generated by $[\alpha * \beta]$.