**Week 12 Final, Math 104** **Name:** *James Wang*
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**Exercise 1 (collaborated as a class):** Prove that if $G$ and $G'$ are homeomorphic finite graphs, then they have the same Euler characteristic.

Suppose $G$ and $G'$ are homeomorphic finite linear graphs. Since $G$ and $G'$ are homeomorphic, we have $\pi_1(G) \cong \pi_1(G')$.

We first discuss the case when $G$ and $G'$ are both connected. By Theorem 84.7, we know the fundamental group of a connected graph is always a free group (we can see the fundamental group of a tree as the free group on 0 generator), so let $m$ and $n$ denote the cardinalities of the systems of free generators of $\pi_1(G)$ and $\pi_1(G')$, respectively. By Lemma 85.2, we know $m = 1 - \chi(G) = n = 1 - \chi(G')$, so it suffices to show $m = n$. Since we already know $F_m \cong F_n$, and a result from seminar shows that abelianizations of these two isomorphic free groups, which are free abelian groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$, must also be isomorphic, we get $\mathbb{Z}^m \cong \mathbb{Z}^n$. Finally, by Theorem 67.8, we know the rank of a free abelian group is uniquely determined. It follows quickly that $m = n$ and $\chi(G) = \chi(G')$, as desired.

We then discuss the case when $G$ and $G'$ are both not connected. Since they are homeomorphic, there is a homeomorphism between their connected components. By the previous argument, we know each homeomorphic pair of connected components have the same Euler characteristics. We can then take the sum of each connected component's Euler characteristic to get $\chi(G)$ and $\chi(G')$, respectively. Clearly, we have $\chi(G) = \chi(G')$, as desired.

*QED*

**Exercise 2 (collaborated as a class):** Let $F = < a, b >$ be the free group on two generators, and let $F' = [F, F]$. We know that $F'$, as a subgroup of a free group, is free. We want to find a set of free generators for $F'$ by using covering space theory.

Let $B$ be the wedge of two circles, and let $E$ be the integer grid lattice of $\mathbb{R}^2$, then $p : E \to B$ is a covering map that wraps all the horizontal lines onto the right circle $\alpha$ and all the vertical lines onto the left circle $\beta$. Let $b_0 \in B$ be the intersection of $\alpha$ and $\beta$, and choose some $e_0 \in p^{-1}(b_0)$. Using the Classification of Covering Spaces, we claim that $E$ is the covering space that corresponds to the commutator subgroup $F' \leqslant \pi_1(B, b_0) = F$ in the Galois Correspondence, that

$$F' = p_*(\pi_1(E, e_0)).$$

To see $F' \subseteq p_*(\pi_1(E, e_0))$, take any $x_1, x_2 \in F$ and consider their commutator $x_1 x_2 x_1^{-1} x_2^{-1}$. Since the powers of $x_1$ and $x_2$ must sum to 0, we know the lifting of every commutator to $E$ must start and end at $e_0$, therefore is a loop in $E$ based at $e_0$. By Theorem 54.6 $(c)$, this means $x_1 x_2 x_1^{-1} x_2^{-1} \in p_*(\pi_1(E, e_0))$.

To see $p_*(\pi_1(E, e_0)) \subseteq F'$, observe the group of covering transformations $\mathcal{C}(E, p, B) \cong \mathbb{Z}^2$. By Corollary 81.3, since for every pair of $e_1, e_2 \in p^{-1}(b_0)$ there is a covering transformation $h : E \to B$ with $h(e_1) = e_2$, we get $\mathbb{Z}^2 \cong \pi_1(B, b_0)/p_*(\pi_1(E, e_0)) \cong F/p_*(\pi_1(E, e_0))$. Moreover, by Theorem 69.4, we have the abelianization $F/F' \cong \mathbb{Z}^2$. Putting these two results together and define the projection map $\varphi : F/F' \to F/p_*(\pi_1(E, e_0))$, we see that $\ker(\varphi) = p_*(\pi_1(E, e_0))/F'$. Yet since $\varphi$ is an isomorphism, it must be that $p_*(\pi_1(E, e_0))/F' = 1$. Together with the last part, we can conclude $F' = p_*(\pi_1(E, e_0))$.

Thus, finding a set of free generators for $F'$ is equivalent to finding a set of free generators for $p_*(\pi_1(E, e_0))$. By Exercise 85.3, we know one such set is $\{\beta^m \alpha^n \beta \alpha^{-n} \beta^{-(m+1)} : m, n \in \mathbb{Z}\}$. This can be verified by taking the maximal tree in $E$ consisting of all the horizontal lines and only the $y$-axis, and just reading images of edges that are not in the maximal tree.

$$QED$$

**85.2:**

Let $F$ be a free group on two free generators $\alpha$ and $\beta$. Let $H$ be the subgroup generated by $\alpha$. We want to show that $H$ has infinite index in $F$.

Clearly, $H$ is a free group on one generator. Suppose for a contradiction that $H$ has finite index in $F$, that $[F : H] = n \in \mathbb{N}$. Then by Theorem 85.3, $H$ has $(n)(1)+1 = 1$ free generator, which means $n = 0$ and $F = H$. This is a contradiction with our initial assumption of $H$ being a free group on one generator, because by Corollary 69.5, we know any system of free generators for a free group on two generators must have two elements.

$$QED$$