Week 6 Final, Math B504 **Name:** *James Wang*
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<u>Problem 1:</u>

We want to show that the $\mathbb{Z}$-module $\mathbb{Q}$ is flat by using Exercise 10.4.8.

Note: I had no idea how to solve this problem, so I looked at Professor Kara's solution at first. But I then realized this is an example on page 401. I read it over again, and now feel much better about it.

Part c) of 10.4.8 tells us that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

Let $\psi : L \to M$ be an injective $\mathbb{Z}$-module homomorphism, we want to show that

$$1 \otimes \psi : \mathbb{Q} \otimes_{\mathbb{Z}} L \to \mathbb{Q} \otimes_{\mathbb{Z}} M$$

defined by $(1/d) \otimes l \mapsto (1/d) \otimes \psi(l)$ is injective. Take any $(1/d) \otimes l \in \ker(1 \otimes \psi)$, then by the definition of our map we see that $1 \otimes \psi((1/d) \otimes l) = (1/d) \otimes \psi(l) = 0 \in \mathbb{Q} \otimes_{\mathbb{Z}} M$. By 10.4.8 c), this means $z\psi(l) = 0$ for some nonzero $z \in \mathbb{Z}$. By linearity we get $\psi(zl) = 0$, so $zl \in \ker(\psi)$. Recall that ψ is injective by assumption, so $zl = 0 \in L$, or equivalently $(1/d) \otimes l = 0$. Hence $1 \otimes \psi$ is injective, and the $\mathbb{Z}$-module $\mathbb{Q}$ is flat.

$$QED$$

Problem 2:

Let P_1 and P_2 be R-modules. We want to show that $P_1 \oplus P_2$ is flat if and only if P_1 and P_2 are flat.

The hint on page 403 states that tensor product commutes with arbitrary direct sums. Hence take any R-module K, we have the isomorphism $(P_1 \oplus P_2) \otimes K \cong (P_1 \otimes K) \oplus (P_2 \otimes K)$.

$(\Longrightarrow)$:

Suppose $P_1 \oplus P_2$ is flat. This means given any injective R-module homomorphism $\psi : L \to M$, then $1 \otimes \psi : (P_1 \oplus P_2) \otimes L \to (P_1 \oplus P_2) \otimes M$ is also injective.

Hence P_1 and P_2 must be flat, because the commutativity of tensor product and direct sums would give injective homomorphisms

$$1 \otimes \psi_1 : P_1 \otimes_R L \to P_1 \otimes_R M,$$
$$1 \otimes \psi_2 : P_2 \otimes_R L \to P_2 \otimes_R M.$$

$(\Longleftarrow)$:

Conversely, suppose P_1 and P_2 are flat. Then we can get the injective homomorphisms $1 \otimes \psi_1$ and $1 \otimes \psi_2$, and putting them together would give the injective homomorphism $1 \otimes \psi$. Hence $P_1 \otimes P_2$ is also flat.

QED

Problem 3:

We want to prove the following

a)

The polynomial ring $R[x]$ over the commutative ring R is a flat R-module.

We show that $R[x]$ is a free R-module by showing it has a basis. Clearly, $B = \{1, x, x^2, \dots\}$ is a generating set of $R[x]$, as all polynomials with coefficients in R can be written as a R-linear combination of elements in B. We claim that B is also linearly independent. This is because suppose we have a finite R-linear combination such that $r_0+r_1x+r_2x^2+r_nx^n = 0$, then it must be that each $r_i = 0$. Recall also that all free modules are flat modules. Hence $R[x]$ is flat.

$$QED$$

b)

$R[x, y]/(xy)$ is not a flat R-module.

We consider the homomorphism $\phi : R[x] \to R[x]$ defined by $f(x) \mapsto xf(x)$. Take arbitrary $f_1, f_2 \in R[x]$ and consider $xf_1 = xf_2$, then it must be that $f_1 = f_2$. Hence ϕ is injective. Now take the tensor product and consider how ϕ induces

$$\varphi : R[x, y]/(xy) \otimes_{R[x]} R[x] \to R[x, y]/(xy) \otimes_{R[x]} R[x]$$
$$g_1 \otimes g_2 \mapsto g_1 \otimes xg_2.$$

Recall we have the isomorphism $R[x, y]/(xy) \otimes_{R[x]} R[x] \cong R[x, y]/(xy)$, and that $g_1 \otimes xg_2 = xg_1 \otimes g_2$, so the induced φ is essentially

$$\varphi : R[x, y]/(xy) \to R[x, y]/(xy)$$
$$g_1 \mapsto xg_1.$$

Notice that under φ, $y \neq 0 \in R[x, y]/(xy)$, yet $\varphi(y) = xy = 0 \in R[x, y]/(xy)$. This means we do not get an injective function, so $R[x, y]/(xy)$ is not a flat R-module.

$$QED$$