Week 5 Final, Math B504 **Name:** *James Wang*
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<u>Problem 1:</u>

Suppose that

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "g_1"] \arrow[d, "\beta"'] & C \arrow[r, "h_1"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\
A' \arrow[r, "f_2"'] & B' \arrow[r, "g_2"'] & C' \arrow[r, "h_2"'] & D'
\end{tikzcd}$$

is a commutative diagram of groups and that rows are exact. We want to prove that

a) If α is surjective, and β, δ are injective, then γ is injective.

Take $c \in \ker(\gamma)$. We want to show that it must be $c = 0$.

Since $\gamma(c) = 0$ and 0 must always map to 0, we have $h_2(\gamma(c)) = \delta(h_1(c)) = 0$ by the commutative diagram. Since δ is also injective by assumption, it must be that $h_1(c) = 0$, so $c \in \ker(h_1) = \operatorname{im}(g_1)$ because the rows are exact, and there exists $b \in B$ such that $g_1(b) = c$. Similarly, using the commutative diagram, we have $g_2(\beta(b)) = \gamma(g_1(b)) = \gamma(c) = 0$, which means $\beta(b) \in \ker(g_2) = \operatorname{im}(f_2)$ because the rows are exact, so there exists $a' \in A'$ such that $f_2(a') = \beta(b)$. Since α is surjective by assumption, there exists $a \in A$ such that $\alpha(a) = a'$. By the commutative diagram, $\beta(f_1(a)) = f_2(\alpha(a)) = f_2(a') = \beta(b)$. Recall that β is also injective by assumption, so it must be $f_1(a) = b$. Putting everything we have for the top row together and using exactness, we have $c = g_1(b) = g_1(f_1(a))$. By exactness, this means $f_1(a) \in \operatorname{im}(f_1) = \ker(g_1)$, so $g_1(f_1(a)) = 0 = c$.

We have proved what we want. Thus γ is injective.

QED

b) If δ is injective, and α, γ are surjective, then β is surjective.

Take arbitrary $b' \in B'$, we want to show $b' \in \mathrm{im}(\beta)$. Since γ is surjective, there exists $c \in C$ such that $\gamma(c) = g_2(b')$. By commutativity, $h_2(\gamma(c)) = \delta(h_1(c))$. Since the rows are exact, we know $\mathrm{im}(g_2) = \ker(h_2)$. This expands the expression to $\delta(h_1(c)) = h_2(\gamma(c)) = h_2(g_2(b')) = 0$. Recall that δ is injective, so we obtain $h_1(c) = 0$. By exactness, this means $c \in \ker(h_1) = \mathrm{im}(g_1)$, and there exists $b \in B$ such that $g_1(b) = c$. By commutativity, we have $g_2(\beta(b)) = \gamma(g_1(b)) = \gamma(c) = g_2(b')$. By linearity, we can shift the terms to get $g_2(b' - \beta(b)) = 0$, so $b' - \beta(b) \in \ker(g_2) = \mathrm{im}(f_2)$. Hence there exists some $a' \in A'$ such that $f_2(a') = b' - \beta(b)$. Since α is also surjective, there exists some $a \in A$ such that $\alpha(a) = a'$. By commutativity, we get $\beta(f_1(a)) = f_2(\alpha(a)) = f_2(a') = b' - \beta(b)$. By linearity again, we can shift the terms to get $b' = \beta(f_1(a) + b)$. This proves $b' \in \mathrm{im}(\beta)$, which is what we want.

QED

Problem 2:

We want to show the short exact sequence of R-modules $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is split if and only if there is an R-module homomorphism $\mu : C \to B$ such that $\varphi \circ \mu = \mathrm{Id}_C$.

I read the Proof on page 384. If given μ, we can define $C' = \mu(C) \subseteq B$, and if given C', we can define $\mu = \varphi^{-1} : C \cong C' \subseteq B$. The result follows from the definitions.

QED

Problem 3:

We want to show if $0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \to 0$ is exact for every R-module D, then $0 \to L \overset{\psi}{\to} M \overset{\varphi}{\to} N \to 0$ is a split short exact sequence.

We use the statement we just proved in Problem 2. Let $D = N$, and define $\phi : N \to M$. Then we see that $\mathrm{Id}_N = \varphi \circ \phi$, which shows the sequence is a split short exact sequence.

QED

Problem 4:

We want to show the following.

a) $\mathbb{Z}$ is a projective $\mathbb{Z}$-module.

By Corollary 31 on page 390, we know free modules are projective. We know $\mathbb{Z}$ is a free $\mathbb{Z}$-module, so it must be a projective $\mathbb{Z}$-module.

$$QED$$

b) The $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.

We use the hint on page 404. Let Q be a nonzero divisible $\mathbb{Z}$-module. We show that Q is not a projective $\mathbb{Z}$-module, which automatically gives us that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. Note also that if F is any free module, then $\cap_{n=1}^{\infty} nF = 0$. This is because take any $x \in \cap_{n=1}^{\infty} nF$, then for any $n \in \mathbb{Z}$, $nx = 0 \in nF$, meaning $x = 0$.

Suppose for a contradiction that Q is a projective $\mathbb{Z}$-module, so there exists a $\mathbb{Z}$-module M such that $Q \oplus M$ is free. Yet recall since Q by assumption is nonzero divisible, we have $nQ = Q$ for $n \in \mathbb{Z}$. This means $\cap_{n=1}^{\infty} n(Q \oplus M) = \cap_{n=1}^{\infty} nQ \oplus nM = \cap_{n=1}^{\infty} Q \oplus nM = Q$, thereby contradicting our previous observation. Hence Q is not a projective $\mathbb{Z}$-module.

Since $\mathbb{Q}$ by itself is a nonzero divisible $\mathbb{Z}$-module, it follows that it is not projective.

$$QED$$

Problem 5:

Let P_1 and P_2 be R-modules. We want to show the following.

a) $P_1 \oplus P_2$ is projective if and only if P_1 and P_2 are projective.

Recall that by Proposition 30.4 we know if P is a direct summand of a free R-module, then P is a projective R-module.

$(\Longrightarrow)$:

Suppose $P_1 \oplus P_2$ is projective, then there exists some R-module M such that $(P_1 \oplus P_2) \oplus M$ is a free R-module. Now clearly, P_1 and P_2 are direct summands of a free R-module, so P_1 and P_2 are projective.

$(\Longleftarrow)$:

Suppose P_1 and P_2 are projective, then there exist some R-modules M_1 and M_2 such that $P_1 \oplus M_1$ and $P_2 \oplus M_2$ are free R-modules. Since a direct sum of free modules is still free, and $(P_1 \oplus M_1) \oplus (P_2 \oplus M_2) \cong (P_1 \oplus P_2) \oplus (M_1 \oplus M_2)$, we see that $P_1 \oplus P_2$ is a direct summand of a free R-module, so $P_1 \oplus P_2$ is projective.

QED

b) $P_1 \oplus P_2$ is injective if and only if P_1 and P_2 are injective.

We use Baer's Criterion extensively: The module Q is injective if and only if for every left ideal I of R any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.

$(\Longrightarrow)$:

Suppose $P_1 \oplus P_2$ is injective, and take a left ideal I of R to consider an R-module homomorphism $f_1 : I \to P_1$. We can then include f_1 into an R-module homomorphism $f_1' : I \to P_1 \oplus P_2$ defined by $i \mapsto f_1(i) \oplus 0$. Since $P_1 \oplus P_2$ is injective, we can now extend f_1' to an R-module homomorphism $F_1' : R \to P_1 \oplus P_2$. Finally, consider $F_1 : R \to P_1$ defined by $r \mapsto \pi_1(F_1'(r))$, where π_1 is the projection map onto the first coordinate. By observation, F_1 is the desired extension of f_1 to invoke Baer's Criterion, so P_1 is injective. Using the same construction, we can conclude that P_2 is also injective.

$(\Longleftarrow):$

Suppose P_1 and P_2 are injective, and take a left ideal I of R to consider an R-module homomorphism of the form $f : I \to P_1 \oplus P_2$. We can then break it into two distinct R-module homomorphisms and denote them as $f_1 : I \to P_1$ and $f_2 : I \to P_2$. This means f is defined as $i \mapsto f_1(i) \oplus f_2(i)$. Since P_1 and P_2 are injective, Baer's Criterion can extend f_1 and f_2 into $f_1' : R \to P_1$ and $f_2' : R \to P_2$, respectively. Finally, consider the homomorphism $F : R \to P_1 \oplus P_2$ defined by $r \mapsto f_1'(r) \oplus f_2'(r)$. By observation, F is the desired extension of f to invoke Baer's Criterion, so $P_1 \oplus P_2$ is injective.

QED