Week 3 Final, Math B504 **Name:** *James Wang*
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Problem 1:

Prove that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$

where $d = \gcd(m, n)$ with $d \neq 1$. If $d = 1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Note: We discussed this, and I read Example 3 on page 369 of Dummit & Foote.

Take $a \in \mathbb{Z}/m\mathbb{Z}$ and $b \in \mathbb{Z}/n\mathbb{Z}$. By the relations of simple tensors, we have

$$a \otimes b = a \otimes (b \cdot 1) = (ab) \otimes 1 = ab(1 \otimes 1),$$

which means $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is a cyclic group and $1 \otimes 1$ is its generator. By the operations on tensors, we see that

$$m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$$
$$n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0,$$

which means for $d(1 \otimes 1) = 0$, it must be that $d \in m\mathbb{Z} \cup n\mathbb{Z}$, so $d = \gcd(\text{m,n})$. This shows that the order of $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ divides d.

We now consider $\varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ defined by $\varphi(a \bmod m, b \bmod n) = ab \bmod d$. This map is well-defined because d divides both m and n. This map is also $\mathbb{Z}$-bilinear, which we just verify the first factor because the second factor is essentially the same:

$$\begin{aligned} \varphi(r_1a_1 \bmod m + r_2a_2 \bmod m, b \bmod n) &= \varphi(r_1a_1 + r_2a_2 \bmod m, b \bmod n) \\ &= (r_1a_1 + r_2a_2)b \bmod d \\ &= r_1a_1b \bmod d + r_2a_2b \bmod d \\ &= r_1\varphi(a_1 \bmod m, b \bmod n) + r_2\varphi(a_2 \bmod m, b \bmod n). \end{aligned} \tag{1}$$

By Corollary 12 on page 368, we get an R-module homomorphism $\psi : \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ that maps $1 \otimes 1$ to $1 \in \mathbb{Z}/d\mathbb{Z}$, which has order d. This shows that the order of $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ must be at least d.

We have thus obtained the isomorphism $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

QED

Problem 2:

a) Prove that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic as left $\mathbb{Q}$-modules. (Show that they are both 1-dimensional vector spaces over $\mathbb{Q}$.)

By Corollary 19 on page 374, we know $R^s \otimes_R R^t \cong R^{st}$, which tells us that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$. This means $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ is a 1-dimensional vector space over $\mathbb{Q}$. Furthermore, by the defined relation we see that any simple tensor $q_1 \otimes q_2 = 1 \otimes q_1 q_2 = 1 \otimes q$ for some $q \in \mathbb{Q}$, which means $1 \otimes 1$ generates the vector space, so it is the only basis element.

We then show that $1 \otimes 1$ is also the only basis element for the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. We claim that it generates all elements in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider arbitrary $\frac{a_1}{b_1} \otimes \frac{a_2}{b_2}$:

$$\frac{a_1}{b_1} \otimes \frac{a_2}{b_2} = \frac{a_1}{b_1} \otimes \frac{b_1 a_2}{b_1 b_2} = \frac{a_1 b_1}{b_1} \otimes \frac{a_2}{b_1 b_2} = a_1 \otimes \frac{a_2}{b_1 b_2} = 1 \otimes \frac{a_1 a_2}{b_1 b_2},$$

and $\frac{a_1 a_2}{b_1 b_2} \in \mathbb{Q}$. This means every simple tensor has the form $1 \otimes q$, so every simple tensor is generated by $1 \otimes 1$. Furthermore, since this is only 1 element, it must be linearly independent. Hence $1 \otimes 1$ is a basis for $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic as left $\mathbb{Q}$-modules.

QED

b) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

By Corollary 19 on page 374, we know $R^s \otimes_R R^t \cong R^{st}$, which tells us that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$. This means $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a 1-dimensional vector space over $\mathbb{C}$, so a 2-dimensional vector space over $\mathbb{R}$.

We claim $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a 4-dimensional vector space over $\mathbb{R}$. Consider an arbitrary element $a_1 + b_1 i \otimes a_2 + b_2 i$...I want to show $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$ is a basis for this vector space, which I know is true because of the discussions I had with Hongyi, Dalena, and Ellie during problem session, but I do not know how to prove this rigorously, so I'm leaving this blank.

Problem 3:

Let I be a two-sided ideal of R and let N be a left R-module. Recall that R/I is an $(R/I, R)$-bimodule. Define

$$IN = \Big\{ \sum_{\text{finite}} a_i n_i : a_i \in I, n_i \in N \Big\}.$$

Prove that

a) IN is a left R-submodule of N.

We use the Submodule Criterion.

First, since $0_R \in I$ and $0_N \in N$, we have $0_R \cdot 0_N = 0_N \in IN$, which means, $IN \neq \emptyset$.

Next, let $\sum a_i n_i, \sum b_i m_i \in IN$ and $r \in R$, then $r\sum a_i n_i + \sum b_i m_i = \sum r a_i n_i + \sum b_i m_i \in IN$ because the sum of two finite sums must be finite. Moreover, since I is an ideal, by its black hole property we know $ra_i \in I$ for all $r \in R$ and $a_i \in I$.

We have therefore proved IN is a left R-submodule of N.

$$QED$$

b) $(R/I) \otimes_R N \cong N/IN$.

Note: I read Example 8 on page 370 of Dummit & Foote.

By observation, $(R/I) \otimes_R N$ is an abelian group generated by $(r \bmod I) \otimes n = r(1 \otimes n)$ for $r \in R$ and $n \in N$, so $1 \otimes n$ for $n \in N$ generate $(R/I) \otimes_R N$ as an R/I module. Hence the R-module homomorphism $\varphi : N \to (R/I) \otimes_R N$ defined by $\varphi(n) = 1 \otimes n$ is surjective.

Going back to part a) and consider IN, we see that φ maps each $a_i n_i$, where $a_i \in I$ and $n_i \in N$, to $1 \otimes a_i n_i = a_i \otimes n_i = 0$, which means $IN \subseteq \ker(\varphi)$. By the Isomorphism Theorems, we have a homomorphism $f : N/IN \to (R/I) \otimes_R N$ defined by $f(n \bmod I) = 1 \otimes n$.

We finally claim f is an isomorphism. Notice that $\lambda : (R/I) \times N \to N/IN$ defined by $\lambda(r \bmod I, n) = rn \bmod IN$ is well-defined: Suppose $r_1 \bmod I = r_2 \bmod I$ such that $r_1 - r_2 \in I$, then

$$\lambda(r_1 \bmod I, n) = r_1 n \bmod IN \quad \text{and} \quad \lambda(r_2 \bmod I, n) = r_2 n \bmod IN$$

Consider $r_1n - r_2n = (r_1 - r_2)n$, since $r_1 - r_2 \in I$ and $n \in N$, it must be that $(r_1 - r_2)n \in IN$, so $r_1n \bmod IN = r_2n \bmod IN$.

λ is also R-balanced:

$$\begin{aligned}
\lambda(r_1 \bmod I + r_2 \bmod I, n) &= \lambda((r_1 + r_2) \bmod I, n) \\
&= (r_1 + r_2)n \bmod IN \\
&= r_1n \bmod IN + r_2n \bmod IN \\
&= \lambda(r_1 \bmod I, n) + \lambda(r_2 \bmod I, n)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\lambda(r \bmod I, n_1 + n_2) &= r(n_1 + n_2) \bmod IN \\
&= rn_1 \bmod IN + rn_2 \bmod IN \\
&= \lambda(r \bmod I, n_1) + \lambda(r \bmod I, n_2)
\end{aligned} \tag{3}$$

$$\begin{aligned}
\lambda(r \bmod I, r'n) &= rr'n \bmod IN \\
&= \lambda(rr' \bmod I, n)
\end{aligned} \tag{4}$$

By Theorem 10 on page 365, we then have a group homomorphism $g : (R/I) \otimes N \to N/IN$ defined by $g((r \bmod I) \otimes n) = rn \bmod IN$. We can see that $g \circ f$ and $f \circ g$ are both the identity. Hence we have proved $(R/I) \otimes_R N \cong N/IN$.

Problem 4:

Suppose R is a commutative ring and M, N are left R-modules considered with the standard R-module structures. Prove that there is a unique R-module isomorphism

$$M \otimes_R N \cong N \otimes_R M$$

mapping $m \otimes n$ to $n \otimes m$.

Note: This does not mean $a \otimes b = b \otimes a$ when $M = N$.

Note: I read Proposition 20 on page 374 of Dummit & Foote.

We consider the map $\varphi_1 : M \times N \to N \otimes M$ defined by $(m, n) \mapsto n \otimes m$. We see that this map is R-balanced because

$$\varphi_1(m_1 + m_2, n) = n \otimes (m_1 + m_2) = n \otimes m_1 + n \otimes m_2 = \varphi_1(m_1, n) + \varphi_1(m_2, n)$$
$$\varphi_1(m, n_1 + n_2) = (n_1 + n_2) \otimes m = n_1 \otimes m + n_2 \otimes m = \varphi_1(m, n_1) + \varphi_1(m, n_2)$$
$$\varphi_1(m, rn) = rn \otimes m = nr \otimes m = n \otimes rm = n \otimes mr = \varphi_1(mr, n)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

By Theorem 10 on page 365, we then have a unique group homomorphism $\psi_1 : M \otimes N \to N \otimes M$ defined by $m \otimes n \mapsto n \otimes m$. We claim ψ_1 is the unique R-module isomorphism. Since $M \otimes N$ and $N \otimes M$ are abelian groups, ψ_1 as a group homomorphism takes care of the addition operation needed in an R-module homomorphism. To show ψ respects scalar multiplication by $r \in R$:

$$\psi(r(m \otimes n)) = \varphi(rm \otimes n) = n \otimes rm = nr \otimes m = rn \otimes m = r(n \otimes m) = r\psi(m \otimes n).$$

By the exact same process, we can construct a map $\varphi_2 : N \otimes M \to M \otimes N$ and get a unique R-module homomorphism ψ_2. It is easy to see that $\psi_2 \circ \psi_1$ and $\psi_1 \circ \psi_2$ both give the identity. Hence they are bijctive and we have constructed the isomorphism.

$$QED$$

Problem 5:

Let R be a commutative ring and A, B be R-algebras. Show that the multiplication

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

makes $A \otimes_R B$ into an R-algebra.

Note: I was very confused about showing this multiplication is well-defined until I read Dummit & Foote and found out it provided a sketch, which I just moved to here. Furthermore, I used Corollary 16 on page 372, which I believe is something we didn't talk about in class but did mention briefly during problem session.

We first show the multiplication is well-defined. Observe that the map $\varphi : A \times B \times A \times B \to A \otimes B$ is multilinear over R. This is easy to see so we only check one case:

$$\begin{aligned} f(a, r_1 b_1 + r_2 b_2, a', b') &= aa' \otimes (r_1 b_1 + r_2 b_2) b' \\ &= aa' \otimes r_1 b_1 b' + aa' \otimes r_2 b_2 b' \\ &= r_1 f(a, b_1, a', b') + r_2 f(a, b_2, a', b'). \end{aligned} \tag{5}$$

By Corollary 16, we get a R-module homomorphism $\psi : A \otimes B \otimes A \otimes B \to A \otimes B$ defined by $\psi(a \otimes b \otimes a \otimes b) = aa' \otimes bb'$. We can also see $A \otimes B \otimes A \otimes B = (A \otimes B) \otimes (A \otimes B)$, and use Corollary 16 again to get another R-bilinear map $\varphi' : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ defined by $\varphi'(a \otimes b, a' \otimes b') = aa' \otimes bb'$. We have thus shown the multiplication is well-defined.

We then show the multiplication makes $A \otimes_R B$ into an R-algebra. Recall that $A \otimes_R B$ is an abelian group, and that that the multiplication being well-defined means it is bilinear, so we get the distributivity between addition and multiplication. Hence to show $A \otimes_R B$ is a ring, we just need to show the multiplication is associative: Take $a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3 \in A \otimes_R B$

and use the assumption A, B are rings

$$\begin{aligned}((a_1 \otimes b_1)(a_2 \otimes b_2))(a_3 \otimes b_3) &= (a_1 a_2 \otimes b_2 b_2)(a_3 \otimes b_3) \\ &= (a_1 a_2) a_3 \otimes (b_1 b_2) b_3 \\ &= a_1(a_2 a_3) \otimes b_1(b_2 b_3) \\ &= (a_1 \otimes b_1) \otimes (a_2 a_3 \otimes b_2 b_3) \\ &= (a_1 \otimes b_1)((a_2 \otimes b_2)(a_3 \otimes b_3)).\end{aligned} \tag{6}$$

We claim that $A \otimes_R B$ is a ring with $1_A \otimes 1_B$, because

$$(a \otimes b)(1_A \otimes 1_B) = a \otimes b = (1_A \otimes 1_B)(a \otimes b).$$

We then proceed to consider the map $\lambda : R \to A \otimes_R B$ defined by $\lambda(r) = i(r) \otimes 1_B$, where i is the ring homomorphism making A into an R-algebra (we use this fact multiple times in the following proof). We claim λ makes $A \otimes_R B$ into an R-algebra.

We check that λ is a ring homomorphism:

$$\begin{aligned}\lambda(r_1 + r_2) &= i(r_1 + r_2) \otimes 1_B \\ &= (i(r_1) + i(r_2)) \otimes 1_B \\ &= i(r_1) \otimes 1_B + i(r_2) \otimes 1_B \\ &= \lambda(r_1) + \lambda(r_2)\end{aligned} \tag{7}$$

$$\begin{aligned}\lambda(r_1 r_2) &= i(r_1 r_2) \otimes 1_B \\ &= (i(r_1) i(r_2)) \otimes 1_B \\ &= (i(r_1) \otimes 1_B)(i(r_2) \otimes 1_B) \\ &= \lambda(r_1)\lambda(r_2).\end{aligned} \tag{8}$$

λ also maps 1_R to $1_A \otimes 1_B$:

$$\lambda(1_R) = i(1_R) \otimes 1_B = 1_A \otimes 1_B.$$

We finally show $\lambda(R)$ is contained in the center of $A \otimes_R B$: Take $r \in R$ and $a \otimes b \in A \otimes_R B$,

$$\begin{aligned}(\lambda(r))(a \otimes b) &= (i(r) \otimes 1_B)(a \otimes b) \\ &= i(r)a \otimes 1_B b \\ &= ai(r) \otimes b1_B \\ &= (a \otimes b)(i(r) \otimes 1_B) \\ &= (a \otimes b)(\lambda(r)).\end{aligned} \tag{9}$$

QED

Problem 6:

Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

Note: I was told that we should use Problem 5 for this.

We see $\mathbb{Z}[i], \mathbb{R}$, and $\mathbb{C}$ as left $\mathbb{Z}$-modules. Consider the map $\varphi : \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C}$ defined by $\varphi(z, r) = zr$. Since φ is clearly $\mathbb{Z}$-bilinear:

$$\varphi(m_1 z_1 + m_2 z_2, r) = (m_1 z_1 + m_2 z_2)r = m_1 z_1 r + m_2 z_2 r = m_1 \varphi(z_1, r) + m_2 \varphi(z_2, r)$$
$$\varphi(z, m_1 r_1 + m_2 r_2) = z(m_1 r_1 + m_2 r_2) = z m_1 r_1 + z m_2 r_2 = m_1 \varphi(z, r_1) + m_2 \varphi(z, r_2)$$

we get a $\mathbb{Z}$-module homomorphism $\psi : \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}$ defined by $\psi(z \otimes r) = zr$.

We first show that ψ is a ring homomorphism, and we only need to show that multiplication preserves the structure, as ψ being a $\mathbb{Z}$-module homomorphism already takes care of addition and the additive inverse. We define multiplication the same way we did in Problem 5:

$$\begin{aligned} \psi((z_1 \otimes r_1)(z_2 \otimes r_2)) &= \psi((z_1 z_2) \otimes (r_1 r_2)) \\ &= (z_1 z_2)(r_1 r_2) \\ &= (z_1 r_1)(z_2 r_2) \\ &= \psi(z_1 \otimes r_1)\psi(z_2 \otimes r_2). \end{aligned} \tag{10}$$

We then show that ψ is surjective. Consider an element $a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$. By our construction of ψ, we see that

$$\psi((1 \otimes a) + (i \otimes b)) = \psi(1 \otimes a) + \psi(i \otimes b) = a + bi.$$

Since $1, i \in \mathbb{Z}[i]$ and $a, b \in \mathbb{R}$, we know $1 \otimes a, i \otimes b \in \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$. And since $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ is an abelian group, $(1 \otimes a) + (i \otimes b) \in \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$.

We finally show that ψ is injective. Consider $\psi(z \otimes r) = zr = 0$. It follows immediately that $z \otimes r = 0$.

$$QED$$