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<u>**Problem 1:**</u> Determine whether the following R-modules are finitely generated or free, with explanations. Give a basis for M if M is a free R-module.

(a) $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Consider $(1, 0)$ and $(0, 1)$, which are generators of M. This means M is finitely generated.

We claim M is not a free R-module, because $2 \cdot (1, 0) = (2, 0) = (0, 0)$, meaning there is a linear dependence, so M does not have a basis.

(b) $R = \mathbb{Z}$, $M = \mathbb{Z}[x]$.

We claim M is a free R-module, because $\{1, x, x^2, \dots\}$ is a basis for M. We can convince ourselves by thinking back to the polynomial vector space. The set we provide is linearly independent, and every $m \in M$ can be written as a $\mathbb{Z}$-linear combination of its elements.

M is not finitely generated because the basis we just found is not finite.

(c) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]$.

We claim M is a free R-module, because $\{1, y, y^2, \dots\}$ is a basis for M. We can convince ourselves because $\mathbb{R}[x, y]$ is all polynomials of the form $a_0 + a_1 x^{i_1} y^{j_1} + a_2 x^{i_2} y^{j_2} + \cdots$, and our ring $\mathbb{R}[x]$ takes care of all the $a_1 x^{i_1}, a_2 x^{i_2}, \dots$.

M is not finitely generated because the basis we just found is not finite.

(d) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]/(y^2 - x)$.

We claim M is finitely generated. We can convince ourselves because elements of $\mathbb{R}[x, y]$ are the same as what we listed in part (c), but the ideal $(y^2 - x)$ adds the relation $y^2 - x = 0$ in the quotient ring, or equivalently $y^2 = x$. Hence $\{1, y\}$ are generators of M.

M is a free R-module because the generating set we found is linearly independent.

(e) $R = \mathbb{R}[x, y]$, $M = (x, y)$.

Clearly, M is finitely generated because the generators are $\{x, y\}$.

But M is not a free R-module because $\{x, y\}$ is not a basis. This is demonstrated through taking $y \in R$ and $-x \in R$, which gives the linear dependence $y \cdot x + (-x) \cdot y = 0_M$, as R is commutative.

Problem 2: Take R as a commutative ring.

(a) Show that $R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \cong R[x]/(x^2-2)$ as $\mathbb{Z}$-algebras.

We use Corollary 12 on page 368.

Since R is a commutative ring, and R and $\mathbb{Z}[\sqrt{2}]$ are both abelian groups ($\mathbb{Z}$-modules), we get $R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ and the map $\iota : R \times \mathbb{Z}[\sqrt{2}] \to R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ defined by $(r, a+b\sqrt{2}) \mapsto r \otimes (a+b\sqrt{2})$.

Furthermore, $R[x]/(x^2-2)$ is a ring, so also an abelian group ($\mathbb{Z}$-module).

We then define the map $\varphi : R \times \mathbb{Z}[\sqrt{2}] \to R[x]/(x^2-2)$ by $(r, a+b\sqrt{2}) \mapsto ra + rbx$. We claim φ is $\mathbb{Z}$-bilinear:

$$\begin{aligned}
\varphi(z_1r_1 + z_2r_2, a+b\sqrt{2}) &= (z_1r_1 + z_2r_2)a + (z_1r_1 + z_2r_2)bx \\
&= z_1r_1a + z_2r_2a + z_1r_1bx + z_2r_2bx \\
&= z_1(r_1a + r_1bx) + z_2(r_2a + r_2bx) \\
&= z_1\varphi(r_1, a+b\sqrt{2}) + z_2\varphi(r_2, a+b\sqrt{2}).
\end{aligned} \tag{1}$$

$$\begin{aligned}
\varphi(r, z_1(a_1+b_1\sqrt{2}) + z_2(a_2+b_2\sqrt{2})) &= \varphi(r, z_1a_1 + z_2a_2 + (z_1b_1 + z_2b_2)\sqrt{2}) \\
&= r(z_1a_1 + z_2a_2) + r(z_1b_1 + z_2b_2)x \\
&= rz_1a_1 + rz_2a_2 + rz_1b_1x + rz_2b_2x \\
&= z_1(ra_1 + rb_1x) + z_2(ra_2 + rb_2x) \\
&= z_1\varphi(r, a_1+b_1\sqrt{2}) + z_2\varphi(r, a_2+b_2\sqrt{2}).
\end{aligned} \tag{2}$$

Thus, we get a $\mathbb{Z}$-module homomorphism $\Phi : R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \to R[x]/(x^2-2)$, and obtain the following commutative diagram: $\varphi = \Phi \circ \iota$. Hence Φ is defined as $r \otimes (a+b\sqrt{2}) \mapsto ra + rbx$.

$$\begin{tikzcd}
\boldsymbol{R \times \mathbb{Z}[\sqrt{2}]} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \boldsymbol{R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]} \arrow[d, "\Phi"] \\
& \boldsymbol{R[x]/(x^2-2)}
\end{tikzcd}$$

We claim that Φ is a $\mathbb{Z}$-algebra homomorphism. Note that since $R\otimes_{\mathbb{Z}}\mathbb{Z}[\sqrt{2}]$ and $R[x]/(x^2-2)$ are rings with identity, they must be $\mathbb{Z}$-algebras.

We proceed with the definition on page 343. To show Φ is a ring homomorphism, it suffices to show multiplication preserves the structure, as Φ being a $\mathbb{Z}$-module homomorphism already takes care of addition. We use the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ defined on our homework, and recall that $x^2 = 2$ in the quotient ring $R[x]/(x^2 - 2)$:

$$\begin{aligned}
\Phi((r_1 \otimes (a_1 + b_1\sqrt{2}))(r_2 \otimes (a_2 + b_2\sqrt{2}))) &= \Phi(r_1 r_2 \otimes (a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2)) \\
&= \Phi(r_1r_2 \otimes (a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2})) \\
&= r_1r_2(a_1a_2 + 2b_1b_2) + r_1r_2(a_1b_2 + a_2b_1)x \\
&= r_1r_2a_1a_2 + 2r_1r_2b_1b_2 + r_1r_2a_1b_2x + r_1r_2a_2b_1x.
\end{aligned} \tag{3}$$

$$\begin{aligned}
\Phi(r_1 \otimes (a_1 + b_1\sqrt{2}))\Phi(r_2 \otimes (a_2 + b_2\sqrt{2})) &= (r_1a_1 + r_1b_1x)(r_2a_2 + r_2b_2x) \\
&= r_1a_1r_2a_2 + r_1a_1r_2b_2x + r_1b_1r_2a_2x + r_1b_1r_2b_2x^2 \\
&= r_1r_2a_1a_2 + r_1r_2a_1b_2x + r_1r_2a_2b_1x + 2r_1r_2b_1b_2.
\end{aligned} \tag{4}$$

Hence $\Phi((r_1 \otimes (a_1 + b_1\sqrt{2}))(r_2 \otimes (a_2 + b_2\sqrt{2}))) = \Phi(r_1 \otimes (a_1 + b_1\sqrt{2}))\Phi(r_2 \otimes (a_2 + b_2\sqrt{2}))$.

We also need to verify scalar multiplication:

$$\begin{aligned}
\Phi(z \cdot (r \otimes (a + b\sqrt{2}))) &= \Phi(zr \otimes (a + b\sqrt{2})) \\
&= zra + zrbx \\
&= z(ra + rbx) \\
&= z \cdot \Phi(r \otimes (a + b\sqrt{2}))
\end{aligned} \tag{5}$$

We finally show Φ is bijective.

Φ is surjective:

By the Euclidean Algorithm, all elements in $R[x]/(x^2-2)$ are of the form $m+nx$. Thus, for an arbitrary $m+nx$ in the codomain, we can choose $1\otimes(m+n\sqrt{2})$ in the domain. We see that $\Phi(1\otimes(m+n\sqrt{2}))=m+nx$.

Φ is injective:

Suppose $r_1a_1+r_1b_1x=r_2a_2+r_2b_2x$, then clearly $r_1\otimes(a_1+b_1\sqrt{2})=r_2\otimes(a_2+b_2\sqrt{2})$.

Hence $R\otimes_{\mathbb{Z}}\mathbb{Z}[\sqrt{2}]\cong R[x]/(x^2-2)$ as $\mathbb{Z}$-algebras.

$$QED$$

(b) Determine whether $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ is an integral domain, with explanations.

Using what we proved in part (a), since $\mathbb{Z}[\sqrt{2}]$ is a commutative ring, we know $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \cong \mathbb{Z}[\sqrt{2}][x]/(x^2 - 2)$ as $\mathbb{Z}$-algebras. Now we forget about the ring homomorphisms and only see them as rings, in particular consider $\mathbb{Z}[\sqrt{2}][x]/(x^2 - 2)$.

Since $(x + \sqrt{2})(x - \sqrt{2}) = x^2 - 2$, which is in the ideal $(x^2 - 2)$, we know $x^2 - 2 = 0$ by the quotient ring construction. We have thus found zero divisors in the ring $\mathbb{Z}[\sqrt{2}][x]/(x^2 - 2)$. Since $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \cong \mathbb{Z}[\sqrt{2}][x]/(x^2 - 2)$ as rings, $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ CANNOT be an integral domain.

$$QED$$

Problem 3:

This Snake killed my eyes and my patience, but I will wish myself a happy birthday here.

Suppose

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of R-modules with exact rows. Prove there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$$

where $\operatorname{coker}(\alpha)$ (the cokernel of α) is $A'/(\text{image }\alpha)$ and similarly for $\operatorname{coker}(\beta)$ and $\operatorname{coker}(\gamma)$.

We first define each map in the sequence

$$\ker(\alpha) \xrightarrow{\varphi} \ker(\beta) \xrightarrow{\phi} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{\psi} \operatorname{coker}(\beta) \xrightarrow{\lambda} \operatorname{coker}(\gamma),$$

and then show it is exact.

We first define $\varphi : \ker(\alpha) \to \ker(\beta)$. Consider some $a \in A$ such that $a \in \ker(\alpha)$. By the commutative diagram, we have $\beta(f(a)) = f'(\alpha(a)) = f'(0) = 0$. This means $f(a) \in \ker(\beta)$, so we can define φ as $a \mapsto f(a)$ for all $a \in \ker(\alpha)$.

We then define $\phi : \ker(\beta) \to \ker(\gamma)$ similarly. Consider some $b \in B$ such that $b \in \ker(\beta)$. By the commutative diagram, we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. This means $g(b) \in \ker(\gamma)$, so we can define ϕ as $b \mapsto g(b)$ for all $b \in \ker(\beta)$.

We next define $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$. Consider some $c \in C$ such that $c \in \ker(\gamma)$. Since each row is exact, by Proposition 22.(2) on page 379, g must be surjective. Thus, there exists $b \in B$ such that $g(b) = c$. By the commutative diagram, we have $g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$, so $\beta(b) \in \ker(g')$. Recall that $\operatorname{coker}(\alpha) = A'/\operatorname{im}(\alpha)$, so we want to associate $g(b)$ with something in A'. Since each row is exact, by the definition on page 378, we know $\ker(g') = \operatorname{im}(f')$. This means there exists $a' \in A'$ such that $f'(a') = \beta(b)$, so let's define δ as $c \mapsto [a']$, which is an equivalence class.

We next define $\psi : \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$ by the canonical option $[a'] \mapsto [f'([a'])]$.

Similarly, we finally define $\lambda : \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$ by $[b'] \mapsto [g'([b'])]$.

Note: I will check that this is well-defined if I still have brain power.

We now proceed to showing the sequence is exact.

We first show the exactness at $\ker(\beta)$ by showing $\operatorname{im}(\varphi) = \ker(\phi)$. Recall we have defined $\varphi(a) = f(a)$ if $a \in \ker(\alpha)$ and $\phi(b) = g(b)$ if $b \in \ker(\beta)$. We first show $\operatorname{im}(\varphi) \subseteq \ker(\phi)$. Take arbitrary $a \in A$ such that $a \in \ker(\alpha)$, then $\phi(\varphi(a)) = g(f(a)) = 0$ because $f(a) \in \ker(\beta)$, and $f(a) \in \ker(g)$ by the exactness at B. Hence $\operatorname{im}(\varphi) \subseteq \ker(\phi)$. We then show $\ker(\phi) \subseteq \operatorname{im}(\varphi)$. Take arbitrary $b \in \ker(\beta)$ such that $\phi(b) = 0$, by the exactness at B, we know there exists $a \in A$ such that $f(a) = b$. By commutativity. we have $f'(\alpha(a)) = \beta(f(a)) = \beta(0) = 0$, so $\alpha(a) \in \ker(f')$. By Proposition 22.(2) on page 379, we know f' is injective. This means $\alpha(a) = 0$ and $a \in \ker(\alpha)$, which shows for every $b \in \ker(\phi)$, there exists $a \in \ker(\alpha)$ such that $f(a) = \varphi(a) = b$. Hence $\ker(\phi) \subseteq \operatorname{im}(\varphi)$.

We then show the exactness at $\ker(\gamma)$ by showing $\operatorname{im}(\phi) = \ker(\delta)$. We first show $\operatorname{im}(\phi) \subseteq \ker(\delta)$. Take arbitrary $b \in B$ such that $b \in \ker(\beta)$...

Note: I tried everything I could to show the exactness at $\ker(\gamma), \operatorname{coker}(\alpha)$, and $\operatorname{coker}(\beta)$, but I didn't figure them out. I suspect it is because I didn't define the maps ψ and λ correctly, but I don't see other definitions that make sense.

Problem 4:

Let R be a ring and M an R-module.

(a) Show $\mathrm{Hom}_{\mathbb{Z}}$ is a left R-module under the action defined by $r \cdot \varphi(x) = \varphi(rx)$ with $r, x \in R$

We first show $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an abelian group. Here $\mathbb{Z}$ is the ring and R is a $\mathbb{Z}$-module.

Take $\varphi_1, \varphi_2 \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, and define $(\varphi_1 + \varphi_2)(r) := \varphi_1(r) + \varphi_2(r)$ for all $r \in R$. We claim that the operation is well-defined, that $(\varphi_1 + \varphi_2) \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$:

$$\begin{aligned}(\varphi_1 + \varphi_2)(zr_1 + r_2) &= \varphi_1(zr_1 + r_2) + \varphi_2(zr_1 + r_2) \\ &= z\varphi_1(r_1) + \varphi_1(r_2) + z\varphi_2(r_1) + \varphi_2(r_2) \\ &= z(\varphi_1(r_1) + \varphi_2(r_1)) + \varphi_1(r_2) + \varphi_2(r_2) \\ &= z(\varphi_1 + \varphi_2)(r_1) + (\varphi_1 + \varphi_2)(r_2).\end{aligned} \tag{6}$$

Associativity:

Take $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, and consider $(\varphi_1 + \varphi_2) + \varphi_3$ and $\varphi_1 + (\varphi_2 + \varphi_3)$. This is straightforward because for any $r \in R$, $\varphi_1(r) + \varphi_2(r) + \varphi_3(r)$ is associative, because they are in the $\mathbb{Z}$-module M, which is an abelian group by definition.

Identity:

Consider the zero map $\varphi_0(r) := 0_M$ for all $r \in R$. Take $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, we have

$$(\varphi + \varphi_0)(r) = \varphi(r) + \varphi_0(r) = \varphi(r) = \varphi_0(r) + \varphi(r) = (\varphi_0 + \varphi)(r).$$

Inverses:

Take $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, define $-\varphi(r) := -\varphi(r)$ for all $r \in R$. Clearly $-\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$:

$$-\varphi(zr_1 + r_2) = -(z\varphi(r_1) + \varphi(r_2)) = -z\varphi(r_1) - \varphi(r_2).$$

And

$$\begin{aligned}(\varphi + (-\varphi))(r) &= \varphi(r) - \varphi(r) = 0_M, \\ ((-\varphi) + \varphi)(r) &= -\varphi(r) + \varphi(r) = 0_M.\end{aligned}$$

We then verify the module axioms.

Take $r, r_1, r_2, 1_R \in R$ and $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, then

$$\begin{aligned}(r_1 + r_2) \cdot \varphi(x) &= \varphi((r_1 + r_2)x) \\ &= \varphi(r_1 x + r_2 x) \\ &= \varphi(r_1 x) + \varphi(r_2 x) \\ &= r_1 \cdot \varphi(x) + r_2 \cdot \varphi(x) \\ &= (r_1 \cdot \varphi + r_2 \cdot \varphi)(x)\end{aligned} \tag{7}$$

$$(r_1 r_2) \cdot \varphi(x) = \varphi(r_1 r_2 x) = r_1 \cdot \varphi(r_2 x) = r_1 \cdot (r_2 \cdot \varphi(x)).$$

$$\begin{aligned}(r \cdot (\varphi_1 + \varphi_2)(x)) &= (\varphi_1 + \varphi_2)(rx) \\ &= \varphi_1(rx) + \varphi_2(rx) \\ &= r \cdot \varphi_1(x) + r \cdot \varphi_2(x) \\ &= (r \cdot \varphi_1 + r \cdot \varphi_2)(x).\end{aligned} \tag{8}$$

$$1_R \cdot \varphi(x) = \varphi(1_R x) = \varphi(x).$$

Hence we have proved $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the given action.

$$QED$$

(b) Show if Q is an injective R-module, then $\text{Hom}_{\mathbb{Z}}(R, Q)$ is also an injective R-module.

We use Proposition 34.(2) on page 395.

Suppose L and M are R-modules and $0 \to L \xrightarrow{\psi} M$ is exact, and take arbitrary $f \in \text{Hom}_R(L, \text{Hom}_{\mathbb{Z}}(R, Q))$, it suffices to show that there is a lift $F \in \text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, Q))$ such that $f = F \circ \psi$.

Now we define $g : L \to Q$ as $l \mapsto f(l)(1_R)$. Note that $f(l) \in \text{Hom}_{\mathbb{Z}}(R, Q)$, so $f(l)(1_R) \in Q$.

We claim $g \in \text{Hom}_R(L, Q)$, and use the addition of maps we defined in the last part:

$$\begin{aligned} g(l_1 + l_2) &= f(l_1 + l_2)(1_R) \\ &= (f(l_1) + f(l_2))(1_R) \\ &= f(l_1)(1_R) + f(l_2)(1_R) \\ &= g(l_1) + g(l_2). \end{aligned} \tag{9}$$

$$g(rl) = f(rl)(1_R) = (rf(l))(1_R) = r(f(l)(1_R)) = rg(l).$$

Since Q is an injective R-module by assumption, and $g \in \text{Hom}_R(L, Q)$, we know there is a lift $G \in \text{Hom}_R(M, Q)$ such that $g = G \circ \varphi$.

We now define $F : M \to \text{Hom}_{\mathbb{Z}}(R, Q)$ as $F(m)(r) := G(rm)$. Note that $F(m)$ is a map from R to Q, so $F(m)(r) \in Q$. Indeed, $G(rm) \in Q$.

Everything is best illustrated by the commutative diagram below:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\psi"] \arrow[d, "f"] \arrow[dd, "g"'] & M \arrow[dl, dashed, "F"'] \arrow[ddl, dashed, "G"] \\
& Hom_{\mathbb{Z}}(R, Q) & \\
& Q &
\end{tikzcd}$$

We finally verify that $f = F \circ \psi$, which is just putting all of our results together, including the ring action defined in part (a). Note that the input of both maps is some $l \in L$ and the output of both maps should be an element of $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$, so take arbitrary $l \in L$ and $r \in R$. Note that $\psi(l) \in M$:

$$\begin{aligned}
(F \circ \psi(l))(r) &= F(\psi(l))(r) \\
&= G(r\psi(l)) \\
&= G(\psi(rl)) \\
&= g(rl) \\
&= rg(l) \\
&= r(f(l)(1_R)) \\
&= f(l)(r)
\end{aligned} \tag{10}$$

Hence $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module.

QED

Problem 5: Let S^1 be the circle group, which is a subgroup of $\mathbb{C}^\times$ defined by $S^1 = \{e^{i\theta} \in \mathbb{C}^\times : \theta \in R\}$. Show that S^1 is an injective $\mathbb{Z}$-module.

We use Baer's Criterion on page 396. Take an arbitrary left ideal I of $\mathbb{Z}$, and consider an arbitrary $\mathbb{Z}$-module homomorphism $g : I \to S^1$, we want to extend it to an $\mathbb{Z}$-module homomorphism $G : \mathbb{Z} \to S^1$.

The definition we are given is $S^1 = \{e^{i\theta} \in \mathbb{C}^\times : \theta \in \mathbb{Z}\}$, so we can define $G : \mathbb{Z} \to S^1$ as

$$G(z) = e^{iz} = \cos(z) + i\sin(z).$$

We verify that G is a $\mathbb{Z}$-module homomorphism (abelian group homomorphism). Take $z_1, z_2 \in \mathbb{Z}$:

$$\begin{aligned} G(z_1 + z_2) &= e^{i(z_1+z_2)} \\ &= e^{iz_1}e^{iz_2} \\ &= G(z_1)G(z_2). \end{aligned} \tag{11}$$

We claim that $G|_I = g$. Since the domain of G is $\mathbb{Z}$, there is only one possible group homomorphism G, because 1 has to be sent to the identity element of S^1, and where 1 gets sent to determines where all $n \in \mathbb{Z}$ gets sent to. Given that $I \subseteq \mathbb{Z}$, $G|_I$ must also be unique. This means that if g really is a $\mathbb{Z}$-module homomorphism, then it must be that $G|_I = g$.

$$QED$$

Note: This is probably far from the right track, but it was the best that I could do :(