Week 2 Final, Math B504 **Name:** *James Wang*
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Problem 1:

An R-module M is called a torsion module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$ where r may depend on m.

a) Prove that every finite abelian group is a torsion $\mathbb{Z}$-module.

Recall that every abelian group can be viewed as a $\mathbb{Z}$-module. Let G be a finite abelian group. Then for all $g \in G$, $|g| < \infty$. Note that $|g| \in \mathbb{Z}$, and we claim that $|g| \cdot g = 0$ for all $g \in G$ when using addition as the group operation:

$$|g| \cdot g = \underbrace{g + g + \cdots + g}_{|g| \text{ times}} = 0.$$

QED

b) Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

We claim the infinite product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$ is an infinite group that is a torsion $\mathbb{Z}$-module. This group is infinite because it has infinitely many elements. This group is abelian because it is a direct product of abelian groups. Finally, this group is a torsion $\mathbb{Z}$-module because our construction from part a) by using the order of each group element as the element of the ring $\mathbb{Z}$ acting on the module to get to 0 still works, since all elements in this group have either order 1 or 2 (finite order).

Problem 2:

Let R be an integral domain.

a) Prove that every finitely generated torsion R-module has a nonzero annihilator, i.e. there is a nonzero element $r \in R$ such that $rm = 0$ for all $m \in M$ (here r does not depend on m).

Let M be a finitely generated torsion R-module, then there exists a finite set $A \subseteq M$ such that $M = RA$. By the definition of torsion R-modules, moreover, for each $a_i \in A$ there exist a nonzero element r_i such that $r_i a_i = 0$ (where r_i may depend on a_i). We claim the product $r = \prod_{\text{finite}} r_i$ is a nonzero annihilator of M.

First note that since R is an integral domain and each r_i is nonzero by construction, r must also be nonzero. It remains to verify that r is indeed an annihilator. Recall that M is finitely generated, which means every $m \in M$ can be written as $m = \sum_{\text{finite}} r_j a_j$ where $r_j \in R$ and $a_j \in A$. Thus,

$$rm = r \sum_{\text{finite}} r_j a_j = \sum_{\text{finite}} r r_j a_j.$$

Since R as an integral domain must be commutative, observe that for each $r r_j a_j$, we just need to pull the factor r_i of r so that $r_i a_j = 0$ all the way next to a_j, then each term will become 0, so $rm = 0$.

$$QED$$

b) Give an example of a torsion R-module whose annihilator is the zero ideal.

I am very confused by this question. Isn't the zero ideal of R always an annihilator of every R-module?

<u>Problem 3:</u>

Let R be a commutative ring and let A, B and M be R-modules. Prove the following isomorphisms of R-modules:

a) $Hom_R(A \times B, M) \cong Hom_R(A, M) \times Hom_R(B, M)$

We define the map $\varphi : Hom_R(A, M) \times Hom_R(B, M) \to Hom_R(A \times B, M)$ by $(f, g) \mapsto \lambda$, where for all $(a, b) \in A \times B$, $\lambda(a, b) := f(a) + g(b)$.

We first show that this map is well-defined, that $\lambda \in Hom_R(A \times B, M)$. Let $(a_1, b_1), (a_2, b_2) \in (A \times B)$, and $r \in R$. Then, since $f \in Hom_R(A, M)$ and $g \in Hom_R(B, M)$,

$$\begin{aligned} \lambda(r(a_1, b_1) + (a_2, b_2)) &= \lambda(ra_1 + a_2, rb_1 + b_2) \\ &= f(ra_1 + a_2) + g(rb_1 + b_2) \\ &= rf(a_1) + f(a_2) + rg(b_1) + g(b_2) \\ &= r(f(a_1) + g(b_1)) + f(a_2) + g(b_2) \\ &= r\lambda(a_1, b_1) + \lambda(a_2, b_2). \end{aligned} \tag{1}$$

We then show that φ is an R-module homomorphism. Let $f_1, f_2 \in Hom_R(A, M)$ and $g_1, g_2 \in Hom_R(B, M)$. Then for all $a \in A$ and $b \in B$,

$$\begin{aligned} \varphi(r(f_1, g_1) + (f_2, g_2))(a, b) &= \varphi(rf_1 + f_2, rg_1 + g_2)(a, b) \\ &= (rf_1 + f_2)(a) + (rg_1 + g_2)(b) \\ &= rf_1(a) + f_2(a) + rg_1(b) + g_2(b) \\ &= r(f_1(a) + g_1(b)) + f_2(a) + g_2(b) \\ &= r\varphi(f_1, g_1)(a, b) + \varphi(f_2, g_2)(a, b). \end{aligned} \tag{2}$$

We next show that φ is injective. Let $\lambda_1, \lambda_2 \in Hom_R(A \times B, M)$ and suppose $\lambda_1 = \lambda_2$, or equivalently $\varphi(f_1, g_1)(a, b) = \varphi(f_2, g_2)(a, b)$ for all $(a, b) \in A \times B$. Then for all $(0_A, b) \in A \times B$,

$$g_1(b) = \varphi(f_1, g_1)(0_A, b) = \varphi(f_2, g_2)(0_A, b) = g_2(b).$$

Similarly, for all $(a, 0_B) \in (A \times B)$,

$$f_1(a) = \varphi(f_1, g_1)(a, 0_B) = \varphi(f_2, g_2)(a, 0_B) = f_2(a).$$

Hence $(f_1, g_1) = (f_2, g_2)$.

We finally show that φ is surjective. Let $\lambda \in Hom_R(A \times B, M)$. Recall that by definition

$$\lambda(a, b) = \varphi(f, g)(a, b) = f(a) + g(b),$$

so for any $(a, b) \in A \times B$, we can just define $\lambda(a, 0_B) = f(a) \in Hom_R(A, M)$ and $\lambda(0_A, b) = g(b) \in Hom_R(B, M)$, and

$$\lambda(a, b) = \lambda(a, 0_B) + \lambda(0_A, b) = f(a) + g(b).$$

$$QED$$

b) $Hom_R(M, A \times B) \cong Hom_R(M, A) \times Hom_R(M, B)$

We define the map $\varphi : Hom_R(M, A \times B) \to Hom_R(M, A) \times Hom_R(M, B)$ by $\lambda \mapsto (f, g)$, where for all $m \in M$, $f(\lambda(m))$ is the 1st coordinate of $\lambda(m)$ and $g(\lambda(m))$ is the 2nd coordinate of $\lambda(m)$.

We first show that this map is well-defined, that $f \in Hom_R(M, A)$ and $g \in Hom_R(M, B)$. This is straightforward because every $\lambda \in Hom_R(M, A \times B)$ is already a homomorphism, so restricting the B coordinate must give some homomorphism in $Hom_R(M, A)$ and restricting the A coordinate must give some homomorphism in $Hom_R(M, B)$.

We then show that φ is an R-module homomorphism. Let $\lambda_1, \lambda_2 \in Hom_R(M, A \times B)$ and $r \in R$. Then for all $m \in M$,

$$\begin{aligned}
\varphi(r\lambda_1 + \lambda_2)(m) &= (f((r\lambda_1 + \lambda_2)(m)), g((r\lambda_1 + \lambda_2)(m))) \\
&= (rf(\lambda_1)(m) + f(\lambda_2)(m), rg(\lambda_1)(m) + g(\lambda_2)(m)) \\
&= r(f(\lambda_1(m)), g(\lambda_1(m))) + (f(\lambda_2(m)), g(\lambda_2(m))) \\
&= r\varphi(\lambda_1)(m) + \varphi(\lambda_2)(m)
\end{aligned} \tag{3}$$

We next show that φ is injective. This is straightforward, because for $\varphi(\lambda)(m) = (0, 0)$ for all $m \in M$, it must be $f(\lambda(m)) = 0$ and $g(\lambda(m)) = 0$ for all $m \in M$, meaning λ can only be the zero transformation.

We finally show that φ is surjective. This is even more straightforward. For any $(f, g) \in Hom_R(M, A) \times Hom_R(M, B)$, just define $\lambda(m) = (f(\lambda(m)), g(\lambda(m)))$.

$$QED$$

Note: I really dislike this problem, the notations took me several hours.

Problem 4:

Let R be a commutative ring and let F be a free R-module of finite rank. Prove the following isomorphism of R-modules: $Hom_R(F,R) \cong F$.

Since F is a free R-module of finite rank, we can denote the rank of F by n. By the universal property of free modules (Theorem 1.22), we know $F \cong R^n$. Together with Problem 3 part a), we can therefore establish

$$Hom_R(F,R) \cong Hom_R(R^n,R) \cong (Hom_R(R,R))^n.$$

By a remark on the handout, moreover, we know there exists a ring homomorphism

$$\zeta : R \to End_R(R) \text{ defined by } r \mapsto rI, \text{ where } I \text{ is the identity.}$$

We claim that ζ is actually an isomorphism because it is invertible. We can define

$$\zeta^{-1} : End_R(R) \to R \text{ by } i \mapsto i(1_R),$$

which is a ring homomorphism because

$$\zeta^{-1}(i_1+i_2) = (i_1+i_2)(1_R) = i_1(1_R) + i_2(1_R) = \zeta^{-1}(i_1) + \zeta^{-1}(i_2),$$
$$\zeta^{-1}(i_2 \circ i_1) = i_2 \circ i_1(1_R) = i_2(i_1(1_R)) = \zeta^{-1}(i_2) \circ \zeta^{-1}(i_1).$$

We can verify that the composition gives back the identity:

$$\zeta^{-1} \circ \zeta(r) = \zeta^{-1}(rI) = r\zeta^{-1}(I) = rI(1_R) = r \cdot 1_R = r.$$

Thus, we can conclude that $(Hom_R(R,R))^n \cong R^n \cong F$.

Putting the two parts together, we have proved $Hom_R(F,R) \cong F$.

QED

Problem 4':

In general, for an R-module M, prove that $Hom_R(R, M) \cong M$ as R-modules.

The idea is similar to our answer to Problem 4. We define a map

$$\varphi : Hom_R(R, M) \to M \text{ by } i \mapsto i(1_R).$$

This map is an R-module homomorphism because

$$\varphi(i_1 + i_2) = (i_1 + i_2)(1_R) = i_1(1_R) + i_2(1_R) = \varphi(i_1) + \varphi(i_2),$$
$$\varphi(ri) = (ri)(1_R) = r(i(1_R)) = r\varphi(i).$$

This map is also injective. Consider $\ker(\varphi)$. For $i \in \ker(\varphi)$, it must be that $i(1_R) = 0_M$, which means $\varphi(r) = 0_M$ for all $r \in R$, implying that i is actually the zero map. Hence the only $i \in \ker(\varphi)$ is the identity element of $Hom_R(R, M)$.

Finally, this map is surjective. For each $m \in M$, we can just choose the $i \in Hom_R(R, M)$ such that $i(r) = rm$ for all $r \in R$. This way, $\varphi(i) = i(1_R) = 1_R \cdot m = m$.

We have thus proved that $Hom_R(R, M) \cong M$ as R-modules.

$$QED$$

Problem 5 (I read the textbook):

Show that the tensor product $S \otimes_R N$ is a left S-module under the action of S defined by

$$s\Big(\sum_{\text{finite}} s_i \otimes n_i\Big) = \sum_{\text{finite}} (ss_i) \otimes n_i.$$

We first check that this operation is well-defined. I used page 361 of Dummit & Foote. Suppose that $\sum s_i \otimes n_i = \sum s_i' \otimes n_i'$ are two representatives of the same element in $S \otimes_R N$, then we have $\sum(s_i, n_i) - \sum(s_i', n_i') \in H$ by the coset recognition lemma. Note that left multiplication by any $s \in S$ to any of the 3 generators of H will make it stay in H (another generator), so $\sum(ss_i, n_i) - \sum(ss_i', n_i') \in H$. It follows quickly that $\sum ss_i \otimes n_i = \sum s, s_i' \otimes n_i'$, so the operation is well-defined.

Since $S \otimes_R N$ is already an abelian quotient group, we just need to check the 4 axioms of the ring action $S \times (S \otimes_R N) \to S \otimes_R N$.

1) Let $s, s' \in S$ and $s_i \otimes n_i \in S \otimes_R N$, then

$$\begin{aligned}
(s+s')(s_i \otimes n_i) &= ((s+s')s_i) \otimes n_i \\
&= (ss_i + s's_i) \otimes n_i \\
&= ss_i \otimes n_i + s's_i \otimes n_i \\
&= s(s_i \otimes n_i) + s'(s_i \otimes n_i).
\end{aligned} \tag{4}$$

2) Let $s, s' \in S$ and $s_i \otimes n_i \in S \otimes_R N$, then

$$\begin{aligned}
(ss')(s_i \otimes n_i) &= (ss's_i) \otimes n_i \\
&= s(s's_i \otimes n_i) \\
&= s(s'(s_i \otimes n_i))
\end{aligned} \tag{5}$$

3) Let $s \in S$ and $s_i \otimes n_i, s_i' \otimes n_i' \in S \otimes_R N$, then

$$\begin{aligned}
s((s_i \otimes n_i) + (s_i' \otimes n_i')) &= ((ss_i \otimes n_i) + (ss_i' \otimes n_i')) \\
&= s(s_i \otimes n_i) + s(s_i' \otimes n_i')
\end{aligned} \tag{6}$$

4) Let $1_S \in S$ and $s_i \otimes n_i \in S \otimes_R N$, then

$$\begin{aligned} 1_S(s_i \otimes n_i) &= (1_S s_i) \otimes n_i \\ &= s_i \otimes n_i \end{aligned} \tag{7}$$

Hence the tensor product $S \otimes_R N$ is a left S-module under the action of S by the given definition.

QED