Week 9 Final, Math B504 **Name:** *James Wang*
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Problem 1 (Hint from Hongyi and Problem Session):

We want to show a finitely generated module over a PID is projective if and only if it is free.

$(\Longrightarrow)$:

Suppose R is a PID and M is a finitely generated projective R-module. By Corollary 31 on page 390, we know M is a direct summand of a finitely generated free module, so $M(S) = M \oplus N$, where N is also an R-module. Now by the Fundamental Theorem of Finitely Generated Modules over PIDs (page 462-463), we get the isomorphism

$$M(S) \cong R^r \oplus \mathrm{Tor}(M) \oplus N.$$

Since $M(S)$ is free, it must also be torsion free. Hence M is also torsion free and free.

$(\Longleftarrow)$:

By Corollary 31 on page 390, we know that free modules are projective.

$$QED$$

Problem 2:

We may use the Proposition: Let R be a PID. Then any submodule of a free R-module is also free.

(a) We want to show that over a PID, a module M is projective if and only if it is free.

$(\Longrightarrow)$:

Suppose R is a PID and M is a projective R-module. By Corollary 31 on page 390, we know M is a direct summand of a finitely generated free module. By the given Proposition, we know any submodule of a free R-module is also free. Hence M is free.

$(\Longleftarrow)$:

By Corollary 31 on page 390, we know that free modules are projective.

$$QED$$

(b) We want to show that as a $\mathbb{Z}$-module, $\mathbb{Q}$ is not projective.

Using our result from part (a), since $\mathbb{Z}$ is a PID, it suffices to show $\mathbb{Q}$ is not a free $\mathbb{Z}$-module. This is quick because take any nonzero $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in \mathbb{Q}$, which means nonzero $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, we can then take $x = b_1 a_2, y = -a_1 b_2 \in \mathbb{Z}$, and obtain the linear dependence

$$x(\frac{a_1}{b_1}) + y(\frac{a_2}{b_2}) = b_1 a_2(\frac{a_1}{b_1}) - a_1 b_2(\frac{a_2}{b_2}) = a_2 a_1 - a_1 a_2 = 0.$$

Hence $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

$$QED$$

Problem 3:

We want to show that if R is an integral domain and M is any non-principal ideal of R, then as an R-module, M is torsion-free of rank 1, but is not a free R-module.

Suppose R is an integral domain and M is a non-principal ideal of R. We first show M is torsion-free of rank 1. Take nonzero $r \in R$ and some $m \in M \subseteq R$. Since R is an integral domain, if $rm = 0$, it must be that $m = 0$. Hence M is torsion-free. Since M is non-principal, we know it has a rank greater than 1. Yet take any nonzero $x, y \in M$, we can get an R-linear combination such that $(-y)(x) + (x)(y) = 0$, which gives a linear dependence. Hence M has a rank of exactly 1.

We then show M is not a free R-module. Recall we have already shown that M has rank 1. This means if M has a basis, then there is only one element in it, say x, such that all $m \in M$ can be expressed as $m = rx$ for some $r \in R$. This would contradict with our assumption that M is a non-principal ideal. Hence M does not have a basis.

$$QED$$

Problem 4:

Let $V \cong \mathbb{Q}^3$ with the $\mathbb{Q}[x]$-module structure given by the transformation T with matrix

$$A = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix}.$$

We want to use the Invariant Factor Decomposition Algorithm to find the invariant factor decomposition for V and the rational canonical form for T. Hence we use row and column operations in $\mathbb{Q}[x]$ to reduce the matrix

$$xI - A = \begin{bmatrix} x & 4 & -85 \\ -1 & x-4 & 30 \\ 0 & 0 & x-3 \end{bmatrix}.$$

$$\begin{bmatrix} x & 4 & -85 \\ -1 & x-4 & 30 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & x-4 & 30 \\ x & 4 & -85 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{-R_1 \mapsto R_1} \begin{bmatrix} 1 & 4-x & -30 \\ x & 4 & -85 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{R_2 - xR_1 \mapsto R_2}$$

$$\begin{bmatrix} 1 & 4-x & -30 \\ 0 & (x-2)^2 & 30x-85 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{C_2-(4-x)C_1 \mapsto C_2} \begin{bmatrix} 1 & 0 & -30 \\ 0 & (x-2)^2 & 30x-85 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{C_3+30C_1 \mapsto C_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (x-2)^2 & 30x-85 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{R_2-30R_3 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x-2)^2 & 5 \\ 0 & 0 & x-3 \end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & (x-2)^2 \\ 0 & x-3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5}(x-2)^2 \\ 0 & x-3 & 0 \end{bmatrix} \xrightarrow{-(x-3)R_2+R_3 \mapsto R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5}(x-2)^2 \\ 0 & 0 & -\frac{1}{5}(x-3)(x-2)^2 \end{bmatrix} \xrightarrow{(x-3)R_2 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x-3 & \frac{1}{5}(x-2)^2(x-3) \\ 0 & 0 & -\frac{1}{5}(x-3)(x-2)^2 \end{bmatrix} \xrightarrow{R_2+R_3 \mapsto R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & -\frac{1}{5}(x-3)(x-2)^2 \end{bmatrix} \xrightarrow{-5R_3 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & (x-3)(x-2)^2 \end{bmatrix}$$

This determines the invariant factors $x-3$, $x^3-7x^2+16x-12$ for this matrix. Let now V be a 3-dimensional vector space over $\mathbb{Q}$ with basis e_1, e_2, e_3, and let T be the corresponding linear transformation which defines the action of x on V:

$$xe_1 = T(e_1) = e_2$$
$$xe_2 = T(e_2) = -4e_1 + 4e_2$$
$$xe_3 = T(e_3) = 85e_1 - 30e_2 + 3e_3.$$

We now start with the basis $[e_1, e_2, e_3]$ for V and change it according to our row operations:

$$[e_1, e_2, e_3] \xrightarrow{R_1 \leftrightarrow R_2} [e_2, e_1, e_3] \xrightarrow{-R_1 \mapsto R_1} [-e_2, e_1, e_3] \xrightarrow{R_2 - xR_1 \mapsto R_2} [-e_2 + xe_1, e_1, e_3] \xrightarrow{R_2 - 30R_3 \mapsto R_2}$$
$$[-e_2 + xe_1, e_1, e_3 + 30e_2] \xrightarrow{\frac{1}{5}R_2 \mapsto R_2} [-e_2 + xe_1, 5e_1, e_3 + 30e_2] \xrightarrow{-(x-3)R_2 + R_3 \mapsto R_3}$$
$$[-e_2 + xe_1, 5e_1 + (x-3)(e_3 + 30e_2), e_3 + 30e_2] \xrightarrow{(x-3)R_2 \mapsto R_2}$$
$$[-e_2 + xe_1, e_3 + 30e_2, e_3 + 30e_2] \xrightarrow{R_2 + R_3 \mapsto R_2} [-e_2 + xe_1, e_3 + 30e_2, 0] \xrightarrow{-5R_3 \mapsto R_3}$$
$$[-e_2 + xe_1, e_3 + 30e_2, 0].$$

Using the formulas above for the action of x, we see that these last elements are the elements $[0, e_3 + 20e_2, 0]$ of V corresponding to the elements 1, $x-3$, and $(x-3)(x-2)^2$ in the diagonalized form of $xI - A$, respectively. The element $f_1 = e_3 + 20e_2$ is therefore a $\mathbb{Q}[x]$-module generator for the two cyclic factors of V in its invariant factor decomposition as a $\mathbb{Q}[x]$-module. The corresponding $\mathbb{Q}$-vector space bases for these two factors are then

$$f_1 = e_3 + 20e_2$$

$$Tf_1 = 5e_1 + 50e_2 + 3e_3$$

$$T^2 f = 55e_1 + 115e_2 + 9e_3$$

which are the columns of our matrix P. Using online matrix calculator we see that

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 12 \\ 1 & 0 & -16 \\ 0 & 1 & 7 \end{bmatrix},$$

which is precisely what the textbook says.

Problem 5:

Let F be a field.

(a) We want to show that two 2×2 matrices over F which are not scalar multiple of the identity matrix are similar if and only if they have the same characteristic polynomial.

$(\Longrightarrow)$:

Suppose two 2×2 matrices over F which are not scalar multiple of the identity matrix are similar. This means they must have the same invariant factors. Since the characteristic polynomial of a matrix is the product of its invariant factors, these two matrices must have the same characteristic polynomials.

$(\Longleftarrow)$:

Suppose $A, B \in M_{2\times 2}(F)$ are not scalar multiples of the identity matrix, and they have the same characteristic polynomials $c_A(x) = c_B(x)$. To show A and B are similar, we claim that each of them only has one invariant factor. We know A has at most two invariant factors, $a_1(x)$ and $a_2(x)$, so we suppose this is the case for a contradiction. Since $a_1(x)\big|a_2(x)$, and their product gives the quadratic $c_A(x)$, it must be that $a_1(x) = a_2(x)$, which is a linear term. This is the invariant factor of a scalar multiple of the identity matrix, which is only similar to a scalar multiple of the identity matrix. This contradicts our assumption that A is not a such matrix. Hence A only has one invariant factor $a_1(x)$. Using the same argument, we see that B also only has one invariant factor $b_1(x)$. We then get $a_1(x) = c_A(x) = c_B(x) = b_1(x)$. Since A and B have the same invariant factor, we can conclude they are similar.

$$QED$$

(b) We want to show that two 3×3 matrices over F are similar if and only if they have the same characteristic and minimal polynomials.

$(\Longrightarrow)$:

Suppose two 3×3 matrices over F are similar. This means they must have the same invariant factors. Since the characteristic polynomial of a matrix is the product of its invariant factors, these two matrices must have the same characteristic polynomials. Since the minimal polynomial of a matrix is the first invariant factor, these two matrices must have the same minimal polynomials.

$(\Longleftarrow)$:

Suppose $A, B \in M_{3\times 3}(F)$ have the same characteristic polynomials $c_A(x) = c_B(x)$ and the same minimal polynomials $m_A(x) = m_B(x)$. We know the characteristic polynomials are cubics, and the minimal polynomials could be cubic, quadratic, or linear.

If $m_A(x) = m_B(x)$ is cubic, then there is only one invariant factor, so $a_1(x) = c_A(x) = c_B(x) = b_1(x)$. If $m_A(x) = m_B(x)$ is quadratic, then there are two invariant factors. This means $a_2(x) = m_A(x) = m_B(x) = b_2(x)$, and $a_1(x) = c_A(x)/m_A(x) = c_B(x)/m_B(x) = b_1(x)$. If $m_A(x) = m_B(x)$ is linear, then there are three invariant factors, which are all linear. Since they must all divide one of them, they have to be the same. This means $(a_1(x))^3 = (a_2(x))^3 = (a_3(x))^3 = c_A(x) = c_B(x) = (b_3(x))^3 = (b_2(x))^3 = (b_1(x))^3$. Since in all three cases, we can conclude A and B have the same invariant factors, we know they are similar.

$$QED$$

(c) We want to give an explicit counterexample to the statement of (b) for 4×4 matrices. Suppose $A, B \in M_{4\times 4}(F)$. Suppose A has the invariant factors x^2, x^2, and B has the invariant factors x, x, x^2. Then clearly, A and B have the same characteristics polynomials and minimal polynomials, but the matrices are

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which are clearly not similar.