Week 8 Final, Math B504 **Name:** *James Wang*
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Problem 1:

Suppose that V is an n-dimensional vector space over the field F where $\mathrm{char}(F) \neq 2$. We want to show the following isomorphisms of F-modules (vector spaces).

(a) $V \otimes_F V \cong M_n(F)$

Let $\mathcal{B}_1 = \{v_i\}$ be a basis of V, then the set $\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ is a basis of $V \otimes_F V$. Let $\mathcal{B}_2 = \{E_{ij}\}$ be the standard ordered basis of $M_n(F)$, which is the set of elementary matrices (1 in the ij^{th} entry and 0 elsewhere). Then the map $\varphi : v_i \otimes v_j \mapsto E_{ij}$ extends linearly to an isomorphism. Alternatively, we can easily see that $\dim_F(V \otimes_F V) = \dim_F(M_n(F)) = n^2$.

(b) $S^2(V) \cong Sym_n(F)$

Take arbitrary $M \in Sym_n(F) \subseteq M_n(F)$, then we have $M = M^T$, which means $M_{ij} = M_{ji}$ for each entry where $i \neq j$. Given the standard ordered basis we chose in the codomain and the map φ we defined in part (a), we know the entry at M_{ij} is completely determined by the elementary matrix E_{ij} (a scalar multiple of it), and by no other basis elements. This means $M_{ij} = M_{ji}$ implies that $v_i \otimes v_j = v_j \otimes v_i$ in the domain, which belongs to $S^2(V)$. Alternatively, we can see that $\dim_F(S^2(V)) = \binom{2+n-1}{n-1} = \binom{n+1}{n-1} = \binom{n+1}{2} = \frac{n(n+1)}{2} = \dim_F(Sym_n(F))$.

(c) $\bigwedge^2(V) \cong Skew_n(F)$

Take arbitrary $M \in Skew_n(F) \subseteq M_n(F)$, then we have $M = -M^T$, which means $M_{ij} = -M_{ji}$ for each entry where $i \neq j$. Given the standard ordered basis we chose in the codomain and the map φ we defined in part (a), we know the entry at M_{ij} is completely determined by the elementary matrix E_{ij} (a scalar multiple of it), and by no other basis elements. This means $M_{ij} = -M_{ji}$ implies that $v_i \otimes v_j = -(v_j \otimes v_i)$ in the domain, which belongs to $\bigwedge^2(V)$. Alternatively, we can see that $\dim_F(\bigwedge^2(V)) = \binom{n}{2} = \frac{n(n-1)}{2} = \dim_F(Skew_n(F))$.

QED

Problem 2:

In $S = R[x_1, \dots, x_n]$, we want to prove an ideal is a graded ideal if and only if it is generated by homogeneous polynomials.

($\Longrightarrow$):

Suppose an ideal I in the polynomial ring is a graded ideal. By definition, this means

$$I = \bigoplus_{k=0}^{\infty} (I \cap S_k).$$

Clearly, I is generated by the set $\{I \cap S_k\}$. Since each S_k is a homogeneous component, and each $I \cap S_k$ is a subspace of S_k, we can conclude I is generated by homogeneous polynomials.

($\Longleftarrow$):

Suppose an ideal I in the polynomial ring is generated by homogeneous polynomials, that $I = (f_1, f_2, \dots f_n)$ where each f_i is homogeneous. Take arbitrary $g \in I$, we can express as

$$g = h_1 f_1 + h_2 f_2 + \cdots h_n f_n.$$

Since each $h_i \in S$, we can write it as

$$h_i = h_{i,0} + h_{i,1} + \cdots h_{i,m},$$

where each $h_{i,j}$ is the homogeneous component of S of degree j.

We can then expand the expression of g with each h_i, and group together homogeneous components of the same degree, which gives us

$$g = \sum \Big(\sum h_{i,j} f_i \Big).$$

This means each summand of g can be expressed as a linear combination of f_i, which are elements already in I. Hence each summand is also in I. By a remark on the bottom of page 443, we can conclude the ideal I is graded.

$$QED$$

Problem 3:

Consider the graded ring $S = \mathbb{Q}[x, y]$ and the ideal $I = (x^4, y^4)$. We want to give a basis for each of the following $\mathbb{Q}$-vector spaces.

(a) S_5

Since $S = \bigoplus_{k=0}^{\infty} S_k$, where S_k contains homogeneous elements of degree k, we know $\mathcal{B}_1 = \{x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\}$ is a basis for S_5 because these are all the monic monomials of degree 5 in S.

(b) I_5

We know by definition that $I_5 = I \cap S_5$, which means I_5 is a subgroup of S_5. If $\mathcal{B}_2$ is a basis of I_5, then it must be that $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Recall that $I = (x^4, y^4)$, so we know elements of $\mathcal{B}_2$ must be divisible by either x^4 or y^4. This gives us $\mathcal{B}_2 = \{x^5, x^4y, xy^4, y^5\}$.

(c) $(S/I)_5$

We know S/I is naturally a graded ring whose homogeneous component $(S/I)_k$ is isomorphic to S_k/I_k. If $\mathcal{B}_3$ is a basis of $(S/I)_5 \cong S_5/I_5$, then its elements are those in $\mathcal{B}_1$ that do not get killed after quotienting by elements in $\mathcal{B}_2$. Hence $\mathcal{B}_3 = \{x^3y^2, x^2y^3\}$.

QED

Problem 4:

Let F be any field of characteristic $\text{char}(F) \neq 2$ and let V be any vector space over F. We want to show that $V \otimes_F V = S^2(V) \oplus \bigwedge^2(V)$. In other words, every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.

By the isomorphisms from Problem 1, we can see $S^2(V)$ and $\bigwedge^2(V)$ as subspaces of $M_2(F)$, hence also elements of $V \otimes_F V$. Specifically, elements of $S^2(V)$ have the relation $v_i \otimes v_j = v_j \otimes v_i$, and elements of $\bigwedge^2(V)$ have the relation $v_i \otimes v_j = -(v_j \otimes v_i)$.

Take an arbitrary 2-tensor $v_i \otimes v_j \in S^2(V) \cap \bigwedge^2(V)$, then it must be that $v_j \otimes v_i = -(v_j \otimes v_i)$. Clearly, the only possibility for $v_i \otimes v_j$ is the trivial element, which means $S^2(V)$ and $\bigwedge^2(V)$ intersect trivially.

We finally claim that $\dim_F(V \otimes_F V) = \dim_F(S^2(V) \oplus \bigwedge^2(V))$. We already know that $\dim_F(V \otimes_F V) = n^2$. By the counting formulas provided, we know $\dim_F(S^2(V)) = \binom{2+n-1}{n-1} = \binom{n+1}{n-1} = \binom{n+1}{2}$, and $\dim_F(\bigwedge^2(V)) = \binom{n}{2}$. We can use a combinatorial argument to show

$$\binom{n+1}{2} + \binom{n}{2} = n^2.$$

Hence we have proved that $V \otimes_F V = S^2(V) \oplus \bigwedge^2(V)$.

QED