Nominally Robust Model Predictive Control With State Constraints

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***Abstract*—In this paper, we present robust stability results for constrained discrete-time nonlinear systems using a finite-horizon model predictive control (MPC) algorithm for which we do not require the terminal cost to have any particular properties. We introduce a definition that attempts to characterize the robustness properties of the MPC optimization problem. We assume the systems under consideration satisfy this definition (for which we give sufficient conditions) and make two further assumptions. These are that the value function is bounded by a $\mathcal{K}_\infty$ function of a state measure (related to the distance from the state to some target set) and that this measure is detectable from the stage cost used in the MPC algorithm. We show that these assumptions lead to stability that is robust to sufficiently small disturbances. While in general the results are semiglobal and practical, when the detectability and upper bound assumptions are satisfied with linear $\mathcal{K}_\infty$ functions, the stability and robustness are either semiglobal or global (with respect to the feasible set). We discuss algorithms employing terminal inequality constraints and also provide a specific example of an algorithm that employs a terminal equality constraint.**

***Index Terms*—Discrete-time systems, nonlinear model predictive control (MPC), robustness, receding horizon control.**

I. Introduction

GREAT progress has been made regarding the stabilization of state and input-constrained nonlinear systems with model predictive control (MPC), while progress on the robustness has been less successful; see [1] and [2] for discussions. In [3], the authors have given conditions when MPC algorithms generate a closed-loop systems with zero robustness (that is, any size of disturbance destabilizes the system). One step removed from this type of result are those that consider nominal robustness, or what the authors of [1] call inherent robustness. These results ask how large a disturbance a closed-loop system, designed ignoring uncertainty, can tolerate and still maintain stability. Such results are the focus of this paper; previous results have been given in [4]–[7], and more recently in [8]. We do not discuss here other types of results that are classified in [1] as open loop (in which the design uses all possible realizations of the uncertainty; see [9] and [10]) or feedback (in which design compensates for uncertainty, see [11] and [12]).

In [5], the authors show that robust stability in the presence of decaying perturbations is achievable using a terminal equality constraint and assuming that the MPC control law is locally Lipschitz. In [4], the authors show that robust stability in the presence of gain and additive disturbances is achievable again with a terminal equality constraint but instead assuming that the value function is twice continuously differentiable. The authors of [8] use this idea of conservative sets to show that under appropriate assumptions, an MPC algorithm that ignores uncertainty achieves a type of input-to-state stability with respect to additive disturbances and hence nominal robustness for constrained discrete-time nonlinear (locally Lipschitz) systems. Their method employs a terminal constraint set that is a sublevel set of a control Lyapunov function terminal cost (as is typically done; see [1]) but adds a requirement for the constraint sets at each time step. Imposing a sequence of nested constraint sets and assuming that the incremental cost is Lipschitz continuous, they show that the value function is an ISS-Lyapunov function for sufficiently small disturbances.

In [13], the authors state that a terminal cost need not be a local control Lyapunov function to ensure closed-loop asymptotic stability of the origin of an *unconstrained* discrete-time nonlinear system. The purpose of this paper is to use the ideas presented in [13] to generalize the result in [8] to include cases when a control Lyapunov function terminal cost is not employed and still ensure robust stability of a constrained discrete-time nonlinear system for sufficiently small disturbances. Our results assert semiglobal practical robust stability in general, but when stricter assumptions are met, we can assert semiglobal and global (that is, the basin of attraction is the entire feasible set) results.

Our results do not require any continuity assumptions on the value function or the MPC control law. In fact, as shown in [3], when either the feedback control law or the value function is continuous, the robustness we seek in this paper is automatically guaranteed. Another result of [3] is that linear systems with convex constraints also exhibit such robustness (since the value function in this case is continuous); therefore, we necessarily focus on nonlinear systems here. Our results also do not require any terminal constraints to be imposed for the purposes of stability. Still, our formulation applies to cases where such constraints are present, perhaps being required by the system (for example, hard state constraints) or desirable from other considerations. In particular, we show that, in contrast to [8], if the terminal constraint set is a sublevel set of *some* local control

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Lyapunov function, that is, it does not need to be related to the terminal cost, the stability induced by our MPC algorithm is robust provided the horizon is sufficiently long. We demonstrate our results for the case of terminal equality constraints using a system that is controllable to the origin in two steps. We impose a terminal equality constraint and choose an identically zero terminal cost. We show that although, for a small horizon, the closed-loop system exhibits no robustness to additive disturbances, when the horizon is increased according to the prescription of this paper, the closed-loop system becomes robust to sufficiently small disturbances. We note that in this paper we are primarily concerned with robust stability under general assumptions as opposed to performance. The horizon lengths required can be very large in general, and the disturbance bounds are conservative.

A. Preliminaries

For any $c \in \mathbb{R}$, we use the notation $\mathbb{R}_{\geq c}(\mathbb{R}_{>c})$ to refer to the subset of real numbers $\{s \in \mathbb{R} \,|\, s \geq c\,(s > c)\}$. The corresponding notation applies to the set of integers $\mathbb{Z}$. We denote the closed unit ball in $\mathbb{R}^n$ by $\mathcal{B} := \{s \in \mathbb{R}^n \,|\, |s| \leq 1\}$. We define $\mathrm{Id}(s) := s$ for $s \in \mathbb{R}_{\geq 0}$. Given a sequence $\mathbf{d} = \{d_0, d_1, \ldots\}$, the supremum norm is denoted $\|\mathbf{d}\| := \sup\{|d_k| \,|\, k \geq 0\}$($\|\mathbf{e}\|$ will be used similarly).

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{G}(\alpha \in \mathcal{G})$ if it is continuous, nondecreasing, and zero at zero; to belong to class $\mathcal{K}(\alpha \in \mathcal{K})$ if $\alpha \in \mathcal{G}$ and is strictly increasing; and to belong to class $\mathcal{K}_\infty(\alpha \in \mathcal{K}_\infty)$ if $\alpha \in \mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}(\beta \in \mathcal{KL})$ if, for each fixed $k, \beta(\cdot, k) \in \mathcal{K}$ and for each fixed s, $\beta(s, \cdot)$ tends to zero at infinity.

Instead of a norm as a state measure, we use a continuous, positive definite function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. This function is a design choice. The general control objective will be to drive the state towards the set $\{x \,|\, \sigma(x) = 0\}$. If we are given a set $\mathcal{A}$ as a desired attractor, we measure the distance to a set $\mathcal{A} \subset \mathbb{R}^n$ using

$$|x|_{\mathcal{A}} := \inf_{s \in \mathcal{A}} |x - s|$$

where $|\cdot|$ can be any norm. This allows us to study regulation to sets more general than a point. In order to assert stability results on $\mathcal{A}$, we require σ to be a *proper indicator function for* $\mathcal{A}$, that is, that there exist $\underline{\alpha}_\sigma, \bar{\alpha}_\sigma \in \mathcal{K}_\infty$ such that $\underline{\alpha}_\sigma(|x|_{\mathcal{A}}) \leq \sigma(x) \leq \bar{\alpha}_\sigma(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$. However, readers distracted by this generality can take $\mathcal{A}$ as the origin, which implies that $|x|_{\mathcal{A}} = |x|$, and σ as a norm.

B. System Description and Problem Formulation

We consider a *system* of the form $x(k+1) = f(x(k), u(k))$ or, more succinctly

$$x^+ = f(x, u) \tag{1}$$

where the *state* $x \in \mathbb{R}^n$ and the *control input* $u \in \mathcal{U} \subseteq \mathbb{R}^m$. We consider f nonlinear and continuous in both of its arguments. We assume that $\mathcal{U}$ is closed and that for any $x \in \mathcal{A}$, there exists $u \in \mathcal{U}$ such that $f(x, u) \in \mathcal{A}$. We denote by $\mathbf{u} \in \mathbb{R}^{m \times N}$ a control input sequence enumerated as $\{u_0, u_1, \ldots, u_{N-1}\}$. By $[\mathbf{u}]_i^j, 0 \leq i \leq j \leq N-1$, we mean the truncated $\mathbf{u}$ as $\{u_i, u_{i+1}, \ldots, u_j\}$. The *solution* of the system (1) k steps into the future, starting from an initial state x and under the influence of a control input sequence $\mathbf{u}$, will be denoted $\phi(k, x, \mathbf{u})$, with $\phi(0, x, \mathbf{u}) = x$. When $\mathbf{u} \in \mathbb{R}^{m \times N}$, then $\phi(k, x, \mathbf{u})$ is defined for $k \in \{0, 1, \ldots, N\}$.

We use the *cost function*

$$J_N(x, \mathbf{u}) := g(\phi(N, x, \mathbf{u})) + \sum_{k=0}^{N-1} \ell(\phi(k, x, \mathbf{u}), u_k) \tag{2}$$

parameterized by the *horizon length* $N \in \mathbb{Z}_{\geq 1}$ and constructed from the *terminal cost* $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and the *stage cost* $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. We consider the constrained optimization

$$V_N(x) := \inf_{\mathbf{u}} J_N(x, \mathbf{u}) \tag{3}$$

subject to $u_k \in \mathcal{U}$ for $k \in \{0, 1, \ldots, N-1\}$ and $\phi(k, x, \mathbf{u}) \in \mathcal{X}_{k,N}$ for $k \in \{0, 1, \ldots, N\}$, where $V_N(x)$ is the *value function* and the state constraint sets $\mathcal{X}_{k,N} \subseteq \mathbb{R}^n$ for all $k \in \{0, 1, \ldots, N\}$ are closed and contain the set $\mathcal{A}$ in their interior. This formulation includes optimizations without state constraints when $\mathcal{X}_{k,N} = \mathbb{R}^n$ for all $k \in \{0, 1, \ldots, N\}$, optimization with hard constraints when $\mathcal{X}_{k,N} = \mathcal{X}$ for all $k \in \{0, 1, \ldots, N\}$, optimization with terminal set constraint when $\mathcal{X}_{N,N} \subset \mathbb{R}^n$, optimization with terminal equality constraint when $\mathcal{X}_{N,N} = \{s\} \subset \mathbb{R}^n$ (usually $s = 0$), and any meaningful combination of these. A control input sequence is said to be *admissible* if it satisfies the control constraints and the corresponding solution of (1) satisfies the state constraints. For a given horizon length N, we denote the set of points for which an admissible control input sequence exists by $\mathcal{F}_N$. We assume all the control input sequences considered below are admissible.

For a given initial state x, when the infimum of (3) is achieved by an admissible control input sequence $\mathbf{u}(x)$, the MPC *feedback law* $\kappa_N : \mathbb{R}^n \to \mathcal{U}$ is a function that returns the first element of the control input sequence given the current state, that is, $\kappa_N(x) = u_0(x)$, where $J_N(x, \mathbf{u}(x)) = V_N(x)$.

For our robustness results, we refer to the closed-loop system

$$x^+ = f(x, \kappa_N(x + e)) + d \tag{4}$$

where κ_N is an MPC feedback law. Given $\delta > 0$, we denote by ϕ_δ any solution of (4) starting at x for which $\max\{\|\mathbf{e}\|, \|\mathbf{d}\|\} \leq \delta$. We denote the set of such solutions by $\mathcal{S}_\delta(x)$. Also, $\mathrm{int}(\mathcal{R})$ denotes the interior of the set $\mathcal{R}$.

Definition 1: Let $\mathcal{A}$ be compact and let $\mathcal{R}$ contain a neighborhood of $\mathcal{A}$. For (4), the set $\mathcal{A}$ is said to be *semiglobally practically* (in the parameter N) *robustly asymptotically stable on* $\mathrm{int}(\mathcal{R})$ if there exists $\beta \in \mathcal{KL}$, for each compact set $\mathcal{C} \subset \mathrm{int}(\mathcal{R})$ and $\varepsilon > 0$ there exists $N^* \in \mathbb{Z}_{>0}$, and for each $N \geq N^*$ there exists $\delta > 0$ (also depending on $\mathcal{C}$ and ε) such that for each $x \in \mathcal{C}$ and $\phi_\delta \in \mathcal{S}_\delta(x)$, the solution satisfies $|\phi_\delta(k, x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, k) + \varepsilon$ for all $k \geq 0$. ■

Definition 2: Let $\mathcal{A}$ be compact and let $\mathcal{R}$ contain a neighborhood of $\mathcal{A}$. For (4), the set $\mathcal{A}$ is said to be semiglobally (in the parameter N) robustly asymptotically stable on $\mathrm{int}(\mathcal{R})$ if there exists $\beta \in \mathcal{KL}$, for each compact set $\mathcal{C} \subset \mathrm{int}(\mathcal{R})$ there exists $N^* \in \mathbb{Z}_{>0}$, and for each $\varepsilon > 0$ and $N \geq N^*$ there exists $\delta > 0$ (also depending on $\mathcal{C}$) such that for each $x \in \mathcal{C}$ and

$\phi_\delta \in \mathcal{S}_\delta(x)$, the solution satisfies $|\phi_\delta(k,x)|_{\mathcal{A}} \le \beta(|x|_{\mathcal{A}},k)+\varepsilon$ for all $k \ge 0$. ■

Definition 3: Let $\mathcal{A}$ be compact and let $\mathcal{R}$ contain a neighborhood of $\mathcal{A}$. For (4) with a given N, the set $\mathcal{A}$ is said to be robustly asymptotically stable on $\mathrm{int}(\mathcal{R})$ if there exists $\beta \in \mathcal{KL}$ and for each compact set $\mathcal{C} \subset \mathrm{int}(\mathcal{R})$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in \mathcal{C}$ and $\phi_\delta \in \mathcal{S}_\delta(x)$, the solution satisfies $|\phi_\delta(k,x)|_{\mathcal{A}} \le \beta(|x|_{\mathcal{A}},k)+\varepsilon$ for all $k \ge 0$. ■

Remark 1: Note the subtle differences of the previous three definitions. The set $\mathcal{C}$ is the set of initial points of trajectories of interest, and ε represents the desired target set of these trajectories at infinite time. The first definition indicates that the minimum required horizon length N^* for trajectories starting in $\mathcal{C}$ to end up in the target set given by ε is dependent on both the initial and target sets. Note that the "practical" part of the definition is that N^* depends on the target set ε; the ε in each definition is due to the fact that there are (nonvanishing) disturbances present. The second definition indicates that N^* is dependent on the initial set but independent of the target set. In these definitions, the bound δ on the disturbances depends on the initial and targets sets as well as the horizon $N \ge N^*$ chosen. Finally, the third definition is for a particular N and the basin of attraction is all of $\mathrm{int}(\mathcal{R})$. Here the bound on disturbances does not depend on the horizon length. Clearly, if the system is robustly asymptotically stable on $\mathrm{int}(\mathcal{R})$ for a given N, then $\mathcal{R} \subseteq \mathcal{F}_N$. ■

We now give the main assumptions of this paper. We call them standing assumptions to indicate that they are assumed in all subsequent results.

Standing Assumption 1: The functions g and ℓ are continuous. ■

Standing Assumption 2: Given a horizon length N and $x \in \mathcal{F}_N$, there exists an admissible control input sequence $\mathbf{u}(x)$ such that $J_N(x,\mathbf{u}(x))$ is finite. Furthermore, for each compact set $\mathcal{C} \subset \mathcal{F}_N$, horizon length N, admissible control input sequence $\mathbf{u}(x)$, and positive real constant η, there exists $\mu > 0$ such that $x \in \mathcal{C}$ and $J_N(x,\mathbf{u}(x)) \le \eta$ imply that $\sup_{i\in\{0,1,\ldots,N-1\}} |u_i(x)| \le \mu$. Finally, $\mathcal{F}_{N+1} \supseteq \mathcal{F}_N$ for all N. ■

Remark 2: The requirement that $\mathcal{F}_{N+1} \supseteq \mathcal{F}_N$ for all N constrains the system function f as well as the choice of constraint sets. It requires that the feasible region not get smaller for longer horizon lengths. ■

Standing Assumptions 1 and 2 are sufficient to guarantee the existence of an optimal solution $\mathbf{u}^*(x)$ of (3) such that $J_N(x,\mathbf{u}^*(x)) = V_N(x)$; see [14].

For generality, we do not require ℓ to be positive definite in the state; instead, we employ some measure of the state σ and require it and ℓ to satisfy the following detectability definition.

Definition 4: Consider (1) and functions $\sigma : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\ge 0}$. The function σ is said to be *detectable from ℓ with respect to* $(\bar{\alpha}_W, \alpha_W, \gamma_W)$ if $\alpha_W, \gamma_W \in \mathcal{K}_\infty, \bar{\alpha}_W \in \mathcal{G}$, and there exists a continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that

$$W(x) \le \bar{\alpha}_W(\sigma(x))$$
$$W(f(x,u)) - W(x) \le -\alpha_W(\sigma(x)) + \gamma_W(\ell(x,u))$$

for all $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$. ■

Standing Assumption 3: For system (1) and cost function (2), σ is detectable from ℓ with respect to some $(\bar{\alpha}_W, \alpha_W, \gamma_W)$. ■

Remark 3: Standing Assumption 3 is greatly simplified in the case of a positive definite stage cost. Specifically, if there exists $\underline{\alpha}_\ell \in \mathcal{K}_\infty$ such that $\ell(x,u) \ge \underline{\alpha}_\ell(\sigma(x))$ for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$, then we can choose $W(x) \equiv 0$ to show that σ is detectable from ℓ with respect to $(0 \cdot \mathrm{Id}, \alpha_W, \gamma_W)$ for any pair (α_W, γ_W) satisfying $\gamma_W^{-1} \circ \alpha_W(s) \le \underline{\alpha}_\ell(s)$ for all $s \in \mathbb{R}_{\ge 0}$. One such pair is $(\alpha_W, \gamma_W) = (\underline{\alpha}_\ell, \mathrm{Id})$. ■

We make an assumption that the value function can be upper bounded by a $\mathcal{K}_\infty$ function of $\sigma(x)$, where σ is from Standing Assumption 3; this assumption is related to the controllability of the system to the set $\{\sigma(x) = 0\}$ (see [13]).

Standing Assumption 4: There exist $\bar{N} \ge 1$ and $\bar{\alpha} \in \mathcal{K}_\infty$ such that, for all $N \ge \bar{N}$, Standing Assumption 2 holds and

$$V_N(x) \le \bar{\alpha}(\sigma(x))$$

for all $x \in \mathcal{F}_N$. ■

For analysis purposes, we introduce supplementary value functions that capture the tail of a full horizon value function. For $L \in \{1,2,\ldots,N\}$, define the constrained optimization

$$V_{N,L}(x) := \inf_{\mathbf{u}} J_L(x,\mathbf{u}) \tag{5}$$

subject to $u_k \in \mathcal{U}$ for $k \in \{0,1,\ldots,L-1\}$ and $\phi(k,x,\mathbf{u}) \in \mathcal{X}_{N-L+k,N}$ for $k \in \{0,1,\ldots,L\}$.

Remark 4: When $\mathcal{X}_{k,N} = \mathcal{X}$ for all $k, V_{N,L}(x) = V_L(x)$. Note also that $V_{N,N}(x) = V_N(x)$. ■

II. Main Results

Before stating our main results, we introduce a definition that characterizes the robustness and feasibility of an MPC optimization problem subject to disturbances. The definition treats the horizon as being made up of two pieces. The first piece measures the robustness of the optimal control sequence in relation to the constraint sets. It is equal to the number of steps for which a trajectory perturbed at $k=1$, under the influence of the truncated optimal control sequence $\{u_1^*(x), u_2^*(x), \ldots, u_{N-1}^*(x)\}$, remains within the constraints of the previous time step (that is, $\mathcal{X}_{k-1,N}$). The second piece is a measure of the feasibility from the end of this first piece to the terminal step using *any* minimizing control sequence. The length of the second piece can be thought of as an inherent property of the system. For example, if the terminal constraint set is the origin, and all others are the whole space, this piece would give the number of steps required to drive the trajectory to the origin.

Definition 5: The MPC formulation (3) is said to be *robustly feasible* (RF) *with respect to* (N,M) if, for each compact subset $\mathcal{C} \subset \mathcal{F}_N$, there exists $\varepsilon > 0$ such that, for each $x \in \mathcal{C}$, each $z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}$, and each $k \in \{0,1,\ldots,N-M\}$, the points $\psi_k := \phi(k,z,[\mathbf{u}(x)]_1^{N-M})$ satisfy $\psi_k \in \mathcal{X}_{k,N}$ and ψ_{N-M} is feasible for $V_{N,M}$, where $\mathbf{u}(x)$ satisfies $V_N(x) = J_N(x,\mathbf{u}(x))$. ■

Remark 5: If an MPC formulation is RF with respect to (i,j), then it is also RF with respect to $(i,\tilde{j})$ for all $\tilde{j} \in \{j, j+1,\ldots,i\}$. ■

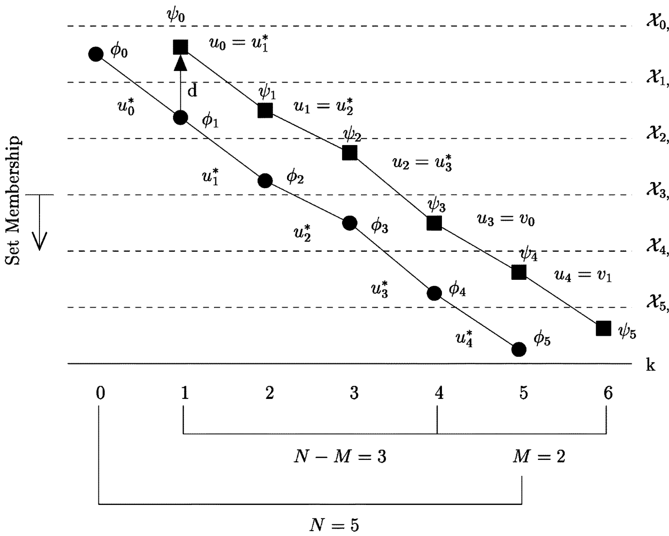

Fig. 1. The system above is robustly feasible with respect to (5,2). The optimizing control input sequence is $\mathbf{u}^*(x)$, $\psi_0 = f(x, \kappa_5(x)) + d$, and the control sequence $\mathbf{v} = \{v_0, v_1\}$ comes from $V_{5,2}(\psi_3) = J_2(\psi_3, \mathbf{v})$.

See Fig. 1 for an illustration of the definition. In Section III, we show that one way to ensure that a system satisfies the requirements of the definition is that the constraint sets be nested in a certain way depending on the size of the disturbance. This is related to the conditions imposed in [8].

Standing Assumption 5: There exists $M \in \{1, 2, \ldots, \bar{N}\}$ (where $\bar{N}$ is from Standing Assumption 4) such that, for all $N \geq \bar{N}$, the MPC formulation (3) is robustly feasible with respect to (N, M). ■

In the following results, we make one of the following two assumptions (we remove the standing designation to indicate that these assumptions must be explicitly invoked).

Assumption 6: In Standing Assumption 5, $M > 1$, and there exists $\bar{\alpha} \in \mathcal{K}_\infty$ such that

$$V_{N,L}(x) \leq \bar{\alpha}(\sigma(x)) \tag{6}$$

for all $N \geq M$ and all $L \in \{M, M+1, \ldots, N\}$, provided that x is feasible for $V_{N,L}$. ■

When $M = 1$ in Standing Assumption 5, we require the terminal cost to act like a local control Lyapunov function, as stated in the next assumption.

Assumption 7: In Standing Assumption 5, $M = 1$ and for any horizon $N \geq \bar{N}$, the terminal cost can be decomposed as $g(x) = g_N(x) = \Gamma(N) \cdot G(x)$, where the function $\Gamma : \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 1}$ is nondecreasing and unbounded. Furthermore, there exist functions $(\underline{\alpha}_G, \bar{\alpha}_\ell) \in \mathcal{K}_\infty$ such that $\underline{\alpha}_G(\sigma(x)) \leq G(x)$ and, for any $x \in \mathcal{X}_{N-1,N}$, there exists $u \in \mathbb{R}^m$ such that $f(x,u) \in \mathcal{X}_{N,N}$, $G(f(x,u)) - G(x) \leq 0$, and $\ell(x,u) \leq \bar{\alpha}_\ell(\sigma(x))$. ■

We now list our main results. We assert results for two different but related systems. The first set of results is for a closed-loop system with an additive disturbance attenuated by a strictly positive function $H_N : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ parameterized by the horizon length

$$x^+ \in f(x, \kappa_N(x)) + H_N(x, N-M) \cdot d \cdot \mathcal{B} \tag{7}$$

where $d \in [0,1]$; the solution of (7) will be denoted by $\phi_{H_N}(k,x)$. The function H_N acts to cancel the effects of the state and horizon on the disturbance and also causes the effective disturbance on the system to be small when the state is near the boundary of $\mathcal{F}_N$. The results for (7) are limited in the sense that they do not apply to a "natural" system due to the attenuation function H_N; therefore, we make an assumption on the attractor $\mathcal{A}$ in order to assert a second set of results. These results pertain to the system (4) and the robustness definitions defined above.

Theorem 1 states that there is a function Y_N that will serve as a (robust) Lyapunov function for the closed-loop system (7) provided that $N - M$ is sufficiently large. If we regard M as a property of the system, this implies that the total horizon N should be chosen sufficiently large.

Theorem 1: Consider (7), and let the functions $\underline{\alpha}_Y, \bar{\alpha}_Y, \alpha_Y \in \mathcal{K}_\infty$ and $\Upsilon_Y, \hat{\Upsilon}_Y \in \mathcal{KL}$ be constructed according to Algorithm 1 below. Then, for each $N \geq \bar{N}$, there exist $Y_N : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, H_N : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$, and $\gamma_Y \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_Y(\sigma(x)) \leq Y_N(x) \leq \bar{\alpha}_Y(\sigma(x)) \tag{8}$$

and, for all $x \in \mathcal{F}_N$, all $d \in [0,1]$, and all $z \in f(x, \kappa_N(x)) + H_N(x, N-M) \cdot d \cdot \mathcal{B}$, if Assumption 6 holds, then Y_N satisfies

$$\begin{aligned} &Y_N(z) - Y_N(x) \\ &\quad \leq -\alpha_Y(\sigma(x)) + \Upsilon_Y(\sigma(x), N-M+1) + \gamma_Y(d) \end{aligned} \tag{9}$$

and $z \in \mathcal{F}_N$; otherwise, if Assumption 7 holds, then Y_N satisfies

$$Y_N(z) - Y_N(x) \leq -\alpha_Y(\sigma(x)) + \hat{\Upsilon}_Y(\sigma(x), \Gamma(N)) + \gamma_Y(d) \tag{10}$$

and $z \in \mathcal{F}_N$. ■

Algorithm 1: Construction of Theorem 1 functions.

Given functions $\bar{\alpha}_W, \alpha_W$, and γ_W from Standing Assumption 3, $\bar{\alpha}$ from Standing Assumption 4, and $\underline{\alpha}_G$ and $\bar{\alpha}_\ell$ (if applicable) from Assumption 7, we have the following.

Step 1) If $\gamma_W(s) \leq s$ for all $s \in \mathbb{R}_{\geq 0}$, then select

$$\begin{aligned} \underline{\alpha}_Y(s) &:= \alpha_W(s) \\ \bar{\alpha}_Y(s) &:= \bar{\alpha}(s) + \bar{\alpha}_W(s) \\ \alpha_Y(s) &:= \alpha_W(s) \\ \Upsilon_V(s,t) &:= \bar{\alpha} \circ \alpha_W^{-1}\left(\frac{\bar{\alpha}(s) + \bar{\alpha}_W(s)}{\max(t,1)}\right) \\ \Upsilon_Y(s,t) &:= \Upsilon_V(s,t) \\ \hat{\Upsilon}_V(s,t) &:= \bar{\alpha}_\ell \circ \underline{\alpha}_G^{-1}\left(\frac{\bar{\alpha}(s)}{\max(t,1)}\right) \\ \hat{\Upsilon}_Y(s,t) &:= \hat{\Upsilon}_V(s,t) \end{aligned}$$

for $s, t \in \mathbb{R}_{\geq 0}$. Otherwise, define the following.

Step 2)

$$\begin{aligned}
\tilde{\alpha}_0(s) &:= \gamma_W(s) + \bar{\alpha}_W(\alpha_W^{-1}(2\gamma_W(s)))\\
\tilde{\alpha}_1(s) &:= \tilde{\alpha}_0^{-1}\left(\frac{1}{4}\alpha_W(s)\right)\cdot\frac{1}{8}\alpha_W(s)\\
q_W(s) &:= \frac{1}{2}\tilde{\alpha}_0^{-1}(s)\\
\rho_W(s) &:= \int_0^s q_W(t)dt\\
\tilde{\alpha}_2(s) &:= \rho_W(\bar{\alpha}_W(s)) + \gamma_W(\bar{\alpha}(s))\cdot\bar{\alpha}(s)\\
q_V(s) &:= 2\gamma_W(2s)\\
\rho_V(s) &:= \int_0^s q_V(t)\,dt
\end{aligned}$$

in order to define

$$\begin{aligned}
\underline{\alpha}_Y(s) &:= \min\left\{\rho_V\left(\gamma_W^{-1}(\alpha_W(s)/2)\right),\right.\\
&\qquad \left.\rho_W(\alpha_W(s)/2)\right\}\\
\bar{\alpha}_Y(s) &:= \rho_V(\bar{\alpha}(s)) + \rho_W(\bar{\alpha}_W(s))\\
\alpha_Y(s) &:= \tilde{\alpha}_1(s)\\
\Upsilon_V(s,t) &:= \bar{\alpha}\circ\tilde{\alpha}_1^{-1}\left(\frac{\tilde{\alpha}_2(s)}{\max(t,1)}\right)\\
\Upsilon_Y(s,t) &:= q_V(\bar{\alpha}(s) + \Upsilon_V(s,t))\cdot\Upsilon_V(s,t)\\
\hat{\Upsilon}_V(s,t) &:= \bar{\alpha}_\ell\circ\underline{\alpha}_G^{-1}\left(\frac{\bar{\alpha}(s)}{\max(t,1)}\right)\\
\hat{\Upsilon}_Y(s,t) &:= q_V(\bar{\alpha}(s) + \hat{\Upsilon}_V(s,t))\cdot\hat{\Upsilon}_V(s,t)
\end{aligned}$$

for $s,t\in\mathbb{R}_{\geq 0}$. ■

For the following corollaries, we need the following fact [15, Lemma B.1].

Fact 1: For any $\alpha\in\mathcal{K}_\infty$, there exists $\hat{\alpha}\in\mathcal{K}_\infty$ satisfying $\hat{\alpha}(s)\leq\alpha(s)$ for all $s\geq 0$ and $\mathrm{Id}-\hat{\alpha}\in\mathcal{K}$. ■

Corollary 1 relies on Theorem 1 to assert semiglobal practical stability and gives a bound on the disturbance (dependent on the desired region of attraction) for this stability.

Corollary 1: Assume that either Assumption 6 or 7 is satisfied. Let $\alpha_Y, \bar{\alpha}_Y$, and γ_Y come from Theorem 1 and let $\hat{\alpha}_1$ come from applying Fact 1 to $\alpha_1(s) := (1/2)\alpha_Y(\bar{\alpha}_Y^{-1}(s))$. Then there exist functions $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$; given $\Delta>\delta>0$, if $\|\mathbf{d}\|\leq\min\{1,\gamma_Y^{-1}((1/2)\hat{\alpha}_1(\bar{\alpha}_Y(\Delta)))\}$, there exists $N^*\geq\bar{N}$; and, for each $N\geq N^*$, there exists $H_N:\mathbb{R}^n\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{>0}$ such that for (7)

$$\sigma(\phi_{H_N}(k,x))\leq\beta(\sigma(x),k)+\max\{\delta,\gamma(\|\mathbf{d}\|)\} \tag{11}$$

for all $x\in\{s\,|\,\sigma(s)\leq\Delta\}\cap\mathcal{F}_N$ and all $k\geq 0$.

In addition, given $\mathcal{A}$, let σ be a proper indicator function for $\mathcal{A}$ with functions $\underline{\alpha}_\sigma,\bar{\alpha}_\sigma$. Then the set $\mathcal{A}$ is semiglobally practically asymptotically stable for (7); that is, there exist functions $\beta_\mathcal{A}\in\mathcal{KL}$ and $\gamma_\mathcal{A}\in\mathcal{K}_\infty$; given $\Delta_\mathcal{A}>\delta_\mathcal{A}>0$, if $\|\mathbf{d}\|\leq\min\{1,\gamma_Y^{-1}((1/2)\hat{\alpha}_1(\bar{\alpha}_Y(\bar{\alpha}_\sigma(\Delta_\mathcal{A}))))\}$, there exists $N^*\geq\bar{N}$; and, for each $N\geq N^*$, there exists $H_N:\mathbb{R}^n\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{>0}$ such that for (7)

$$|\phi_{H_N}(k,x)|_\mathcal{A}\leq\beta_\mathcal{A}(|x|_\mathcal{A},k)+\max\{\delta_\mathcal{A},\gamma_\mathcal{A}(\|\mathbf{d}\|)\} \tag{12}$$

for all $x\in\{s\,|\,|s|_\mathcal{A}\leq\Delta_\mathcal{A}\}\cap\mathcal{F}_N$ and all $k\geq 0$. ■

The "practical" part of Corollary 1 can be removed when the functions in Standing Assumptions 3 and 4 are linear near zero, as we assert in the following corollary.

Corollary 2: Let either Assumption 6 or 7 be satisfied. For some $\Delta_\ell>0$, suppose the functions of Standing Assumptions 3 and 4 satisfy $\bar{\alpha}_W(s)\leq\bar{a}_W\cdot s, \alpha_W(s)\leq a_W\cdot s, \gamma_W(s)\leq s$, and $\bar{\alpha}(s)\leq\bar{a}\cdot s$ for $s\leq\Delta_\ell$, where $\bar{a}_W\geq 0, a_W,\bar{a}>0$. Let $\alpha_Y,\bar{\alpha}_Y$, and γ_Y come from Theorem 1 and let $\hat{\alpha}_1$ come from applying Fact 1 for $\alpha_1(s):=(1/2)\alpha_Y(\bar{\alpha}_Y^{-1}(s))$. Then there exist functions $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$; given $\Delta>0$, if $\|\mathbf{d}\|\leq\min\{1,\gamma_Y^{-1}((1/2)\hat{\alpha}_1(\bar{\alpha}_Y(\Delta)))\}$, there exists $N^*\geq\bar{N}$; and for each $N\geq N^*$, there exists $H_N:\mathbb{R}^n\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{>0}$ such that for (7)

$$\sigma(\phi_{H_N}(k,x))\leq\beta(\sigma(x),k)+\gamma(\|\mathbf{d}\|) \tag{13}$$

for all $x\in\{s\,|\,\sigma(s)\leq\Delta\}\cap\mathcal{F}_N$ and all $k\geq 0$.

In addition, given $\mathcal{A}$, let σ be a proper indicator function for $\mathcal{A}$ with functions $\underline{\alpha}_\sigma,\bar{\alpha}_\sigma$. Then the set $\mathcal{A}$ is semiglobally asymptotically stable for (7); that is, there exist functions $\beta_\mathcal{A}\in\mathcal{KL}$ and $\gamma_\mathcal{A}\in\mathcal{K}_\infty$; given $\Delta_\mathcal{A}>0$, if $\|\mathbf{d}\|\leq\min\{1,\gamma_Y^{-1}((1/2)\hat{\alpha}_1(\bar{\alpha}_Y(\bar{\alpha}_\sigma(\Delta_\mathcal{A}))))\}$, there exists $N^*\geq\bar{N}$; and, for each $N\geq N^*$, there exists $H_N:\mathbb{R}^n\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{>0}$ such that for (7)

$$|\phi_{H_N}(k,x)|_\mathcal{A}\leq\beta_\mathcal{A}(|x|_\mathcal{A},k)+\gamma_\mathcal{A}(\|\mathbf{d}\|) \tag{14}$$

for all $x\in\{s\,|\,|s|_\mathcal{A}\leq\Delta_\mathcal{A}\}\cap\mathcal{F}_N$ and all $k\geq 0$. ■

When the functions in Standing Assumptions 3 and 4 can be taken as linear everywhere, we can assert a stability result for all of $\mathcal{F}_N$ as follows.

Corollary 3: Suppose Standing Assumptions 3 and 4 are satisfied with $\bar{\alpha}_W=\bar{a}_W\cdot\mathrm{Id}, \alpha_W=a_W\cdot\mathrm{Id}, \gamma_W=\mathrm{Id}$, and $\bar{\alpha}=\bar{a}\cdot\mathrm{Id}$, where $\bar{a}_W\geq 0, a_W>0$, and $\bar{a}>0$. Then there exist constants $K,\lambda,g\in\mathbb{R}_{>0}$ and, for any $N\geq\bar{N}$, there exists $H_N:\mathbb{R}^n\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{>0}$ such that if $\|\mathbf{d}\|\leq 1$ and either:

1) Assumption 6 holds and $N>M-1+(\bar{a}(\bar{a}+\bar{a}_W))/(a_W^2)$;
2) Assumption 7 holds and $\Gamma(N)>(\bar{a}_\ell\bar{a})/(\underline{a}_G a_W)$.

Then (7) satisfies

$$\sigma(\phi_{H_N}(k,x))\leq Ke^{-\lambda k}\sigma(x)+g\cdot\gamma_Y(\|\mathbf{d}\|) \tag{15}$$

for all $x\in\mathcal{F}_N$ and for all $k\geq 0$.

Moreover, given $\mathcal{A}$, if σ is a proper indicator function for $\mathcal{A}$, then the set $\mathcal{A}$ is asymptotically stable for (7) on $\mathcal{F}_N$; that is, there exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$ such that, for (7)

$$|\phi_{H_N}(k,x)|_\mathcal{A}\leq\beta(|x|_\mathcal{A},k)+\gamma(\|\mathbf{d}\|) \tag{16}$$

for all $x\in\mathcal{F}_N$ and for all $k\geq 0$. If $\sigma(\cdot)=|\cdot|_\mathcal{A}^p, p>0$, then there exist constants $\tilde{K},\tilde{\lambda}\in\mathbb{R}_{>0}$ and $\tilde{\gamma}\in\mathcal{K}_\infty$ such that for (7)

$$|\phi_{H_N}(k,x)|_\mathcal{A}\leq\tilde{K}e^{-\tilde{\lambda}k}|x|_\mathcal{A}+\tilde{\gamma}(\|\mathbf{d}\|) \tag{17}$$

for all $x\in\mathcal{F}_N$ and for all $k\geq 0$. ■

When $\mathcal{A}$ is compact and σ is a proper indicator for $\mathcal{A}$, we can use the above corollaries in turn to assert that (4), without an attenuation function, satisfies the corresponding robust stability definitions given above. For the following corollaries, we use the definition $\mathcal{F}_\infty := \bigcup_N \mathrm{int}(\mathcal{F}_N)$.

Corollary 4: Let the set $\mathcal{A}$ be compact, let σ be a proper indicator for $\mathcal{A}$, and let $\mathcal{F}_{\bar{N}}$ contain a neighborhood of $\mathcal{A}$. Let the conditions of Corollary 1 hold. Then for (4), the set $\mathcal{A}$ is semiglobally practically robustly asymptotically stable on $\mathcal{F}_{\infty}$. ■

Corollary 5: Let the set $\mathcal{A}$ be compact, let σ be a proper indicator for $\mathcal{A}$, and let $\mathcal{F}_{\bar{N}}$ contain a neighborhood of $\mathcal{A}$. Let the conditions of Corollary 2 hold. For (4), the set $\mathcal{A}$ is semiglobally robustly asymptotically stable on $\mathcal{F}_{\infty}$. ■

Corollary 6: Let the set $\mathcal{A}$ be compact, let σ be a proper indicator for $\mathcal{A}$, and let $\mathcal{F}_{\bar{N}}$ contain a neighborhood of $\mathcal{A}$. Let the conditions of Corollary 3 hold and let N^* be the smallest horizon length for which they hold. Then for each $N \geq N^*$, the set $\mathcal{A}$ is robustly asymptotically stable on $\mathcal{F}_N$. ■

III. Sufficient Conditions for Being Robustly Feasible

In this section, we discuss conditions sufficient to guarantee that an MPC formulation is robustly feasible. We focus on the sequence of sets $\mathcal{X}_{k,N}$ and suppose the following (see [8, Lemma 2]).

Supposition 1: Consider a horizon length $N \geq 1$ and $M \in \{1, 2, \ldots, N\}$. There exists $\varepsilon > 0$ such that

$$\mathcal{X}_{k+1,N} + \varepsilon\mathcal{B} \subseteq \mathcal{X}_{k,N} \tag{18}$$

for all $k \in \{0, 1, \ldots, N-M\}$, and for each $x \in \mathcal{X}_{N-M+1,N} + \varepsilon\mathcal{B} \subseteq \mathcal{X}_{N-M,N}$, there exists an admissible control input sequence $\mathbf{u}$ of length M such that

$$\phi(k, x, \mathbf{u}) \in \mathcal{X}_{k+N-M,N} \tag{19}$$

for all $k \in \{0, 1, \ldots, M\}$. ■

Remark 6: If $\mathcal{X}_{k,N} = \mathbb{R}^n$ for all $k \in \{0, 1, \ldots, N\}$, this holds vacuously. ■

Theorem 2: Consider a pair (N, M) that satisfies Supposition 1 with $\varepsilon_1 > 0$. Then the MPC formulation (3) with horizon N is RF with respect to (N, M). ■

Proof: Let $\mathbf{u}^*(x)$ satisfy $V_N(x) = J_N(x, \mathbf{u}^*(x))$. Then $\phi(k, x, \mathbf{u}^*(x)) \in \mathcal{X}_{k,N}$ for all $k \in \{0, 1, \ldots, N\}$ and hence $\phi(k, f(x, \kappa_N(x)), [\mathbf{u}^*(x)]_1^{N-1}) \in \mathcal{X}_{k+1,N}$ for all $k \in \{0, 1, \ldots, N-1\}$. The continuity of f implies that, given $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$ such that, for all $z \in f(x, \kappa_N(x)) + \varepsilon_2\mathcal{B}$, we have $\psi_k := \phi(k, z, [\mathbf{u}^*(x)]_1^{N-M}) \in \mathcal{X}_{k+1,N} + \varepsilon_1\mathcal{B} \subseteq \mathcal{X}_{k,N}$ for all $k \in \{0, 1, \ldots, N-M\}$. Finally, since $\psi_{N-M} \in \mathcal{X}_{N-M+1,N} + \varepsilon_1\mathcal{B} \subseteq \mathcal{X}_{N-M,N}$, there exists an admissible control input sequence $\mathbf{u}$ such that $\phi(k, \psi_{N-M}, \mathbf{u}) \in \mathcal{X}_{k+N-M,N}$ for all $k \in \{0, 1, \ldots, M\}$. Hence ψ_{N-M} is feasible for $V_{N,M}$.

IV. Robustness of Unconstrained MPC Using Terminal Constraints

Lemma 1: Suppose that the set $\mathcal{X}_f$ is reachable from $\mathbb{R}^n$ in M steps using admissible control values. Then if $\mathcal{X}_{k,N} = \mathbb{R}^n$ for all $k \in \{0, 1, \ldots, N-1\}$ and $\mathcal{X}_{N,N} = \mathcal{X}_f$, Supposition 1 holds for the pair (N, M) for any $N \geq M$. ■

Proof: Since $\mathcal{X}_{k,N} = \mathbb{R}^n$ for all $k \in \{0, 1, \ldots, N-1\}$ and $M \geq 1$, we trivially have that, for all $k \in \{0, 1, \ldots, N-M\}$, $\mathcal{X}_{k+1,N} + \varepsilon\mathcal{B} \subseteq \mathcal{X}_{k,N}$. Then, since $\mathcal{X}_{N-M,N} = \mathbb{R}^n$ and the set $\mathcal{X}_f$ is reachable from $\mathbb{R}^n$ in M steps, there exists an admissible control input sequence of length M such that $\phi(k, x, \mathbf{u}) \in \mathcal{X}_{k+N-M,N}$ for all $k \in \{0, 1, \ldots, M\}$. ■

Definition 6: Let $c > 0$ and let $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be continuous. The function $\mathcal{V}$ is said to be a *continuous control Lyapunov function (CLF) on the level* c if $L_{\mathcal{V}}(c) := \{s \in \mathbb{R}^n \,|\, \mathcal{V}(s) \leq c\}$ is compact and there exists a continuous, positive definite function $\ell : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and, for each $x \in L_{\mathcal{V}}(c)$, there exists $u(x)$ such that

$$\mathcal{V}(f(x, u(x))) \leq V(x) - \ell(x). \tag{20}$$

■

Lemma 2: Given $N \geq 2$ and a function $\mathcal{V}$ that is a continuous CLF on the level $\bar{c}$, Supposition 1 holds with $(N, 1)$ if $X_{k,N} = \mathbb{R}^n$ for $k \in \{0, 1, \ldots, N-1\}$ and $X_{N,N} = L_{\mathcal{V}}(c)$, where $c < \bar{c}$. ■

Proof: Let $M = 1$ and $N \geq 2$. Let $\varepsilon_2 := \min_{x \in \{s : c \leq V(s) \leq \bar{c}\}} \ell(x)$. Since $\mathcal{V}$ is continuous, we can select $\varepsilon_1 > 0$ such that, for all $x \in L_{\mathcal{V}}(c) + \varepsilon_1\mathcal{B}$, we have that $x \in L_{\mathcal{V}}(\bar{c})$ and $\mathcal{V}(x) \leq c + \varepsilon_2$. This implies that, for all $x \in X_{N,N} + \varepsilon_1\mathcal{B}$, there exists $u(x)$ such that $\mathcal{V}(f(x, u(x))) \leq \mathcal{V}(x) - \ell(x)$. Then if $\mathcal{V}(x) \leq c$, then $\mathcal{V}(f(x, u(x))) \leq c$ since ℓ is positive definite. If $c < \mathcal{V}(x) \leq \bar{c}$, then $\mathcal{V}(x) - \ell(x) \leq c + \varepsilon_2 - \varepsilon_2 \leq c$, and again $\mathcal{V}(f(x, u(x))) \leq c$. Hence $f(x, u(x)) \in \mathcal{X}_{N,N}$, that is, (19) is satisfied. Finally, we have (18) since $\mathbb{R}^n + \varepsilon\mathcal{B} \subseteq \mathbb{R}^n$. ■

Corollary 7: For $\tilde{N} \geq \max\{\bar{N}, 2\}$, if $X_{k,\tilde{N}} = \mathbb{R}^n$ for $k \in \{0, 1, \ldots, \tilde{N}-1\}$ and $X_{\tilde{N},\tilde{N}} = L_{\mathcal{V}}(c)$, where $\mathcal{V}$ is a continuous local CLF on the level $\bar{c} > c$, then there exist functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that, given $\Delta > \delta > 0$, if $\|\mathbf{d}\| \leq \min\{1, \gamma_Y^{-1}((1/2)\hat{\alpha}_1(\bar{\alpha}_Y(\Delta)))\}$, then there exists $N^* \geq \tilde{N}$ and, for each $N \geq N^*$, there exists $H_N : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ such that, for all $x \in \{s \,|\, \sigma(s) \leq \Delta\} \cap \mathcal{F}_N$, (7) satisfies

$$\sigma(\phi_{H_N}(k, x)) \leq \beta(\sigma(x), k) + \max\{\delta, \gamma(\|\mathbf{d}\|)\}$$

for all $k \geq 0$. ■

Proof: The proof is a consequence of Lemma 2, Theorem 2, and Corollary 1. ■

V. Examples

A. Origin as Terminal Constraint

In this section, we present a nominally asymptotically stable closed-loop system formed by an MPC feedback law that initially has zero robustness and show that increasing the horizon length makes the closed loop robust to sufficiently small disturbances. Consider the system (see [3])

$$x^+ = \begin{bmatrix} x_1(1-u) \\ |x|u \end{bmatrix} =: f(x, u) \tag{21}$$

where $u \in [0, 1] =: \mathcal{U}$ and $x := [x_1 \; x_2]^T \in \mathbb{R}^2 =: \mathcal{X}$. The possible values for x^+ are on the line segment between the points $(0, |x|)$ and $(x_1, 0)$. The global asymptotic stability, but nonrobustness, of the closed-loop system employing MPC (with horizon length two and $\mathcal{X}_{N,N} = \{0\}$) is shown in [3]; we show here that choosing a longer horizon renders the origin of the closed-loop system robust. We consider the optimization problem

$$V_N(x) = \inf_{\mathbf{u}} \sum_{k=0}^{N-1} |\phi(k,x,\mathbf{u})|$$

$$\text{subject to} \quad \begin{cases} u_k \in [0,1] & k \in \{0,1,\ldots,N-1\} \\ \phi(N,x,\mathbf{u}) = 0 \end{cases}.$$

Since any state can be driven to the origin in two steps, the MPC formulation (22) is RF with respect to (N,M) for all $N > M \geq 2$. Note that $\mathcal{F}_N = \mathcal{F}_2$ for all $N > 2$. Let $\sigma(x) := |x|, \bar{\alpha}_W(s) := 0, \alpha_W(s) := s$, and $\gamma_W(s) := s$; then σ is detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W)$ with $W(x) = 0$. Then we can take $Y_N = V_N$. The existence of a two-step control input sequence yields $|x| \leq V_N(x) \leq 2|x|$. Hence, $\bar{\alpha}_Y(s) = 2s$ in (8). This upper bound is tight, since for any $N \geq 2$, initial conditions starting on the x_1 axis cannot improve on this cost. This is because the smallest value that can be achieved after one step in this case is $(1/\sqrt{2})|x|$. Since the state must then be mapped to the x_2 axis before being mapped to the origin, this will incur larger cost than $2|x|$ for any $N > 2$. Also, using the upper bound, we can let $\gamma_Y(s) = 2s$ in Theorem 1; this means that the worst a disturbance can affect the decrease of the value function is $2\|\mathbf{d}\|$. For this simple example, we can also take the attenuation function $H_N \equiv 1$.

Now, if we choose $M = 2$, since the only constraint is on $\mathcal{X}_{N,N}$, we have that $V_{N,2} = V_{N,M+1} = \cdots = V_N$ and Assumption 6 holds. Let $N = 4$. Then from Corollary 3, (15) holds with $K = 2, \lambda = \ln(1.2)$, and $g = 6$ (see the proof of Corollary 3 for these calculations). Thus, by choosing $N = 4$ instead of $N = 2$, we render the origin of the closed-loop system robust. Fig. 2 shows a sample trajectory with a constant disturbance and the corresponding stage costs. Also plotted are the bounds on trajectories from Corollary 3. For $N = 4$, the bound is very loose. As shown in the proof of Corollary 3, longer horizon lengths can help achieve tighter bounds (that is, larger λ and smaller g), at the expense of computational burden.

B. State Constraints

Consider the system (due to Artstein and discussed in [3])

$$x_1^+ = \frac{-\left(x_1^2 + x_2^2\right)u + x_1}{1 + \left(x_1^2 + x_2^2\right)u^2 - 2x_1 u}$$
$$x_2^+ = \frac{x_2}{1 + \left(x_1^2 + x_2^2\right)u^2 - 2x_1 u}.$$

We wish to stabilize the origin with controls $u \in \mathcal{U} := [-1,1]$ while constraining the state to the set $\mathcal{X} := \{x \,|\, x_1 \leq c\}$. In [3], a standard MPC formulation (as described by [1]) applied to this example is shown to be nonrobust. This nonrobustness has nothing to do with the horizon length chosen. To make the closed loop robust, let $\mathcal{X}_{k,N} = \{x \,|\, x_1 \leq c_k\}$, where $c_0 = c, c_{k+1} = c_k - N^{-1}\nu$ with $\nu > 0$ sufficiently small. Then $\mathcal{F}_N = \mathcal{X}_{0,N}$ for all N. Since these sets are nested with margin $N^{-1}\nu$ and, for ν sufficiently small, for any $x \in \mathcal{X}_{N-1,N}$, there exists $u \in \mathcal{U}$ such that $x^+ \in \mathcal{X}_{N,N}$, we satisfy Supposition 1 and therefore the system is RF with respect to (N,M) for $M = 1$ and all $N \geq 1$. If we choose $\sigma(s) := |s|$ and choose any g and

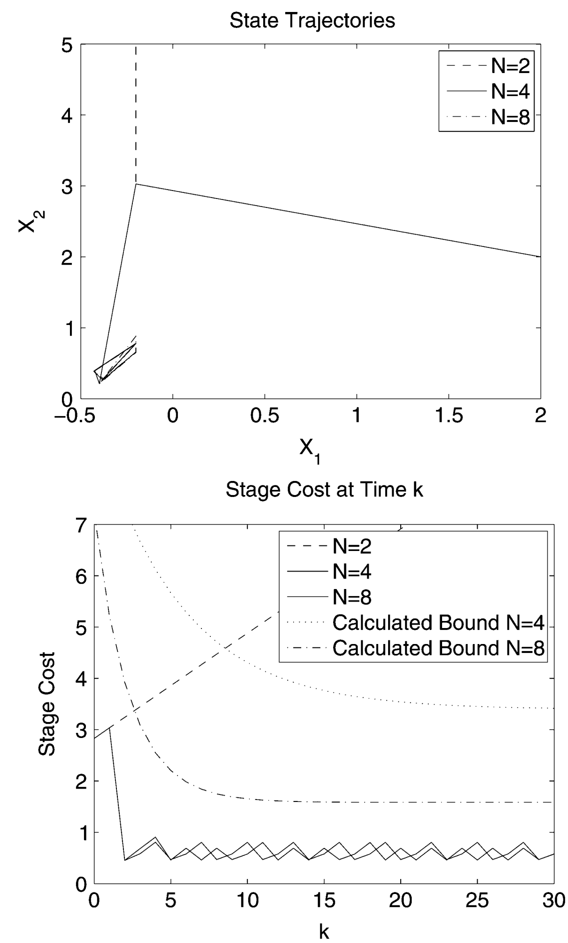

Fig. 2. Comparison of MPC-generated trajectories of system (21) for $N = 2, 4, 8$ under additive disturbance of $[-0.2, 0.2]^T$. The second plot shows the stage costs and trajectory bounds for $N = 4, 8$ calculated from Corollary 3.

ℓ to satisfy the Standing Assumptions, Corollary 4 asserts that the system is semiglobally practically robustly asymptotically stable on $\{x \,|\, x_1 < c\}$.

VI. Conclusion

We have presented constrained discrete-time nonlinear MPC results on the stability of general attractors that are robust to sufficiently small disturbances. Our formulation does not require any particular properties of the terminal cost. We assume the system satisfies certain detectability conditions, the value function can be upper bounded by a $\mathcal{K}_\infty$ function of the state, and the MPC formulation satisfies a definition that we introduce to characterize robustness. These assumptions yield semiglobal practical stability results; however, when our detectability and upper bound assumptions hold with linear functions, the results are semiglobal or cover the entire feasible region. We show that these results hold for MPC formulations that use either terminal inequality or terminal equality constraints and include an example that demonstrates the latter case.

Appendix I
Proof of Y_N Decrease

In order to prove Theorem 1, we first state and prove a theorem for the value function V_N.

Theorem 3: Given $x \in \mathcal{F}_N$, let $\bar{\varepsilon}_x > 0$ be generated from Definition 5 (RF) for $\mathcal{C} = \{x\}$. Then the value function satisfies

$$\ell(x, \kappa_N(x)) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)). \tag{23}$$

Furthermore, there exist $\Upsilon_V, \hat{\Upsilon}_V \in \mathcal{KL}$ and $\delta_N^V : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\delta_N^V(s_1, s_2, \cdot)$ continuous and zero at zero for any fixed (s_1, s_2), such that, for every nonnegative $\varepsilon \leq \bar{\varepsilon}_x$ and all $z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}$, if Assumption 6 holds, then V_N satisfies

$$\begin{aligned} V_N(z) - V_N(x) \leq& -\ell(x, \kappa_N(x)) \\ &+ \Upsilon_V(\sigma(x), N - M + 1) + \delta_N^V(x, N - M, \varepsilon) \end{aligned} \tag{24}$$

and $z \in \mathcal{F}_N$; otherwise, if Assumption 7 holds, then V_N satisfies

$$\begin{aligned} V_N(z) - V_N(x) \leq& -\ell(x, \kappa_N(x)) \\ &+ \hat{\Upsilon}_V(\sigma(x), \Gamma(N)) + \delta_N^V(x, N - 1, \varepsilon) \end{aligned} \tag{25}$$

and $z \in \mathcal{F}_N$. ■

Proof: The bounds in (23) come from the definition of V_N, the fact that ℓ is nonnegative, and Standing Assumption 4. Let $\mathbf{u}^*(x)$ be an optimal control input sequence such that $V_N(x) = J_N(x, \mathbf{u}^*(x))$ and consider $\varepsilon \leq \bar{\varepsilon}_x$. For $k \in \{0, 1, \ldots, N - M\}$ and $z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}$, define $\psi_k := \phi(k, z, [\mathbf{u}^*(x)]_1^{N-M})$ and $\phi_k := \phi(k, x, \mathbf{u}^*(x))$. By the definition of RF, $z \in \mathcal{F}_N, \psi_k \in \mathcal{X}_{k,N}$, and ψ_{N-M} are feasible for $V_{N,M}$. In order to simplify later equations, we define, for $L \in \{0, 1, \ldots, N - M\}$, the equation shown at the bottom of the page, where $\bar{\alpha}_\ell$ is from Assumption 7 and is taken as identically zero if the assumption does not hold. Note that the subscript N denotes the implicit dependence of the terms on the horizon length through the feedback function κ_N. We also define for $s_1, s_3 \in \mathbb{R}_{\geq 0}$ and $s_2 \in \{0, 1, \ldots, N - M\}$

$$\begin{aligned} \delta_N^{ab}(s_1, s_2, s_3) :=& \sup_{L \in [0, s_2]} \delta_N^a(s_1, L, s_3) \\ &+ \sup_{L \in [0, s_2]} \delta_N^b(s_1, L, s_3) \\ \delta_N^{acd}(s_1, s_2, s_3) :=& \delta_N^a(s_1, s_2, s_3) + \delta_N^c(s_1, s_2, s_3) \\ &+ \delta_N^d(s_1, s_2, s_3) \\ \delta_N^V(s_1, s_2, s_3) :=& \max\{0, \delta_N^{ab}(s_1, s_2, s_3), \delta_N^{acd}(s_1, s_2, s_3)\}. \end{aligned}$$

Note that $\delta_N^V(\cdot, \cdot, 0) = 0$. Since $\mathbf{u}^*(x)$ is a locally bounded function of the state, δ_N^V is bounded on compact sets. Since $f, \bar{\alpha}, \bar{\alpha}_\ell, \ell, g$, and σ are continuous, $\delta_N^V(s_1, s_2, \cdot)$ is continuous for fixed s_1, s_2.

Proof of (24): Consider $M > 1$ and let Assumption 6 hold. For $z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}$ and each $j \in \{0, 1, \ldots, N - M\}$

$$\begin{aligned} &V_N(z) - V_N(x) \leq -g(\phi_N) \\ &\quad - \sum_{k=0}^{N-1} \ell(\phi_k, u_k^*(x)) + \sum_{k=0}^{N-M-j-1} \ell(\psi_k, u_{k+1}^*(x)) \\ &\quad + \inf_{\mathbf{v}} \Bigg\{ g(\phi(M + j, \psi_{N-M-j}, \mathbf{v})) \\ &\qquad + \sum_{k=0}^{M+j-1} \ell(\phi(k, \psi_{N-M-j}, \mathbf{v}), v_k) \Bigg\} \\ &\text{subject to} \\ &v_k \in \mathcal{U}, k \in \{0, \ldots, M + j - 1\} \\ &\phi(k, \psi_{N-M-j}, \mathbf{v}) \in \mathcal{X}_{N-M-j+k,N}, k \in \{0, \ldots, M + j\} \\ &\leq -\ell(x, \kappa_N(x)) + V_{N,M+j}(\psi_{N-M-j}) \\ &\quad + \delta_N^a(x, N - M - j, \varepsilon) \\ &\leq -\ell(x, \kappa_N(x)) + \bar{\alpha}(\sigma(\psi_{N-M-j})) \\ &\quad + \delta_N^a(x, N - M - j, \varepsilon) \\ &\leq -\ell(x, \kappa_N(x)) + \bar{\alpha}(\sigma(\phi_{N-M-j+1})) \\ &\quad + \delta_N^{ab}(x, N - M - j, \varepsilon). \end{aligned} \tag{26}$$

By Standing Assumption 3, σ is detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W)$; therefore, there exists a function W satisfying the conditions of Definition 4. We aim to reach (24) from (26) using this function. We study two possible cases.

Case 1) $\gamma_W(s) \leq s$ for all $s \in \mathbb{R}_{\geq 0}$.

We first write the following difference inequality (since $M > 1$, it is well defined):

$$\begin{aligned} &W(\phi_{N-M+2}) - W(x) \\ &\quad \leq \sum_{k=0}^{N-M+1} \{W(\phi_{k+1}) - W(\phi_k)\} \\ &\quad \leq \sum_{k=0}^{N-M+1} \{-\alpha_W(\sigma(\phi_k)) + \gamma_W(\ell(\phi_k, u_k^*(x)))\} \\ &\quad \leq -\sum_{k=0}^{N-M+1} \alpha_W(\sigma(\phi_k)) + \sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)). \end{aligned} \tag{27}$$

The inequalities $W(x) \leq \bar{\alpha}_W(\sigma(x))$ and $W(\phi_{N-M+2}) \geq 0$ hold since W satisfies the conditions of Definition 4. By Standing Assumption 4, $\sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \leq \bar{\alpha}(\sigma(x))$. Using

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$$\begin{aligned} \delta_N^a(x, L, \varepsilon) &:= \begin{cases} \sup_{z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}} \sum_{k=0}^{L-1} \{\ell(\psi_k, u_{k+1}^*(x)) \\ \qquad -\ell(\phi_{k+1}, u_{k+1}^*(x))\}, & \text{L} > 0 \\ 0, & \text{L} = 0 \end{cases} \\ \delta_N^b(x, L, \varepsilon) &:= \sup_{z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}} \bar{\alpha}(\sigma(\psi_L)) - \bar{\alpha}(\sigma(\phi_{L+1})) \\ \delta_N^c(x, L, \varepsilon) &:= \sup_{z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}} g(\psi_L) - g(\phi_{L+1}) \\ \delta_N^d(x, L, \varepsilon) &:= \sup_{z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}} \bar{\alpha}_\ell(\sigma(\psi_L)) - \bar{\alpha}_\ell(\sigma(\phi_{L+1})) \end{aligned}$$

these facts and (27), we write (we are able to drop the $k=0$ term since all terms are positive)

$$\sum_{k=1}^{N-M+1} \alpha_W(\sigma(\phi_k)) \leq \bar{\alpha}_W(\sigma(x)) + \bar{\alpha}(\sigma(x)) \tag{28}$$

which implies that for at least one index $k \in \{1, 2, \ldots, N-M+1\}$,

$$\sigma(\phi_k) \leq \alpha_W^{-1}\left(\frac{\bar{\alpha}_W(\sigma(x)) + \bar{\alpha}(\sigma(x))}{N-M+1}\right). \tag{29}$$

Now, choose $j = N - M + 1 - k$. Note that, for $k \in \{1, 2, \ldots, N-M+1\}$, we have $j \in \{0, 1, \ldots, N-M\}$, as desired. Substituting for k, we have

$$\sigma(\phi_{N-M-j+1}) \leq \alpha_W^{-1}\left(\frac{\bar{\alpha}_W(\sigma(x)) + \bar{\alpha}(\sigma(x))}{N-M+1}\right). \tag{30}$$

Define Υ_V as in Step 1) of Algorithm 1 and δ_N^V as above. Using these definitions and combining (30) and (26), we obtain (24).

Case 2) $\gamma_W(s) > s$ for some $s \in \mathbb{R}_{\geq 0}$.
Define $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, q_W$, and ρ_W as in Step 2) of Algorithm 1. As given in [13, Lemma 4], for all $k \in \{0, 1, \ldots, N-1\}$, we have

$$\begin{aligned}\rho_W(W(\phi_{k+1})) &- \rho_W(W(\phi_k)) \\ &\leq -\tilde{\alpha}_1(\sigma(\phi_k)) + \ell(\phi_k, u_k^*(x)) \cdot \gamma_W(\ell(\phi_k, u_k^*(x)))\end{aligned} \tag{31}$$

which implies that

$$\begin{aligned}\rho_W(W(\phi_{N-M+2})) &- \rho_W(W(x)) \\ &= \sum_{k=0}^{N-M+1} \{\rho_W(W(\phi_{k+1})) - \rho_W(W(\phi_k))\} \\ &\leq \sum_{k=0}^{N-M+1} \{-\tilde{\alpha}_1(\sigma(\phi_k)) \\ &\quad \ell(\phi_k, u_k^*(x)) \cdot \gamma_W(\ell(\phi_k, u_k^*(x)))\}.\end{aligned}$$

From Assumption 4, we have $\sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \leq \bar{\alpha}(\sigma(x))$. Hence

$$\begin{aligned}&\sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \cdot \gamma_W(\ell(\phi_k, u_k^*(x))) \\ &\leq \gamma_W\left(\max_k \ell(\phi_k, u_k^*(x))\right) \cdot \sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \\ &\leq \gamma_W\left(\sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x))\right) \cdot \sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \\ &\leq \gamma_W(\bar{\alpha}(\sigma(x))) \cdot \bar{\alpha}(\sigma(x)).\end{aligned}$$

Moreover, $W(x) \leq \bar{\alpha}_W(\sigma(x))$ and $W(\phi_{N-M+2}) \geq 0$ by Definition 4, which yields (again dropping the $k=0$ term)

$$\begin{aligned}&\sum_{k=1}^{N-M+1} \tilde{\alpha}_1(\sigma(\phi_k)) \\ &\leq \rho_W(W(x)) + \sum_{k=0}^{N-M+1} \ell(\phi_k, u_k^*(x)) \\ &\quad \cdot \gamma_W(\ell(\phi_k, u_k^*(x))) \\ &\leq \rho_W(\bar{\alpha}_W(\sigma(x))) + \gamma_W(\bar{\alpha}(\sigma(x))) \\ &\quad \cdot \bar{\alpha}(\sigma(x)) =: \tilde{\alpha}_2(\sigma(x))\end{aligned}$$

which implies that, for at least one index $k \in \{1, 2, \ldots, N-M+1\}$, we have

$$\sigma(\phi_k) \leq \tilde{\alpha}_1^{-1}\left(\frac{\tilde{\alpha}_2(\sigma(x))}{N-M+1}\right). \tag{32}$$

Again choosing $j = N - M + 1 - k$, we can write

$$\sigma(\phi_{N-M-j+1}) \leq \tilde{\alpha}_1^{-1}\left(\frac{\tilde{\alpha}_2(\sigma(x))}{N-M+1}\right). \tag{33}$$

Define Υ_V as in Step 2) of Algorithm 1 and let δ_N^V be defined as above. Using these definitions and combining (33) and (26), we obtain (24).

Proof of (25): Now, consider $M = 1$ and let Assumption 7 hold. For $z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}$, we write

$$\begin{aligned}&V_N(z) - V_N(x) \\ &\leq -g(\phi_N) - \sum_{k=0}^{N-1} \ell(\phi_k, u_k^*(x)) + \sum_{k=0}^{N-2} \ell(\psi_k, u_{k+1}^*(x)) \\ &\quad + \inf_{\substack{v \in \mathcal{U} \\ f(\psi_{N-1}, v) \in \mathcal{X}_{N,N}}} \{g(f(\psi_{N-1}, v)) + \ell(\psi_{N-1}, v)\} \\ &\leq -\ell(x, \kappa_N(x)) - g(\phi_N) + \delta_N^a(x, N-1, \varepsilon) \\ &\quad + \inf_{\substack{v \in \mathcal{U} \\ f(\psi_{N-1}, v) \in \mathcal{X}_{N,N}}} \{g(f(\psi_{N-1}, v)) + \ell(\psi_{N-1}, v)\}.\end{aligned} \tag{34}$$

Since Assumption 7 holds, there exists a control input u such that, from (34), we have

$$\begin{aligned}&V_N(z) - V_N(x) \\ &\leq -\ell(x, \kappa_N(x)) + \delta_N^a(x, N-1, \varepsilon) - g(\phi_N) \\ &\quad + g(\psi_{N-1}) \\ &\quad - g(\psi_{N-1}) + g(f(\psi_{N-1}, u)) + \ell(\psi_{N-1}, u) \\ &\leq -\ell(x, \kappa_N(x)) + \delta_N^a(x, N-1, \varepsilon) \\ &\quad + \delta_N^c(x, N-1, \varepsilon) + \bar{\alpha}_\ell(\sigma(\psi_{N-1})) \\ &\leq -\ell(x, \kappa_N(x)) + \bar{\alpha}_\ell(\sigma(\phi_N)) \\ &\quad + \delta_N^{acd}(x, N-1, \varepsilon).\end{aligned} \tag{35}$$

Note that ϕ_N satisfies

$$\Gamma(N) \cdot \underline{\alpha}_G(\sigma(\phi_N)) \leq \Gamma(N) \cdot G(\phi_N) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)).$$

Hence

$$\sigma(\phi_N) \leq \underline{\alpha}_G^{-1}\left(\frac{\bar{\alpha}(\sigma(x))}{\Gamma(N)}\right). \tag{36}$$

Define $\hat{\Upsilon}_V$ as in Step 2) of Algorithm 1 and δ_N^V as above. Then, using these definitions and combining (36) with (35), we obtain (25). ■

In order to prove Theorem 1, we use the following lemma to generate a bound on the disturbance that does not depend explicitly on the state or the horizon length.

Lemma 3: Consider a function $\delta : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\delta(x, n, \cdot)$ is continuous, $\delta(\cdot, \cdot, 0) = 0$, and $\delta(\cdot, \cdot, \cdot)$ is bounded on compact sets. Then there exist a continuous function $h : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\delta(x, n, h(x, n) \cdot s) \leq \gamma(s) \tag{37}$$

for all $s \geq 0$. ■

Proof: We begin with defining $\tilde{\alpha}_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\tilde{\alpha}_k(s) := \sup_{|x|+n \in [k-1, k]} \delta(x, n, s) \quad k \in \{1, 2, \ldots\}.$$

Since $\delta(x, n, \cdot)$ is continuous for each fixed x and n, $\tilde{\alpha}_k$ is continuous. Furthermore, $\tilde{\alpha}_k$ is zero at zero and nonnegative everywhere. Then, for each k, there exists $\bar{\alpha}_k \in \mathcal{K}_\infty$ such that $\tilde{\alpha}_k(s) \leq \bar{\alpha}_k(s)$ for all $s \geq 0$. Now, define a positive coefficient π_k as follows:

$$\pi_k := \frac{1}{2^k \bar{\alpha}_k(k)} \quad k \in \{1, 2, \ldots\}. \tag{38}$$

Given any $s > 0$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \pi_k \cdot \bar{\alpha}_k(s) &= \sum_{k=1}^{\lceil s \rceil - 1} \pi_k \cdot \bar{\alpha}_k(s) + \sum_{k=\lceil s \rceil}^{\infty} \pi_k \cdot \bar{\alpha}_k(s) \\ &\leq \sum_{k=1}^{\lceil s \rceil - 1} \pi_k \cdot \bar{\alpha}_k(s) + \sum_{k=\lceil s \rceil}^{\infty} \pi_k \cdot \bar{\alpha}_k(k) \\ &= \sum_{k=1}^{\lceil s \rceil - 1} \pi_k \cdot \bar{\alpha}_k(s) + \sum_{k=\lceil s \rceil}^{\infty} \frac{1}{2^k} \\ &\leq \sum_{k=1}^{\lceil s \rceil - 1} \pi_k \cdot \bar{\alpha}_k(s) + 1.\end{aligned}$$

Hence $\sum_{k=1}^{\infty} \pi_k \cdot \bar{\alpha}_k(s) < \infty$ for all $0 \leq s < \infty$. We define α_2 as

$$\alpha_2(s) := \sum_{k=1}^{\infty} \pi_k \cdot \bar{\alpha}_k(s) \quad s \geq 0. \tag{39}$$

Notice that $\alpha_2 \in \mathcal{K}_\infty$ since it is a pointwise convergent sum (finite for all $s < \infty$) of $\mathcal{K}_\infty$ functions. Then choose any $\alpha_1 \in \mathcal{K}_\infty$ such that

$$\alpha_1(k) \geq \frac{1}{\pi_k} \quad k \in \{1, 2, \ldots\}. \tag{40}$$

Finally, given (x, n, s), let $\bar{k} = 1 + \lfloor |x| \rfloor + n$. Then

$$\begin{aligned}&\delta(x, n, s) \leq \tilde{\alpha}_{\bar{k}}(s) \leq \bar{\alpha}_{\bar{k}}(s) \\ &\leq \frac{1}{\pi_{\bar{k}}}(\pi_1 \cdot \bar{\alpha}_1(s) + \pi_2 \cdot \bar{\alpha}_2(s) + \cdots + \pi_{\bar{k}} \cdot \bar{\alpha}_{\bar{k}}(s) + \cdots) \\ &\leq \frac{1}{\pi_{\bar{k}}} \cdot \sum_{k=1}^{\infty} \pi_k \cdot \bar{\alpha}_k(s) \leq \alpha_1(\bar{k}) \cdot \alpha_2(s) \\ &\leq \alpha_1(1 + |x| + n) \cdot \alpha_2(s).\end{aligned}$$

By [16, Corollary 10], for any $\mathcal{K}_\infty$-function α_2, there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that $\alpha_2(r \cdot s) \leq \gamma_1(r) \cdot \gamma_2(s)$ for all $r, s \geq 0$. Let $h(x, n) := \gamma_1^{-1}((1)/(\alpha_1(1 + |x| + n)))$ and $\gamma := \gamma_2$. Then we have

$$\begin{aligned}&\delta(x, n, h(x, n) \cdot s) \\ &\quad \leq \alpha_1(1 + |x| + n) \cdot \alpha_2(h(x, n) \cdot s) \\ &\quad \leq \alpha_1(1 + |x| + n) \cdot \gamma_1(h(x, n)) \cdot \gamma_2(s) \\ &\quad \leq \gamma(s)\end{aligned}$$

which gives us the result. ■

Proof of Theorem 1: We prove Theorem 1 using the calculations of Theorem 1 and a function W that satisfies the conditions of Definition 4. As above, we study two possible cases.

Case 1) $\gamma_W(s) \leq s$ for all $s \in \mathbb{R}_{\geq 0}$.

Define $Y_N(x) := V_N(x) + W(x)$. Then

$$\begin{aligned}Y_N(x) &\leq \bar{\alpha}(\sigma(x)) + \bar{\alpha}_W(\sigma(x)) = \bar{\alpha}_Y(\sigma(x)) \\ Y_N(x) &\geq \ell(x, \kappa_N(x)) + (\alpha_W(\sigma(x)) - \ell(x, \kappa_N(x))) \\ &= \alpha_W(\sigma(x)) = \underline{\alpha}_Y(\sigma(x)).\end{aligned}$$

Hence (8) is proven.

Let δ_N^V be defined as above. Define $\delta_N^W(x, \varepsilon) := \sup_{z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}} W(z) - W(f(x, \kappa_N(x)))$; then $\delta_N^W(\cdot, 0) = 0$ and $\delta_N^W(s_1, \cdot)$ is continuous for fixed s_1. Now, define

$$\begin{aligned}&\delta_N(s_1, s_2, s_3) \\ &\qquad := \max\left\{0, \delta_N^V(s_1, s_2, s_3) + \delta_N^W(s_1, s_3), \bar{\delta}_V(s_1, s_3)\right\}\end{aligned}$$

where $\bar{\delta}_V$ is the maximum of the δ_V functions coming from applying Lemma 7 in Appendix F to the decrease (24) and (25) when $\gamma_W(s) > s$ for some s (this allows us to use the same function in Case 2 below).

We seek to define an upper bound on δ_N without the implicit N dependence. Let $\bar{\delta}(s_1, L, s_3) := \sup_{n \in [0, L]} \delta_N(s_1, n, s_3)$. Then, by Lemma 3, there exist h and γ_Y such that $\bar{\delta}(x, N - M, h(x, N - M) \cdot d) \leq \gamma_Y(d)$. Finally, define $H_N(y, n) := \min\{h(y, n), \bar{\varepsilon}_y\}$, where $\bar{\varepsilon}_y$ comes from Definition 5 (RF) for $\mathcal{C} = \{y\}$.

We can then use Theorem 1 to conclude that, for all $x \in \mathcal{F}_N$, all $d \in [0,1]$, and all $z \in f(x,\kappa_N(x)) + H_N(x, N-M)\cdot d \cdot \mathcal{B}$, if Assumption 6 holds, then

$$\begin{aligned} Y_N(z) - Y_N(x) &= V_N(z) - V_N(x) + W(z) - W(x) \\ &\le -\ell(x,\kappa_N(x)) + \Upsilon_V(\sigma(x), N-M+1) \\ &\quad + \delta_N^V(x, N-M, H_N(x,N-M)\cdot d) \\ &\quad - \alpha_W(\sigma(x)) + \gamma_W(\ell(x,\kappa_N(x))) \\ &\quad + \delta_W(x, H_N(x,N-M)\cdot d) \\ &\le -\alpha_W(\sigma(x)) + \Upsilon_V(\sigma(x), N-M+1) \\ &\quad + \delta_N(x, N-M, H_N(x,N-M)\cdot d) \\ &\le -\alpha_Y(\sigma(x)) + \Upsilon_Y(\sigma(x), N-M+1) + \gamma_Y(d) \end{aligned}$$

which is (9) with α_Y and Υ_Y defined as in Step 1) of Algorithm 1; otherwise, if Assumption 7 holds, then

$$\begin{aligned} &Y_N(z) - Y_N(x) \\ &\quad \le -\ell(x,\kappa_N(x)) + \hat{\Upsilon}_V(\sigma(x), \Gamma(N)) \\ &\qquad + \delta_N^V(x, N-1, H_N(x,N-1)\cdot d) \\ &\qquad - \alpha_W(\sigma(x)) + \gamma_W(\ell(x,\kappa_N(x))) \\ &\qquad + \delta_W(x, H_N(x,N-1)\cdot d) \\ &\quad \le -\alpha_W(\sigma(x)) + \hat{\Upsilon}_V(\sigma(x), \Gamma(N)) \\ &\qquad + \delta_N(x, N-1, H_N(x,N-1)\cdot d) \\ &\quad \le -\alpha_Y(\sigma(x)) + \hat{\Upsilon}_Y(\sigma(x), \Gamma(N)) + \gamma_Y(d) \end{aligned}$$

which is (10) with $\hat{\Upsilon}_Y$ defined as in Step 1) of Algorithm 1.

Case 2) $\gamma_W(s) > s$ for some $s \in \mathbb{R}_{\ge 0}$.
Let q_V and ρ_V be defined as in Step 2) of Algorithm 1. Let $Y_N(x) := \rho_V(V_N(x)) + \rho_W(W(x))$. Then

$$Y_N(x) \le \rho_V(\bar{\alpha}(\sigma(x))) + \rho_W(\bar{\alpha}_W(\sigma(x))) = \bar{\alpha}_Y(\sigma(x))$$

which constitutes the upper bound. Also

$$\begin{aligned} Y_N(x) \ge\ & \rho_V(\ell(x,\kappa_N(x))) \\ & + \rho_W(\max\{\alpha_W(\sigma(x)) - \gamma_W(\ell(x,\kappa_N(x))),\ 0\}). \end{aligned}$$

If $\alpha_W(\sigma(x))/2 \le \gamma_W(\ell(x,\kappa_N(x)))$, then $Y_N(x) \ge \rho_V(\gamma_W^{-1}(\alpha_W(\sigma(x))/2))$. If $\alpha_W(\sigma(x))/2 \ge \gamma_W(\ell(x,\kappa_N(x)))$, then $Y_N(x) \ge \rho_W(\alpha_W(\sigma(x))/2)$. Thus

$$\begin{aligned} &Y_N(x) \\ &\quad \ge \min\left\{\rho_V(\gamma_W^{-1}(\alpha_W(\sigma(x))/2)), \rho_W(\alpha_W(\sigma(x))/2)\right\} \\ &\quad = \underline{\alpha}_Y(\sigma(x)) \end{aligned}$$

yields the lower bound. Hence (8) is proven.
Let

$$\delta_W(x,\varepsilon) := \sup_{z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}} \rho_W(W(z)) - \rho_W(W(f(x,\kappa_N(x)))).$$

Since both ρ_W and W are continuous, so is $\delta_W(s_1,\cdot)$ for fixed s_1 and $\delta_W(\cdot,0) = 0$. Combining this with (31), we have that, for $z \in f(x,\kappa_N(x)) + \varepsilon\mathcal{B}$

$$\begin{aligned} &\rho_W(W(z)) - \rho_W(W(x)) \\ &\quad \le -\tilde{\alpha}_1(\sigma(x)) + \ell(x,\kappa_N(x)) \\ &\qquad \cdot \gamma_W(\ell(x,\kappa_N(x))) + \delta_W(x,\varepsilon). \end{aligned} \tag{41}$$

Let Υ_Y and α_Y be defined as in Step 2) of Algorithm 1, and let $\delta_N, \bar{\delta}, H_N$, and γ_Y be defined as in Case 1 with δ_W defined in the previous paragraph. Then using Theorem 1 and Lemma 7 in Appendix F, for all $x \in \mathcal{F}_N, d \in [0,1]$, and $z \in f(x,\kappa_N(x)) + H_N(x,N-M)\cdot d \cdot \mathcal{B}$, if Assumption 6 holds, then

$$\begin{aligned} &\rho_V(V_N(z)) - \rho_V(V_N(x)) \\ &\quad \le -\ell(x,\kappa_N(x)) \cdot \gamma_W(\ell(x,\kappa_N(x))) \\ &\qquad + \Upsilon_Y(\sigma(x), N-M+1) + \delta_V(x, H_N(x,N-M)\cdot d) \end{aligned}$$

and, from the combination of (41) and (42)

$$\begin{aligned} &Y_N(z) - Y_N(x) \\ &\quad \le -\alpha_Y(\sigma(x)) + \Upsilon_Y(\sigma(x), N-M+1) \\ &\qquad + \delta_V(x, H_N(x,N-M)\cdot d) \\ &\qquad + \delta_W(x, H_N(x,N-M)\cdot d) \\ &\quad \le -\alpha_Y(\sigma(x)) + \Upsilon_Y(\sigma(x), N-M+1) + \gamma_Y(d) \end{aligned} \tag{42}$$

which is (9). Otherwise, suppose $M = 1$ and Assumption 7 holds. Then define $\hat{\Upsilon}_Y$ as in Step 2) of Algorithm 1, and we similarly have that

$$Y_N(z) - Y_N(x) \le -\alpha_Y(\sigma(x)) + \hat{\Upsilon}_Y(\sigma(x), \Gamma(N)) + \gamma_Y(d)$$

which is (10). ■

APPENDIX II
PROOF OF COROLLARY 1

The proof is a slight modification of the proof given in [15, Lemma 3.5]. Let Assumption 6 hold and M be fixed (the proof for Assumption 7 is very similar and is therefore omitted). Given δ and Δ, choose $N^* \ge \bar{N}, \tilde{\delta}$, and $\tilde{\Delta}$ such that the following conditions are satisfied:

$$\tilde{\Delta} \ge \underline{\alpha}_Y^{-1}(\bar{\alpha}_Y(\Delta)) \tag{43}$$

$$\bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N^* - M + 1) \le \frac{1}{2}\underline{\alpha}_Y(\delta) \tag{44}$$

$$\Upsilon_Y(s, N^* - M + 1) \le \frac{1}{2}\alpha_Y(s), \quad \forall s \in [\tilde{\delta}, \tilde{\Delta}]. \tag{45}$$

Now, choose any $N \ge N^*$. From Theorem 1, there exist Y_N, H_N, and γ_Y satisfying (8) and (9). By the definition of $\alpha_1, \hat{\alpha}_1$, and (45), for all $x \in \{s : \tilde{\delta} \le \sigma(s) \le \tilde{\Delta}\}$ and all $z \in f(x,\kappa_N(x)) + H_N(x,N-M)\cdot d \cdot \mathcal{B}$

$$\begin{aligned} Y_N(z) - Y_N(x) &\le -\frac{1}{2}\alpha_Y(\sigma(x)) + \gamma_Y(d) \\ &\le -\alpha_1(Y_N(x)) + \gamma_Y(d) \\ &\le -\hat{\alpha}_1(Y_N(x)) + \gamma_Y(d). \end{aligned} \tag{46}$$

Consider the set $\mathcal{D} := \{s : Y_N(s) \leq b\}$, where $b := \hat{\alpha}_1^{-1}(2\gamma_Y(\|\mathbf{d}\|))$. By (8), the assumed bound on $\|\mathbf{d}\|$, and (43), we have

$$\begin{aligned}\mathcal{D} &\subseteq \left\{s \,|\, \sigma(s) \leq \underline{\alpha}_Y^{-1}\left(\hat{\alpha}_1^{-1}(2\gamma_Y(\|\mathbf{d}\|))\right)\right\}\\ &\subseteq \left\{s \,|\, \sigma(s) \leq \underline{\alpha}_Y^{-1}(\bar{\alpha}_Y(\Delta))\right\} \subseteq \{s \,|\, \sigma(s) \leq \tilde{\Delta}\}. \quad (47)\end{aligned}$$

Then $x \in \mathcal{D}$ implies that $\sigma(x) \leq \tilde{\Delta}$.

Consider $x \in \{s \,|\, \delta \leq \sigma(s) \leq \Delta\} \cap \mathcal{F}_N$, and define $\phi_k := \phi_{H_N}(k, x)$. We now state and prove a preliminary claim.

Claim 1: If there exists k_0 such that $\phi_{k_0} \in \mathcal{D}$, then $\phi_k \in \mathcal{D}$ for all $k > k_0$ if $\sigma(\phi_j) \geq \tilde{\delta}$ for all $j \in \{k_0, k_0+1, \ldots, k-1\}$.■

Proof: Assume $\sigma(\phi_{k_0}) \geq \tilde{\delta}$ and $\phi_{k_0} \in \mathcal{D}$. Then by (47), $\tilde{\delta} \leq \sigma(\phi_{k_0}) \leq \tilde{\Delta}$ and (46) holds. For $\phi_{k_0} \in \mathcal{D}, Y_N(\phi_{k_0}) \leq b$, and therefore $(1/2)\hat{\alpha}_1(Y_N(\phi_{k_0})) \leq \gamma_Y(\|\mathbf{d}\|)$. By (46)

$$Y_N(\phi_{k_0+1}) \leq -\frac{1}{2}\hat{\alpha}_1(Y_N(\phi_{k_0})) + \tilde{\alpha}_1(Y_N(\phi_{k_0})) + \gamma_Y(\|\mathbf{d}\|)$$

where $\tilde{\alpha}_1 := \mathrm{Id} - (1/2)\hat{\alpha}_1$. Note that if $Y_N(\phi_{k_0}) = b$, then $\tilde{\alpha}_1(Y_N(\phi_{k_0})) + \gamma_Y(\|\mathbf{d}\|) = Y_N(\phi_{k_0}) = b$, and observe that $\tilde{\alpha}_1 \in \mathcal{K}$ for $\mathrm{Id} - \hat{\alpha}_1 \in \mathcal{K}$ and is therefore strictly increasing. Since $Y_N(\phi_{k_0}) \leq b$, then $Y_N(\phi_{k_0+1}) \leq -(1/2)\hat{\alpha}_1(Y_N(\phi_{k_0})) + b \leq b$. Using induction, we have $Y_N(\phi_{k_0+j}) \leq b$ as long as (46) holds for all $k \in \{k_0, k_0+1, \ldots, k_0+j-1\}$. Then $\phi_k \in \mathcal{D}$ if $\sigma(\phi_j) \geq \tilde{\delta}$ for all $j \in \{k_0, k_0+1, \ldots, k-1\}$. ■

We now proceed to prove the corollary. Let $m_0 = \min\{k \geq 0 \,|\, \{\phi_k \in \mathcal{D}\}$ or $\{\sigma(\phi_k) < \tilde{\delta}\}\}$. For $k < m_0, (1/2)\hat{\alpha}_1(Y_N(\phi_k)) > \gamma_Y(\|\mathbf{d}\|)$ and, from (46), $Y_N(\phi_{k+1}) - Y_N(\phi_k) \leq -(1/2)\hat{\alpha}_1(Y_N(\phi_k))$. Let $\hat{\beta}(s,k)$ be the solution of the scalar difference equation

$$\eta(k+1) = \eta(k) - \frac{1}{2}\hat{\alpha}_1(\eta(k))$$

with initial condition $\eta(0) = s$. Observe that, for any $s > 0$, the sequence $\eta(k)$ decreases to zero (it never crosses zero, because $\mathrm{Id} - \hat{\alpha}_1 \geq 0$); thus, $\hat{\beta} \in \mathcal{KL}$. It follows by induction on k that $Y_N(\phi_k) \leq \hat{\beta}(Y_N(x), k)$ for all $0 \leq k \leq m_0$. This says that if $\phi_k \notin \mathcal{D}$ and $\sigma(\phi_k) \geq \tilde{\delta}$ for all $k < m_0$, then $Y_N(\phi_k)$ will approach $\mathcal{D}$. We now look at what happens if the trajectory enters either $\mathcal{D}$ or $\{s \,|\, \sigma(s) < \tilde{\delta}\}$. First, suppose that $\sigma(\phi_j) \geq \tilde{\delta}$ for all $0 \leq j \leq k$. Then by Claim 1, we can assert that once the trajectory enters $\mathcal{D}$ (we know it will by the above discussion if k is big enough and $\|\mathbf{d}\| \neq 0$), it will remain in $\mathcal{D}$ up to time k. Therefore, $Y_N(\phi_k) \leq \max\{\hat{\beta}(Y_N(x), k), b\}$. Instead suppose that, for some j, $0 \leq j \leq k, \sigma(\phi_j) < \tilde{\delta}$. Then from (8) and (9)

$$\begin{aligned}Y_N(\phi_{j+1}) &\leq Y_N(\phi_j) - \alpha_Y(\sigma(\phi_j))\\ &\quad + \Upsilon_Y(\sigma(\phi_j), N-M+1) + \gamma_Y(d)\\ &\leq \bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N-M+1) + \gamma_Y(\|\mathbf{d}\|). \quad (48)\end{aligned}$$

If $\bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N-M+1) + \gamma_Y(\|\mathbf{d}\|) < b$, then, from (48), $Y_N(\phi_{j+1}) < b$ and hence $Y_N(\phi_k) \leq \max\{\hat{\beta}(Y_N(x), k), b\}$ for all k. Otherwise, if $\bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N-M+1) + \gamma_Y(\|\mathbf{d}\|) \geq b$, then

$$\begin{aligned}&\bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N-M+1)\\ &\quad \geq -\gamma_Y(\|\mathbf{d}\|) + \hat{\alpha}_1^{-1}(2\gamma_Y(\|\mathbf{d}\|))\\ &\quad \geq -\gamma_Y(\|\mathbf{d}\|) + 2\gamma_Y(\|\mathbf{d}\|) = \gamma_Y(\|\mathbf{d}\|) \quad (49)\end{aligned}$$

where we have used that $\mathrm{Id} - \hat{\alpha} \in \mathcal{K}$. Now, we can combine (48) with (49) to obtain

$$Y_N(\phi_{j+1}) \leq 2(\bar{\alpha}_Y(\tilde{\delta}) + \Upsilon_Y(\tilde{\delta}, N-M+1)).$$

Then by (44), we have $Y_N(\phi_{j+1}) \leq \underline{\alpha}_Y(\delta)$ and hence by (8) we have $\sigma(\phi_{j+1}) \leq \delta$. For $\sigma(\phi_j) \leq \delta$, one of the above cases applies, and we can show that $\sigma(\phi_{j+1}) \leq \delta$. Consequently, $\sigma(\phi_k) \leq \max\{\underline{\alpha}_Y^{-1}(\hat{\beta}(Y_N(x), k)), \underline{\alpha}_Y^{-1}(\hat{\alpha}_1^{-1}(2\gamma_Y((\|\mathbf{d}\|)))), \delta\}$ for all k, from which (11) follows with $\gamma(s) := \underline{\alpha}_Y^{-1}(\hat{\alpha}_1^{-1}(2\gamma_Y(s)))$ and $\beta(s,k) := \underline{\alpha}_Y^{-1}(\hat{\beta}(\bar{\alpha}_Y(s), k))$.

Now, if σ is a proper indicator function for the set $\mathcal{A}$ with functions $(\underline{\alpha}_\sigma, \bar{\alpha}_\sigma)$, we can define $\delta := \underline{\alpha}_\sigma(\delta_\mathcal{A}), \Delta := \bar{\alpha}_\sigma(\delta_\mathcal{A})$ and follow the proof above to obtain (12) with $\beta_\mathcal{A}(s,k) := \underline{\alpha}_\sigma^{-1}(\beta(\bar{\alpha}_\sigma(s), k))$ and $\gamma_\mathcal{A}(s) := \underline{\alpha}_\sigma^{-1}(\gamma(s))$. ■

APPENDIX III
PROOF OF COROLLARY 2

This proof specializes that of Corollary 1. Let $\delta = 0$ and suppose Assumption 6 holds (the proof for the case where Assumption 7 holds is very similar). Given Δ, choose $\tilde{\delta} = 0$ and $\tilde{\Delta}$ such that $\tilde{\Delta} \geq \max\{\underline{\alpha}_Y^{-1}(\bar{\alpha}_Y(\Delta)), \Delta_\ell\}$. Suppose that $\tilde{\Delta} \leq \Delta_\ell$. Then, using the definitions from Step 1) of Algorithm 1, if $\tilde{N}_1 \geq M + 1 + (2\bar{a}(\bar{a} + \bar{a}_W))/(a_W^2)$, then $\Upsilon_Y(s, \tilde{N}_1 - M + 1) \leq (1/2)\alpha_Y(s)$ for all $s \leq \Delta_\ell$. Then, choose $N^* = \max\{\tilde{N}_1, \bar{N}\}$ and (43)–(45) hold. Instead, suppose that $\tilde{\Delta} > \Delta_\ell$. Then choose $\tilde{N}_2$ such that $\Upsilon_Y(s, \tilde{N}_2 - M + 1) \leq (1/2)\alpha_Y(s)$ for all $\Delta_\ell \leq s \leq \tilde{\Delta}$. Then choose $N^* = \max\{\tilde{N}_1, \tilde{N}_2, \bar{N}\}$ and again (43)–(45) hold. The proof then follows from that of Corollary 1, letting $\delta = \tilde{\delta} = 0$. ■

APPENDIX IV
PROOF OF COROLLARY 3

Proof:

Case 1) Since $\gamma_W = \mathrm{Id}$, from Theorem 1 we have

$$\begin{aligned}&a_W \cdot \sigma(x) \leq Y_N(x) \leq (\bar{a} + \bar{a}_W) \cdot \sigma(x)\\ &Y_N(x^+) - Y_N(x)\\ &\quad \leq \left(-a_W + \frac{\bar{a}}{a_W} \cdot \frac{\bar{a} + \bar{a}_W}{N - M + 1}\right) \cdot \sigma(x) + \gamma_Y(d). \quad (50)\end{aligned}$$

Hence, if $N > M - 1 + (\bar{a}(\bar{a} + \bar{a}_W))/(a_W^2)$, there exists $\tilde{\epsilon} \in (0, \alpha_W)$ such that

$$Y_N(x^+) - Y_N(x) \leq -\tilde{\epsilon} \cdot \sigma(x) + \gamma_Y(d)$$

which, by (50), becomes

$$Y_N(x^+) - Y_N(x) \leq -\frac{\tilde{\epsilon}}{\bar{a} + \bar{a}_W} \cdot Y_N(x) + \gamma_Y(d).$$

Note that $\tilde{\epsilon}$ can be chosen independently of N by using the assumed lower bound. Let $\epsilon := (\tilde{\epsilon})/(\bar{a} +$

$\bar{a}_W$) [note that $\epsilon < 1$ by the definition of $\tilde{\epsilon}$ and (50)], then we have

$$Y_N(x^+) - Y_N(x) \leq -\epsilon \cdot Y_N(x) + \gamma_Y(d)$$
$$Y_N(x^+) \leq (1-\epsilon) \cdot Y_N(x) + \gamma_Y(\|\mathbf{d}\|). \quad (51)$$

If we iterate (51), we get

$$Y_N(\phi_{H_N}(k,x)) \leq (1-\epsilon)^k \cdot Y_N(x) + \frac{1}{\epsilon} \cdot \gamma_Y(\|\mathbf{d}\|).$$

Now we use (50) to get

$$\sigma(\phi_{H_N}(k,x)) \leq \frac{\bar{a} + \bar{a}_W}{a_W} \cdot (1-\epsilon)^k \cdot \sigma(x) + \frac{1}{\epsilon a_W} \cdot \gamma_Y(\|\mathbf{d}\|).$$

Thus the result (15) follows by choosing $K = (\bar{a} + \bar{a}_W)/(a_W)$, $\lambda = -\ln(1-\epsilon)$, and $g = (1)/(\epsilon a_W)$.

Case 2) We have that (50) holds from the argument in the previous case. Using Assumption 7 and Theorem 1, we have

$$Y_N(x^+) - Y_N(x) \leq \left(-a_W + \frac{\bar{a}_\ell \bar{a}}{\underline{a}_G \cdot \Gamma(N)}\right) \cdot \sigma(x) + \gamma_Y(d).$$

Hence, if $\Gamma(N) > (\bar{a}_\ell \bar{a})/(\underline{a}_G a_W)$, then there exists $\tilde{\epsilon} > 0$ such that

$$Y_N(x^+) - Y_N(x) \leq -\tilde{\epsilon} \cdot \sigma(x) + \gamma_Y(d).$$

The rest follows as in the previous case. Now, if σ is a proper indicator function for the set $\mathcal{A}$ with functions $(\underline{\alpha}_\sigma, \bar{\alpha}_\sigma)$, then, from (15), we have

$$|\phi_{H_N}(k,x)|_{\mathcal{A}} \leq \underline{\alpha}_\sigma^{-1}(Ke^{-\lambda k}\bar{\alpha}_\sigma(|x|_{\mathcal{A}})) + \underline{\alpha}_\sigma^{-1}(g \cdot \gamma_Y(\|\mathbf{d}\|)). \quad (52)$$

Therefore, the result (16) follows with $\beta(s,k) = \underline{\alpha}_\sigma^{-1}(Ke^{-\lambda k}\bar{\alpha}_\sigma(s))$ and $\gamma(s) = \underline{\alpha}_\sigma^{-1}(g \cdot \gamma_Y(s))$. If $\sigma(\cdot) = |\cdot|_{\mathcal{A}}$, then (52) becomes

$$|\phi_{H_N}(k,x)|_{\mathcal{A}} \leq \sqrt[p]{K} e^{-\frac{\lambda}{p}k}|x|_{\mathcal{A}} + \sqrt[p]{g \cdot \gamma_Y(\|\mathbf{d}\|)}$$

from which the result (17) follows with $\tilde{K} = \sqrt[p]{K}$, $\tilde{\lambda} = (\lambda/p)$, and $\tilde{\gamma}(s) = \sqrt[p]{g \cdot \gamma_Y(s)}$. ■

Appendix V
Proof of Corollaries 4–6

The proofs of the corollaries are similar. We prove Corollary 4, since it is the most general, and then make comments on the changes required to prove the other corollaries. For the proof, we will need the following lemmas.

Lemma 4: If an MPC formulation is RF with respect to (N, M), then for each compact set $\mathcal{C} \subset \mathcal{F}_N$, there exists $\varepsilon > 0$ such that $f(\mathcal{C}, \kappa_N(\mathcal{C})) + \varepsilon\mathcal{B} \subset \mathcal{F}_N$. ■

Proof: It follows from the definition of RF that, for each compact $\mathcal{C} \subset \mathcal{F}_N$, there exists $\varepsilon > 0$ such that, given $x \in \mathcal{C}$, for each $z \in f(x, \kappa_N(x)) + \varepsilon\mathcal{B}$, there exists an admissible input sequence of length N. The result follows by the definition of $\mathcal{F}_N$. ■

In what follows, the notation $\bar{\mathcal{S}}$ means the closure of the set $\mathcal{S}$.

Lemma 5: Let f be continuous, κ_N be locally bounded, and $\mathcal{C}$ be a compact set. Then, for $\varepsilon > 0$, $f(\mathcal{C}, \kappa_N(\mathcal{C})) + \varepsilon\mathcal{B} \subset \mathcal{F}_N$ implies that $\overline{f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}} \subset \mathrm{int}(\mathcal{F}_N)$. ■

Proof: Consider $x \in \overline{f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}}$. We have two cases. If $x \in f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}$, then $x + (1/2)\varepsilon\mathcal{B} \subset \mathcal{F}_N$, which implies $x \in \mathrm{int}(\mathcal{F}_N)$. If $x \notin f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}$ (that is, x is in the boundary of $f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}$), then there exists a sequence $\{x_i\}$ with $x_i \in f(\mathcal{C}, \kappa_N(\mathcal{C})) + (1/2)\varepsilon\mathcal{B}$ and $\lim_{i\to\infty} x_i = x$. Then for some i large enough, $x \in x_i + (1/2)\varepsilon\mathcal{B}$. Since $x_i + (1/2)\varepsilon\mathcal{B} \subset \mathcal{F}_N$, we have that $x \in \mathrm{int}(\mathcal{F}_N)$. ■

Lemma 6: Let an MPC formulation be RF with respect to (N, M) and define $H(y,n) := \min\{h(y,n), \varepsilon_y\}$, where h is any strictly positive continuous function and ε_y comes from Definition 5 (RF) for the set $\{y\}$. Then for any compact set $\mathcal{C} \subset \mathrm{int}(\mathcal{F}_N)$, there exists $\underline{H} > 0$ such that $x \in \mathcal{C}$ implies that $H(x,n) \geq \underline{H}$. ■

Proof: Since h is strictly positive and continuous and $\mathcal{C}$ is compact, there exists $\underline{h} > 0$ such that $x \in \mathcal{C}$ implies $h(x,n) \geq \underline{h}$. Given $\mathcal{C} \subset \mathrm{int}(\mathcal{F}_N)$, there exists a compact set $\bar{\mathcal{C}} \subset \mathrm{int}(\mathcal{F}_N)$ such that $\mathcal{C} \subset \mathrm{int}(\bar{\mathcal{C}})$. Let $\varepsilon_{\bar{\mathcal{C}}}$ come from Definition 5 for the set $\bar{\mathcal{C}}$. Define $\underline{H} := \min\{\underline{h}, \varepsilon_{\bar{\mathcal{C}}}\}$. Then, for any $x \in C$, $\varepsilon_x \geq \varepsilon_{\bar{\mathcal{C}}}$. Then for $x \in \mathcal{C}$, $H(x,n) \geq \min\{\underline{h}, \varepsilon_{\bar{\mathcal{C}}}\} =: \underline{H} > 0$. ■

Proof of Corollary 4: Given a compact set $\mathcal{C} \subset \mathrm{int}(\mathcal{F}_\infty)$ and $\varepsilon > 0$, define $\mathcal{C}_0 := \mathcal{C} + \delta_c\mathcal{B} \subset \mathrm{int}(\mathcal{F}_\infty)$, with $\delta_c > 0$ chosen sufficiently small. Choose $\hat{N} \geq \bar{N}$ such that $\mathcal{C}_0 \subset \mathrm{int}(\mathcal{F}_{\hat{N}})$ for all $N \geq \hat{N}$. Such $\hat{N}$ exists by definition of $\mathcal{F}_\infty$, since $\delta_c > 0$ and since we assume that $\mathcal{F}_{N+1} \supseteq \mathcal{F}_N$ for all N. Choose $a_1 > 0$ such that $\mathcal{S}_1 := \{s \in \mathbb{R}^n \,|\, |s|_{\mathcal{A}} \leq a_1\} \subset \mathrm{int}(\mathcal{F}_{\hat{N}})$; a_1 exists since $\mathrm{int}(\mathcal{F}_{\hat{N}})$ includes a neighborhood of $\mathcal{A}$. Define $\tilde{\Delta} := \max_{s\in\mathcal{C}_0} |s|_{\mathcal{A}}$ and $\tilde{\delta} := \min\{\varepsilon/4, \tilde{\Delta}/2, a_1/2\}$. From Corollary 1, there exist functions $\tilde{\beta} \in \mathcal{KL}, \tilde{\gamma}, \tilde{\alpha} \in \mathcal{K}_\infty$ (note that these functions do not depend on $\mathcal{C}$ or ε) and for this pair $\tilde{\delta}, \tilde{\Delta}$ there exists $\tilde{N}$ and for each $N \geq \tilde{N}$ there exists $H_N : \mathbb{R}^n \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ such that $\|\mathbf{d}\| \leq \min\{1, \tilde{\alpha}(\tilde{\Delta})\}$ and $x \in \{s \in \mathbb{R}^n \,|\, |s|_{\mathcal{A}} \leq \tilde{\Delta}\} \cap \mathcal{F}_N$ imply, for the system $x^+ = f(x, \kappa_N(x)) + H_N(x, N-M)d$, that

$$|\phi_{H_N}(k,x)|_{\mathcal{A}} \leq \tilde{\beta}(|x|_{\mathcal{A}}, k) + \max\{\tilde{\delta}, \tilde{\gamma}(\|\mathbf{d}\|)\}.$$

If the condition $\|\mathbf{d}\| \leq \min\{1, \tilde{\alpha}(\tilde{\Delta}), \tilde{\gamma}^{-1}(\tilde{\delta})\}$ is imposed instead, we can restate the result, since $a + b \leq \max\{2a, 2b\}$, in the following form:

$$|\phi_{H_N}(k,x)|_{\mathcal{A}} \leq \max\{2\tilde{\beta}(|x|_{\mathcal{A}}, k), 2\tilde{\delta}\}. \quad (53)$$

Choose $a_2 > 0$ such that $4\tilde{\beta}(2a_2, 0) \leq \varepsilon$. Define $N^* := \max\{\hat{N}, \tilde{N}\}$ and let some $N \geq N^*$ be given. Given a compact set $\tilde{\mathcal{C}} \subset \mathcal{F}_N$, let the function $\mathcal{E} : 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \to \mathbb{R}_{>0}$ satisfy $f(\tilde{\mathcal{C}}, \kappa_N(\tilde{\mathcal{C}})) + \mathcal{E}(\mathcal{F}_N, \tilde{\mathcal{C}})\mathcal{B} \subset \mathcal{F}_N$; such a function exists by Lemma 4. Let $\bar{K} \in \mathbb{Z}_{\geq 0}$ be such that $2\tilde{\beta}(\tilde{\Delta}, \bar{K}) \leq a_1$. For $k = \{0, 1, \ldots, \bar{K}\}$, define $\varepsilon_k := \mathcal{E}(\mathcal{F}_N, \mathcal{C}_k)$ and $\mathcal{C}_{k+1} := \overline{f(\mathcal{C}_k, \kappa_N(\mathcal{C}_k)) + (1/2)\varepsilon_k\mathcal{B}}$. For all $k = \{0, 1, \ldots, \bar{K}\}$, $\mathcal{C}_k$ is compact and, by Lemma 5, $\mathcal{C}_k \subset \mathrm{int}(\mathcal{F}_N)$. Define $\mathcal{S}_2 :=$

$\{\bigcup_{k\leq\bar{K}} \mathcal{C}_k\} \cup \mathcal{S}_1$; then $\mathcal{S}_2 \subset \text{int}(\mathcal{F}_N)$ and is compact. By Lemma 6, there exists $\underline{H} > 0$ such that $s \in \mathcal{S}_2$ implies that $H_N(s, N-M) \geq \underline{H}$. Define $\underline{\varepsilon} := (1/2)\min_{k\leq\bar{K}} \varepsilon_k$. Finally, choose $0 < \delta \leq \min\{\delta_c, a_2, \varepsilon/2\}$ such that $s \in \mathcal{S}_2$ and $\max\{\|\mathbf{e}\|, \|\mathbf{d}\|\} \leq \delta$ imply $|d + e^+ + f(s-e, \kappa_N(s)) - f(s, \kappa_N(s))| \leq \min\{\underline{H} \cdot \min\{1, \tilde{\alpha}(\tilde{\Delta}), \tilde{\gamma}^{-1}(\tilde{\delta})\}, \underline{\varepsilon}\}$. This δ exists since $\mathcal{S}_2$ is compact, κ_N is locally bounded, and f is continuous. We now show that this δ satisfies the conditions of Definition 1.

Define $\eta(k) := \phi_\delta(k, x) + e(k)$, where $e(k)$ is the value of the measurement error at time k. Then from (4)

$$\begin{aligned}\eta^+ &= x^+ + e^+\\ &= f(\eta, \kappa_N(\eta)) + d + e^+ + f(\eta - e, \kappa_N(\eta)) - f(\eta, \kappa_N(\eta))\\ &= f(\eta, \kappa_N(\eta)) + H_N(\eta, N-M)\tilde{d}\\ &= f(\eta, \kappa_N(\eta)) + \hat{d}\end{aligned} \tag{54}$$

where $\tilde{d} := (d + e^+ + f(\eta - e, \kappa_N(\eta)) - f(\eta, \kappa_N(\eta)))/H_N(\eta, N-M)$ and $\hat{d} := H_N(\eta, N-M)\tilde{d}$ (we drop the time step arguments to avoid clutter). For $x \in \mathcal{C}$, $\eta(0) \in \mathcal{C}_0 \subset \mathcal{S}_2$ since $\|\mathbf{e}\| \leq \delta_c$. When $\eta \in \mathcal{C}_k$, $\hat{d} \leq \underline{\varepsilon}$ and therefore $\eta^+ \in \mathcal{C}_{k+1}$. Then for $k \in \{0, 1, \ldots, \bar{K}\}$, $\eta(k) \in \mathcal{C}_k \subset \mathcal{S}_2$ and $H(\eta, N-M) \leq \underline{H}$; therefore, $|\tilde{d}(k)| \leq \min\{1, \tilde{\alpha}(\tilde{\Delta}), \tilde{\gamma}^{-1}(\tilde{\delta})\}$. Then from (53), $\eta(\bar{K}) \in \mathcal{S}_1 \subset \mathcal{S}_2$. Since $2\tilde{\delta} \leq a_1, \eta \in \mathcal{S}_1$ and $|\tilde{d}| \leq \min\{1, \tilde{\alpha}(\tilde{\Delta}), \tilde{\gamma}^{-1}(\tilde{\delta})\}$ imply that $\eta^+ \in \mathcal{S}_1$ and $|\tilde{d}^+| \leq \min\{1, \tilde{\alpha}(\tilde{\Delta}), \tilde{\gamma}^{-1}(\tilde{\delta})\}$. Then the bound on $\|\tilde{\mathbf{d}}\|$ holds for all $k \geq 0$ and from (53)

$$|\eta(k)|_{\mathcal{A}} \leq \max\{2\tilde{\beta}(|\eta(0)|_{\mathcal{A}}, k), 2\tilde{\delta}\}.$$

Note that $|\phi_\delta(k,x)|_{\mathcal{A}} - \|\mathbf{e}\| \leq |\eta(k)|_{\mathcal{A}} \leq |\phi_\delta(k,x)|_{\mathcal{A}} + \|\mathbf{e}\|$ and $\tilde{\delta} \leq \varepsilon/4$. Therefore, we can write

$$|\phi_\delta(k,x)|_{\mathcal{A}} \leq \max\{2\tilde{\beta}(|x|_{\mathcal{A}} + \|\mathbf{e}\|, k), \varepsilon/2\} + \|\mathbf{e}\|.$$

Since $\|\mathbf{e}\| \leq \varepsilon/2, \|\mathbf{e}\| \leq a_2$, and $a + b \leq \max\{2a, 2b\}$

$$\begin{aligned}|\phi_\delta(k,x)|_{\mathcal{A}} &\leq \max\{4\tilde{\beta}(|x|_{\mathcal{A}} + \|\mathbf{e}\|, k), \varepsilon\}\\ &\leq \max\{4\tilde{\beta}(2|x|_{\mathcal{A}}, k), \varepsilon\}\\ &\leq \beta(|x|_{\mathcal{A}}, k) + \varepsilon\end{aligned}$$

where $\beta(s,k) := 4\tilde{\beta}(2s, k)$ and we have the result. ∎

Proof of Corollary 5: To change the proof so that it is applicable to Corollary 5, we rely on Corollary 2 to generate $\tilde{\beta}, \tilde{\alpha}, \tilde{\gamma}$, and $\tilde{N}$. The proof is then exactly the same, but since $\tilde{N}$ is independent of $\tilde{\delta}$ (and hence ε), we obtain the semiglobal result. Here, $\tilde{\delta}$ is then just a notational convenience and can be replaced by making an appropriate change to the bound on $\|\mathbf{d}\|$. ∎

Proof of Corollary 6: To change the proof so that it is applicable to Corollary 6, we assume that we are given N that satisfies the conditions of Corollary 3 and only look at $\mathcal{F}_N$ instead of $\mathcal{F}_\infty$, rely on Corollary 3 to generate $\tilde{\beta}$ and $\tilde{\gamma}$, and ignore all terms involving $\tilde{\Delta}$. As in the case of the previous corollary, $\tilde{\delta}$ becomes a notational convenience. ∎

Appendix VI
Changing Supply Functions in the Presence of Disturbance

The following lemma is similar to results found in [17] but specialized for our use.

Lemma 7: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with δ continuous and zero at zero, be such that

$$\begin{aligned}h(x) \leq V(x) &\leq \alpha_1(\sigma(x))\\ V(z) - V(x) &\leq \alpha_2(\sigma(x)) - h(x) + \delta(\varepsilon)\end{aligned} \tag{55}$$

for all $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and all $z \in f(x) + \varepsilon\mathcal{B}$ for some $\varepsilon > 0$. Let $\rho \in \mathcal{K}_\infty$ be such that $q(s) := (d\rho/ds)(s)$ is well defined, continuous, and nondecreasing. Then there exists a continuous function $\delta_V : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\delta_V(s, 0) = 0$ for all s and

$$\begin{aligned}\rho(V(z)) - \rho(V(x)) \leq\ & q\{\alpha_1(\sigma(x)) + \alpha_2(\sigma(x))\} \cdot \alpha_2(\sigma(x))\\ & - q\left\{\frac{1}{2}h(x)\right\} \cdot \frac{1}{2}h(x) + \delta_V(x, \varepsilon)\end{aligned} \tag{56}$$

for all $x \in \mathcal{X}$. ∎

Proof: By the fact that q is nondecreasing and the mean value theorem, we have

$$\rho(a) - \rho(b) \leq q(a) \cdot (a - b)$$

for all nonnegative a and b. We also have

$$V(z) \leq \alpha_1(\sigma(x)) + \alpha_2(\sigma(x)) + \delta(\varepsilon)$$

from (55). We study two possible cases.

Case 1) $V(z) \leq (1/2)V(x)$

$$\begin{aligned}\rho(V(z)) - \rho(V(x)) &\leq \rho\left(\frac{1}{2}V(x)\right) - \rho(V(x))\\ &\leq q\left(\frac{1}{2}V(x)\right) \cdot \left(-\frac{1}{2}V(x)\right)\\ &\leq q\left(\frac{1}{2}h(x)\right) \cdot \left(-\frac{1}{2}h(x)\right).\end{aligned}$$

Case 2) $V(z) \geq (1/2)V(x)$

$$\begin{aligned}&\rho(V(z)) - \rho(V(x))\\ &\quad \leq q(V(z)) \cdot (\alpha_2(\sigma(x)) - h(x) + \delta(\varepsilon)\\ &\quad = q(V(z)) \cdot \alpha_2(\sigma(x)) - q(V(z)) \cdot h(x)\\ &\qquad + q(V(z)) \cdot \delta(\varepsilon)\\ &\quad \leq q(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x)) + \delta(\varepsilon)) \cdot \alpha_2(\sigma(x))\\ &\qquad - q\left(\frac{1}{2}V(x)\right) \cdot h(x)\\ &\qquad + q(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x)) + \delta(\varepsilon)) \cdot \delta(\varepsilon).\end{aligned}$$

Define $\tilde{\delta}(s,\varepsilon) := q(\alpha_1(\sigma(s)) + \alpha_2(\sigma(s)) + \delta(\varepsilon)) - q(\alpha_1(\sigma(s)) + \alpha_2(\sigma(s)))$. Since $q \in \mathcal{K}_\infty$ and δ is continuous and zero at zero, $\tilde{\delta}$ is nonnegative, continuous, and zero at zero in its second argument. Then

$$\begin{aligned}
\rho(V(z)) - \rho(V(x))
&\le q(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x))) \cdot \alpha_2(\sigma(x)) \\
&\quad - q\left(\frac{1}{2}V(x)\right) \cdot h(x) + \tilde{\delta}(s,\varepsilon) \cdot \alpha_2(\sigma(x)) \\
&\quad + q(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x)) + \delta(\varepsilon)) \cdot \delta(\varepsilon) \\
&\le q(\alpha_1(\sigma(x)) + \alpha_2(\sigma(x))) \cdot \alpha_2(\sigma(x)) \\
&\quad - q\left(\frac{1}{2}h(x)\right) \cdot h(x) + \delta_V(\sigma(x),\varepsilon)
\end{aligned}$$

where $\delta_V(s,\epsilon) := \tilde{\delta}(s,\epsilon) \cdot \alpha_2(\sigma(s)) + q(\alpha_1(\sigma(s)) + \alpha_2(\sigma(s)) + \delta(\epsilon)) \cdot \delta(\epsilon)$. In both cases, (56) holds. ■

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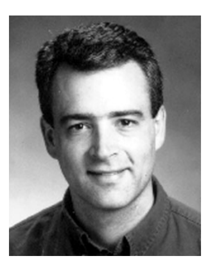

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