MTH9831 HW1 Team 5

Theoretical problems break-down: Zhu, Bin 3.4
Twahir, Omar 3.6, 3.7
Wu, Chengxun (James) 3.4, 3.7, 3.11

3.4

Reflection at time S.

① $\mathbb{E}[\tilde{B_t}] = \begin{cases} \mathbb{E}[B_t] & \text{if } t \le S \\ 2\mathbb{E}[B_S] - \mathbb{E}[B_t] & \text{if } t > S \end{cases}$ Since $(B_t, t \ge 0)$ is a standard Brownian motion, we have $\mathbb{E}B_S = \mathbb{E}B_t = 0$, thus $\mathbb{E}(\tilde{B_t}) = 0$.

② Pick $t_1 < t_2$. We consider the following cases:

i. $t_1 < t_2 < S$. Hence, we have $\mathrm{COV}(\tilde{B}_{t_1}, \tilde{B}_{t_2}) = \mathrm{COV}(B_{t_1}, B_{t_2}) = t_1$ by BM property.

ii. $S < t_1 < t_2$. $\mathrm{COV}(\tilde{B}_{t_1}, \tilde{B}_{t_2}) = \mathrm{COV}(2B_S - B_{t_1}, 2B_S - B_{t_2})$

$= 4\mathrm{COV}(B_S, B_S) - 2\mathrm{COV}(B_{t_1}, B_S) - 2\mathrm{COV}(B_{t_2}, B_S) + \mathrm{COV}(B_{t_1}, B_{t_2})$

$\hookrightarrow \mathrm{cov}(B_S, B_t) = S \wedge t$ $\quad = 4S - 2S - 2S + t_1 = t_1 = t_1 \wedge t_2$.

iii. $t_1 < S < t_2$. Then $\mathrm{COV}(\tilde{B}_{t_1}, \tilde{B}_{t_2}) = \mathrm{COV}(B_{t_1}, 2B_S - B_{t_2}) = 2\mathrm{COV}(B_{t_1}, B_S) - \mathrm{COV}(B_{t_1}, B_{t_2})$

$= 2t_1 - t_1$

$= t_1 = t_1 \wedge t_2$.

Thus, $\forall t_1 < t_2$, $\mathrm{COV}(\tilde{B}_{t_1}, \tilde{B}_{t_2}) = t_1 \wedge t_2$.

③ Continuity. when $t < S$, $\tilde{B_t} = B_t$ is continuous with probability 1

$t > S$, $\tilde{B_t} = 2B_S - B_t$ is also continuous with probability 1

Moreover, when $t = S$, we have $B_t = 2B_t - B_t = 2B_S - B_t$. This means the left limit and right limit of $\tilde{B}_t$ approximating $t = S$ is the same $\Rightarrow$ $\tilde{B_t}$ also continuous at $t = S$.

Hence $\tilde{B_t}$ is continuous for all t, with probability one

By ①②③, we have shown that $\tilde{B_t}$ is a standard Brownian motion.

3.7

a) proof. $B_t = (1+t)\, U_{\frac{t}{1+t}}$.

- $\mathbb{E}[B_t] = (1+t)\,\mathbb{E}[U_{\frac{t}{1+t}}] = (1+t)\,\mathbb{E}[B_{\frac{t}{1+t}} - \frac{t}{1+t} B(1)]$

$= (1+t)\,\mathbb{E}[B_{\frac{t}{1+t}}] - t\,\mathbb{E}[B_1]$

$= 0$

- $\mathrm{Cov}(B_S, B_t) = \mathrm{Cov}((1+S)\, U_{\frac{S}{1+S}}, (1+t)\, U_{\frac{t}{1+t}})$

$= (1+S)(1+t)\, \mathrm{Cov}(U_{\frac{S}{1+S}}, U_{\frac{t}{1+t}})$.

Since we know that U_t, Brownian bridge, has covariance $S(1-t)$ for $S \le t$.
WLOG assume $S < t \Rightarrow \mathrm{Cov}(U_{\frac{S}{1+S}}, U_{\frac{t}{1+t}}) = \frac{S}{1+S}\left(\frac{1}{1+t}\right)$ since $\frac{S}{1+S} < \frac{t}{1+t}$.
Therefore, $\mathrm{Cov}(B_S, B_t) = (1+S)(1+t)\frac{S}{1+S}\cdot\frac{1}{1+t} = S = t \wedge S$

- continuity. $(1+t)\, U_{\frac{t}{1+t}} = (1+t)\, B_{\frac{t}{1+t}} - t B_1$ is continuous as it's an combination of continuous functions.

We hence conclude that $B_t = (1+t)\, U_{\frac{t}{1+t}},\ t \in [0,1]$ is a standard Brownian motion.

b) $\frac{B_t}{t} = \frac{1+t}{t} U_{\frac{t}{1+t}} \Rightarrow \lim_{t\to\infty} \frac{B_t}{t} = \lim_{t\to\infty}(1+\frac{1}{t}) \cdot \lim_{t\to\infty} U_{\frac{t}{1+t}}$

$= 1 \cdot U_1$

$= U_1 = 0$, by the definition of Brownian bridge.

Thus, we conclude that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely. $\square$

3.11 proof. Let $S_n := \sum_{k=1}^{n} X_i$. Since $X_n \geq 0\ \forall n$, we have $S_n \geq 0$ and $S_{n+1} - S_n = X_{n+1} \geq 0 \Rightarrow S_n$ is increasing.

By the Monotone Convergence Theorem, we have:

$\lim_{n\to\infty} \mathbb{E}[S_n] = \mathbb{E}[\lim_{n\to\infty} S_n]$, which is equivalent to

$$\lim_{n\to\infty} \mathbb{E}[\textstyle\sum_{k=1}^{n} X_k] = \mathbb{E}[\lim_{n\to\infty} \textstyle\sum_{k=1}^{n} X_k]$$

$$\Leftrightarrow \quad \sum_{n\geq 1} \mathbb{E}[X_n] = \mathbb{E}[\textstyle\sum_{n\geq 1} X_n]. \qquad \square$$

Omar Twahir

HW 1

3.6 Let $(B_t, t \geq 0)$ be a Brownian motion. We consider the process $X_t = t B_{1/t}$ for $t > 0$.

This property of Brownian motion relates the behavior for t large to the behavior for t small.

(a) Show that $(X_t, t > 0)$ has the distribution of a Brownian motion on $t > 0$.

(b) Argue that X_t converges to 0 as $t \to 0$ in the sense of L^2-convergence. It is possible to show convergence almost surely so that $(X_t, t \geq 0)$ is really a Brownian motion for $t \geq 0$.

(c) Use this property of Brownian motion to show that the law of larger numbers for Brownian motion $\lim_{t \to \infty} \frac{X_t}{t} = 0$ almost surely.

a) X_t is a Brownian motion iff its a Gaussian process, $\mathbb{E}[X_t] = 0$ for all t, $\mathbb{E}[X_s X_t] = (s \wedge t)$, on a set Ω of probability one, the paths $t \mapsto B_t(\omega)$ are continuous.

If $(X_{t_1}, \dots, X_{t_n})$ is a Gausssian vector for any $t_1 < \dots < t_n \Rightarrow X_t$ is a Gaussian process

Consider $(X_{t_1} - 0, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$

i.e. $\left(t_1 B_{1/t_1}, \; t_2 B_{\frac{1}{t_2}} - t_1 B_{\frac{1}{t_1}}, \dots \; t_n B_{\frac{1}{t_n}} - t_{n-1} B_{\frac{1}{t_{n-1}}} \right)$

It is a Gaussian vector since its a linear transformation of a Gaussian vector

$$\mathbb{E}[X_t] = \mathbb{E}[t B_{\frac{1}{t}}] = t \, \underbrace{\mathbb{E}[B_{1/t}]}_{=0} = 0$$

since $\mathbb{E}[B_t] = 0$ for all t

$$\mathbb{E}[X_s X_t] = \mathbb{E}[s B_{1/s} \, t B_{1/t}] = st \, \mathbb{E}[B_{1/s} B_{1/t}]$$

$$= st \cdot \left(\frac{1}{s} \wedge \frac{1}{t}\right) = (s \wedge t)$$

$f(t) = 1/t$ and $g(x) = tx$

Since X_t is a composition of continuous functions $\Rightarrow$ $t \mapsto X_t(\omega)$ is a continuous function for a set ω of probability one.

b) WTS: $\mathbb{E}[(X_t)^2] \to 0$

$$\mathbb{E}[(X_t)^2] = \mathbb{E}[(t B_{1/t})^2] = \mathbb{E}[t^2 B^2_{1/t}] = t^2 \mathbb{E}[B^2_{1/t}]$$

$$\mathbb{E}[B^2_{1/t}] = \frac{1}{t} \Rightarrow t^2 \mathbb{E}[B^2_{1/t}] = t^2 \cdot \frac{1}{t} = t \Rightarrow \text{if } t \to 0$$

$\Rightarrow X_t$ converges to 0 in the sense of L^2 convergence.

c) WTS: $\lim_{t \to \infty} \frac{X_t}{t} = 0$ almost surely

i.e. $\lim_{t \to \infty} \frac{t B_{1/t}}{t} = 0$ almost surely

$\Leftrightarrow$ $\lim_{t \to \infty} B_{1/t} = 0$ almost surely

$\Leftrightarrow$ $B_0 = 0$ by defn of Brownian motion