Final Exam

Wednesday December 21 6-8 pm 9-140

3 problems 10 T or F Review sheet

- Problems: Chapter 6-7-8-9-10
- T or F : All chapters + 1 Jump Processes

Chapter 11 : Jump Processes

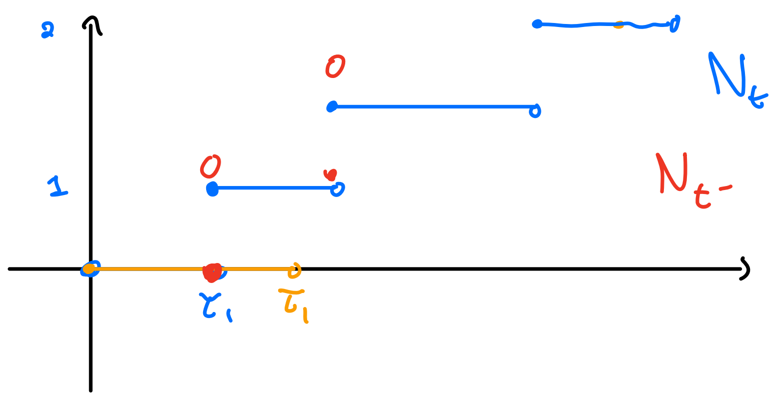
We already saw the Poisson process $(N_t, t\geq 0)$ with rate $\lambda > 0$

- $N_0 = 0$
- increments are independent and

$$N_t - N_s \sim \text{Poisson}\left(\lambda(t-s)\right)$$

Poisson paths

Increasing so paths have bounded variation.



? Can the process jump by more than 1 at a given time?

$$\lim_{t\to 0} \frac{\mathbb{P}\left(N_t = 2\right)}{t} = \lim_{t\to 0} \frac{(\lambda t)^2 e^{-\lambda t}}{t} = 0$$

In fact let τ_k IID $\text{Exp}(\lambda)$

$$\boxed{N_t := \#\left\{ k : \tau_1 + \dots + \tau_k \leq t \right\}}$$

τ_k : waiting time between the $(k-1)^{th}$ and k^{th} jumps.

Why? A sum of k IID $\mathrm{Exp}(\lambda)$ is distributed like a $\mathrm{Gamma}(\lambda)$.

PDF of $\tau_1 + \dots + \tau_k$ is $\dfrac{(\lambda t)^{k-1} \lambda e^{-\lambda t}}{\Gamma(k)}$, $\quad \Gamma(k) = (k-1)!$ integer k

Thus $\quad P(N_t \geq k+1) = P(\tau_1 + \dots + \tau_{k+1} \leq t)$

$$= \int_0^t \frac{(\lambda s)^k \lambda e^{-\lambda s}}{\Gamma(k+1)} ds \qquad \Gamma(k+1) = k\Gamma(k)$$

integration by parts

$$= \frac{(\lambda s)^k e^{-\lambda s}}{\Gamma(k+1)}\Big|_0^t + \int_0^t \frac{k(\lambda s)^{k-1} \lambda e^{-\lambda s}}{\Gamma(k+1)} ds$$

$$\frac{(\lambda t)^k e^{-\lambda t}}{\Gamma(k+1)} + P(N_t \geq k)$$

$\Rightarrow$

$$P(N_t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \text{Poisson r.v.}$$

For independence of increments, see notes. Use memory loss property. $\blacksquare$

Note that the Poisson process is time-homogeneous Markov!

$$E[g(N_t) \mid \mathcal{F}_s]$$

$$= E[g(N_s + \underbrace{N_t - N_s) \mid \mathcal{F}_s}_{\text{independent}}] = \sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!} g(N_s + k)$$

What is the generator?

$$\partial_t f = Af(x) = \lim_{t\to 0} \frac{E\left[f(x+N_t) \mid \overset{=0}{N_0} \right] - f(x+N_0)}{t}$$

$$\frac{E\left[f(N_t) \mid N_0 = x \right] - f(N_0)}{t}$$

$$\lim_{t\to 0} \frac{\sum_{k=0}^{\infty} f(x+k) \frac{(\lambda t)^k e^{-\lambda t}}{k!} - f(x)}{t}$$

$$= \lim_{t\to 0} \frac{f(x) e^{-\lambda t} - f(x) + f(x+1) \lambda t e^{-\lambda t}}{t}$$

$$= \boxed{\lambda \left(f(x+1) - f(x) \right) = Af}$$

!

We have the backward-forward eqns as before

$$\partial_t f = Af$$

$$\partial_t f = A^* f$$

More generally we have the class of Lévy processes

Definition 8.1. *A process $(X_t, t \geq 0)$ is called a Lévy process if it has independent and stationary increments, that is*

(1) *for all n and all $0 \leq t_1 < t_2 < \cdots \leq t_n < \infty$ the increments $X_{t_{j+1}} - X_{t_j}$ are independent;*

(2) *the increment $X_{t+s} - X_s$ has the same distribution as X_t for all $t, s \geq 0$.*

t Infinite divisibility

The Lévy processes are completely classified

- Lévy-Khinchine formula
- Markov processes!

Essentially a Lévy process is a linear combination of BM with drift and a Compound Poisson process.

continuous part

jump part

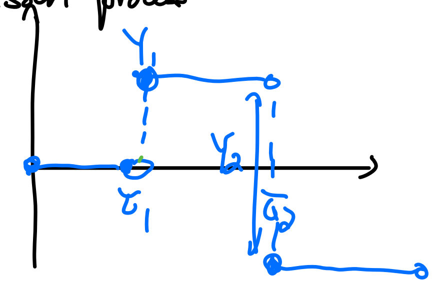
Compound Poisson process:

Let (Y_k) be IID r.v.

Let (N_t) be a Poisson process of rate λ.

Then define the compound Poisson process

$$\boxed{Q_t = \sum_{k=1}^{N_t} Y_k}$$



Lévy process?

Generator? $A = \lambda \int_{\mathbb{R}} \left(f(x+y) - f(x) \right) \underbrace{g_Y(y)}_{\text{PDF of } Y} dy.$

Constructing martingales using Poisson Process

As for BM we can construct "easy martingales"

Examples:

- $N_t - \lambda t$
- Geometric: $\exp\left(\alpha N_t - \lambda(e^{\alpha} - 1) t \right)$

Do martingale transform to construct more martingales!

Stochastic Integrals and Doeblin-Itô's formula

Important remarks

- Paths of (N_t) have bounded variation so we can define $\int_0^t X_s \, dN_s$ as usual

- What strategy can we integrate? (X_t) adapted However jumps are tricky...

Example. Consider $M_s = N_s - \lambda s$

$$\int_0^t X_s \, dM_s = \underbrace{\int_0^t X_s \, dN_s}_{①} - \lambda \underbrace{\int_0^t X_s \, ds}_{②}$$

$$X_s = \underbrace{N_s - N_{s^-}}_{= \Delta N_{s^-}} = \begin{cases} 1 & \text{if jump at } s \\ 0 & \text{else} \end{cases}$$

Not a martingale

② $= 0$

① $= \sum_{0 \leq s \leq t} X_s (N_s - N_{s^-}) = \sum_{0 \leq s \leq t} (N_s - N_{s^-})^2 = N_t$

Conclusion : We need the strategy to be left-continuous $\mathcal{F}_{s^-}$

We want X_s to be $\mathcal{F}_{s^-}$-mble.

Definition 8.5. *Let $(J_t, t \geq 0)$ be a pure jump process with only finitely many jumps on a finite interval of time a.s. We suppose that J_t is constant between jumps and that it is right-continuous with left limits* CÀDLÀG. Continue à droite, limite à gauche

Let (Φ_s) be a left-continuous adapted process for the filtration of (J_s). Then

$$\int_0^t \Phi_s \, \mathrm{d}J_s = \sum_{0 \le s \le t} \Phi_s \Delta J_s$$

where $\Delta J_s = J_s - J_{s^-}$ is the size of the jumps at s. Note that the sum is well-defined as it is a finite sum!

Towards Itô's formula

Definition 8.6. *A semi-martingale (X_t) is a process of the form*

$$X_t = X_0 + \int_0^t \Gamma_s \, \mathrm{d}B_s + \int_0^t \Theta_s \, \mathrm{d}s + J_t$$

where (J_t) is a jump process as above. We denote by X_t^c the continuous part of X_t, i.e.

$$\mathrm{d}X_t^c = \Gamma_t \, \mathrm{d}B_t + \Theta_t \, \mathrm{d}t \ .$$

We assume X_t^c is an Itô process.

Itô's formula (In fact Doeblin-Itô)

Theorem 8.9. *Let $f \in \mathcal{C}^2(\mathbb{R})$ and (X_t) a semi-martingale as above. Then we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, \mathrm{d}X_s^c + \frac{1}{2} \int_0^t f''(X_s)(\, \mathrm{d}X_s^c \cdot \, \mathrm{d}X_s^c) + \sum_{0 \le s \le t} \{f(X_s) - f(X_{s^-})\} \ .$$

<u>Example</u>: $S_t = S_0 \exp\left(\alpha N_t - \lambda(e^\alpha - 1)t\right)$

Let's prove again that (S_t) is a martingale

$f(x) = e^x \qquad X_t = \alpha N_t - \lambda(e^\alpha - 1)t$

$(dX_t^c)^2 = 0 \qquad dX_t^c = -\lambda(e^\alpha - 1)dt$

$dS_t = \overbrace{e^{X_t}}^{S_t}\left(-\lambda(e^\alpha - 1)\right)dt$

if jump at u $\quad f(X_u) - f(X_{u-})$

$$f(X_u) = e^{\alpha(N_{u^-}+1) - \lambda(e^\alpha - 1)u}$$

$$- \quad f(X_{u^-}) = e^{\alpha N_{u^-} - \lambda(e^\alpha-1)u}$$

$$\underbrace{e^{\alpha N_{u^-} - \lambda(e^\alpha - 1)u}}_{S_{u^-}} \left(e^\alpha - 1\right)$$

$$dS_t = -\lambda(e^\alpha - 1) S_t dt + (e^\alpha - 1) S_{t^-} \Delta N_{t^-}$$

$$S_t = S_0 - \int_0^t \lambda(e^\alpha-1) \underbrace{S_u}_{\downarrow} du + (e^\alpha - 1) \sum_{0 \le u \le t} S_{u^-} \Delta N_u$$

$$\int_0^t \underbrace{S_{u^-}}_{} dN_u$$

might be not left-continuous but Riemann! S_{u^-}

left-continuous version of S_u

Martingale

$$S_t = S_0 + (e^\alpha - 1) \int_0^t S_{u^-} dM_u$$

Martingale

$\hookrightarrow$ compensated.

$$dS_t = (e^\alpha - 1) S_{u^-} dM_u$$

RN pricing for jump processes

Let (N_t) be a Poisson process with rate λ.

Change of probability

(c) **Biasing a Poisson random variable.** Let N be a Poisson random variable with parameter $\lambda > 0$.

(i) Show that the MGF of N is

$$\mathbf{E}[e^{aN}] = \exp(\lambda(e^a - 1)).$$

(ii) Use the above to show that under the probability $\widetilde{\mathbf{P}}$ defined by $\widetilde{\mathbf{P}}(\mathcal{E}) = \mathbf{E}[M\mathbf{1}_{\mathcal{E}}]$, where $M = \frac{e^{aN}}{\mathbf{E}[e^{aN}]}$, the random variable N is also Poisson-distributed with parameter λe^a.

(1) Exercise 9.3

Solutions. (a) Using the distribution of a Poisson variable, we have

$$\mathbf{E}[e^{aN}] = \sum_{k=0}^{\infty} e^{ak}\frac{\lambda^k}{k!}e^{-\lambda} = \exp(\lambda(e^a - 1)),$$

by the Taylor expansion of exponential.

(b) It suffices to compute the MGF under $\widetilde{\mathbf{P}}$. We get for $b \in \mathbb{R}$

$$\widetilde{\mathbf{E}}[e^{bN}] = \mathbf{E}\left[\frac{e^{aN}}{\mathbf{E}[e^{aN}]}e^{bN}\right] = \frac{\mathbf{E}[e^{(a+b)N}]}{\mathbf{E}[e^{aN}]} = \exp(\lambda e^a(e^b - 1)),$$

by the above. But this is the MGF of a Poisson variable of parameter λe^a. □

More generally

Proposition 8.12. *Let (X_t) be a semi-martingale. Consider the process*

$$Z_t = \exp(X_t^c - \frac{1}{2}\int_0^t \mathrm{d}X_t^c \cdot \mathrm{d}X_t^c) \prod_{0\leq s\leq t}(1 + \Delta X_s)\ .$$

Then Z_t satisfies

$$Z_t = 1 + \int_0^t Z_{s^-}\ \mathrm{d}X_s\ .$$

In particular, if (X_t) is a martingale then so is Z_t (under the assumption of Theorem 8.8).

Pricing

Consider a market model with

- Risky asset: $S_t = S_0 \exp\left(\alpha N_t - \lambda t (e^{\alpha}-1) + \mu t\right)$

- Discounting: $D_t = e^{-rt}$

Existence of the RNP?

$$d\left(e^{-rt} S_t\right) = -re^{-rt} S_t dt + e^{-rt} dS_t \quad + 0$$

$$dS_t = S_{t-}\,(e^{\alpha}-1)\left(dM_t\right) + \mu S_t dt$$

Thus $$d\tilde{S}_t = (e^{\alpha}-1)\,\tilde{S}_{t-}\, dM_t - r\,\tilde{S}_t dt + \mu \tilde{S}_t dt$$

$$= (e^{\alpha}-1)\tilde{S}_{t-} dN_t - \underbrace{\left(\lambda(e^{\alpha}-1) + r - \mu\right)}_{\tilde{\lambda}} \tilde{S}_t dt$$

$$= (e^{\alpha}-1)\tilde{S}_{t-}\left(\underset{\text{rate } \lambda \neq \tilde{\lambda}}{dN_t} - \underbrace{\left(\lambda + \frac{r-\mu}{e^{\alpha}-1}\right)}_{\tilde{\lambda}} dt\right)$$

We want (N_t) to be a Poisson process with rate $\tilde{\lambda}$.

Which a? $\quad \tilde{\lambda} = \lambda e^{a} \Rightarrow a = \log \frac{\tilde{\lambda}}{\lambda}$

$$M_T = \frac{e^{aN_T}}{E\left[e^{aN_T}\right]}$$

$$\tilde{\lambda} = \lambda + \frac{r-\mu}{e^{\alpha}-1}$$

$$\boxed{\text{We need } \tilde{\lambda} > 0 \text{ for existence of RNP}}$$

condition $\lambda + \frac{r-\mu}{e^{\alpha}-1} > 0$

$d\tilde{S}_t = (e^{\alpha}-1)\tilde{S}_{t-}\, dM_t$

Geometric PP with rate $\tilde{\lambda}$

$\downarrow$ comp with rate λ.

Pricing of European options

Price of a Call?

$$O_t = e^{-r(T-t)} \tilde{E}\left[O_T \mid \mathcal{F}_t \right]$$

$$O_t = e^{-r(T-t)} \tilde{E}\left[(S_T - K)^+ \mid \mathcal{F}_t \right]$$

Explicit formula

$$O_t = e^{-r(T-t)} \tilde{E}\left[\left(e^{rT} \tilde{S}_T - K \right)^+ \mid \mathcal{F}_t \right]$$

Under $\tilde{\mathbb{P}}$ $\tilde{S}_t = S_0 \exp\left(\alpha \tilde{N}_t - \tilde{\lambda}(e^{\alpha}-1)t \right)$

$$\tilde{S}_T = \tilde{S}_t \exp\left(\alpha(\tilde{N}_T - \tilde{N}_t) - \tilde{\lambda}(e^{\alpha}-1)(T-t) \right)$$

$$= e^{-r(T-t)} \tilde{E}\left[\left(e^{r(T-t)} S_t e^{\alpha(\tilde{N}_T - \tilde{N}_t) - \tilde{\lambda}(e^{\alpha}-1)(T-t)} - K \right)^+ \mid \mathcal{F}_t \right]$$

$$O_t = \sum_{j=0}^{\infty} \left(S_t e^{\alpha j - \tilde{\lambda}(e^{\alpha}-1)(T-t)} - K e^{-r(T-t)} \right)^+ \frac{\left(\tilde{\lambda}(T-t)\right)^j}{j!} e^{-\tilde{\lambda}(T-t)}$$

PDE formula

$$dS_t = (e^{\alpha}-1) S_{t-}\, \underbrace{d\tilde{M}_t}_{= d\tilde{N}_t - \tilde{\lambda} dt} + r S_{t-}\, dt$$

Use Itô - Doeblin

Assume $\Theta_t = f(t, S_t)$

$$d\left(e^{-rt} f(t,S_t)\right) = -r e^{-rt} f + \partial_0 f\, e^{-rt} dt + e^{-rt}\partial_1 f\, dS_t^c + e^{-rt}\tfrac{1}{2}\partial_1^2 f\, \cancel{d\langle S^c \rangle_t}^{\,0}$$
$$+ e^{-rt}\left(f(t,S_t) - f(t,S_{t-})\right)$$

$$=$$

$$dS_t = (e^{\alpha}-1) S_{t-}\, d\tilde{N}_t + \underbrace{\left(r S_t - \tilde{\lambda}(e^{\alpha}-1) S_t\right) dt}_{S_t^c}$$

$$e^{-rt}\left\{ -rf + \partial_0 f + \partial_1 f\left(r S_t - \tilde{\lambda}(e^{\alpha}-1) S_t\right) \right\} dt$$
$$+ \quad e^{-rt}\left(f(t,S_t) - f(S_{t-}) \right)$$

Rewrite $\displaystyle\sum_{0 \le u \le t} f(u,S_u) - f(u,S_{u-})$

At jump $f(u, S_u) = f(u, e^{\alpha} S_{u-})$

$$= \sum_{0 \le u \le t} \Big(f(u, e^{\alpha} S_{u-}) - f(u, S_{u-}) \Big) \Delta \tilde{N}_u$$

martingale

$$= \int_0^t \underbrace{f(u, e^{\alpha} S_{u-}) - f(u, S_{u-})}_{\text{left-continuous!}} \, d\tilde{M}_u$$

$$+ \int_0^t \tilde{\lambda} \Big(f(u, e^{\alpha} S_{u-}) - f(u, S_{u-}) \Big) dt.$$

$$dt.$$

$$-rf + \partial_0 f + \partial_1 f \Big(r S_t - \tilde{\lambda}(e^{\alpha} - 1) S_t \Big) + \tilde{\lambda} \Big(f(t, e^{\alpha} S_{t-}) - f(t, S_{t-}) \Big)$$

$$f(t, x)$$

$$\boxed{\partial_t f + \partial_x f \Big(rx - \tilde{\lambda}(e^{\alpha} - 1) x \Big) - rf + \tilde{\lambda} \Big\{ f(t, e^{\alpha} x) - f(t, x) \Big\} = 0}$$