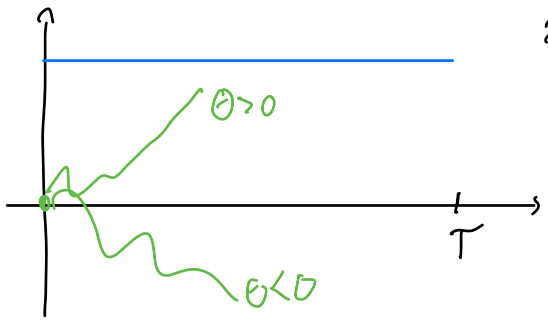
Another application of Cameron-Martin-Girsanov

$$\left\{\max_{t\in[0,T]} \widetilde{B}_t > a\right\} = \left\{\widetilde{\tau}_a \leq T\right\}$$



$$\widetilde{B}_t = B_t + \theta t$$

$$\max_{t\in[0,T]} \widetilde{B}_t$$

$$\widetilde{\tau}_a = \min\{t \geq 0 : \widetilde{B}_t \geq a\}$$

Proposition 9.7 (Bachelier formula for Brownian motion with constant drift). *Let $\widetilde{B}_t = B_t + \theta t$ be a Brownian motion with drift θ. Let $\tilde{\tau}_a = \min\{t \geq 0 : \widetilde{B}_t > a\}$. Then we have*

$$\mathbf{P}\left(\max_{t\in[0,T]} \widetilde{B}_t > a\right) = \mathbf{P}(\tilde{\tau}_a \leq T) = \int_0^T \frac{a}{s^{3/2}} \frac{e^{-\frac{(a-\theta s)^2}{2s}}}{\sqrt{2\pi}}\, ds.$$

$a>0$

$\theta<0$

In particular, the PDF of $\tilde{\tau}_a$ is

$$f_{\tilde{\tau}_a}(t) = \frac{a}{t^{3/2}} \frac{e^{-\frac{(a-\theta t)^2}{2t}}}{\sqrt{2\pi}}. \tag{9.11}$$

Proof Beautiful!

$$\mathcal{E}_a = \left\{\max_{t\leq T} \widetilde{B}_t > a\right\} = \left\{\widetilde{\tau}_a \leq T\right\}$$

$$P\left(\max_{t\leq T} \widetilde{B}_t > a\right) = E\left[\mathbb{1}(\mathcal{E}_a)\right]$$

$$= E\left[\left(\frac{d\widetilde{P}}{dP}\right)\left(\frac{d\widetilde{P}}{dP}\right)^{-1} \mathbb{1}(\mathcal{E}_a)\right]$$

$$\frac{d\widetilde{P}}{dP} = e^{-\theta \widetilde{B}_T + \frac{1}{2}\theta^2 T}$$

$$= \widetilde{E}\left[\underbrace{e^{\theta \widetilde{B}_T - \frac{1}{2}\theta^2 T}}_{M_T} \mathbb{1}(\widetilde{\tau}_a \leq T)\right]$$

$$= \widetilde{E}\left[M_T \underbrace{\overbrace{\mathbb{1}(\widetilde{\tau}_a \leq T)}^{\text{fct of } \widetilde{\tau}_a}}_{\mathcal{F}_{\widetilde{\tau}_a}\text{-mble}}\right] \qquad \widetilde{E}\left[M_T \mid \mathcal{F}_{\widetilde{\tau}_a}\right] = M_{\widetilde{\tau}_a}$$

$$= \widetilde{E}\left[M_{\widetilde{\tau}_a} \mathbb{1}(\widetilde{\tau}_a \leq T)\right] \qquad M_{\widetilde{\tau}_a} = e^{a\theta - \frac{1}{2}\theta^2 \widetilde{\tau}_a}$$

$$= \tilde{E}\left[e^{a\theta - \frac{1}{2}\theta^2 \tilde{\tau}_a} 1(\tilde{\tau}_a \le T) \right]$$

We know the distribution of $\tilde{\tau}_a$ for std BM!

$$f(t) = \frac{a}{\sqrt{2\pi}} \frac{1}{t^{3/2}} e^{-a^2/2t}$$

$$= \int_0^T e^{a\theta - \frac{1}{2}\theta^2 s} \frac{a}{\sqrt{2\pi}} \frac{e^{-a^2/2s}}{s^{3/2}} \, ds$$

$$= \int_0^T \frac{a}{\sqrt{2\pi}} \frac{e^{-(a-\theta s)^2/2s}}{s^{3/2}} \, ds$$

6. Multivariate Itô Calculus

We generalize Itô calculus to several variables.

This is useful when considering market models with:

1) several risky assets
2) random interest rates

6.1 Multidimensional Brownian motion

Definition

A d-dimensional standard Brownian motion is a process in $\mathbb{R}^d$ of the form

$$t \longmapsto \left(B_t^{(1)}, \ldots, B_t^{(d)}\right) = \vec{B}_t$$

where $\left(B_t^{(j)}, t \geq 0\right)$, $j \leq d$, are IID standard BM's.

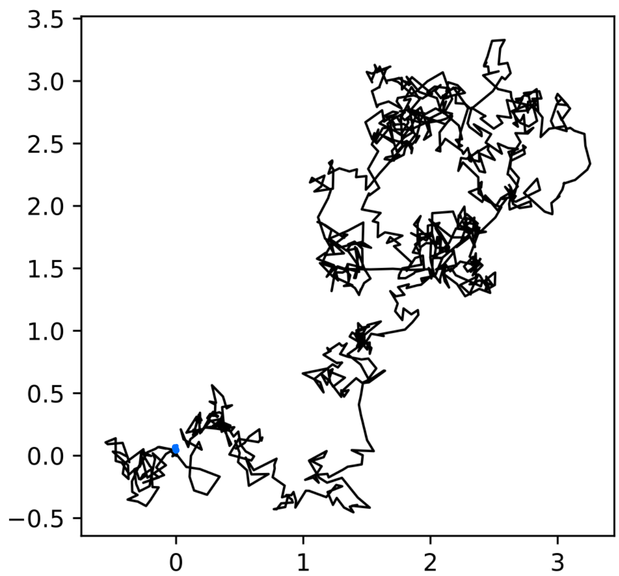


Figure 6.1. A simulation of a path of two-dimensional Brownian motion starting at (0, 0) from time 0 to time 5 for a time discretization of 0.005. See Numerical Project 6.1.

Numerical Project!

Example We can define BM's as follows:

$$\begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_t^{(1)} \\ B_t^{(2)} \end{pmatrix} \quad \sim X = AZ$$

More generally $\vec{W}_t = \mathcal{O} \vec{B}_t$ where $\mathcal{O}$ is an orthogonal matrix.

Exercise 6.1 and 6.2

6.2 Itô's formula

For a smooth function $f: \mathbb{R}^d \to \mathbb{R}$, we can define a process $t \longmapsto f(B_t^{(1)}, \dots, B_t^{(d)})$.

1) Can we write this process in differential form?
2) Can we find conditions on f for $f(\vec{B}_t)$ to be a martingale?

Same questions as for 1D!

Theorem 6.6 (Itô's formula). *Let $(B_t, t \geq 0)$ be a d-dimensional Brownian motion. Consider $f \in \mathcal{C}^2(\mathbb{R}^d)$. Then we have with probability one that for all $t \geq 0$,*

$f: \mathbb{R}^d \to \mathbb{R}$

$$f(B_t) - f(B_0) = \underbrace{\sum_{i=1}^{d} \int_0^t \partial_i f(B_s) \, dB_s^{(i)}}_{\int_0^t \nabla f(B_s) \cdot dB_s} + \frac{1}{2} \int_0^t \underbrace{\sum_{i=1}^{d} \partial_i^2}_{\Delta = \sum_{i=1}^{d} \partial_i^2} f(B_s) \, ds. \tag{6.6}$$

Corollary 6.8 (Brownian martingales). *Let $(B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$. Consider $f \in \mathcal{C}^2(\mathbb{R}^d)$ such that the processes $(\partial_i f(B_t), t \leq T) \in \mathcal{L}_c^2(T)$ for every $i \leq d$. Then the process*

$$f(B_t) - \int_0^t \frac{1}{2} \Delta f(B_s) \, ds, \quad t \leq T,$$

is a martingale for the Brownian filtration.

Remark: If $\Delta f = 0$, then $f(B_t)$ is a martingale

Harmonic! $\frac{d^2 f}{dx^2} = 0 \Rightarrow f = ax+b$

Why does the formula make sense?

Taylor + Quadratic Variation

$$f(B_t) - f(B_0) = \sum_{j=0}^{n-1} f(B_{t_{j+1}}) - f(B_{t_j})$$

Itô integrals!

$$= \sum_{k=1}^{d} \left(\sum_{j=0}^{n-1} \partial_k f(B_{t_j}) \left(B^{(k)}_{t_{j+1}} - B^{(k)}_{t_j} \right) \right) + \frac{1}{2} \sum_{j=1}^{n-1} \sum_{k,\ell} \partial_k \partial_\ell f(B_{t_j}) \left(B^{(k)}_{t_{j+1}} - B^{(k)}_{t_j}\right)\left(B^{(\ell)}_{t_{j+1}} - B^{(\ell)}_{t_j}\right) + \text{Error}$$

$$\nabla f(B_{t_j}) \cdot (B_{t_{j+1}} - B_{t_j}) + \frac{1}{2}(B_{t_{j+1}} - B_{t_j})^T \nabla^2 f(B_{t_j})(B_{t_{j+1}} - B_{t_j})$$

$$\sum_{j=1}^{n-1} \partial_k \partial_\ell f(B_{t_j}) \left(B^{(k)}_{t_{j+1}} - B^{(k)}_{t_j}\right)\left(B^{(\ell)}_{t_{j+1}} - B^{(\ell)}_{t_j}\right)$$

$k=\ell$ $\quad \frac{1}{2} \sum_{k=1}^{d} \partial_k^2 f(B_{t_j}) \left(B^{(k)}_{t_{j+1}} - B^{(k)}_{t_j}\right)^2 \longrightarrow \frac{1}{2}\int_0^t \Delta f(B_s)\,ds$

$k \neq \ell$ $\quad \sum_{j=1}^{n-1} \left(B^{(k)}_{t_{j+1}} - B^{(k)}_{t_j}\right)\left(B^{(\ell)}_{t_{j+1}} - B^{(\ell)}_{t_{j+1}}\right) \xrightarrow[\text{in } L^2]{} 0$

$$\boxed{\begin{array}{c}\text{Rule of Itô} \\ dB^{(i)}_t \, dB^{(j)}_t = 0 \\ i \neq j\end{array}}$$

6.5. **Cross-variation of $B_t^{(1)}$ and $B_t^{(2)}$.** Let $(t_j, j \leq n)$ be a sequence of partitions of $[0,t]$ such that $\max_j |t_{j+1} - t_j| \to 0$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \sum_{j=0}^{n} (B^{(1)}_{t_{j+1}} - B^{(1)}_{t_j})(B^{(2)}_{t_{j+1}} - B^{(2)}_{t_j}) = 0 \quad \text{in } L^2.$$

This justifies the rule $dB_t^{(1)} \cdot dB_t^{(2)} = 0$.
Hint: Just compute the second moment of the sum.

Problem Set

Warning: cross variation is not always 0!

As usual we have the extension to $f(t, B_t)$

Theorem 6.11 (Itô's formula). *Let $(B_t, t \leq T)$ be a d-dimensional Brownian motion. Consider a function $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$. Then we have with probability one for all $t \leq T$,*

$$f(t,B_t) - f(0,B_0) = \sum_{i=1}^{d} \int_0^t \partial_i f(s,B_s)\,\mathrm{d}B_s^{(i)} + \int_0^t \left(\partial_0 f(s,B_s) + \frac{1}{2}\Delta f(s,B_s)\right)\mathrm{d}s,$$

where $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$.

Martingale condition: $\boxed{\frac{1}{2}\Delta f = -\partial_t f}$

~~Example~~ Write the following process in differential form:

(1) $X_t = \left(B_t^{(1)}\right)^2 + \left(B_t^{(2)}\right)^2$

(2) $Y_t = \exp\left(B_t^{(1)}\right) \cos\left(B_t^{(2)}\right)$

(1) $f(x,y) = x^2 + y^2$

$\partial_1 f(x,y) = 2x \qquad \partial_1^2 f = 2$

$\partial_2 f(x,y) = 2y \qquad \partial_2^2 f = 2$

$dX_t =$

$X_t = \int_0^t 2B_t^{(1)}\,dB_t^{(1)} + \int_0^t 2B_t^{(2)}\,dB_t^{(2)} + 2t$

$X_t - 2t$ is a martingale

(2) $f(x,y) = e^x \cos y \qquad Y_t = 1 + \int_0^t e^{B_s^{(1)}} \cos B_s^{(2)}\,dB_s^{(1)}$

$\partial_1 f = e^x \cos y \qquad \partial_1^2 f = e^x \cos y \qquad + \int_0^t -e^{B_s^{(1)}} \sin B_s^{(2)}\,dB_s^{(2)}$

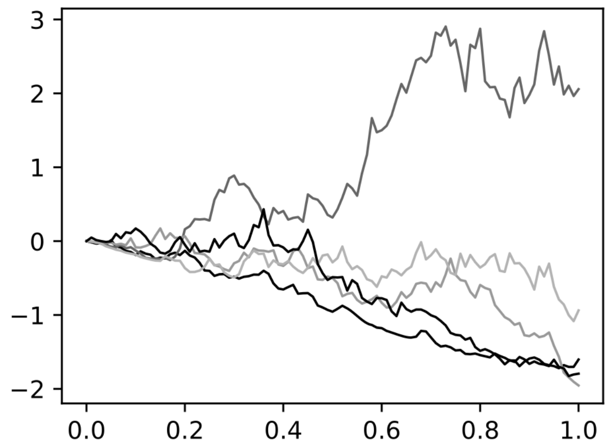
$\partial_2 f = -e^x \sin y \qquad \partial_2^2 f = -e^x \cos y$

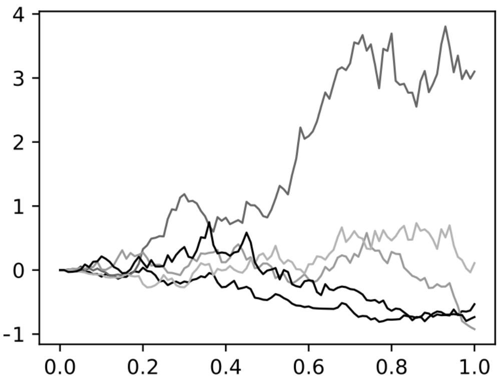
$\Delta f = 0$

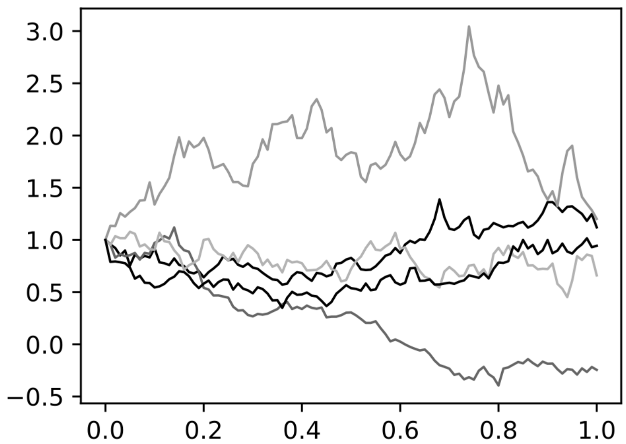
$X_t - 2t$

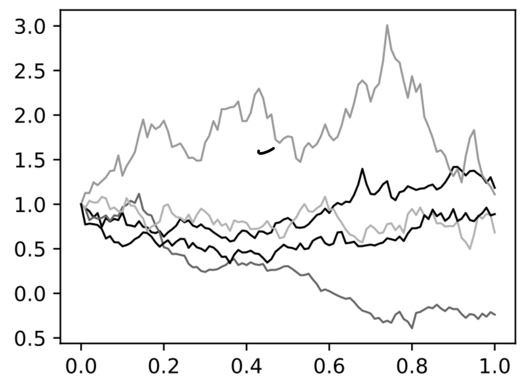
Y_t

Figure 6.2. A simulation of 5 paths of the processes $(X_t - 2t, t \in [0,1])$ and $(Y_t, t \in [0,1])$ defined in equation (6.5).







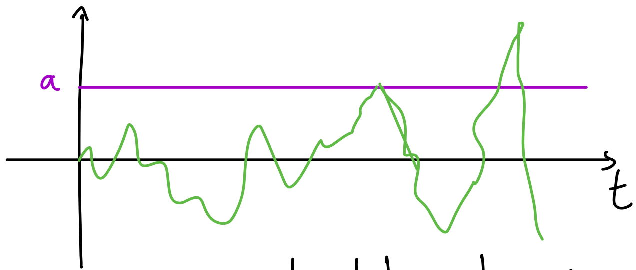


.3. A simulation of 5 paths of the processes $\int_0^t 2B_s^{(1)}\, dB_s^{(1)} + \int_0^t 2B_s^{(2)}\, dB_s^{(2)}$
$\int_0^t \exp(B_s^{(1)}) \cos B_s^{(2)}\ dB_s^{(1)} - \int_0^t \exp(B_s^{(1)}) \sin B_s^{(2)}\ dB_s^{(2)}$. These were generated
same Brownian motion as in Figure 6.2. Note how each path corresponds.
formula here, just Itô's formula. See Numerical Project 6.2.

6.3 Recurrence vs Transience of BM

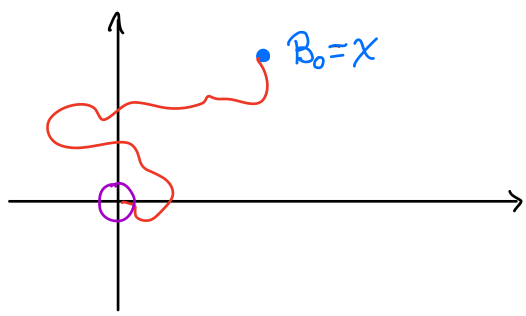
In 1D, we know that a BM path reaches any height a.

In particular, it will come back to a infinitely often.



BM is recurrent is 1D!

Is there an equivalent statement in higher d?



Reccurent

Yes $d=2$ (1) Does a BM path visit a neighborhood of 0 infinitely often?

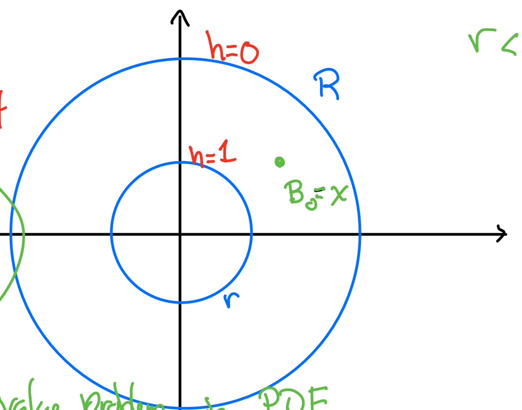
NO $d>2$ (2) Does it actually hit 0? NO!

Transient

As usual, we need to { find a martingale
Apply Doob O.S.

A drunk man will eventually find his way home but a drunk bird may get lost forever.

Setup



$r < \|x\| < R$

Why do we want

$\Delta h = 0$ in annulus

$h = \begin{cases} 1 & \|x\| = r \\ 0 & \|x\| = R \end{cases}$

Boundary-value problem in PDE

$$\tau_R' = \min\{ t \geq 0 : \|B_t\| \geq R \} \qquad \tau_R' < \infty \text{ w.prob } 1$$

$$\tau_r = \min\{ t \geq 0 : \|B_t\| \leq r \} \qquad \tau_r < \infty \text{ w. prob } 1.$$

What is $P(\tau_r < \tau_R')$?

- Let's find a good martingale:

  We need $\Delta f = 0$ in the annulus

$$f(x) = \begin{cases} \log \|x\| & d = 2 \\ \|x\|^{2-d} & d \geq 3 \end{cases}$$

Rotational symmetry

Of course, $h = af + b$ is also harmonic

$$\Delta h = a\Delta f + \Delta b = a\Delta f = 0$$

$h(B_t)$ it is a martingale

$$\tau = \tau_r \wedge \tau_R' \qquad \mathbb{E}[\mathbb{1}(\|B_\tau\| = r)]$$

$$P(\tau_r < \tau_R') = P(\|B_\tau\| = r) = \mathbb{E}[h(B_\tau)]$$

if $h(x) = \begin{cases} 0 & \|x\| = R \\ 1 & \|x\| = r \end{cases}$ Boundary-value problem

- We use $\boxed{\text{Doob OS}}$ · martingale bounded? ✓
  $h(B_t)$ is a martingale, $\Delta h = 0$ · $\tau < \infty$? ✓

$$P(\tau_r < \tau_R) = E_x[h(B_\tau)] = h(x) = h(B_0)$$

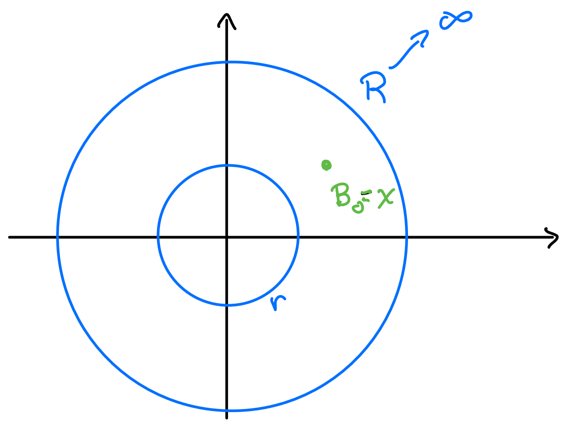
What is $h(x)$?

$af + b$

$$h = \begin{cases} 0 & \|x\| = R \\ 1 & \|x\| = r \end{cases}$$

$$h(x) = \begin{cases} \dfrac{\log\|x\| - \log R}{\log r - \log R} & d=2 \\[2ex] \dfrac{\|x\|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} & d \geq 3 \end{cases}$$

$$= P(\tau_r < \tau_R)$$



$$P(\tau_r < \infty) = \lim_{R \to \infty} \begin{cases} \dfrac{\log\|x\| - \log R}{\log r - \log R} & d=2 & = 1 \\[2ex] \dfrac{\|x\|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} & d \geq 3 & = \left(\dfrac{\|x\|}{r}\right)^{2-d} \end{cases}$$

$$= \left(\frac{r}{\|x\|}\right)^{d-2}$$

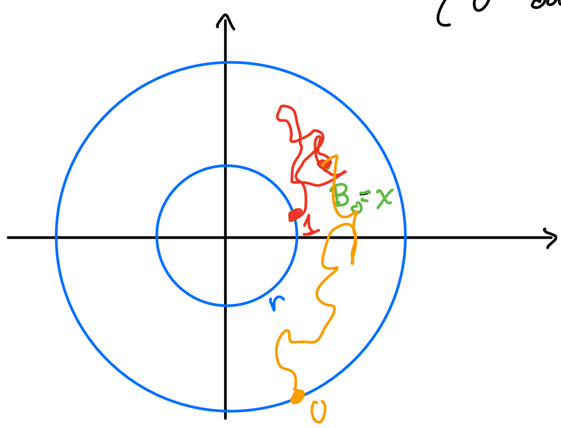
$$0 < \quad < 1$$

$$E_x\left[h(B_\tau)\right] = h(x)$$

↳ representation of a harmonic fct

$\Delta h = 0$ in annulus

$h = \begin{cases} 1 & \text{inner} \\ 0 & \text{outer} \end{cases}$ boundary



$d = 2$ BM paths visits a neighborhood of 0 of any radius i.o.

$d \geq 3$ there are paths that do not visit a neighborhood of 0.

In $d=2$, does the path actually touches 0?

$\frac{\log\|x\| - \log R}{\log r - \log R}$ $d=2$

$\frac{\|x\|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}$ $d \geq 3$

$\lim_{R\to\infty} \lim_{r\to 0} (\quad) = 0$!

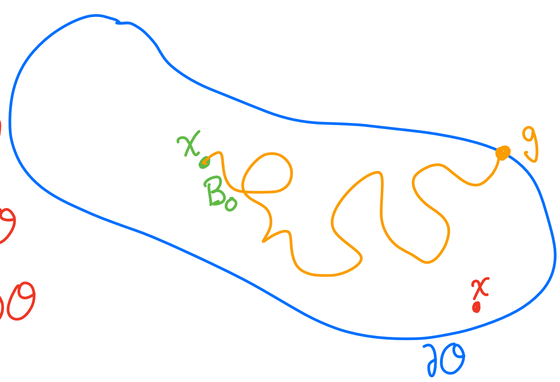
6.4 Dynkin's formula

Consider a BM in a region $\mathcal{O}$ (bounded)

Dirichlet problem

$\Delta h = 0$ in $\mathcal{O}$

$h = g$ on $\partial\mathcal{O}$



$\tau = \min\{t \geq 0 : B_t \in \partial\mathcal{O}\}$

Solution $h(x) = E_x[h(B_\tau)] = E_x[g(B_\tau)]$

$$\boxed{h(x) = E_x[g(B_\tau)]}$$

For f, consider the martingale

$$M_t = f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)\, dB_s$$

The stopped martingale $\cdot$ $M_{t\wedge\tau}$ is bounded

$\cdot$ $\tau < \infty$

We have Doob O.S.

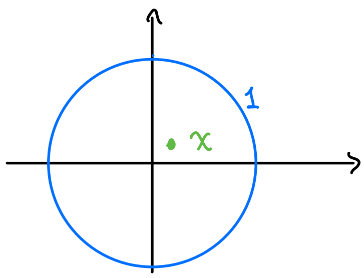
$$\boxed{E[M_\tau] = f(B_0)}$$

In particular if $\Delta f = 0$ in Θ

$$f(x) = E_x\left[f(B_\tau) \right]$$

$f(x)$ is the average of f on $\partial\Omega$!

<u>Example</u>



$\tau = \min\{ t \geq 0 : \|B_t\| \geq 1 \}$

$E[\tau] = ?$