MTH9831 HW3 Team 5

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5.1. **Stopped martingales are martingales.** Let $(M_n, n = 0, 1, 2, \dots)$ be a martingale in discrete time for the filtration $(\mathcal{F}_n, n \geq 0)$. Let τ be a stopping time for the same filtration. Use the martingale transform with the process

$$X_n(\omega) = \begin{cases} +1 & \text{if } n < \tau(\omega), \\ 0 & \text{if } n \geq \tau(\omega) \end{cases}$$

to show that the stopped martingale $(M_{\tau \wedge n}, n \geq 0)$ is a martingale.

For martingale $(M_n, n=0,1,2 \dots)$ for filtration $(F_n, n \geq 0)$ and $(X_n, n \geq 0)$ defined above, we can define the martingale transform I_t as

$$I_t = X_0(w)(M_1 - M_0) + X_1(w)(M_2 - M_1) + \dots + X_{t-1}(w)(M_t - M_{t-1})$$

For $t < \tau(w)$. $I_t = M_1 - M_0 + M_2 - M_1 + \dots + M_t - M_{t-1} = M_t - M_0 = M_t$

For $t \geq \tau(w)$. $I_t = M_1 - M_0 + \dots + M_{\tau(w)} - M_{\tau(w)-1} = M_{\tau(w)} - M_0 = M_{\tau(w)}$

Thus it corresponds to the stopped martingale $(M_{\tau \wedge n}, n \geq 0)$.

As martingale transforms are martingales, the stopped martingale $(M_{\tau \wedge n}, n \geq 0)$ is a martingale.

EX 5.2 we have $I_t = \begin{cases} 10B_t & t\in[0,\frac{1}{3}] \\ 10B_{1/3} + 5(B_t - B_{1/3}) & t\in(1/3, 2/3] \\ 10B_{1/3} + 5(B_{2/3} - B_{1/3}) + 2(B_t - B_{2/3}) & t\in(\frac{2}{3},1] \end{cases}$

a) we have that $(I_{1/3}, I_{2/3}, I_1) = (10B_{1/3}, 10B_{1/3} + 5(B_{2/3}-B_{1/3}), 10B_{1/3} + 5(B_{2/3}-B_{1/3}) + 2(B_1 - B_{2/3}))$

By properties of Brownian motion. let $X = B_{1/3}$, $Y = B_{2/3} - B_{1/3}$, $Z = B_1 - B_{2/3}$

Then $X, Y, Z \sim N(0, 1/3)$ and they are independent.

Hence $(I_{1/3}, I_{2/3}, I_1) = (10X, 10X + 5Y, 10X + 5Y + 2Z)$

And hence for any $(w_1, w_2, w_3) \in \mathbb{R}^3$, $w_1 I_{1/3} + w_2 I_{2/3} + w_3 I_1$

$$= 10(w_1 + w_2 + w_3)X + 5(w_1 + w_2)Y + 2w_3 Z$$

which is also normal (due to independence).

Hence by definition. $(I_{1/3}, I_{2/3}, I_1)$ is Gaussian.

b) For X, Y, Z defined above, the mean vector is $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and covariance matrix

$\Sigma = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$. Now, $\begin{pmatrix} I_{1/3} \\ I_{2/3} \\ I_1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 \\ 10 & 5 & 0 \\ 10 & 5 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, hence the covariance matrix

for $(I_{1/3}, I_{2/3}, I_1)$ is $\Sigma' = \begin{pmatrix} 10 & 0 & 0 \\ 10 & 5 & 0 \\ 10 & 5 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 10 & 10 & 10 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{100}{3} & \frac{100}{3} & \frac{100}{3} \\ \frac{100}{3} & \frac{125}{3} & \frac{125}{3} \\ \frac{100}{3} & \frac{125}{3} & \frac{129}{3} \end{pmatrix}$.

$(\Sigma' = M\Sigma M^T)$

And the mean vector is $\mu' = M\mu = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

c) $\mathbb{E}[B_1 I_1] = \mathbb{E}[B_1(10B_{1/3} + 5B_{2/3} - 5B_{1/3} + 2B_1 - 2B_{\frac{2}{3}})]$

$= 5\mathbb{E}[B_1 B_{\frac{1}{3}}] + 3\mathbb{E}[B_1 B_{\frac{2}{3}}] + 2\mathbb{E}[B_1 B_1]$ $\quad \mathbb{E}[B_s B_t] = s \wedge t.$

$= \frac{5}{3} + \frac{2}{3} \times 3 + 2 \times 1 = \frac{17}{3}$

Since $\mathbb{E}B_1 = 0$, $\mathbb{E}I_1 = 5\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z = 0$, we have $Cov(B_1, I_1) = \mathbb{E}B_1 I_1 = \frac{17}{3}$

Hence B_1, I_1 are Gaussian with non-zero covariance $\Rightarrow$ B_1 and I_1 are not independent.

$\square$

EX 5.4 a) since $t_1 \le t_2 \le t_3 \le t_4$, we have $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}_{t_3} \subseteq \mathcal{F}_{t_4}$.

Thus, we have: $\mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_4}-M_{t_3})]$

$= \mathbb{E}[\mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_4}-M_{t_3}) \mid \mathcal{F}_{t_2}]]$

$= \mathbb{E}[(M_{t_2}-M_{t_1})\mathbb{E}[(M_{t_4}-M_{t_3}) \mid \mathcal{F}_{t_2}]]$ since $M_{t_2}-M_{t_1}$ is $\mathcal{F}_{t_2}$-measurable

$= \mathbb{E}[(M_{t_2}-M_{t_1})\mathbb{E}[(M_{t_4}-M_{t_3})]]$ since $M_{t_4}-M_{t_3}$ independent from $\mathcal{F}_{t_2}$

$= \mathbb{E}[(M_{t_2}-M_{t_1})(\mathbb{E}(M_{t_4})-\mathbb{E}(M_{t_3}))]$

$= 0$, since $\mathbb{E}[M_{t_4}] = \mathbb{E}[M_{t_3}] = \mathbb{E}[M_0]$ by the martingale property.

b) Let $I_t = \int_0^t X_s\, dB_s$. By properties of $\mathcal{L}^2_c(T)$ processes, $\int_0^t \mathbb{E}[X_s^2]\, ds = \mathbb{E}[(\int_0^t X_s\, dB_s)^2]$

under the notation, LHS becomes $\mathbb{E}[I_t I_{t'}]$ $= \mathbb{E}[I_t^2]$ (Isometry)

then, $\mathbb{E}[I_t I_{t'} - I_t^2] = \mathbb{E}[I_t(I_{t'} - I_t + I_t) - I_t^2] = \mathbb{E}[I_t(I_{t'} - I_t)]$

$= \mathbb{E}[(I_t - I_0)(I_{t'} - I_t)]$

since $I_t, I_{t'}$ are martingales. then take $0 \le t \le t \le t'$ and apply a), with $M_{t_1} = 0 = I_0$, $M_{t_2} = I_t$, $M_{t_3} = I_t$, $M_{t_4} = I_{t'}$. Thus we obtain:

$\mathbb{E}[(I_t - I_0)(I_{t'} - I_t)] = 0 \Rightarrow \mathbb{E}[I_t I_{t'} - I_t^2] = 0$

$\Rightarrow \mathbb{E}[I_t I_{t'}] = \mathbb{E}[I_t^2]$

By Itô's isometry. $\Leftrightarrow \mathbb{E}[\int_0^t X_s\, dB_s \cdot \int_0^{t'} X_s\, dB_s] = \int_0^t \mathbb{E}[X_s^2]\, dB_s$

$\square$

EX 5.6 $M_t = \exp(\sigma B_t - \sigma^2 t/2)$

For any $T>0$, we have

① since B_t is adapted to $\mathcal{F}_t$ for $t\in T$ and M_t is a function of B_t, we have that M_t is $\mathcal{F}_t$-measurable. Hence M_t is adapted.

② since B_t is continuous a.s. and M_t is a continuous function of B_t, we have that M_t is continuous almost surely on $[0,T]$ for any T.

③ $\int_0^T \mathbb{E}[M_t^2]\,dt = \int_0^T e^{-\sigma^2 t}\,\mathbb{E}[\exp(2\sigma B_t)]\,dt.$

since $B_t \sim N(0,t)$, then $\mathbb{E}[e^{2\sigma B_t}]$ is the MGF of $N(0,t)$ valued at σ: $\mathbb{E}[e^{2\sigma B_t}] = \exp(\frac{1}{2}t(2\sigma)^2) = \exp(2\sigma^2 t)$.

Hence the integral becomes $\int_0^T e^{\sigma^2 t}\,dt = \frac{1}{\sigma^2}[e^{\sigma^2 t}]_0^T = \frac{1}{\sigma^2}[\exp(\sigma^2 T)-1] < \infty$ for given $T>0$.

thus, we conclude that $(M_t, t\in T)\in \mathcal{L}_c^2(T)$ $\forall\, T>0$. $\square$

Ex 5.7 $M_t = \exp(B_t^2)$, $t\in T$.

we check the case when $T > \frac{1}{4}$.

$$\int_0^T \mathbb{E}[M_t^2]\,dt = \int_0^T \mathbb{E}[\exp(2B_t^2)]\,dt$$

$$= \int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \cdot \exp(2x^2)\,dx\,dt \quad \text{since } B_t\sim N(0,t)$$

$$= \int_0^T \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{(2-\frac{1}{2t})x^2}\,dx\,dt.$$

If $2-\frac{1}{2t}>0$, then the inner integral diverges, leading to $\int_0^T \mathbb{E}[M_t^2]\,dt = \infty$.

since $t\in[0,T]$, then if $T\le\frac{1}{4}$, we have $t\le T \Rightarrow 2-\frac{1}{2t} \le 2-\frac{1}{2\cdot\frac{1}{4}} = 0$

Hence, the integral converges when $T\le\frac{1}{4}$, and when $T>\frac{1}{4}$, for $t\in(\frac{1}{4},T]$, we have that $2-\frac{1}{2t}>0 \Rightarrow \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi t}} e^{(2-\frac{1}{2t})x^2}\,dx = \infty$

$$\Rightarrow \int_0^T \mathbb{E}[M_t^2]\,dt = \infty \Rightarrow M_t \notin \mathcal{L}_c^2(T),$$

since M_t violates the condition that $\int_0^T \mathbb{E}[M_t^2]\,dt < \infty$.

Thus, we conclude that $(e^{B_t^2}, t\le T)$ is not in $\mathcal{L}_c^2(T)$ for $T>\frac{1}{4}$.