MTH9831 Homework 2 Theoretical Questions
Team 5
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4.7. **Gaussian conditioning.** Consider the Gaussian process (X_1, X_2) of mean 0 and covariance

$$\mathcal{C} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

(a) Find IID standard Gaussians (Z_1, Z_2) that are linear combinations of (X_1, X_2).
(b) Write down (X_1, X_2) in terms of (Z_1, Z_2).
(c) Compute $\mathbf{E}[X_2|X_1]$.
(d) Compute $\mathbf{E}[e^{aX_2}|X_1]$ for $a \in \mathbb{R}$. What is the conditional distribution of X_2 given X_1?

a). b). Assume $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. then $MM^T = C$.

$M = U^T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$\begin{cases} X_1 = Z_1 \\ X_2 = -Z_1 + Z_2 \end{cases} \Rightarrow \begin{cases} Z_1 = X_1 \\ Z_2 = X_1 + X_2. \end{cases}$

c). $E[X_2 | X_1] = \dfrac{E[X_2 X_1]}{E[X_1^2]} X_1 = \dfrac{-1}{1} X_1 = -X_1.$

d). $E[e^{aX_2} | X_1] = E[e^{aZ_2 - aZ_1} | Z_1]$ (substitution)

$= e^{-aZ_1} E[e^{aZ_2} | Z_1]$ (property of expectation).

$= e^{-aZ_1} E[e^{aZ_2}]$ (Z_1, Z_2 are independent)

Since the MGF for $X \sim N(\mu, \sigma^2)$ is $M(t) = E[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

We have $E[e^{aZ_2}] = e^{0 \cdot a + \frac{1}{2} \cdot 1 \cdot a^2} = e^{\frac{1}{2}a^2}$.

$E[e^{aX_2} | X_1] = e^{-aZ_1 + \frac{1}{2}a^2} = e^{-aX_1 + \frac{1}{2}a^2}$

Observe that $E[e^{aX_2} | X_1] = e^{(-X_1) \cdot a + \frac{1}{2} \cdot 1 \cdot a^2}$ is the MGF for $N(-X_1, 1)$. Thus the conditional distribution of X_2 given X_1 follows $N(-X_1, 1)$. $f_{X_2|X_1}(x_2 | x_1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(X_2 + X_1)^2}$

4.11. **Geometric Poisson process.** Let $(N_t, t \geq 0)$ be a Poisson process of intensity λ. For $\alpha > 0$, prove that the process $(e^{\alpha N_t - \lambda t(e^\alpha - 1)}, t \geq 0)$ is a martingale for the filtration of the Poisson process $(N_t, t \geq 0)$.

① Given α. $(e^{\alpha N_t - \lambda t(e^\alpha-1)}, t \geq 0)$ is adapted to the filtration of $(N_t . t \geq 0)$

② $E[|e^{\alpha N_t - \lambda t(e^\alpha-1)}|] = E[e^{\alpha N_t - \lambda t(e^\alpha-1)}]$

$$= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \cdot e^{\alpha n - \lambda t(e^\alpha-1)}$$

$$= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!} e^{\alpha n - \lambda t e^\alpha}$$

$$= \sum_{n=0}^{+\infty} \frac{(\lambda t e^\alpha)^n}{n!} \cdot e^{-\lambda t e^\alpha} = e^{\lambda t e^\alpha} \cdot e^{-\lambda t e^\alpha} = 1$$

③ $E[e^{\alpha N_t - \lambda t(e^\alpha-1)} | N_s]$

$$= E[e^{\alpha N_s - \lambda s(e^\alpha-1)} \cdot e^{\alpha(N_t - N_s)} \cdot e^{-\lambda(e^\alpha-1)(t-s)} | N_s]$$

$$= e^{\alpha N_s - \lambda s(e^\alpha-1)} \cdot E[e^{\alpha(N_t - N_s)} \cdot e^{-\lambda(e^\alpha-1)(t-s)}]$$

$$= e^{\alpha N_s - \lambda s(e^\alpha-1)} \cdot E[e^{\lambda(t-s)(e^\alpha-1)} e^{-\lambda(e^\alpha-1)(t-s)}]$$

$$= e^{\alpha N_s - \lambda s(e^\alpha-1)}$$

Thus the process $(e^{\alpha N_t - \lambda t(e^\alpha-1)}, t \geq 0)$ is a martingale for the filtration of the Poisson process $(N_t, t \geq 0)$

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Q12 a) we have $M_t = tB_t - \frac{1}{3}B_t^3$

① Adaptedness: M_t consists of B_t which is an $\mathcal{F}_t$-adapted process. Hence M_t is also $\mathcal{F}_t$-adapted.

② Integrable: $\mathbb{E}[|M_t|] \le \mathbb{E}[|tB_t|] + \frac{1}{3}\mathbb{E}[|B_t^3|]$

$\le |t| \cdot \mathbb{E}[B_t^2]^{1/2} + \frac{1}{3}\mathbb{E}[B_t^6]^{1/2}$

(use that $t \ge 0$) $= t^{3/2} + 5t^{3/2} = 6t^{3/2} < \infty$, for any given $t > 0$.

Hence M_t is integrable.

③ Martingale property. $\forall t > s$, $\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[tB_t - \frac{1}{3}B_t^3 | \mathcal{F}_s]$

$= \mathbb{E}[tB_t | \mathcal{F}_s] - \frac{1}{3}\mathbb{E}[B_t^3 | \mathcal{F}_s]$

$\mathbb{E}[tB_t | \mathcal{F}_s] = t B_s$ since BM is a martingale, t is a constant

$\mathbb{E}[B_t^3 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^3 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^3 | \mathcal{F}_s] + 3\mathbb{E}[(B_t - B_s)^2 B_s | \mathcal{F}_s] + 3\mathbb{E}[(B_t - B_s)B_s^2 | \mathcal{F}_s] + \mathbb{E}[B_s^3 | \mathcal{F}_s]$.

since $(B_t - B_s)$ is independent of B_s (hence $\mathcal{F}_s$), we have

$\mathbb{E}[(B_t - B_s)^3 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^3] = 0$ (third moment of $N(0, t-s)$)

$\mathbb{E}[(B_t - B_s)^2 B_s | \mathcal{F}_s] = B_s \mathbb{E}[(B_t - B_s)^2] = (t-s)B_s$; $\mathbb{E}[(B_t - B_s)B_s^2 | \mathcal{F}_s] = B_s^2 \mathbb{E}[B_t - B_s] = 0$

$\mathbb{E}[B_s^3 | \mathcal{F}_s] = B_s^3$, where we also use that B_s is $\mathcal{F}_s$-measurable. Putting together:

$\mathbb{E}[M_t | \mathcal{F}_s] = tB_s - \frac{1}{3} \cdot 3(t-s)B_s - \frac{1}{3}B_s^3 = sB_s - \frac{1}{3}B_s^3 = M_s$.

Hence, $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ for $s < t$.

By ①, ②, ③, we've shown that $(M_t, t \ge 0)$ is a martingale. □

b) we first show that $\tau < \infty$ a.s.

$\mathbb{P}(\tau < \infty) \ge \mathbb{P}(\bigcup_{n \ge 1} \{|B_{n+1} - B_n| > a+b\}) = 1 - \mathbb{P}(\bigcap_{n \ge 1} \{|B_{n+1} - B_n| \le a+b\}) \overset{\text{continuity of probability}}{=} 1 - \lim_{m \to \infty} \mathbb{P}(\bigcap_{n=1}^{m} \{|B_{n+1} - B_n| \le a+b\})$

(where $p = \mathbb{P}(|B_{i+1} - B_i| \le a+b) < 1$) $\le 1 - \lim_{m \to \infty} p^m = 1$.

Hence $\tau < \infty$ a.s.

Note that, $M_{t \wedge \tau}$ is bounded since $B_{t \wedge \tau} \in [-b, a]$ and $\tau < \infty$ a.s. we can hence apply Doob's Optional Stopping theorem: $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$

$\Leftrightarrow \mathbb{E}[M_\tau] = 0$ since $B_0 = 0$

$\Leftrightarrow \mathbb{E}[\tau B_\tau] = \frac{1}{3}\mathbb{E}[B_\tau^3] = \frac{1}{3}\left(\frac{ba^3}{a+b} - \frac{ab^3}{a+b}\right) = \frac{1}{3}\frac{ab}{a+b} \cdot (a-b)(a+b) = \frac{1}{3}ab(a-b)$.

since from the lecture, we know that $\mathbb{P}(B_\tau = a) = \frac{b}{a+b}$ and $\mathbb{P}(B_\tau = -b) = \frac{a}{a+b}$. □

c) we have $\mathbb{E}[e^{aB_\tau - \frac{a^2\tau}{2}}] = \mathbb{E}[\mathbb{E}[e^{aB_\tau - \frac{a^2}{2}\tau} | \tau = t]]$ ⋯ (*)

Conditioned on $\tau = t$, we have:

$\mathbb{E}[e^{aB_t - \frac{1}{2}a^2 t}] = e^{-\frac{1}{2}a^2 t}\,\mathbb{E}[e^{aB_t}] = e^{-\frac{1}{2}a^2 t}\,\mathbb{E}[e^{aX}]$ for $X \sim N(0,t)$

$\mathbb{E}[e^{aX}]$ is just the M.G.F. for $N(0,t)$ valued at a, hence $\mathbb{E}[e^{aX}] = e^{\frac{1}{2}ta^2}$.

Thus, $e^{-\frac{1}{2}a^2 t}\cdot \mathbb{E}[e^{aB_t}] = e^{-\frac{1}{2}a^2 t}\, e^{\frac{1}{2}a^2 t} = 1,\ \forall a>0.$

Thus, $(*)$ becomes $\mathbb{E}[1]=1$, and hence $\mathbb{E}[e^{aB_\tau - \frac{a^2}{2}\tau}] = 1 \quad \forall a>0.$ $\square$

d) By c), we can write $\exp(aB_\tau - \frac{\tau}{2}a^2)$ as a Taylor series around 0:

$$\mathbb{E}[e^{aB_\tau - \frac{1}{2}a^2\tau}] = 1 \Leftrightarrow 1 = \mathbb{E}[1 + aB_\tau - \tfrac{1}{2}a^2\tau + \tfrac{1}{2}(aB_\tau - \tfrac{1}{2}a^2\tau)^2 + \tfrac{1}{6}(aB_\tau - \tfrac{1}{2}a^2\tau)^3 + \cdots]$$

$$1 = 1 + a\mathbb{E}[B_\tau] + \tfrac{1}{2}a^2\mathbb{E}[B_\tau^2 - \tau] - \tfrac{1}{2}a^3\mathbb{E}[\tau B_\tau] + \tfrac{1}{6}a^3\mathbb{E}[B_\tau^3] + \sum_{n\geq 4} f_n a^n$$

$$\Leftrightarrow \quad 0 = a\mathbb{E}[B_\tau] + \tfrac{1}{2}a^2\mathbb{E}[B_\tau^2 - \tau] + \tfrac{1}{6}\mathbb{E}[B_\tau^3 - 3\tau B_\tau]\cdot a^3 + \sum_{n\geq 4} f_n a^n, \text{ where } f_n \text{ is a function of } B_\tau \text{ and } \tau.$$

LHS $=0 \Leftrightarrow$ all coefficients $=0$. Since B_t and $B_t^2 - t$ are martingales (shown in class)

$\Rightarrow \mathbb{E}[B_\tau] = \mathbb{E}[B_0] = 0$, $\mathbb{E}[B_\tau^2 - \tau] = \mathbb{E}[B_0^2] = 0$. Then we must also have $\mathbb{E}[B_\tau^3 - 3\tau B_\tau] = 0$, which means

$$\mathbb{E}[\tau B_\tau] = \tfrac{1}{3}\mathbb{E}[B_\tau^3] = \tfrac{1}{3}\left(\tfrac{ba^3}{a+b} - \tfrac{ab^3}{a+b}\right) = \tfrac{1}{3}ab(a-b).$$

this is the re ired result. $\square$

Q15. a) τ: $\min_{t\geq 0}\{ : X_t = +1\}$. So this means: stop as soon as winning a game.

Now, $\mathbb{E}[M_\tau] = \mathbb{E}[\sum_{n\geq 1} \mathbb{1}_{\{\tau=n\}} M_n] = \sum_{n\geq 1}\mathbb{E}[\mathbb{1}_{\{\tau=n\}} M_n]$ by Fubini's thm (Ex. 3.11)

when $\tau=n$, we have: $M_\tau = 1\cdot(-1) + 2\cdot(-1) + \cdots + 2^{n-1}\cdot(-1) + 2^n\cdot 1 = 2^n - (1+2+\cdots+2^{n-1}) = 1$.

so we can rewrite it as $\sum_{n\geq 1}\mathbb{E}[\mathbb{1}_{\{\tau=n\}}\cdot 1] = \sum_{n\geq 1}\mathbb{E}[\mathbb{1}_{\{\tau=n\}}] = \sum_{n\geq 1}\mathbb{P}(\tau=n)$

$= \sum_{n\geq 1}(\frac{1}{2})^n$.

since $\sum_{n\geq 1}(\frac{1}{2})^n = \frac{1}{2}\cdot(\sum_{n\geq 0}(\frac{1}{2})^n) = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = 1$, we have $\mathbb{E}[M_\tau] = 1$.

what about $\mathbb{E}[M_0]$? $\mathbb{E}[M_0] = \mathbb{E}[S_0] = 0$. this proves the required result. $\square$

b) we can't apply Optional stopping since $M_{n\wedge\tau}$ is not bounded.

$\forall m \in \mathbb{N}, m>0$, we have when $n>\tau$, $M_{n\wedge\tau} = M_n = -1-2-\cdots 2^n = 1-2^{n+1}$

$\Rightarrow M_{n\wedge\tau} = 1-2^{n+1} < -m$ if $n > \log_2 m + 1$. Thus $M_{n\wedge\tau}$ is not bounded below.

$\Rightarrow$ violation of Doob's Optional Stopping theorem assumptions. $\square$

c) Disadvantages:

① Takes a lot to win (Input might be infinity) A player might lose all his/her money before winning.

② Even if wins, player only wins 1 (Little return).

Q16 proof

① Adaptedness. $M_t = \mathbb{E}[X|\mathcal{F}_t]$ which is $\mathcal{F}_t$-measurable for any $t\geq 0$

Hence M_t is an $\mathcal{F}_t$-adapted process.

② Integrable. we have that $\mathbb{E}|M_t| = \mathbb{E}[|\mathbb{E}[X|\mathcal{F}_t]|]$.

By Triangle Inequality (Jensen's Inequality on $|\cdot|$), $|\mathbb{E}[X|\mathcal{F}_t]| \le \mathbb{E}[|X||\mathcal{F}_t]$. Thus, $\forall\, t \ge 0$
$\mathbb{E}[|M_t|] \le \mathbb{E}[\mathbb{E}[|X||\mathcal{F}_t]] = \mathbb{E}[|X|] < \infty$ as $|X|$ is integrable.
Hence M_t is integrable.

③ Martingale property: $\forall\, t > s > 0$, we have: $\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t] | \mathcal{F}_s]$.
since $(\mathcal{F}_t, t \ge 0)$ is a filtration and $s < t$, we have that $\mathcal{F}_s \subseteq \mathcal{F}_t$. By the tower property,
we have: $\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s] = M_s$.

By ①, ②, ③, we conclude that $M_t = \mathbb{E}[X|\mathcal{F}_t]$ is a martingale. $\square$