DES201: Discrete Mathematics
Final Mock Exam

from James Dean and Google and The Peanuts

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Conditions: Closed Book

Directions:

1. This exam has 20 pages (including this page).
2. Students are encouraged to dramatically sigh every 3 minutes.
3. Write your name, or your preferred superhero alias.
4. Reading the problem is optional but highly recommended.
5. Solutions can be written in English or ASCII.
6. Students may not escape through windows or air vents.

This mock exam is based on what we learned in DES201, Section 3, excluding the Graph chapter.

Problem 1

Show that $2 + 4 + 6 + \ldots + 2n$ is $\Theta(n^2)$

$$
\begin{aligned}
2+4+6+8+\ldots+2n &= 2(1+2+3+\ldots+n) \\
&= \frac{\cancel{2}(n)(n+1)}{\cancel{2}} \\
&= n(n+1)
\end{aligned}
$$

Big-O: $f(n) = n(n+1)$

$$
\begin{aligned}
f(n) &\leq (n+1)(n+1) \\
&\leq n^2+n+n+1 \\
&\leq n^2+2n+1 \\
&\leq n^2+2n^2+n^2 \\
f(n) &\leq 4n^2
\end{aligned}
$$

$\therefore O(n^2)$

Big-Ω: $f(n) = n(n)$

$f(n) \geq n^2$

$\therefore \Omega(n^2)$

$\therefore \Theta(n^2)$ #

Problem 2

Show that $n! = \Theta(n^n)$.

$n! = n(n-1)(n-2)\ldots(2)(1)$

Big-O: $f(n) = n(n-1)(n-2)\ldots(2)(1)$

$f(n) \leq \underbrace{n(n)(n)\ldots(n)(n)}_{n \text{ terms}}$

$f(n) \leq n^n$

$\therefore O(n^n)$

Big-Ω: $f(n) = n(n-1)(n-2)\ldots(2)(1)$ (using the guideline)

$f(n) = \cancel{1}(\cancel{2})\ldots\cancel{}(n-2)(n-1)(n)$

$f(n) \geq \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil \ldots (n-1)(n)$

$\geq \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil \ldots \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$

$f(n) \geq \frac{n^n}{2^n}$

$\therefore \Omega(n^n)$

$\therefore \Theta(n^n)$ #

Problem 3

Find a theta notation for the number of times the statement $x = x + 1$ is executed.

2 256 2^8

```
i = 2
while (i < n) {
    i = i^2
    x = x + 1
}
```

$n = 10$

2	2^{2^1}	2^{2^2}	2^{2^3}
2	2^2	2^4	2^8
2	4	16	256
|	|	|	|
1	1	1	$\times$

$i = 2^{2^k}$

$2^{2^k} \geqslant n$

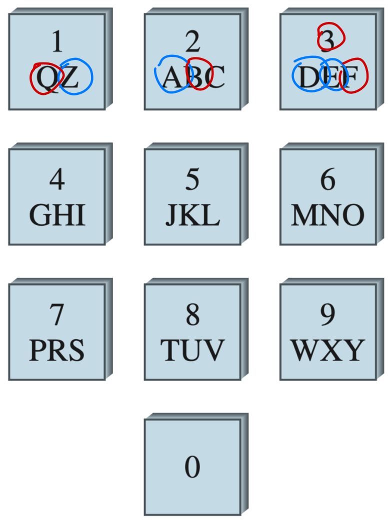
$2^k \geqslant \log_2(n)$

$k \geqslant \log_2(\log_2(n))$

$\therefore O(\log_2(\log_2(n)))$ #

Problem 4

The following diagram shows the keypad for an automatic teller machine. As you can see, the same sequence of keys represents a variety of different PINs. For instance, 2133, AZDE, and BQ3F are all keyed in exactly the same way.



How many different PINs are represented by the same sequence of keys as 5031?

$3 \times 1 \times 4 \times 3 = 36$ #

Problem 5

At a certain company, passwords must be from 3-5 symbols long and composed from the 26 uppercase letters of the Roman alphabet, the ten digits 0-9, and the 14 symbols !, @, #, \$, %, $\wedge$, &, $*$, (,), $-$, $+$, {, }.

$26+10+14=50$

How many passwords are possible if repetition of symbols is allowed?

$$\underline{50}\ \underline{50}\ \underline{50} + \underline{50}\ \underline{50}\ \underline{50}\ \underline{50} + \underline{50}\ \underline{50}\ \underline{50}\ \underline{50}\ \underline{50} = 50^3+50^4+50^5 \ \#$$

How many passwords contain no repeated symbols?

$$\underline{50}\ \underline{49}\ \underline{48} + \underline{50}\ \underline{49}\ \underline{48}\ \underline{47} + \underline{50}\ \underline{49}\ \underline{48}\ \underline{47}\ \underline{46} \ \#$$

How many passwords have at least one repeated symbol?

ทั้งหมด - ไม่มี repeat

$$(50^3+50^4+50^5) - \left(\underline{50}\ \underline{49}\ \underline{48} + \underline{50}\ \underline{49}\ \underline{48}\ \underline{47} + \underline{50}\ \underline{49}\ \underline{48}\ \underline{47}\ \underline{46}\right) \ \#$$

Problem 6

An instructor gives an exam with fourteen questions. Students are allowed to choose any ten to answer.

How many different choices of ten questions are there?

$C_{14,10}$ #

Suppose six questions require proof and eight do not.

- How many groups of ten questions contain four that require proof and six that do not?

$C_{6,4} \cdot C_{8,6}$ #

- How many groups of ten questions contain at least one that requires proof?

$C_{14,10}$ #

(ii) Because there are only eight questions that do not require proof, any group of ten questions contains at least two (not just one) that require proof. Thus the answer is the same as for part (a):

Problem 7

Find how many solutions there are to the given equation that satisfy the given condition.

$$a + b + c + d + e = 500$$

each of a, b, c, d, and e is an integer that is at least 10.

$a \geqslant 10, \quad b \geqslant 10, \quad c \geqslant 10, \quad d \geqslant 10, \quad e \geqslant 10$

$x_1 = a-10 \quad x_2 = b-10 \quad x_3 = c-10 \quad x_4 = d-10 \quad x_5 = e-10$

$$a+b+c+d+e = 500$$

$$x_1+10+x_2+10+x_3+10+x_4+10+x_5+10 = 500$$

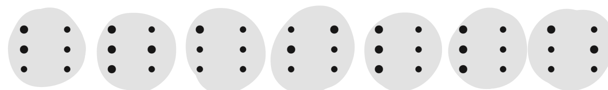
$$x_1+x_2+x_3+x_4+x_5+50 = 500$$

$$x_1+x_2+x_3+x_4+x_5 = 450$$

$$\therefore C_{454,4} \ \#$$

Problem 8

Each symbol in the Braille code is represented by a rectangular arrangement of six dots, each of which may be raised or flat against a smooth background. For instance, when the word Braille is spelled out, it looks like this:



Given that at least one of the six dots must be raised, how many symbols can be represented in the Braille code?

2 2
2 2
2 2

.. $2^6 - 1 = 63$ #

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Problem 9

Find a recurrence relation and initial conditions that generate a sequence that begins with the given terms.

$$15, 12, 9, 6, 3, \ldots$$

$-3 \quad -3 \quad -3 \quad -3$

$a_n = a_{n-1} - 3$

$a_1 = 15$

Iteratively unfold the recurrance relation

$a_n = a_{n-1} - 3$

$= a_{n-2} - 3 - 3$

$= a_{n-3} - 3 - 3 - 3$

$a_n = a_{n-k} - 3k$

Make use of the initial condition

Let $n-k = 1$

$k = n-1$

$a_n = a_1 - 3(n-1)$

$= 15 - 3(n-1)$

$= 15 - 3n + 3$

$a_n = 18 - 3n$ #

Problem 10

$\frac{4}{4} = 1$

Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer n, let $R_n =$ the amount on deposit at the end of the nth quarter, assuming no additional deposits or withdrawals, and let R_0 be the initial amount deposited.

Find a recurrence relation for $R_0, R_1, R_2, \ldots$

$R_n = (1.01) R_{n-1}$ #

If $R_0 = \$5{,}000$, find the amount of money on deposit at the end of one year.

$R_4 = (1.01)^4 (5000) = \$5203.02$ #

Find an explicit formula for A_n.

$R_n = (1.01) R_{n-1}$

$= (1.01)(1.01) R_{n-2}$

$= (1.01)(1.01)(1.01) R_{n-3}$

$R_n = (1.01)^3 R_{n-3}$

$R_n = (1.01)^k R_{n-k}$

let $n - k = 0$

$k = n$

$R_n = (1.01)^n R_0$

$R_n = (1.01)^n (5000)$ #

Problem 11

Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later?

$A_n = (50 \times 10^6)(1.03)^n$

$A_{25} = (50 \times 10^6)(1.03)^{25}$

$= 104.68 \times 10^6$ #

Problem 12

State whether each of the following is a **second-order linear homogeneous recurrence relation with constant coefficients**:

a. $a_k = 3a_{k-1} + 2a_{k-2}$ Re-Al ✓ Fa-Ke ○

b. $b_k = b_{k-1} + b_{k-2} + b_{k-3}$ Re-Al ○ Fa-Ke ✓

order 3

c. $c_k = \frac{1}{2}c_{k-1} - \frac{3}{7}c_{k-2}$ Re-Al ✓ Fa-Ke ○

d. $d_k = d_{k-1}^2 + d_{k-1} \cdot d_{k-2}$ Re-Al ○ Fa-Ke ✓

** e. $e_k = 2e_{k-2}$ $A=0, B=2$ Re-Al ✓ Fa-Ke ○

f. $f_k = 2f_{k-1} + 1$ Re-Al ○ Fa-Ke ✓

g. $g_k = g_{k-1} + g_{k-2}$ Re-Al ✓ Fa-Ke ○

h. $h_k = (-1)h_{k-1} + (k-1)h_{k-2}$ Re-Al ○ Fa-Ke ✓

Problem 13

Suppose a sequence $b_0, b_1, b_2, \ldots$ satisfies the recurrence relation

$$b_k = 4b_{k-1} - 4b_{k-2} \quad \text{for every integer } k \geq 2,$$

with initial conditions

$$b_0 = 1 \quad \text{and} \quad b_1 = 3.$$

Find an explicit formula for $b_0, b_1, b_2, \ldots$

$t^2 - 4t + 4 = 0$

$(t-2)(t-2) = 0$

$t = 2 \quad (r_1 = r_2)$

$a_n = b(2)^n + dn(2)^n$

$$\left.\begin{array}{l} a_0 = b(2)^0 + d(0)(2)^0 \longrightarrow 1 = b \\ a_1 = b(2)^1 + d(1)(2)^1 \longrightarrow 3 = 2b + 2d \end{array}\right\} \begin{array}{l} b = 1 \\ d = 1/2 \end{array}$$

$\therefore b_n = 2^n + \frac{n}{2}(2)^n$ #

Problem 14

Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for every integer } n \geq 0,$$

where C and D are real numbers. Find C and D so that $b_0 = 3$ and $b_1 = 4$. What is b_2 in this case?

$b_n = D(-2)^n + C(3)^n$

$$\left.\begin{array}{l} b_0 = D(-2)^0 + C(3)^0 \longrightarrow 3 = D + C \\ b_1 = D(-2)^1 + C(3)^1 \longrightarrow 4 = -2D + 3C \end{array}\right\} \begin{array}{l} C = 2 \\ D = 1 \end{array}$$

$b_n = (-2)^n + 2(3)^n$ #

$b_2 = (-2)^2 + 2(3)^2$

$= 4 + 18$

$= 22$ #

Additional Task: After finding the values of C and D, calculate $C^{10} \times D$ and convert the result to binary *(just for fun, lol)*.

$(C^2 + D) = 1025 = 100\ 0000\ 0001_2$ #

Problem 15

```
procedure peanut(n)
    if (n = 1) then return (3)
    if (n = 2) then return (7)
    temp := 1
    i := 0
    while i ≤ 3n do
        begin
            temp := (temp * peanut(n-2))
            i := i + 3
        end
    return (temp * peanut(n-1))
end
```

This algorithm takes a positive integer as its input.

- Let a_n be **the number of times the procedure peanut is invoked** when its input is n. Write a recurrence relation and initial conditions that together define $a_1, a_2, a_3, \ldots$
- Let b_n be **the value that this procedure returns** when it is invoked with the input n. Write a recurrence relation and initial conditions that together define $b_1, b_2, b_3, \ldots$

Find both a_n and b_n.

$a_4 = 1 + 5a_2 + a_3$

$a_3 = 1 + 4a_1 + a_2$

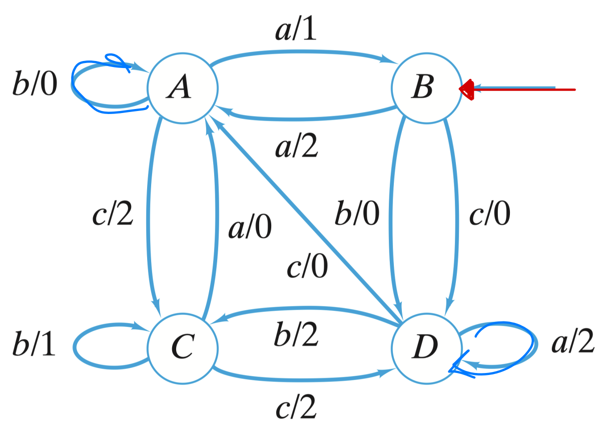
$\therefore a_n = 1 + (n+1)(a_{n-2}) + (a_{n-1})$ #

$b_4 = 1 \cdot (b_2)^5 \cdot b_3$

$\therefore b_n = (b_{n-2})^{n+1} \cdot b_{n-1}$ #

Problem 16

Find the output string for the given input string $cacbccbaabac$



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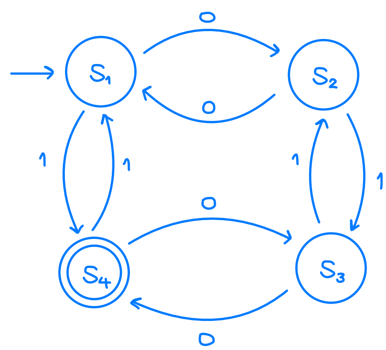
Problem 17

Draw the transition diagram of a finite-state automaton that accepts the set of strings over $\{0, 1\}$ that contain an even number of 0's and an odd number of 1's.

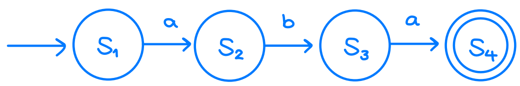


#

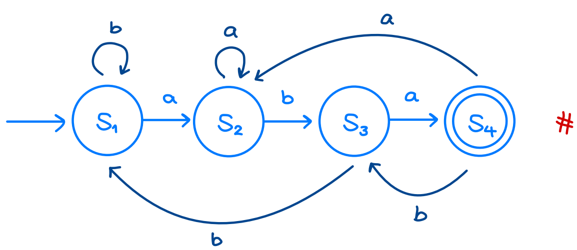
Problem 18

Draw the transition diagram of a finite-state automaton that accepts the given set of strings over {a,b} that ends with *aba*

1) Start with shortest happy path

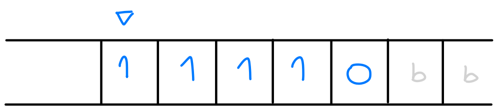


2) Complete every state



Problem 19

Construct a Turing machine that recognizes the set of all bit strings that end with '0'.



$S_0\ 0 : S_0\ 0\ R$

$S_0\ 1 : S_0\ 1\ R$

$S_0\ b : S_1\ b\ L$

$S_1\ 0 : S_{yes}\ 0\ H$

$S_1\ 1 : S_{NO}\ 1\ H$

$S_0\ b$: (empty) #