GTS210: Mathematics for Technologists III
Final Mock Exam

from James Dean and Google and The Peanuts

Name.............ID.............Section.............Seat No.............

Conditions: Open Book, Calculator Allowed

Directions:

1. This exam has 12 pages (including this page).
2. Students are encouraged to sigh whenever the matrix seems unsolvable.
3. Write your name, pseudonym, or favorite eigenvector.
4. Reading the problem is optional but highly recommended.
5. Solutions can be written in English, Thai, or linear combinations of both.
6. Students must remain within the subspace of the exam room for the first two hours—no projecting onto external spaces during this time.

Problem 1

(a) Solve the homogeneous system given by

$$\begin{aligned} x_1 - x_2 - x_3 + x_5 &= 0 \\ x_3 - x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

(b) Find the solution of the system below depending on the parameter a.

Consider:

- Case 1: $\det A = 0$,
- Case 2: $\det A \neq 0$.

The system is:

$$\begin{aligned} ax_1 + x_2 &= 0 \\ (a-1)x_2 &= 1 \end{aligned}$$

Problem 2

(a) Find whether $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 12 \\ 16 \end{pmatrix} \right\}$ is linearly independent in $\mathbb{R}^4$

(b) Find whether $\{1 - x^2, x^2 + 1 + x - x^2\}$ is a basis in $\mathbb{P}_2$

Problem 3

Represent $p(x) = 2 + 3x - 4x^2$ in terms of a new basis in $\mathbb{P}_2$ given by

$$B = \{1 + x + x^2, 1 - x, 1 + x\}$$

Problem 4

Find all fundamental spaces of a matrix given by $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and illustrate the Dimension Theorem.

Problem 5

Vector $v = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ is given in the basis $B = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$. Find v_{B_2} where $B_2 = \{\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$.

Hint: Use properties of the orthonormal basis. Find an inverse matrix is NOT required.

Problem 6

$H = \text{span}\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \}$. Find a basis of H (basis of span) and a basis of the orthogonal complement $H^{\perp}$.

Hint: GSO is NOT required for this problem.

Problem 7

The basis of H $=\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\}$. Find $\text{proj}_{H^\perp} v$, where v $=\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}$.

Find a basis of $H^\perp$.

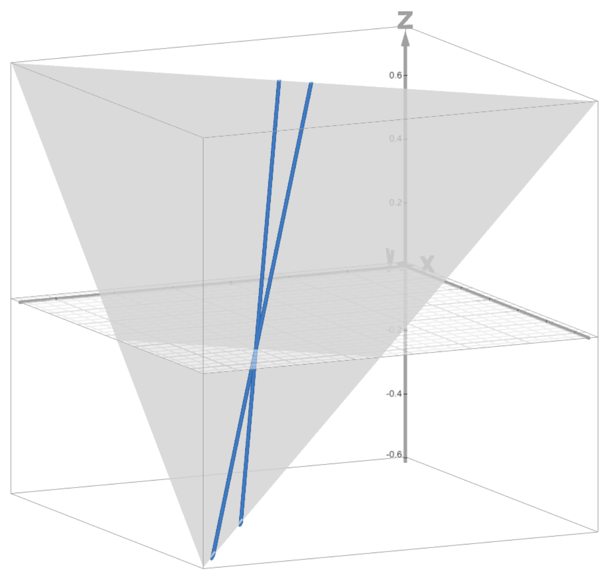
Problem 8

Find the eigenvalues and the eigenvectors of $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ in $\mathbb{R}^4$

Bonus 1

Find the equation $Ax + By + Cz + D = 0$ of a plane P which contains two intersecting lines l_1, l_2 given by

$$l_1(t) = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \qquad l_2(t) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$



Bonus 2

A line in $\mathbb{R}^3$ passes through the points $q_1 = (2, 0, -1)$ and $q_2 = (3, 4, 1)$. Find a plane parallel to this line, i.e., write the equation of that plane as $Ax + By + Cz + D = 0$.

Bonus 3

A line L given by

$$x + 3y + z = 0$$
$$x + y + z = 0$$

Using the idea of a *basis*, prove that the plane given by $-x + z - 1 = 0$ is perpendicular to the line L.