GTS210: Mathematics for Technologists III
Final Mock Exam

from James Dean and Google and The Peanuts

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Conditions: Open Book, Calculator Allowed

Directions:

1. This exam has 12 pages (including this page).
2. Students are encouraged to sigh whenever the matrix seems unsolvable.
3. Write your name, pseudonym, or favorite eigenvector.
4. Reading the problem is optional but highly recommended.
5. Solutions can be written in English, Thai, or linear combinations of both.
6. Students must remain within the subspace of the exam room for the first two hours—no projecting onto external spaces during this time.

Problem 1

(a) Solve the homogeneous system given by

$$x_1 - x_2 - x_3 + x_5 = 0$$
$$x_3 - x_4 = 0$$
$$x_3 + x_4 = 0$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_3' \\ R_3' = R_3/2 \end{array}$$

$$x_1 - x_2 - x_3 + x_5 = 0$$
$$x_3 = 0$$
$$x_4 = 0$$

$$x_1 = x_2 + x_3 - x_5$$
$$x_3 = 0$$
$$x_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 - x_5 \\ x_2 \\ 0 \\ 0 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5 \quad \#$$

(b) Find the solution of the system below depending on the parameter a.

Consider:

- Case 1: $\det A = 0$,
- Case 2: $\det A \neq 0$.

The system is:

$$ax_1 + x_2 = 0$$
$$(a-1)x_2 = 1$$

$$\left[\begin{array}{cc|c} a & 1 & 0 \\ 0 & a-1 & 1 \end{array}\right]$$

$\det(A) = a(a-1)$

$a(a-1) \neq 0$

$a \neq 0, 1 \rightarrow$ unique solution

$a(a-1) = 0$

$a = 0, 1$

$a = 0$

$$\left.\begin{array}{l} x_2 = 0 \\ x_2 = -1 \end{array}\right\} \text{No solution}$$

$a = 1$

$$\left.\begin{array}{l} x_1 + x_2 = 0 \\ 0 = 1 \end{array}\right\} \text{No solution}$$

Problem 2

(a) Find whether $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 12 \\ 16 \end{pmatrix} \right\}$ is linearly independent in $\mathbb{R}^4$

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 4 \\ 3 & 3 & 12 \\ 3 & 1 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 3 & 1 & 16 \end{bmatrix} \quad R_3' = R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 4 \\ 3 & 1 & 16 \end{bmatrix} \longrightarrow \det = -16 \neq 0$$

$\therefore$ Linearly independent in $\mathbb{R}^4$ #

(b) Find whether $\{1 - x^2, x^2, 1 + x - x^2\}$ is a basis in $\mathbb{P}_2 \equiv \mathbb{R}^3$

$$\begin{array}{c} x^0 \\ x^1 \\ x^2 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \longrightarrow \det = -1 \neq 0$$

$\therefore$ Linearly indepedent in $\mathbb{R}^3$

$\therefore$ Constitude a basis in $\mathbb{P}_2$ #

หรือจะใช้วิธี $\left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 0 & 1 & y \\ -1 & 1 & -1 & z \end{array}\right]$ ก็ได้ แต่เสียเวลา!

Problem 3

Represent $p(x) = 2 + 3x - 4x^2$ in terms of a new basis in $\mathbb{P}_2$ given by

$$B = \{1 + x + x^2, 1 - x, 1 + x\}$$

$$A_{B,B_S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$A_{B_S,B}$

$$(P)_B = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1/2 \\ 13/2 \end{bmatrix}$$

$\therefore\ A_{B_S,B} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}$ (using calculator)

$\therefore\ -4(1+x+x^2) - \frac{1}{2}(1-x) + \frac{13}{2}(1+x)$ #

Problem 4

Find all fundamental spaces of a matrix given by $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and illustrate the Dimension Theorem.

Null space of A

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2' = R_2 - 3R_1$$

$$3x_1 = 0$$
$$x_3 + x_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4$$

$$\therefore \ker(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \ \#$$

Column space of A

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2' = R_2 - 3R_1$$

$$\therefore \mathrm{Col}(A) = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \ \#$$

Row space of A

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2' = R_2 - 3R_1$$

$$\therefore \mathrm{Row}(A) = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \ \#$$

Null space of A^T

$$A^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} \quad \begin{matrix} R_1' = R_1 - R_3 \\ \\ \\ R_4' = R_4 - R_3 \end{matrix}$$

$$\left.\begin{matrix} 3x_1 = 0 \\ x_2 = 0 \end{matrix}\right\} \ x_1 = x_2 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

$$\therefore \ker(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \ \#$$

The Dimension Theorem

$$\mathrm{Nullity}(A) + \mathrm{rank}(A) = \#\text{cols}$$
$$2 + 2 = 4 \ \checkmark$$

$$\mathrm{Nullity}(A^T) + \mathrm{rank}(A^T) = \#\text{rows}$$
$$2 + 2 = 4 \ \checkmark$$

Problem 5

Standard basis in B_S

Vector $v = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ is given in the basis $B = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$. Find v_{B_2} where $B_2 = \{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$.

Hint: Use properties of the orthonormal basis. Find an inverse matrix is NOT required.

$$(v_q v_1) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{4\sqrt{5}}{5}$$

$$(v_q v_2) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = -\frac{3\sqrt{5}}{5}$$

$$(v_q v_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -3$$

$$\therefore \; v_{B_2} = \begin{bmatrix} 4\sqrt{5}/5 \\ -3\sqrt{5}/5 \\ -3 \end{bmatrix} \#$$

Check !

Problem 6

$H = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \right\}$. Find a basis of H (basis of span) and a basis of the orthogonal complement $H^\perp$.

Hint: GSO is NOT required for this problem.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - 2R_1 \end{matrix}$$

$$x + 2y + 3z = 0$$
$$x = -2y - 3z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of H #

Find basis of $H^\perp$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \; ; \quad -2x + y = 0$$

$$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \; ; \quad -3x + z = 0$$

$$-2x + y - 3x + z = 0$$
$$-5x + y + z = 0$$
$$x = \frac{y}{5} + \frac{z}{5}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{y}{5} + \frac{z}{5} \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore \left\{ \begin{bmatrix} 1/5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $H^\perp$ #

Problem 7

The basis of $H = \{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$. Find $\text{proj}_{H^\perp} v$, where $v = \{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \}$.

Find a basis of $H^\perp$.

Find basis of $H^\perp$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \; ; \quad y - z = 0$$
$$y = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} z$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $H^\perp$ #

$$\text{proj}_{H^\perp} v = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{proj}_{H^\perp} v = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \quad \#$$

Problem 8

Find the eigenvalues and the eigenvectors of $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ in $\mathbb{R}^4$

$$\det(A-\lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + \lambda^4 = 0$$

$$\lambda^3(-1+\lambda) = 0$$

$$\lambda = 0, 1 \quad \text{(Eigenvalues)} \; \#$$

$\lambda = 0$

$$A\vec{x} = \lambda\vec{x}$$

$$(A-\lambda I)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - \cancel{0I} \right)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 + x_4 = 0$$

$$x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_4$$

$\therefore$ Eigenvector of $\lambda = 0$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} c_2$; $c_1, c_2 \in \mathbb{R}$ #

$\lambda = 1$

$$A\vec{x} = \lambda\vec{x}$$

$$(A-\lambda I)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - (1)I \right)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \quad \begin{array}{l} R_1' = R_1 + R_3 \\ \\ \\ R_4' = R_4 - R_2 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

$\therefore$ Eigenvector of $\lambda = 1$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} c$; $c \in \mathbb{R}$ #

Bonus 1

Find the equation $Ax + By + Cz + D = 0$ of a plane P which contains two intersecting lines l_1, l_2 given by

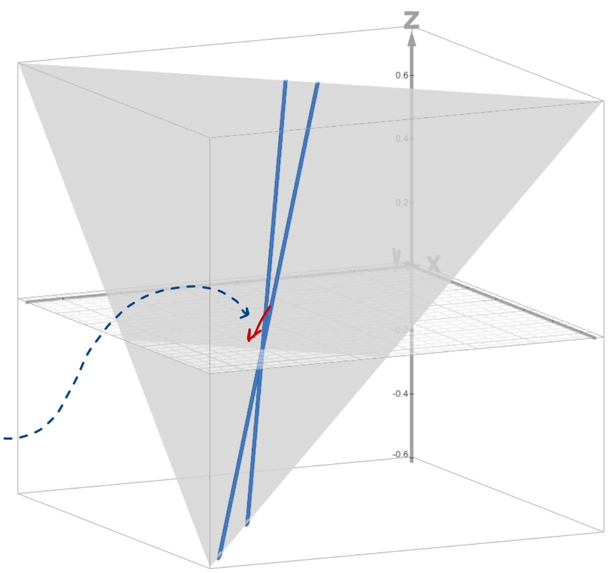
$$l_1(t) = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t \overset{d_1}{\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}}, \qquad l_2(t) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + s \overset{d_2}{\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}}$$

$$\vec{d}_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\vec{d}_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$$

cross product

$$\vec{n} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$



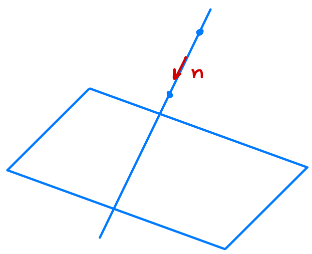
$$Ax + By + Cz + D = 0$$
$$-x - y + z + D = 0$$

$$-(-1) - (-1) + 0 + D = 0$$
$$1 + 1 + D = 0$$
$$D = -2$$

$$-x - y + z - 2 = 0$$
$$x + y - z + 2 = 0 \ \#$$

Bonus 2

A line in $\mathbb{R}^3$ passes through the points $q_1 = (2, 0, -1)$ and $q_2 = (3, 4, 1)$. Find a plane parallel to this line, i.e., write the equation of that plane as $Ax + By + Cz + D = 0$.



$\vec{n} = q_2 - q_1 = \begin{pmatrix} 3-2 \\ 4-0 \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

Plane equation: $(n, r - r_0) = 0$

$r_0 = (2, 0, -1)$

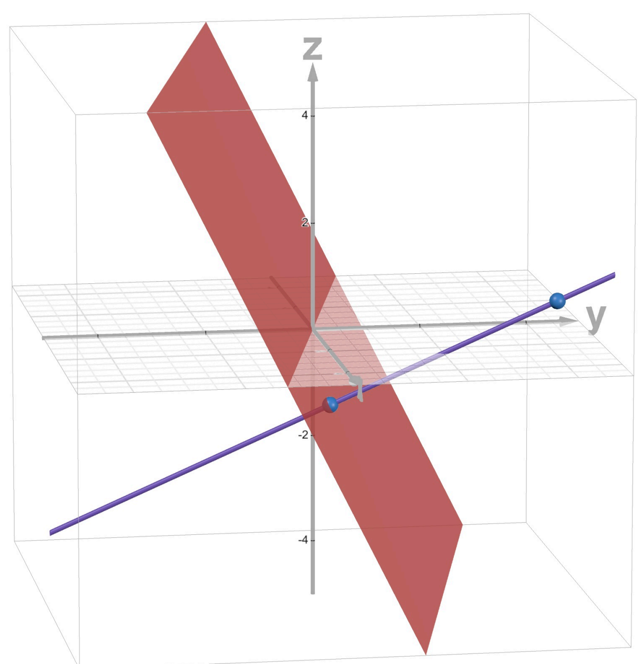
$r - r_0 = \begin{pmatrix} x-2 \\ y \\ z+1 \end{pmatrix}$

$(n, r - r_0) = 0$

$1(x-2) + 4(y) + 2(z+1) = 0$

$x - 2 + 4y + 2z + 2 = 0$

$x + 4y + 2z = 0$ #



Bonus 3

A line L given by

$$\left.\begin{aligned} x + 3y + z &= 0 \\ x + y + z &= 0 \end{aligned}\right\}$$ given by intersection of two planes

Using the idea of a *basis*, prove that the plane given by $-x + z - 1 = 0$ is perpendicular to the line L.

passing through origin

$-x + z = 0$

$[-1 \;\; 0 \;\; 1]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore$ Basis on the plane is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Find a basis on the line

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \quad R_2' = R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \quad \begin{aligned} R_1' &= R_1 - R_2' \\ R_2' &= -R_2 \end{aligned}$$

$x + y + z = 0$

$2y = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$\therefore$ The plane and the line are perpendicular (orthogonal) #