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HW3
A15939512

1 1

$\frac{x}{\|x\|} = \frac{(1,2,3)}{\sqrt{1+4+9}} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

2 2

If a vector v is orthogonal to $(1,1)$, then $(1,1) \cdot v = 0$, $v_1 + v_2 = 0, v_1 = -v_2$. Thus, the vector is parallel to $(1,-1)$. The only unit vectors satisfying this are the unit vector and negative unit vector, or $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$.

3 3

$(x_1, x_2) \cdot (x_1, x_2) = 25$
$x_1^2 + x_2^2 = 25$
I would describe it as a circle centered at the origin with radius 5.

4 4

$f(x) = (w_1, w_2, w_3) \cdot (x_1, x_2, x_3)$
$f(x) = (2, -1, 6) \cdot (x_1, x_2, x_3)$
$w = (2, -1, 6)$

5 5

When doing matrix multiplication, $(m,k) \times (k,n) \to (m,n)$. If A has 30 columns, then $k = 30$. The output $(m,n) = (10,20)$, so $m = 10$,$n = 20$. Thus, A is a 10x30 matrix, B is a 30x20 matrix.

6 6

6.1 a

$n \times d$

6.2 b

$(n,d) \times (d,n) \to (n,n)$
Answer:
$n \times n$

6.3 c

It is the i'th data point dot-producted with the j'th data point

7 7

Assuming a convention that x is a column-vector:
$x^T x = \|x\|^2$
$x^T x x^T x x^T x = \|x\|^6$

8 8

8.1 $x^T x$

$(1, 3, 5) \cdot (1, 3, 5) = 1 * 1 + 3 * 3 + 5 * 5 = 1 + 9 + 25 = 35$

8.2 xx^T

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{bmatrix}$

9 9

$\|x\| = 2, \|y\| = 2, x \cdot y = 2$
$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$
$\cos\theta = \frac{2}{2*2} = \frac{1}{2}$
$\theta = \cos^{-1}(0.5) = \frac{\pi}{3}$

10 10

$x^T \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 6 \end{bmatrix} x$

11 11

11.1 a

AA^T must be symmetric since the i, j entry $(A_{i,j})$ are the dot products of the i'th row and j'th row. Dot product is symmetric, so the $A_{i,j} = A_{j,i} = A_{i,:} \cdot A_{j,:}$

11.2 b

$A^T A$ must be symmetric since it is is the same as $A' = A^T, (A'A'^T)$, and we just showed that AA^T is necessarily symmetric.

11.3 c

$A + A^T$ is necessarily symmetric. For some pair $(i, j), i \neq j$ of the resulting matrix M, we can check $M_{i,j} = A_{i,j} + A^T_{i,j} = A_{i,j} + A_{j,i}$, $M_{j,i} = A_{j,i} + A^T_{j,i} = A_{j,i} + A_{i,j} = M_{i,j}$. Because transposed-corresponding off-diagonal elements are equivalent, the matrix $A + A^T$ must be necessarily symmetric.

11.4 d

$A - A^T$ is not necessarily symmetric. Let us craft a counterexample:
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
$A - A^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
which is clearly not symmetric.

12 12

12.1 a

Determinant of a diagonal matrix is the product of entries, so it is $8! = 40320$

12.2 b

Inverse of a diagonal matrix is a matrix of the diagonal elemenets inverted (according to lecture), so it is $diag(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})$

13 13

13.1 a

If the matrix U is square, has rows of orthonormal vectors, then it is an orthogonal matrix. UU^T is a matrix where the i, j'th entry is a dot product of the i'th and j'th vector. The dot product of a unit vector with itself is 1, so the diagonal entries are all 1. Because the rows form an orthonormal basis, all $U_{i,:} \cdot U_{j,:}, i \neq j = 0$, thus the non-diagonal entries are all zeros. This results in $UU^T = I$.

13.2 b

We can trivially show that $UU^T = I = I^T = (UU^T)^T = U^T U^{TT} = U^T U$. For any invertible square matrix, we also know that $AA^{-1} = A^{-1}A = I$. Matching these equations together, we can see that $U^T = U^{-1}$.

14 14

$\det(A) = 1 * z - 2 * 3 = 0$
$z = 6$

15 15

15.1 a

Training Procedure/Overview:
The dataset was split by randomly sampling 50000 indexes out of the integers in the range [0, 59999] using np.random.choice without replacement, and those indexes correspond to the (sample, label) tuples that comprise the training set. The remaining 10000 non-sampled indexes comprise the validation set.
Next, the code supplied by the jupyter notebook was used to test ranges of values of c, evaluating on the validation set (rather than the test set, which the code was originally set up to do)

- First, a wide variety of power-of-2 c-values were tested, ranging from small powers of 2 to large powers of 2 (min: 2^{-2}, max: 2^{17})
- After finding the best values from the initial broad search (I determined it was from [1024,8192]), a fine=grained search over that interval is performed, sampling 20 equally-spaced points in the interval. From here, the c that resulted in the lowest validation error was chosen as the best hyper-parameter value and was used to evaluate the test set.

Lastly, after finding the best value of c, the model is run through the test set using the method provided by the starter code.
Code will be applied to this question's submission.

15.2 b

c=3664

15.3 c

0.0438 Error Rate on test set

15.4 d

See attached submission pages

Gaussian generative models for handwritten digit classification

Recall that the 1-NN classifier yielded a 3.09% test error rate on the MNIST data set of handwritten digits. We will now see that a Gaussian generative model does almost as well, while being significantly faster and more compact.

1. Set up notebook and load in data

As usual, we start by importing the required packages and data. For this notebook we will be using the *entire* `MNIST` dataset. The code below defines some helper functions that will load `MNIST` onto your computer.

In [1]:
```
%matplotlib inline
import matplotlib.pyplot as plt
import gzip, os, sys
import numpy as np
from scipy.stats import multivariate_normal

if sys.version_info[0] == 2:
    from urllib import urlretrieve
else:
    from urllib.request import urlretrieve
```

In [2]:
```
# Function that downloads a specified MNIST data file from Yann Le Cun's website
def download(filename, source='http://yann.lecun.com/exdb/mnist/'):
    print("Downloading %s" % filename)
    urlretrieve(source + filename, filename)

# Invokes download() if necessary, then reads in images
def load_mnist_images(filename):
    if not os.path.exists(filename):
        download(filename)
    with gzip.open(filename, 'rb') as f:
        data = np.frombuffer(f.read(), np.uint8, offset=16)
    data = data.reshape(-1,784)
    return data

def load_mnist_labels(filename):
    if not os.path.exists(filename):
        download(filename)
    with gzip.open(filename, 'rb') as f:
        data = np.frombuffer(f.read(), np.uint8, offset=8)
    return data
```

Now load in the training set and test set

In [3]:
```
## Load the training set
train_data = load_mnist_images('train-images-idx3-ubyte.gz')
train_labels = load_mnist_labels('train-labels-idx1-ubyte.gz')

## Load the testing set
test_data = load_mnist_images('t10k-images-idx3-ubyte.gz')
test_labels = load_mnist_labels('t10k-labels-idx1-ubyte.gz')
```

```
Downloading train-images-idx3-ubyte.gz
Downloading train-labels-idx1-ubyte.gz
Downloading t10k-images-idx3-ubyte.gz
Downloading t10k-labels-idx1-ubyte.gz
```

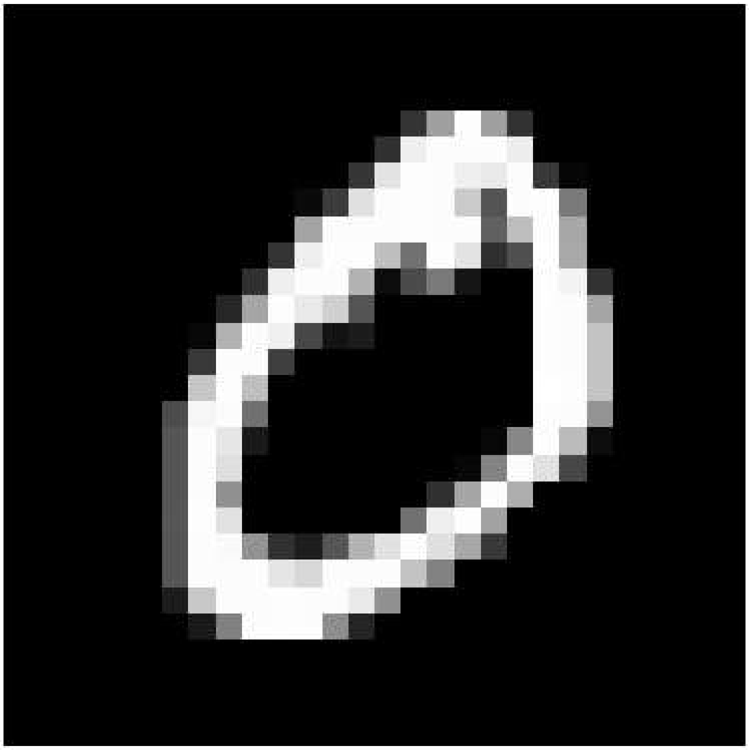
In [10]:
```
# choose 50000 indexes
np.random.seed(123)
idx_tr = np.random.choice(60000, 50000, replace=False)
mask_tr = np.zeros(60000, dtype=bool)
# generate a bit mask for training data
mask_tr[idx_tr] = 1
# generate inverse bitmask for validation data
mask_va = ~mask_tr

tr_X, tr_y = train_data[mask_tr], train_labels[mask_tr]
va_X, va_y = train_data[mask_va], train_labels[mask_va]

print(tr_X.shape, tr_y.shape, va_X.shape, va_y.shape)
```

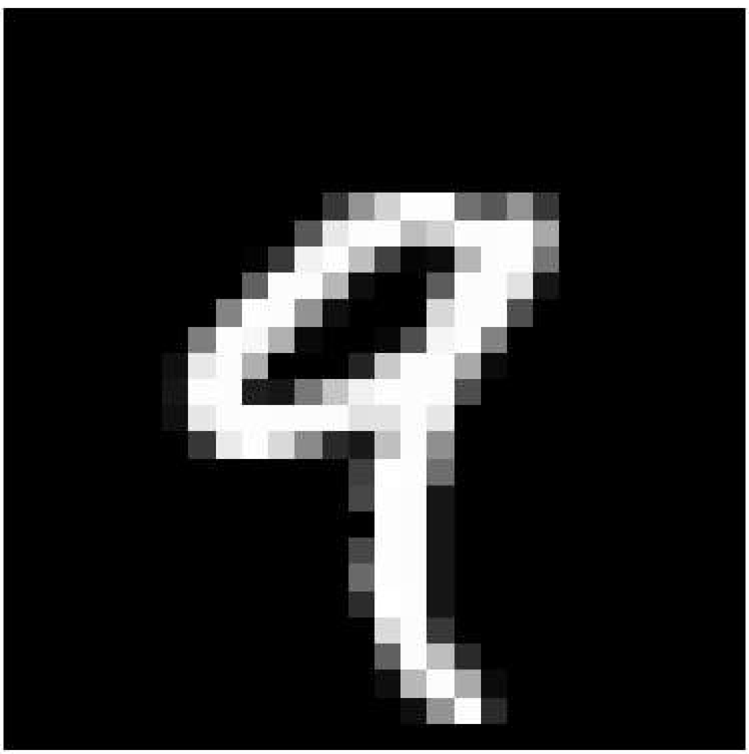
```
(50000, 784) (50000,) (10000, 784) (10000,)
```

The function **displaychar** shows a single MNIST digit. To do this, it first has to reshape the 784-dimensional vector into a 28x28 image.

In [11]:
```
def displaychar(image):
  plt.imshow(np.reshape(image, (28,28)), cmap=plt.cm.gray)
  plt.axis('off')
  plt.show()
```

In [13]:
```
displaychar(tr_X[0])
displaychar(va_X[1])
```





2. Fit a Gaussian generative model to the training data

For you to do: Define a function, **fit_generative_model**, that takes as input a training set (data `x` and labels `y`) and fits a Gaussian generative model to it. It should return the parameters of this generative model; for each label `j = 0,1,...,9`, we have:

- `pi[j]` : the frequency of that label
- `mu[j]` : the 784-dimensional mean vector
- `sigma[j]` : the 784x784 covariance matrix

This means that `pi` is 10x1, `mu` is 10x784, and `sigma` is 10x784x784.

We have already seen how to fit a Gaussian generative model in the Winery example, but now there is an added ingredient. The empirical covariances are very likely to be singular (or close to singular), which means that we won't be able to do calculations with them. Thus it is important to **regularize** these matrices. The standard way of doing this is to add `cI` to them, where `c` is some constant and `I` is the 784-dimensional identity matrix. (To put it another way, we compute the empirical covariances and then increase their diagonal entries by some constant `c`.)

This modification is guaranteed to yield covariance matrices that are non-singular, for any `c > 0`, no matter how small. But this doesn't mean that we should make `c` as small as possible. Indeed, `c` is now a parameter, and by setting it appropriately, we can improve the performance of the model. We will study **regularization** in greater detail over the coming weeks.

Your routine needs to choose a good setting of `c`. Crucially, this needs to be done using the training set alone. So you might try setting aside part of the training set as a validation set, or using some kind of cross-validation.

In [14]:
```
def fit_generative_model(x, y, c):
  k = 10  # labels 0,1,...,k-1
  d = (x.shape)[1]  # number of features
  mu = np.zeros((k,d))
  sigma = np.zeros((k,d,d))
  pi = np.zeros(k)
  ###
  ### Your code goes here
  ###
  for num in range(k):
    idxs = np.nonzero(y == num)[0]
    vals = x[idxs]
    mu[num,:] = np.mean(vals, axis=0)
    sigma[num,:,:] = np.cov(vals.T) + c * np.eye(784)
    pi[num] = len(idxs) / len(x)

  # Halt and return parameters
  return mu, sigma, pi
```

Okay, let's try out your function. In particular, we will use **displaychar** to visualize the means of the Gaussians for the first three digits. You can try the other digits on your own.

In [18]:
```
# Fit `c` parameter
cs = [2**i for i in range(-2, 18)]
def run_vals(cs):
```

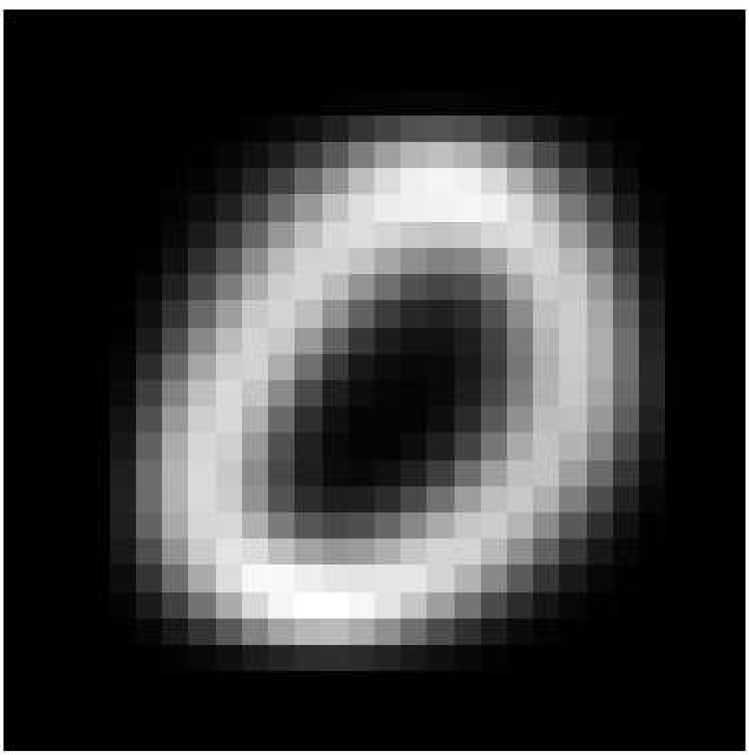
```
  # This is basically copied from the starter code, changing around the datasets
  k = 10
  for c in cs:
    mu, sigma, pi = fit_generative_model(tr_X, tr_y, c = c)
    score = np.zeros((len(va_y),k))
    for label in range(0,k):
        rv = multivariate_normal(mean=mu[label], cov=sigma[label])
        for i in range(0,len(va_y)):
          score[i,label] = np.log(pi[label]) + rv.logpdf(va_X[i,:])
    predictions = np.argmax(score, axis=1)
    # Finally, tally up score
    errors = np.sum(predictions != va_y)
    print(f"c={c}, Err:{str(errors)}/10000, {errors/10000}")
# Get Performance metrics
run_vals(cs)
```

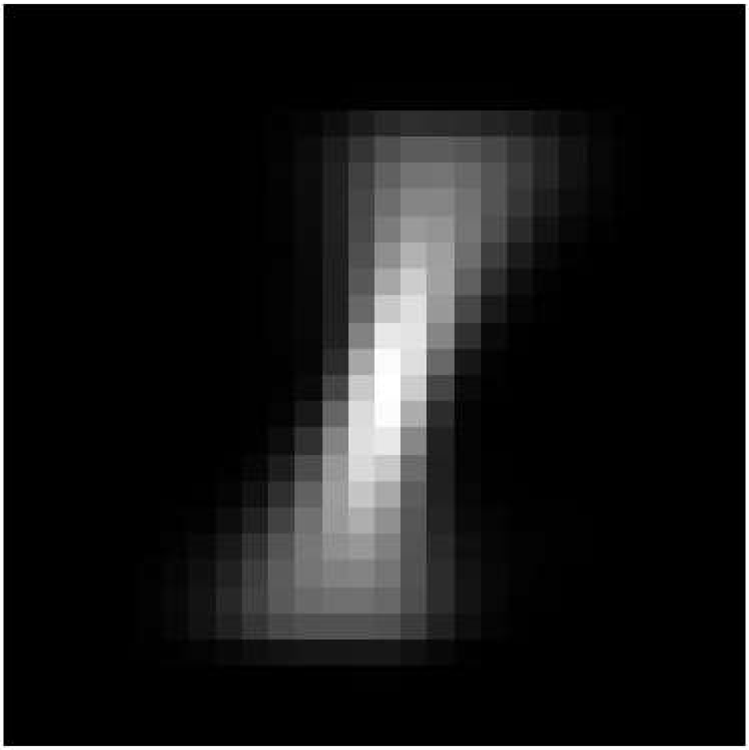
```
c=0.25, Err:1810/10000, 0.181
c=0.5, Err:1732/10000, 0.1732
c=1, Err:1657/10000, 0.1657
c=2, Err:1571/10000, 0.1571
c=4, Err:1471/10000, 0.1471
c=8, Err:1365/10000, 0.1365
c=16, Err:1235/10000, 0.1235
c=32, Err:1110/10000, 0.111
c=64, Err:963/10000, 0.0963
c=128, Err:838/10000, 0.0838
c=256, Err:715/10000, 0.0715
c=512, Err:619/10000, 0.0619
c=1024, Err:551/10000, 0.0551
c=2048, Err:494/10000, 0.0494
c=4096, Err:480/10000, 0.048
c=8192, Err:524/10000, 0.0524
c=16384, Err:651/10000, 0.0651
c=32768, Err:871/10000, 0.0871
c=65536, Err:1228/10000, 0.1228
c=131072, Err:1751/10000, 0.1751
```

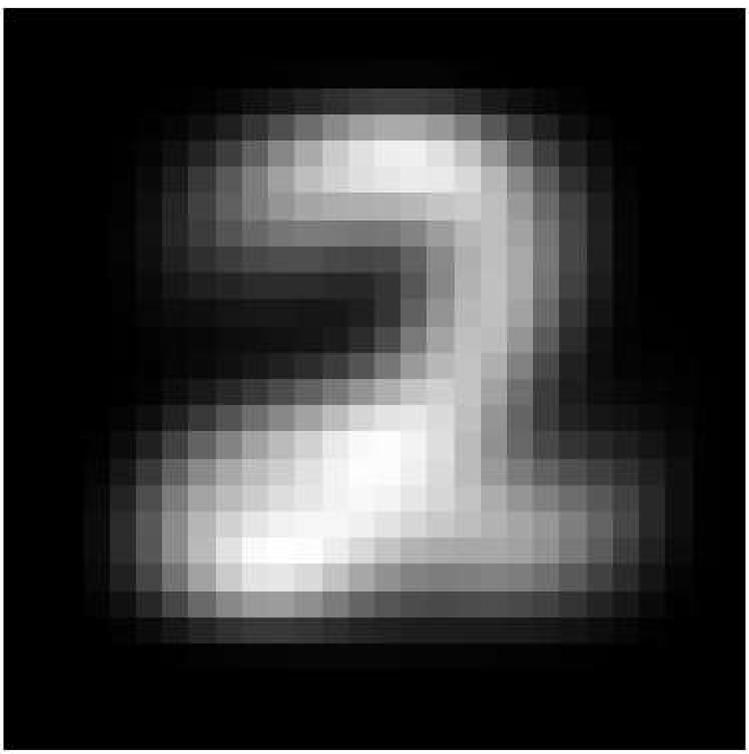
In [19]:
```
cs = np.linspace(1024, 8192, 20).astype(np.int32)
run_vals(cs)
```

```
c=1024, Err:551/10000, 0.0551
c=1401, Err:520/10000, 0.052
c=1778, Err:501/10000, 0.0501
c=2155, Err:492/10000, 0.0492
c=2533, Err:484/10000, 0.0484
c=2910, Err:485/10000, 0.0485
c=3287, Err:478/10000, 0.0478
c=3664, Err:472/10000, 0.0472
c=4042, Err:480/10000, 0.048
c=4419, Err:476/10000, 0.0476
c=4796, Err:477/10000, 0.0477
c=5173, Err:481/10000, 0.0481
c=5551, Err:488/10000, 0.0488
c=5928, Err:496/10000, 0.0496
c=6305, Err:495/10000, 0.0495
c=6682, Err:501/10000, 0.0501
c=7060, Err:510/10000, 0.051
c=7437, Err:519/10000, 0.0519
c=7814, Err:518/10000, 0.0518
c=8192, Err:524/10000, 0.0524
```

In [16]:
```
c = 0.1
mu, sigma, pi = fit_generative_model(tr_X, tr_y, c = 0.1)
displaychar(mu[0])
displaychar(mu[1])
displaychar(mu[2])
```







3. Make predictions on test data

Now let's see how many errors your model makes on the test set.

In [20]:
```
# Compute log Pr(label|image) for each [test image,label] pair.
c = 3664
k = 10
```

```
mu, sigma, pi = fit_generative_model(tr_X, tr_y, c = c)
score = np.zeros((len(test_labels),k))
for label in range(0,k):
    rv = multivariate_normal(mean=mu[label], cov=sigma[label])
    for i in range(0,len(test_labels)):
      score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])
predictions = np.argmax(score, axis=1)
# Finally, tally up score
errors = np.sum(predictions != test_labels)
print(f"c={c}, Err:{str(errors)}/10000, {errors/10000}")
# print("Your model makes " + str(errors) + " errors out of 10000")
```

```
c=3664, Err:438/10000, 0.0438
```

Best Results:

c=3664 = 438/10000, 0.0438 Error Rate

4. Quick exercises

You will need to answer variants of these questions as part of this week's assignment.

Exercise 1: What happens if you do not regularize the covariance matrices?

Answer: An error is thrown saying that the covariance matrix is not positive semidefinite

In [21]:
```
mu, sigma, pi = fit_generative_model(train_data, train_labels, c = 0)
score = np.zeros((len(test_labels),k))
for label in range(0,k):
    rv = multivariate_normal(mean=mu[label], cov=sigma[label])
    for i in range(0,len(test_labels)):
      score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])
predictions = np.argmax(score, axis=1)
# Finally, tally up score
errors = np.sum(predictions != test_labels)
print(f"c={c}, Err:{str(errors)}/10000, {errors/10000}")
```

```
---------------------------------------------------------------------------
LinAlgError                               Traceback (most recent call last)
Cell In[21], line 4
      2 score = np.zeros((len(test_labels),k))
      3 for label in range(0,k):
----> 4     rv = multivariate_normal(mean=mu[label], cov=sigma[label])
      5     for i in range(0,len(test_labels)):
      6       score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])

File c:\Users\James\anaconda3\envs\mlenv\lib\site-packages\scipy\stats\_multivaria
te.py:393, in multivariate_normal_gen.__call__(self, mean, cov, allow_singular, se
ed)
    388 def __call__(self, mean=None, cov=1, allow_singular=False, seed=None):
    389     """Create a frozen multivariate normal distribution.
    390 
    391     See `multivariate_normal_frozen` for more information.
    392     """
--> 393     return multivariate_normal_frozen(mean, cov,
    394                                       allow_singular=allow_singular,
    395                                       seed=seed)

File c:\Users\James\anaconda3\envs\mlenv\lib\site-packages\scipy\stats\_multivaria
te.py:834, in multivariate_normal_frozen.__init__(self, mean, cov, allow_singular,
seed, maxpts, abseps, releps)
    791 """Create a frozen multivariate normal distribution.
    792 
    793 Parameters
   (...)
    830 
    831 """
    832 self._dist = multivariate_normal_gen(seed)
    833 self.dim, self.mean, self.cov_object = (
--> 834     self._dist._process_parameters(mean, cov, allow_singular))
    835 self.allow_singular = allow_singular or self.cov_object._allow_singular
    836 if not maxpts:

File c:\Users\James\anaconda3\envs\mlenv\lib\site-packages\scipy\stats\_multivaria
te.py:417, in multivariate_normal_gen._process_parameters(self, mean, cov, allow_s
ingular)
    410 dim, mean, cov = self._process_parameters_psd(None, mean, cov)
    411 # After input validation, some methods then processed the arrays
    412 # with a `_PSD` object and used that to perform computation.
    413 # To avoid branching statements in each method depending on whether
    414 # `cov` is an array or `Covariance` object, we always process the
    415 # array with `_PSD`, and then use wrapper that satisfies the
    416 # `Covariance` interface, `CovViaPSD`.
--> 417 psd = _PSD(cov, allow_singular=allow_singular)
    418 cov_object = _covariance.CovViaPSD(psd)
    419 return dim, mean, cov_object

File c:\Users\James\anaconda3\envs\mlenv\lib\site-packages\scipy\stats\_multivaria
te.py:172, in _PSD.__init__(self, M, cond, rcond, lower, check_finite, allow_singu
lar)
    169 if len(d) < len(s) and not allow_singular:
    170     msg = ("When `allow_singular is False`, the input matrix must be "
    171            "symmetric positive definite.")
```

```
--> 172     raise np.linalg.LinAlgError(msg)
    173 s_pinv = _pinv_1d(s, eps)
    174 U = np.multiply(u, np.sqrt(s_pinv))

LinAlgError: When `allow_singular is False`, the input matrix must be symmetric po
sitive definite.
```

Exercise 2: What happens if you set the value of `c` too high, for instance to one billion? Do you understand why this happens?

The error skyrockets to around 0.9. This means that the variances are all so insanely high and (relatively) equal that the classifier is just guessing at this point (10% to be right, 1 - 0.1 = 0.9 Error)

In [22]:
```
c = 1E9
mu, sigma, pi = fit_generative_model(train_data, train_labels, c = 1E9)
score = np.zeros((len(test_labels),k))
for label in range(0,k):
    rv = multivariate_normal(mean=mu[label], cov=sigma[label])
    for i in range(0,len(test_labels)):
      score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])
predictions = np.argmax(score, axis=1)
# Finally, tally up score
errors = np.sum(predictions != test_labels)
print(f"c={c}, Err:{str(errors)}/10000, {errors/10000}")
```

```
c=1000000000.0, Err:8865/10000, 0.8865
```

Exercise 3: What value of c did you end up using? How many errors did your model make on the training set?

Best Results:

c=3664 = 438/10000, 0.0438 Error Rate

If you have the time: We have talked about using the same regularization constant `c` for all ten classes. What about using a different value of `c` for each class? How would you go about choosing these? Can you get better performance in this way?

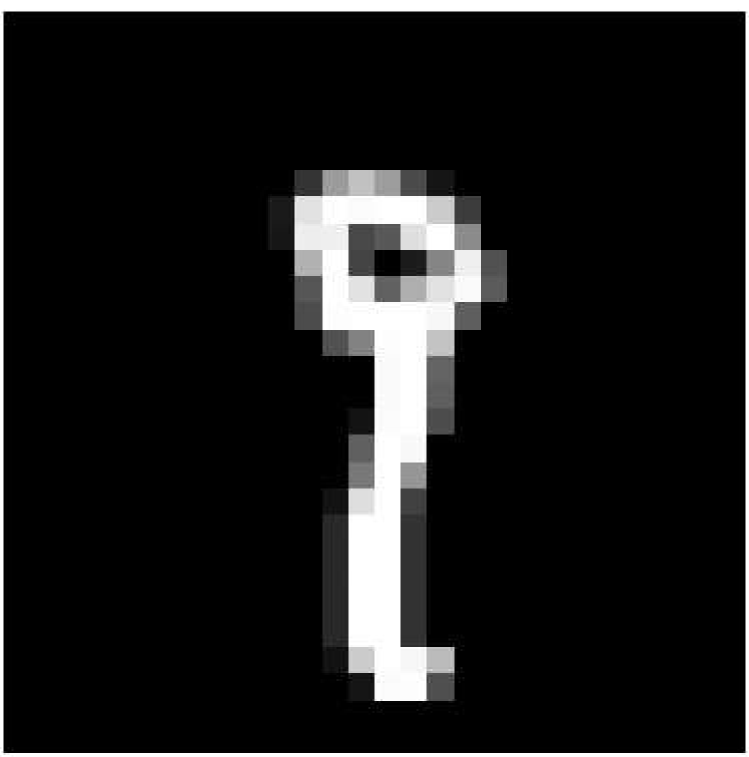
Q15 part d

In [23]:
```
c=3664
mu, sigma, pi = fit_generative_model(train_data, train_labels, c = c)
score = np.zeros((len(test_labels),k))
for label in range(0,k):
    rv = multivariate_normal(mean=mu[label], cov=sigma[label])
    for i in range(0,len(test_labels)):
      score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])
predictions = np.argmax(score, axis=1)
```

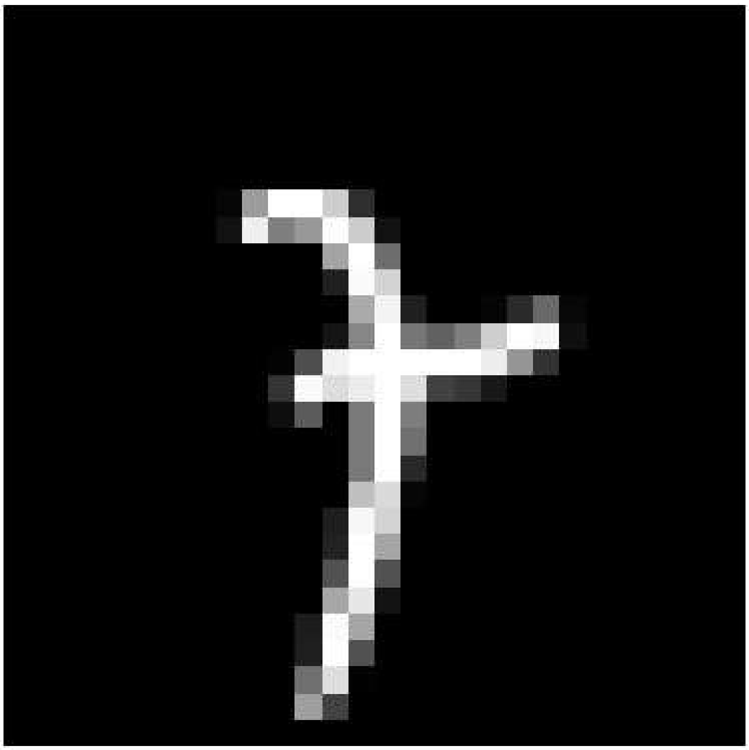
In [24]:
```
idx_wrong = np.nonzero(predictions != test_labels)[0]
```

```python
np.random.seed(123)
# Sample 5 randomly out of those with differing predictions from test_labels
samp = np.random.choice(idx_wrong, 5, replace=False)

for idx in samp:
  log_prob = np.log(pi[label]) + rv.logpdf(test_data[i,:])
  print(f"=== Test Sample {idx}, True label {test_labels[idx]}, Predicted Label {pr
  score_row = score[idx]
  # P(x) = \sum_y P(x,y)
  # log(P(x)) = log(\sum_y P(x,y))
  # P(x,y) = score[x_i,y]
  log_px = -np.inf
  for s in score_row:
    log_px = np.logaddexp(log_px, s)
  for i in range(10):
    # The score[idx,i] is log(P(x|y)P(y)), log(P(x,y))
    # P(y|x) = P(x,y)/P(x), log(P(y|x)) = log(P(x,y)) - log(P(x))
    print(f"\t[{i}], log(P({i}|x)) = {score_row[i] - log_px:10.4f}, P({i}|x) = {np.

  displaychar(test_data[idx])
  print(f"=== END Test Sample {idx}")
```

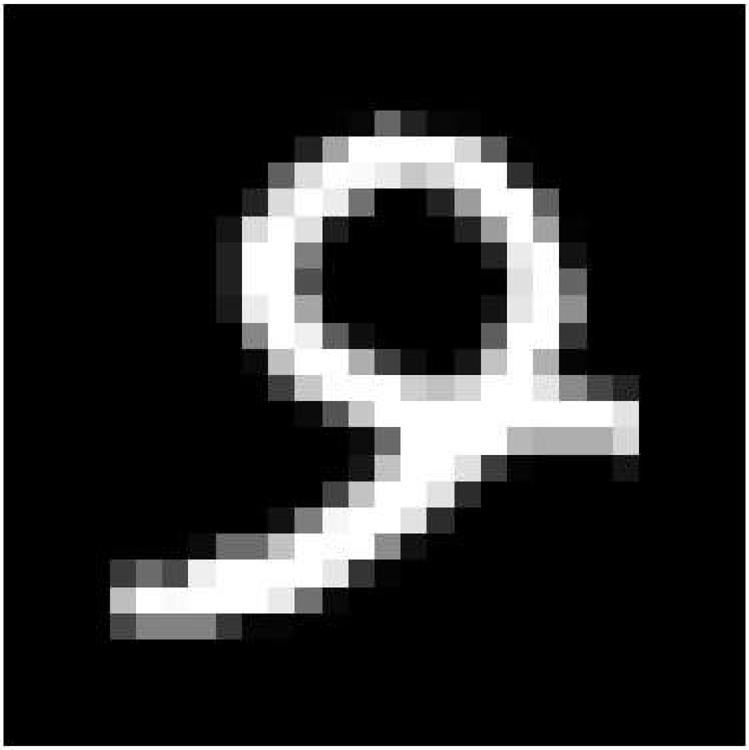
```
=== Test Sample 320, True label 9, Predicted Label 1
        [0], log(P(0|x)) =  -134.0400, P(0|x) = 0.000000
        [1], log(P(1|x)) =    -0.0000, P(1|x) = 0.999985
        [2], log(P(2|x)) =   -86.0809, P(2|x) = 0.000000
        [3], log(P(3|x)) =   -73.4812, P(3|x) = 0.000000
        [4], log(P(4|x)) =   -44.5374, P(4|x) = 0.000000
        [5], log(P(5|x)) =   -79.1180, P(5|x) = 0.000000
        [6], log(P(6|x)) =  -121.4409, P(6|x) = 0.000000
        [7], log(P(7|x)) =   -11.1198, P(7|x) = 0.000015
        [8], log(P(8|x)) =   -30.0164, P(8|x) = 0.000000
        [9], log(P(9|x)) =   -22.9886, P(9|x) = 0.000000
```



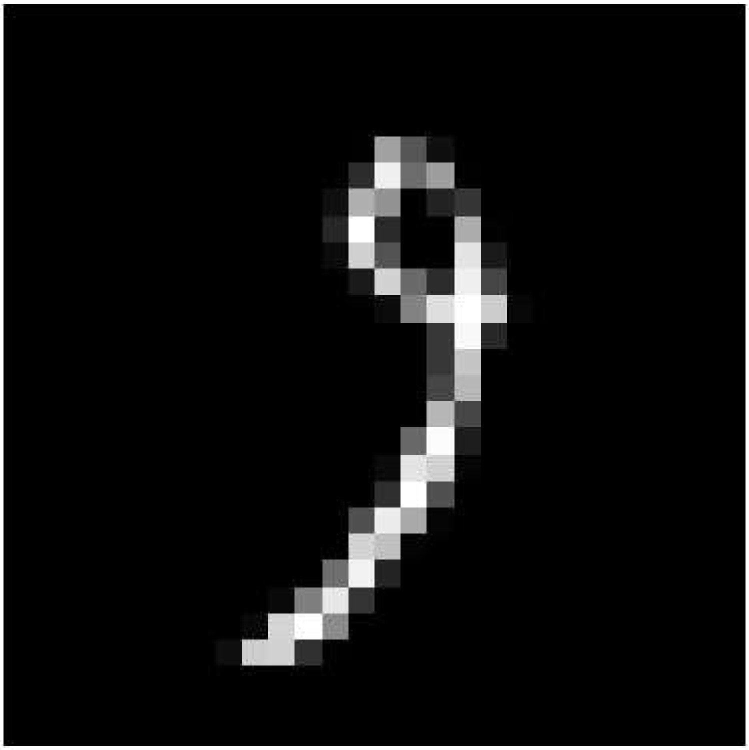
```
=== END Test Sample 320
=== Test Sample 4966, True label 7, Predicted Label 9
        [0], log(P(0|x)) =   -92.3110, P(0|x) = 0.000000
        [1], log(P(1|x)) =    -6.8519, P(1|x) = 0.001057
        [2], log(P(2|x)) =   -49.8557, P(2|x) = 0.000000
        [3], log(P(3|x)) =   -39.9352, P(3|x) = 0.000000
        [4], log(P(4|x)) =   -12.7763, P(4|x) = 0.000003
        [5], log(P(5|x)) =   -41.4150, P(5|x) = 0.000000
        [6], log(P(6|x)) =   -49.6453, P(6|x) = 0.000000
        [7], log(P(7|x)) =    -1.7935, P(7|x) = 0.166378
        [8], log(P(8|x)) =   -13.0910, P(8|x) = 0.000002
        [9], log(P(9|x)) =    -0.1833, P(9|x) = 0.832559
```



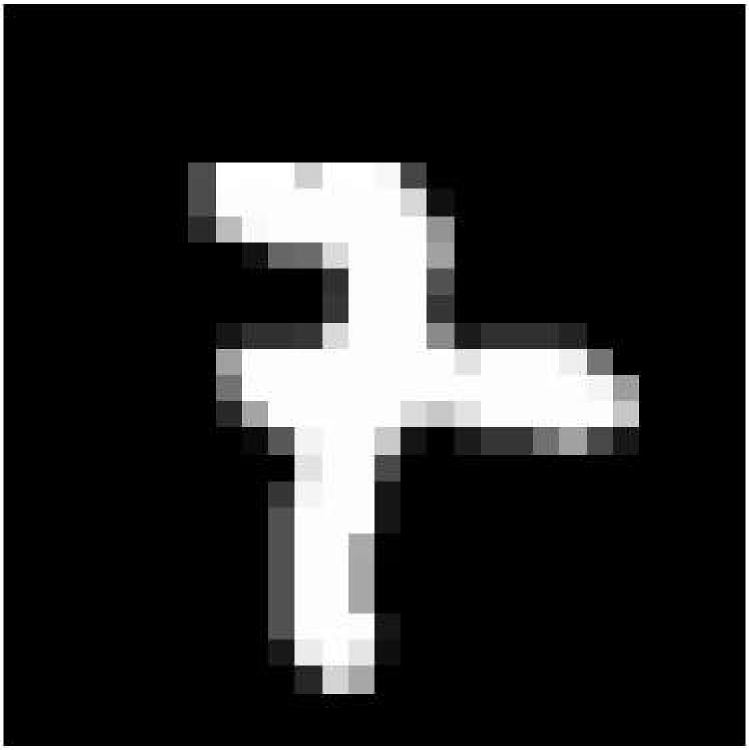
```
=== END Test Sample 4966
=== Test Sample 2129, True label 9, Predicted Label 2
        [0], log(P(0|x)) =   -58.5577, P(0|x) = 0.000000
        [1], log(P(1|x)) =  -277.8698, P(1|x) = 0.000000
        [2], log(P(2|x)) =     0.0000, P(2|x) = 1.000000
        [3], log(P(3|x)) =   -34.4194, P(3|x) = 0.000000
        [4], log(P(4|x)) =   -74.5441, P(4|x) = 0.000000
        [5], log(P(5|x)) =   -64.0238, P(5|x) = 0.000000
        [6], log(P(6|x)) =  -114.5576, P(6|x) = 0.000000
        [7], log(P(7|x)) =   -89.2343, P(7|x) = 0.000000
        [8], log(P(8|x)) =   -30.1542, P(8|x) = 0.000000
        [9], log(P(9|x)) =   -59.2533, P(9|x) = 0.000000
```



```
=== END Test Sample 2129
=== Test Sample 3060, True label 9, Predicted Label 1
        [0], log(P(0|x)) =   -35.9062, P(0|x) = 0.000000
        [1], log(P(1|x)) =    -0.0563, P(1|x) = 0.945272
        [2], log(P(2|x)) =   -38.8763, P(2|x) = 0.000000
        [3], log(P(3|x)) =   -32.3519, P(3|x) = 0.000000
        [4], log(P(4|x)) =   -18.7199, P(4|x) = 0.000000
        [5], log(P(5|x)) =   -42.4339, P(5|x) = 0.000000
        [6], log(P(6|x)) =   -56.9203, P(6|x) = 0.000000
        [7], log(P(7|x)) =    -2.9569, P(7|x) = 0.051981
        [8], log(P(8|x)) =   -35.9937, P(8|x) = 0.000000
        [9], log(P(9|x)) =    -5.8975, P(9|x) = 0.002746
```



```
=== END Test Sample 3060
=== Test Sample 4498, True label 7, Predicted Label 4
        [0], log(P(0|x)) =  -119.1755, P(0|x) = 0.000000
        [1], log(P(1|x)) =  -162.8433, P(1|x) = 0.000000
        [2], log(P(2|x)) =   -33.1990, P(2|x) = 0.000000
        [3], log(P(3|x)) =   -33.0528, P(3|x) = 0.000000
        [4], log(P(4|x)) =    -0.5702, P(4|x) = 0.565393
        [5], log(P(5|x)) =   -56.1085, P(5|x) = 0.000000
        [6], log(P(6|x)) =   -93.6327, P(6|x) = 0.000000
        [7], log(P(7|x)) =    -0.8333, P(7|x) = 0.434607
        [8], log(P(8|x)) =   -17.6050, P(8|x) = 0.000000
        [9], log(P(9|x)) =   -46.9964, P(9|x) = 0.000000
```



```
=== END Test Sample 4498
```