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HW5
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1 Q1

$$L(z) = \sum_{i=1}^{n} \|x^{(i)} - z\|^2$$

$$\frac{\partial L}{\partial z_j} = \sum_{i=1}^{n} \frac{\partial}{\partial z_j} \|x^{(i)} - z\|^2 = \sum_{i=1}^{n} \frac{\partial}{\partial z_j} (x_1^{(i)} - z_1)^2 + ... + \frac{\partial}{\partial z_j} (x_j^{(i)} - z_j)^2 + ...$$

$$= \sum_{i=1}^{n} 0 + 2(x_j^{(i)} - z_j)(-1) + 0 = -2 \sum_{i=1}^{n} x_j^{(i)} - z_j$$

$$= \nabla = \begin{bmatrix} \frac{\partial L}{\partial z_1} \\ \vdots \\ \frac{\partial L}{\partial z_d} \end{bmatrix} = -2 \sum_{i=1}^{n} x^{(i)} - z$$

Solve for gradient = 0 (and cancel the -2):

$$\sum_{i=1}^{n} x^{(i)} - \sum_{i=1}^{n} z = 0$$

$$\sum_{i=1}^{n} x^{(i)} = \sum_{i=1}^{n} z = nz$$

$$z = \frac{\sum_{i=1}^{n} x^{(i)}}{n}$$

2 2

2.1 a

$$\frac{\partial L}{\partial w_j} = \sum_{i=1}^{n} \frac{\partial}{\partial w_j} \left[(w \cdot x^{(i)}) \right] + \frac{\partial}{\partial w_j} \frac{1}{2} c \|w\|^2$$

$$= \sum_{i=1}^{n} x_j^{(i)} + c w_j$$

$$\nabla L(w) = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \vdots \\ \frac{\partial L}{\partial w_d} \end{bmatrix} = \sum_{i=1}^{n} x^{(i)} + cw$$

2.2 b

$$\nabla L(w) = \sum_{i=1}^{n} x^{(i)} + cw = 0$$

$$cw = -\sum_{i=1}^{n} x^{(i)}$$

$$w = -\frac{1}{c} \sum_{i=1}^{n} x^{(i)}$$

3 3

3.1 a

$$\nabla L(w) = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \vdots \\ \frac{\partial L}{\partial w_4} \end{bmatrix} = \begin{bmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ 2w_4 - 2w_3 \end{bmatrix}$$

3.2 b

$$w_{t+1} = w_t - \eta \nabla L(w_t)$$
$$w_{t+1} = 0 - \eta \nabla L(0)$$
$$= \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

3.3 c

$$\nabla L(w) = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \vdots \\ \frac{\partial L}{\partial w_4} \end{bmatrix} = \begin{bmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ 2w_4 - 2w_3 \end{bmatrix} = 0$$
$$w_1 = -1, w_2 = 1, w_3 = w_4$$
$$L((-1, 1, x, x)) = 1 + 2 + x^2 - 2x^2 + x^2 + 2(-1) - 4 + 4$$
$$L((-1, 1, x, x)) = 1$$

Minimum Value = **1**

3.4 d

There is not a unique solution. A solution exists for all $w = (-1, 1, x, x)$.

4 4

4.1 a

$$\frac{\partial L}{\partial w_j} = \sum_{i=1}^{n} \frac{\partial}{\partial w_j}\Big[y^{(i)} - w \cdot x^{(i)}\Big]^2 + \frac{\partial}{\partial w_j}\lambda\|w\|^2$$
$$= \sum_{i=1}^{n} -2\Big[y^{(i)} - w \cdot x^{(i)}\Big]x_j^{(i)} \quad + 2\lambda w_j$$
$$\nabla L(w) = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \vdots \\ \frac{\partial L}{\partial w_d} \end{bmatrix} = -2\sum_{i=1}^{n}\Big[y^{(i)} - w \cdot x^{(i)}\Big]x^{(i)} + 2\lambda w$$

4.2 b

$$w_{t+1} = w_t - \eta_t \nabla L(w_t)$$
$$w_{t+1} = w_t - \eta_t\Big\{ -2\sum_{i=1}^{n}\Big[y^{(i)} - w \cdot x^{(i)}\Big]x^{(i)} + 2\lambda w \Big\}$$

4.3 c

Update Equation:

$$w_{t+1} = w_t - \eta_t \nabla l(w_t; x^{(i)}, y^{(i)})$$
$$w_{t+1} = w_t - \eta_t\Big\{ -2\Big[y^{(i)} - w \cdot x^{(i)}\Big]x^{(i)} + 2\lambda w\Big\}$$

Algorithm:
$w_0 = 0$
Cycle through points $(x^{(i)}, y^{(i)})$ until some stopping condition:

- $w_{t+1} = w_t - \eta_t\Big\{ -2\Big[y^{(i)} - w_t \cdot x^{(i)}\Big]x^{(i)} + 2\lambda w_t\Big\}$

5 5

5.1 a

$\nabla^2 f(x) = 2 \geq 0$, convex.

5.2 b

$\nabla^2 f(x) = -2 \leq 0$, concave.

5.3 c

$\nabla^2 f(x) = 2 \geq 0$, convex.

5.4 d

$\nabla^2 f(x) = 0 = 0$, both.

5.5 e

$\nabla^2 f(x) = 6x$, hessian can change signs, neither.

5.6 f

$\nabla^2 f(x) = 12x^2 \geq 0$, convex.

5.7 g

$\nabla f(x) = \frac{1}{x} = x^{-1}, \nabla^2 f(x) = -x^{-2} \leq 0$, concave.

6 6

6.1 a

For my coordinate descent method, I will repeatedly call a routine until the gradient magnitude is less than a certain threshold. For this routine:

- (1) Initialize the weight vector to a d-dimensional vector sampled from a standard normal (seeded for reproducability)
- (2) while the gradient magnitude is greater than a certain value:
  - (3) Compute the gradient
  - (4) (i) Pick the coordinate with the largest absolute value. This, intuitively, is the component-direction of steepest ascent (and thus the negative is the direction of steepest descent)
  - (5) (ii) Update this coordinate by the magnitude of the gradient. Thus, when the model is close to optimized, the step size will naturally decrease in the corresponding direction.
    $w[\text{idx}] = w[\text{idx}] - \eta * sign(\nabla[\text{idx}]) * norm(\nabla)$

Because this method depends on the gradient, the function must be defined and differentiable at every point in R^d.

6.2 b

In [149…

```python
import pandas as pd
import numpy as np
import os
from pathlib import Path
from sklearn import linear_model, metrics

ITERS=100000

df = pd.read_csv("data/HW4/heart.csv")
df.head()

X = np.array(df.iloc[:, :-1])
y = np.array(df.iloc[:, [-1]])

np.random.seed(123)
idx = np.random.choice(303, 303, replace=False)
tr_idx, te_idx = idx[:200], idx[200:]

X_tr_raw, y_tr_raw = X[tr_idx], y[tr_idx]
X_te_raw, y_te_raw = X[te_idx], y[te_idx]


def normalize(mode, X_tr, X_te):
  if mode == "z":
    # z-score normalize
    X_tr_mu = np.mean(X_tr, axis=0, keepdims=True)
    X_tr_sig = np.std(X_tr, axis=0, keepdims=True)

    X_tr_ = (X_tr - X_tr_mu) / X_tr_sig
    X_te_ = (X_te - X_tr_mu) / X_tr_sig
    return X_tr_, X_te_
  elif mode == "min-max":
    # min-max 0-1 normalization
    X_tr_mi = np.min(X_tr, axis=0, keepdims=True)
    X_tr_ma = np.max(X_tr, axis=0, keepdims=True)
    X_tr_ = (X_tr - X_tr_mi) / (X_tr_ma - X_tr_mi)
    X_te_ = (X_te - X_tr_mi) / (X_tr_ma - X_tr_mi)
    return X_tr_, X_te_
  else:
    return X_tr, X_te


def logit(w, b, X):
  return X @ w + b


def logit_nob(w, X):
  return X @ w


def prob(w, b, X, y):
  logits = logit(w, b, X)
  assert(y.shape == logits.shape)
  return 1 / (1 + np.exp(-y * logits))


def prob_nob(w, X, y):
  logits = logit_nob(w, X)
```

```
  assert(y.shape == logits.shape)
  return 1 / (1 + np.exp(-y * logits))


def loss(w, b, X, y):
  logits = logit(w, b, X)
  assert(y.shape == logits.shape)
  return np.mean(np.log(1 + np.exp(-y * logits)))


def loss_nob(w, X, y):
  logits = logit_nob(w, X)
  assert(y.shape == logits.shape)
  return np.mean(np.log(1 + np.exp(-y * logits)))


def test_err(reg: linear_model.LogisticRegression, X, y):
  y_hat = reg.predict(X)
  err = np.not_equal(y_hat, y.flatten())
  return np.sum(err)


def reg_loss(reg: linear_model.LogisticRegression, X, y):
  w, b = reg.coef_.reshape((-1, 1)), reg.intercept_
  wx_b = reg.decision_function(X).reshape((-1, 1))
  assert(np.allclose(wx_b, logit(w, b, X)))
  return loss(w, b, X, y)


def experiment(mode, X_tr, X_te, y_tr, y_te):
  print(f"=== Using {mode} Normalization ===")
  X_tr_, X_te_ = normalize(mode, X_tr, X_te)
  reg = linear_model.LogisticRegression(C=999999999).fit(X_tr_, y_tr.flatten())
  w, b = reg.coef_.reshape((-1, 1)), reg.intercept_

  print("\tw: ", np.round(w, 4).flatten())
  print("\tb: ", np.round(b, 4))

  # print(f"=== Testing ===")
  # err = test_err(reg, X_te_, y_te)
  # print(f"\tError Rate: {np.sum(err)}/{len(y_te)}={np.sum(err)/len(y_te):0.4f}")
  y_1s_tr = np.where(y_tr == 0, -1, 1)
  y_1s_te = np.where(y_te == 0, -1, 1)

  tr_loss = reg_loss(reg, X_tr_, y_1s_tr)
  te_loss = reg_loss(reg, X_te_, y_1s_te)
  print(f"\tTrain Loss: {tr_loss}, Test Loss: {te_loss}")


  # print(y_tr[:10], y_1s_tr[:10])
  # y_hat = prob(w, b, X_tr_, y_1s_tr)
  # y_hat2 = reg.predict_proba(X_tr_)
  # print(y_hat[:10], y_hat2[:10])
  # calc_loss = metrics.log_loss(y_tr, y_hat2[:,1].flatten())
  # print(f"\tCalculated Loss: {calc_loss}")
  print()
  print("-" * 50)
  print()
  return tr_loss, te_loss
```

In [150…

```
tr_loss, te_loss = experiment(
    "min-max", X_tr_raw, X_te_raw, y_tr_raw, y_te_raw)
```

```
=== Using min-max Normalization ===
        w:  [-0.0916 -2.3859  2.5735 -3.329  -3.3522  0.5152  0.6446  4.345  -0.9711
 -4.1391  1.8231 -3.505  -2.4696]
        b:  [2.4201]
        Train Loss: 0.3138296347754303, Test Loss: 0.44660770010026035

--------------------------------------------------
```

In [154…

```python
import tqdm


def compute_gradient(w, X, y):
  prob_correct = prob_nob(w, X, y)
  prob_wrong = 1 - prob_correct
  assert(y.shape == prob_wrong.shape)
  matrix = y * X * prob_wrong
  return -np.mean(matrix, axis=0).reshape((-1, 1))


def experiment_coordinate(mode, thresh, lr, X_tr, y_tr, X_te, y_te, sel_coord="mag", it
  print(
      f"*** Using {mode} Normalization, thresh {thresh:0.5f}, Coordinate Selector mode
  np.random.seed(123)
  X_tr_, X_te_ = normalize(mode, X_tr, X_te)
  n_tr, n_te = X_tr_.shape[0], X_te_.shape[0]

  # add bias to data
  X_tr_, X_te_ = np.hstack(
      (X_tr_, np.ones((n_tr, 1)))
  ), np.hstack((X_te_, np.ones((n_te, 1))))

  w = np.random.normal(size=(X_tr_.shape[1], 1))
  losses = [loss_nob(w, X_tr_, y_tr)]
  te_losses = [loss_nob(w, X_te_, y_te)]

  for i in range(iters):
    gradient = compute_gradient(w, X_tr_, y_tr)
    gradient_norm = np.linalg.norm(gradient.flatten())
    if gradient_norm <= thresh:
      # print(f"Breaking on iteration [{i}]")
      break

    if sel_coord == "mag":
      idx_big = np.argmax(np.abs(gradient).flatten())
    elif sel_coord == "random":
      idx_big = np.random.randint(0, len(w), size=1).item()

    sign = np.sign(gradient[idx_big, 0])
    w[idx_big, 0] -= lr * sign * gradient_norm
    losses.append(loss_nob(w, X_tr_, y_tr))
    te_losses.append(loss_nob(w, X_te_, y_te))
  print("-" * 50)

  return w, losses, te_losses, X_tr_, X_te_
```

In [155…

```python
import matplotlib.pyplot as plt

def batch_experiment(thresh=1e-1, lr=1e-3, iters=10000):
  _, losses_mag, te_losses_mag, _, _ = experiment_coordinate(
      "min-max", thresh, lr, X_tr_raw, y_tr_raw, X_te_raw, y_te_raw, sel_coord="mag", i
  )
  _, losses_rdm, te_losses_rdm, _, _ = experiment_coordinate(
      "min-max", thresh, lr, X_tr_raw, y_tr_raw, X_te_raw, y_te_raw, sel_coord="random"
  )
```

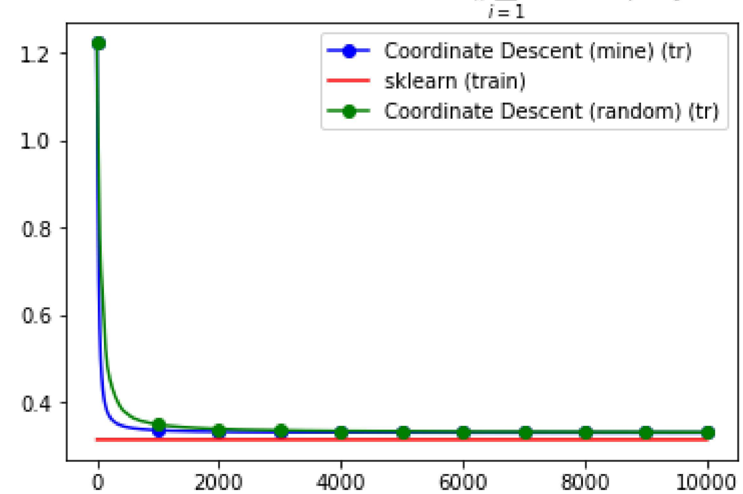
```python
    ct = max(len(losses_mag), len(losses_rdm))
    x_ax = np.arange(0, ct)
    line = np.ones_like(x_ax)
    plt.plot(range(len(losses_mag)), losses_mag, "bo-", markevery=1000, label="Coordinate
    # plt.plot(x_ax, te_losses_mag, "co-", markevery=1000, label="Coordinate Descent (min
    plt.plot(x_ax, line * tr_loss, "r-", label="sklearn (train)")
    # plt.plot(x_ax, line * te_loss, "y-", label="sklearn (test)")
    plt.plot(range(len(losses_rdm)), losses_rdm, "go-", label="Coordinate Descent (random
    # plt.plot(x_ax, te_losses_rdm, "mo-", label="Coordinate Descent (random) (te)", mark
    plt.legend()
    plt.title(f"Loss Plots ($\lambda$={lr},thresh={thresh}). $L(w) = \\frac{{1}}{{n}}\sum
    plt.show()
```

In [156…

```python
batch_experiment(thresh=1e-3,lr=1e-1)
batch_experiment(thresh=1e-3,lr=1e-2)
batch_experiment(thresh=1e-3,lr=1e-3)
batch_experiment(thresh=1e-3,lr=1e-4)
batch_experiment(thresh=1e-3,lr=1e-5)
```

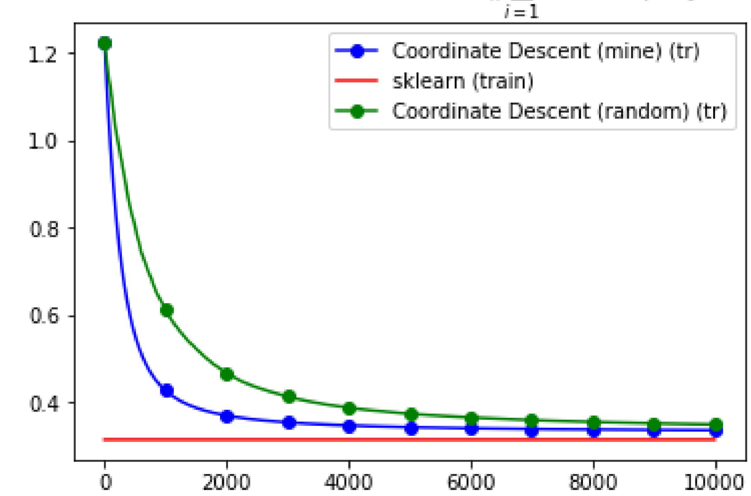
```
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode mag
--------------------------------------------------
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode random
--------------------------------------------------
```

Loss Plots (λ=0.1,thresh=0.001). $L(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)}(w \cdot x^{(i)})))$



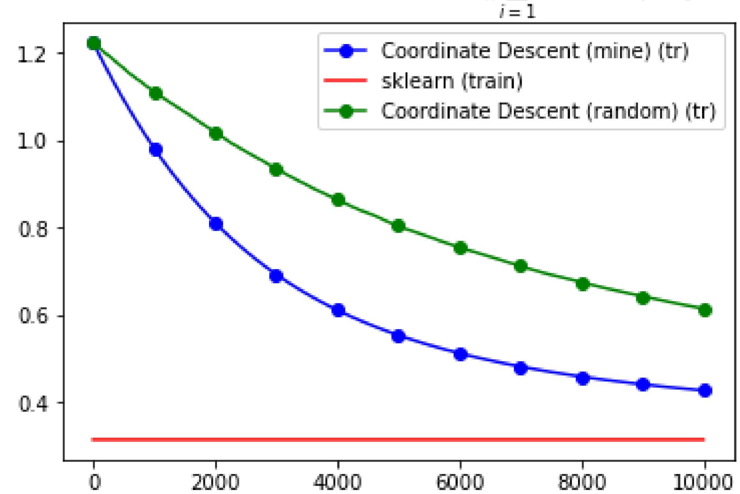

```
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode mag
--------------------------------------------------
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode random
--------------------------------------------------
```

Loss Plots (λ=0.01,thresh=0.001). $L(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)}(w \cdot x^{(i)})))$



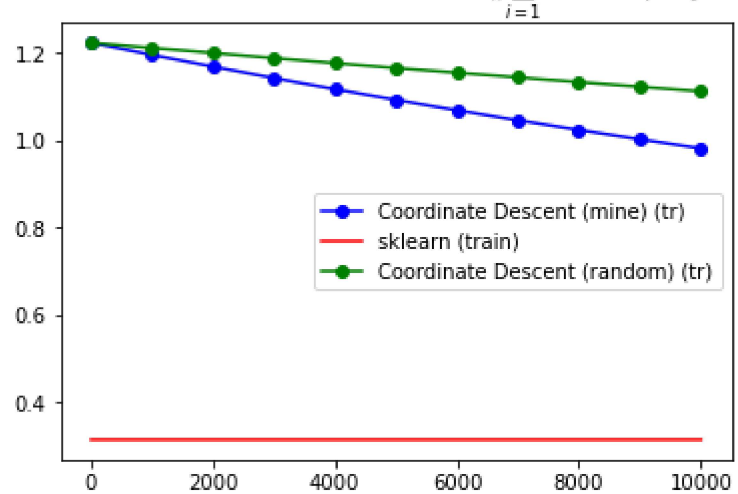

```
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode mag
--------------------------------------------------
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode random
--------------------------------------------------
```

Loss Plots (λ=0.001,thresh=0.001). $L(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)}(w \cdot x^{(i)})))$




```
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode mag
--------------------------------------------------
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode random
--------------------------------------------------
```

Loss Plots (λ=0.0001,thresh=0.001). $L(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)}(w \cdot x^{(i)})))$




```
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode mag
--------------------------------------------------
*** Using min-max Normalization, thresh 0.00100, Coordinate Selector mode random
--------------------------------------------------
```

Loss Plots (λ=1e-05,thresh=0.001). $L(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)}(w \cdot x^{(i)})))$