# Some Corrections to the Original Model

As stated in April 18, we igonre the friction force first. The continuous model will become

$$M(q)\frac{dv}{dt} = n(q)c_n - \nabla V(q) + k(q, v) + u(t) \tag{1}$$

$$\frac{dq}{dt} = v \tag{2}$$

$$0 \leq c_n \perp f(q) \leq 0 \tag{3}$$

There are two ways to discretize the model. One is to discretize the model during the entire motion, as mentioned in [1]. The discrete model shows as follows.

$$M(q_{k+1})(\dot{q}_{k+1} - \dot{q}_k) = n(q_{k+1})c_{n,k+1} + \delta t[-\nabla V(q_k) + C(q_k, \dot{q}_k)\dot{q}_k + u_k] \tag{4}$$

$$q_{k+1} - q_k = \dot{q}_{k+1}\delta t \tag{5}$$

$$0 \leq c_{n,k+1} \perp f(q_{k+1}) \geq 0 \tag{6}$$

Sometimes (6) can be linearized and written as

$$0 \leq c_{n,k+1} \perp J(q_k)q_{k+1} - \alpha_0 \geq 0 \tag{7}$$

This is because

$$f(q_{k+1}) \approx f(q_k) + J(q_k)(q_{k+1} - q_k) \geq 0 \quad \Rightarrow \quad J(q_k)q_{k+1} \geq J(q_k)q_k - f(q_k) = \alpha_0$$

where $J(q_k) = \partial f(q_k)/\partial q_k$ is the Jabobian. This model is proposed only for one-step simulation, thus $q_k$ and $\dot{q}_k$ are known. We can first get an extimated $\hat{q}_{k+1}$ and then solve the LCP to obtain the contact force. Besides, as [1] mentioned, under some conditions, $\hat{q}_{k+1}$ will converge to real $q_{k+1}$. Also, as Mathew mentioned, if we want to get better approximation, we can use $J(\hat{q}_{k+1})$ instead in (7).

Another method only discretizes the model at the contact point, which is mentioned in [2]. The model shows as follows.

$$M(q_{k+1})(\dot{q}_{k+1} - \dot{q}_k) = n(q_{k+1})c_{n,k+1} + \delta t[-\nabla V(q_k) + C(q_k, \dot{q}_k)\dot{q}_k + u_k] \tag{8}$$

$$q_{k+1} - q_k = \dot{q}_{k+1}\delta t \tag{9}$$

$$0 \leq c_{n,k+1} \perp n(q_{k+1})^T\dot{q}_{k+1} \geq 0 \tag{10}$$

The difference between two methods is the difference between (6) and (10). Notice that *Method 2 is only applicable at the contact point because it doesn't consider the constraint of* $f(q) \geq 0$ *when the contact is inactive.* So as [3] mentioned, (8)-(10) are correct when $f(q_{k+1}) < 0$. When $f(q_{k+1}) \geq 0$, we can set $c_{n,k+1} = 0$. This actually introduces a hybrid system with the event switch being $f(q_{k+1}) = 0$. Again, this is easy for one-step simulation, but not good for MPC. Either we use (4)-(6) to build the optimization model, or we introduce extra binary variabes to eliminate the switch and use (8)-(10). Obviously, the latter one is more complex. In [4], the first method is adopted.

So we rewrite (4)-(5) as follows

$$\begin{aligned}\begin{bmatrix} I & -\delta t I \\ 0 & I \end{bmatrix}\begin{bmatrix} q_{k+1} \\ \dot{q}_{k+1} \end{bmatrix} = \begin{bmatrix} q_k \\ \dot{q}_k \end{bmatrix} + \begin{bmatrix} 0 \\ M(q_{k+1})^{-1}\delta t \end{bmatrix} u + \begin{bmatrix} 0 \\ M(q_{k+1})^{-1} n(q_{k+1}) \end{bmatrix} c_{n,k+1} \\ + \begin{bmatrix} 0 \\ M(q_{k+1})^{-1}\delta t \left[C(q_k, \dot{q}_k)\dot{q}_k - \nabla V(q_k)\right] \end{bmatrix}\end{aligned} \tag{11}$$

which can be written as

$$x_{k+1} = Ax_k + Bu_k + Cc_{n,k+1} + F$$

In our model,

$$f(q) = \begin{bmatrix} d_{max} - o_1 - q_1 - q_2 \\ q_1 - q_3 - d_{min} - o_2 \end{bmatrix} \geq 0$$

where $d_{max}, d_{min}$ are boundaries while $o_1, o_2$ are constant offset. Thus (6) can be written as

$$\begin{aligned} & f(q_{k+1}) = f(q_k) + J(q_k)(q_{k+1} - q_k) \geq 0 \\ \Rightarrow \quad & \begin{bmatrix} d_{max} - o_1 - q_1 - q_2 \\ q_1 - q_3 - d_{min} - o_2 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} (q_{k+1} - q_k) \\ \Rightarrow \quad & \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} q_{k+1} + \begin{bmatrix} d_{max} - 1 \\ -d_{min} - 1 \end{bmatrix} \end{aligned}$$

Thus (6) can be written as

$$0 \leq c_{n,k+1} \leq Mz, \quad 0 \leq J(q_k)q_{k+1} + \begin{bmatrix} d_{max} - 1 \\ -d_{min} - 1 \end{bmatrix} \leq M(1 - z)$$

Besides, for this model we also have $F = 0, V = 0$. Thus, the total optimization problem can be written as

$$\begin{aligned} \min_{x,u} \quad & x_f^T Q_f x_f + \sum_{i=0}^{N-1} x^T Q x + u^T R u \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k + Cc_{n,k+1} \\ & 0 \leq c_{n,k+1} \perp Dx_{k+1} + Const \geq 0 \\ & x_k \in \mathcal{X} \\ & u_k \in \mathcal{U} \end{aligned} \tag{11}$$

For simplicity, we can choose $\mathcal{X}$ and $\mathcal{U}$ as ploytops.

# References


[1] D.E. Stewart and J.C. Trinkle. *An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction*. International Journal for Numerical Methods in Engineering, 39(15):26732691, 1996.

[2] M. Anitescu and F.A. Potra. *Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems* Nonlinear Dynamics, 14(3):231247, 1997.

[3] D.E. Stewart *Rigid-body dynamics with friction and impact*. SIAM, 42(1):339, 2000

[4] M. Posa, C. Cantu and R. Tedrake *A direct method for trajectory optimization of rigid bodies through contact*. The International Journal of Robotics Research, 33(1):6981, 2014.