Option Theory - Stochastic Systems

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1 Introduction

The aim of this paper is to deduce the pricing of a stock option through the use of the Black-Scholes model.

Options are useful because they reduce the risk of share speculation. This is due to the fact that options can be used to capitalise on an increase in share price with a small prior investment. In the next section we will elaborate further on the definition of an option.

Black-Scholes is a mathematical model used to determine the price of an option. The model is attributed to the work done by Fischer Black and Myron Scholes, and was developed further by Robert C Merton. This pricing model is largely used by traders who buy or sell options that are lower or higher (respectively) than the Black-Scholes calculated value.

2 Options

In this section, basic option theory terminology is introduced and described, for further information regarding the terminology and option strategies, consult [Tompkins, 2016].

An option is a contract between two parties that enter an agreement for a term, in order to trade shares at a strike price. A term is the period of validity of the contract and the strike price is the predetermined price at which the two parties agreed to trade. There are two types of options:

- A call option, denoted by C, gives the buyer the right to buy shares at the strike price during the term.
- A put option, denoted by P, gives the buyer the right to sell shares at the strike price during the term.

Additionally, an option is a right which means the buyer does not have to exercise the option once the term expires. For example, the buyer suffers a loss when the shares do not meet the expectation and thus the buyer does not exercise their right to buy or sell shares. The risk involved is limited to the premium paid to buy the option which acts as the reward to the seller. This means the price paid for an option is the maximum one can lose. Therefore, in this regard options have unlimited upside potential.

Exercising the right to buy or sell is not the only way to utilise options. Instead, the buyers can also sell the options on the option exchange. This comes with the condition that the option can only be traded during its term. Options can be written with the obligation to deliver or buy the predetermined shares at the strike price.

Furthermore, there is a significant difference between European and American options. For instance, in the first case the option may only be exercised when the term has expired. Whilst, the latter can be exercised at any time on or before the expiration date.

There are many trading strategies that involve options. For example, the short call and the hedging strategy. A short call is when a trader is betting that the price of an asset on which they are placing an option is going to drop. A hedge is an investment that protects your assets from price fluctuations, a put option is a well-known hedging tool.

3 The price of an option

The premium of an option at any time point depends on the following variables:

- Current Stock price, S
- Strike price, K
- Time to expiration, t
- Interest rates, r
- Volatility, σ
- Dividends, D

To simplify the process, each variable stated above will be analysed for the effect on the price of a given option.
First, lets consider the premium of a call option C at expiration or that of an American style option. Here we find that the the call is dependent on the underline share price at that time S^* and also the the strike price K. The call premium at that time is denoted as C^*, which give arise to the expression

$$C^* = \begin{cases} 0, & \text{if} \quad S^* < K. \\ S^* - K, & \text{if} \quad S^* \geq K. \end{cases}$$

Note that there are two possible cases to consider, the first being the case when $S^* < K$, here the option is said to finish out of the money, since the value of the shares is below the strike price, thus it is cheaper to acquire the shares on the market rather than exercising the call for a lose. In this case the call is left unexercised and is worth zero for the buyer of the call. The second case, namely when $S^* \geq K$, occurs when the call option is said to finish in the money, meaning that the call can be exercised to buy an amount of shares below the current market value and then sell them for a profit of $S^* - K$. Hence, the call in this case is worth $S^* - K$ to the buyer. Figure 1 and 2 shows the margin for profit against the share price for both the buyer and seller of the call option. For the American style option, this forms the lower bound for the call

$$C \geq \max(0, S - K), \tag{1}$$

this is because the American style options can be exercised at any time up until the expiration date.

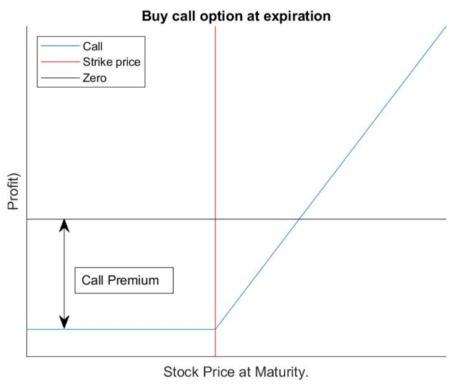

Figure 1: Bought Call option return against the share price.

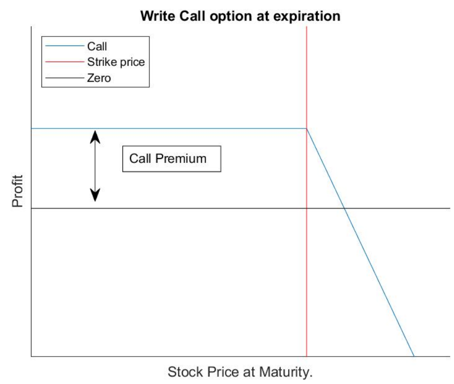

Figure 2: Written Call option return against the share price.

Now lets consider the premium of a put option at the expiration date P^*. The premium is dependent on the value of the underlying shares at that time, along with the strike price. This can be defined as

$$P^* = \begin{cases} K - S^*, & \text{if} \quad S^* < K, \\ 0, & \text{if} \quad S^* \geq K. \end{cases}$$

Firstly, we must consider the case $S^* < K$, here we say that the put finishes in the money, as shares on the market can be purchased at a lower value. Thus, the put is worth $K - S^*$. Secondly, if $S^* \geq K$, the put finishes out of the money, as the shares can be sold via the market for a higher price than that of the strike prise of the put. As a result the put is left unexercised and is worth zero to the buyer.

The intrinsic value of an American put option is given by

$$P \geq \max(0, K - S), \tag{2}$$

as the option can be exercised at anytime up until the expiration date.
Using equations 1 and 2, the relationship between call C^* and put P^* at the expiration date is given by

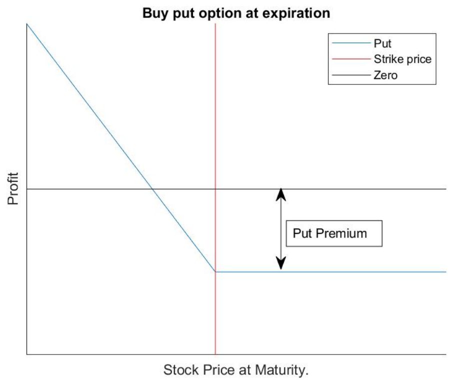

Figure 3: Bought Put option return against the share price.

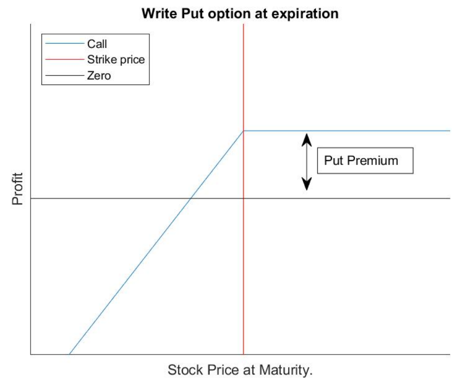

Figure 4: Written Put option return against the share price

$$P^* + C^* = \max(0, K - S^*) + \max(0, S^* - K).$$

Once again, there are the two cases to consider $S^* \geq K$ and $S^* < K$, however, it becomes clear that in either case, that call option finises in the money while the put option finishes out of the money or vice versa. This is shown in the following system of equations

$$\begin{cases} P^* + C^* = 0 + S^* - K, & \text{when} \quad S^* \geq K, \\ P^* + C^* = K - S^* + 0, & \text{when} \quad S^* < K. \end{cases}$$

Therefore, in both cases it implies that $P^* + C^* = |S^* - K|$.

Now, lets introduce the influence of time t and interest rate r for an amount borrowed or loaned from another party, on the premium of an option. Here, we simplify the process by assuming that there are no transacting fees, taxes involved, interest rate stay constant and that it is possible to lend and borrow at the same rate.

Let A_t denote the investment at t years after initial investment (denoted A_0). If the annual interest rate is given by r, but the bank gives interest n times per year, each time, the value of the interest given is r/n. Hence, after the initial interest, the value of the investment is

$$A_{1/n} = A_0 \left(1 + \frac{r}{n}\right).$$

After the second interest, the value will be $A_{2/n} = A_{1/n}(1 + r/n) = A_0(1 + r/n)^2$. It can be easily shown using induction that

$$A_1(n) = A_0 \left(1 + \frac{r}{n}\right)^n.$$

Substituting $r = 0.06$, we find that if interest is applied n times per year, the value of the investment will be

$$A_1(n) = A_0 \left(1 + \frac{0.06}{n}\right)^n .$$

Let us now consider the general form where the time t is also a variable (independent of n). In that case, our equation becomes

$$A = A(n,t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

We would like to verify that this is an increasing function of n. To do so, let us first take the natural logarithm of both sides of the equation

$$\ln A(n,t) = ntA_0 \ln \left(1 + \frac{r}{n}\right),$$

where A_0 is defined as $A(n,0)$. Note that the dependence on n disappears in this case. Furthermore, we will omit the dependence on n and t in notation from now on for the sake of readability. Differentiating both sides with respect to n, we get the following form

$$\frac{1}{A}\frac{dA}{dn} = t \ln \left(1 + \frac{r}{n}\right) - t\frac{r}{n} \cdot \frac{1}{1 + \frac{r}{n}}.$$

Let us now introduce a substitution $u = r/n$, which will make upcoming calculations easier to compute and digest. Doing so, we may rewrite the equation as

$$\frac{1}{A}\frac{dA}{dn} = t \left[\ln(1+u) - \frac{u}{1+u}\right].$$

Hence, we have an expression for the derivative of A_1

$$\frac{dA}{dn} = tA \left[\ln(1+u) - \frac{u}{1+u}\right].$$

Note that u is always non-negative as n is positive and r is taken to be non-negative. Since A is positive for A_0 positive and since $\ln 1 + u > \frac{u}{1+u}$ for all $u \geq 0$, it follows that the derivative dA/dn is always positive. Therefore, $A(n,t)$ is an increasing function of n.

Now that we know that it is an increasing function of n, we might wish to find the limit as n goes to infinity. We would expect that it is equal to a function of t, rather than blowing up to infinity as that would imply the possibility of infinite profit if the interest is compounded continuously. Therefore, we look at the quantity

$$\lim_{n\to\infty} A(n,t) = \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} .$$

If we introduce the substitution $w = n/r$, we get following form

$$\lim_{n\to\infty} A(n,t) = A_0 \lim_{w\to\infty} \left(1 + \frac{1}{w}\right)^{wrt} .$$

Remembering the limit definition of the function e^x and substituting rt for x, we calculate the limit to be equal to

$$\lim_{n\to\infty} A(n,t) = A_0 e^{rt}.$$

Furthermore, we can conclude that Ke^{-rt} will be the amount required to be borrowed at the current time, to ensure a value of K at maturity. Similarly, an amount Ke^{-rt} can be loaned, in order to receive a payment of K at the time of expiration.

An assumption made in this paper is that it will not be possible to form an arbitrage position with the pricing of the options. An arbitrage is defined as the purchase and sale of assets in order to make risk free profit from a difference in the asset's price on the market. The implication of this assumption in the model, is that we can form upper and lower bounds for both call and put option, such that the price of this option eliminates the possibility of risk free trading.

Here, we formulate the boundary condition in a mathematical methodology for the pricing of the European call option.

1. The price of the call is never negative, $C \geq 0$.
   - If $C < 0$, we observe that the C is negative, and thus a risk free profit can be made by buying the call option and holding it to expiration. Since the buyer receives money for "buying" the option and there is no obligation to exercise the option at expiration. Hence the price of the call shall never be negative.
2. The stock price minus the strike price $C \geq S - Ke^{-rt}$.
   - Suppose that $C < S - Ke^{-rt}$, then a arbitrage position can be formed by; shorting a share, pursuing the call option, and placing Ke^{-rt} in the bank with interest rate that gives amount K at time of the expiration date. The result of the these action will result in the positive amount $S - Ke^{-rt} - C$. Maintain this position until the expiration date. If $S^* \geq K$ then the position net worth will be zero and if $S^* < K$, the position will return a profit of $K - S^*$. Hence, we have shown that $C \geq S - Ke^{-rt}$ in order to avoid an arbitrage position.
3. The call price is never greater than the stock price, $S \geq C$.
   - Suppose that $C > S$, then an arbitrage position is formed by pursuing the stock and then selling a call, maintaining this position until the expiration date. If $S^* < K$ the call is worthless and is not exercised, then sell the stock for profit; if $S^* \geq K$ then the call is exercised, thus netting zero.

Thus, the call option is bounded by $S \geq C \geq \max(0, S - Ke^{-rt})$.

Similarly, the same analytical process as above can be used to establish the boundaries for the price of a put option.

1. The price of the put is never negative, $P \geq 0$.
   - The proof of which, is identical to that of the call option.
2. The price of the put is greater than that of the value of the strike price minus that of the current stock price $P \geq Ke^{-rt} - S$.

- If $P < Ke^{-rt} - S$, then arbitrage is possible by the following method; purchase the put option, buy the share of the stock and issuing a loan price K. This result in a position of $P + S - Ke^{-rt}$. Maintain this position until the expiration date. Suppose at expiration $S^* \geq K$, in this cases the put closes out of the money and is not exercised, leaving the shares to be sold and the bond repaid, for a profit of $S^* - K$. Alternatively, if $S^* < K$, then the put option is exercised, the shares are sold and the bond is repaid for a net of zero, $(K - S^*) + S^* - K = 0$

3. The put price is never greater than the stock price, $S \geq P$.

- Suppose that $P > S$, then an arbitrage position is formed by shorting the stock and then selling a put, maintaining this position until the expiration date. If $S^* < K$ the put option is exercised for a net of zero; if $S^* \geq K$ then the put option finishes out the money and is not exercised, the short stock return a profit.

Thus, a put option is bounded by $S \geq P \geq \max(0, Ke^{-rt} - S)$.

Further more, both the call and put option statements can be combined. Consider the following option position; Sell a call, purchase a put, buy a share of the underlying stock and also borrow Ke^{-rt} at a rate r. The result of which is a position of $C - P - S + Ke^{-rt}$. Suppose at the expiration date that $S^* \leq K$, then the put option can be exercised for a profit of $K - S^*$, while the call option is unexercised. Now, instead, consider the case at expiration that $k < S^*$, this implies that the call option is exercised for $S^* - K$, while the put is left unexercised.

	Current Date	Expiration	
		S* ≤ K	K < S*
Sell Call	C	0	K-S*
Buy Put	-P	K-S*	0
Buy Stock	-S	S*	S*
Borrow	Ke^{-rt}	-K	-K
Net	0	0	0

Table 1: Option Arbitrary table

In both cases, the original amount can be paid off with the gains made at expiration by hedging both call and put options. Thus, in all possible out comes the net profit is zero. This aligns with our assumptions that arbitrage is not possible. This formulates mathematical to the model as the put-call parity relationship, denoted as

$$C - P = S - Ke^{-rt} \tag{3}$$

4 A binomial model for option prices

The aim of this section is to introduce a very simple mathematical framework of option theory, namely, to model shares and options in a discrete time manner. Taking inspiration from the extensive work completed by Cox, Ross and Rubinstein's model [Cox and Rubinstein, 1985], this section presents key concepts of this discrete model and will prove mathematical framework of it.

In order to focus on the mathematical concepts at hand, the overall model will be simplified with certain assumptions, which are aimed at removing the ambiguity of real market trading. These assumptions are that interest rates are constant, ignore any taxes and fees on transactions, can short stock as required, and that we can partake in the market without affecting the prices. A more in depth model and literature about binomial distribution in regards to finance can be found in [Van der Hoek and Elliott, 2006].

Now, lets assume that trading of shares and options can only occur at specified uniform time intervals t over a time period T, this can be thought of in terms of either days, hours or minutes. Utilising this framework, observe the model with one trade period before the expiration date. Using the binomial model, we can assume that the stock price S over each t can either increase in value to uS with a probability of q or that it decreases to dS with the probability $1 - q$. Where $p \in [0, 1]$ and $u, d \in \mathbf{R}$ which remains constant throughout the model. This is visualised as a tree diagram below.

$$S \begin{array}{l} \nearrow \; uS \;\; (q) \\ \\ \searrow \; dS \;\; (1-q). \end{array}$$

Furthermore, this can be used to evaluate the premium of an option, let us now consider the case of a call option in the same condition stated above. Here, we denote C to be the current call premium, if the stock price increases over the next trade period the premium will be C_u, similarly, if the stock decreases the premium will be C_d. Using the equations 1 and 2, the premium of these two states are given by $C_u = \max(0, uS - K)$ and $C_d = \max(0, dS - K)$. Hence,

$$C \begin{array}{l} \nearrow \; C_u = \max(0, uS - K) \\ \\ \searrow \; C_d = \max(0, dS - K). \end{array}$$

To analysis these potential changes in values, we introduce the notion of a portfolio, which consists of an amount of shares Δ and an amount borrowed B at an interest rate r. The purpose of this is that we can select, as required the values of Δ and B to match the premium of the call at any given time t, moreover, it allows us to chose an amount of shares that are not whole .

4.1 The value of the portfolio

Considering the conditions stated in the beginning of this section, two possibilities for the portfolio are:

1. the price of the underlying stock goes up and so

$$\begin{aligned} S\Delta + B &\to uS\Delta + \rho B \\ C &\to C_u = \max\{0, uS - K\} \end{aligned}$$

2. the price of the underlying stock goes down and so

$$\begin{aligned} S\Delta + B &\to dS\Delta + \rho B \\ C &\to C_d = \max\{0, dS - K\} \end{aligned}$$

Here, we obtain two equations with two unknowns, Δ and B:

$$\begin{aligned} uS\Delta + \rho B = C_u \\ dS\Delta + \rho B = C_d \end{aligned}$$

Solving for Δ, we obtain

$$\Delta = \frac{C_u - C_d}{S(u-d)}, \tag{4}$$

then for B

$$B = \frac{1}{\rho} \cdot \frac{uC_d - dC_u}{u-d}. \tag{5}$$

Inline with the assumption that an arbitrage position should not be possible, it must hold that

$$C = S\Delta + B. \tag{6}$$

4.2 Applying the portfolio

Suppose that $C < S\Delta + B$, then an arbitrage is possible by buying the call since it is valued less than the portfolio and selling the assets of the portfolio, which are valued higher than the call, this will ensure a risk free profit. Now, consider the case $C > S\Delta + B$, the temptation here is to create a reverse strategy of the previous cases; buy the assets of the portfolio and write a call option. However, the buyer of your written call has the right to exercise it immediately. This will put you, as the writer of call option, as the source of an arbitrage position and thus other investors will make a profit, while you will make a loss. In order to avoid this arbitrage position, the writer of the option will ensure that the call is valued the same as the portfolio. Thus, we conclude that $C = S\Delta + B$.

Rewriting using the expressions for Δ and B, we get

$$C = \frac{1}{u-d}\left(C_u - C_d + \frac{uC_d - dC_u}{\rho}\right)$$

Rewriting the term in the brackets and reorganizing the denominators, we get

$$C = \frac{1}{\rho}\left[\frac{(\rho - d)C_u - (\rho - u)C_d}{u-d}\right]$$

Introducing $p = \frac{\rho-d}{u-d}$, we get the premium of the call one trade period before the expiration as

$$C = \frac{1}{\rho}\big[pC_u + (1-p)C_d\big]. \tag{7}$$

Observe that the equation does not depend on the probability of the stock increasing q, instead, the call premium depends on the parameter p. Since the sum of expected shares is equal to to the amount returned from leading, after a trading period. This gives rise to the expression

$$q(uS) + (1-q)(dS) = \rho S.$$

It follows that $q = p$, so then p is bounded between $[0, 1]$, thus it inherits the property of being a probability.

4.3 Pricing trends

To illustrate the points made in more detail, lets consider an example. In this example we observe the system with two trading periods before the expiration date, we assume that $\rho = 1$ and let $S = 36$,$K = 30$, $u = 3/2$ and $d = 2/3$. Below, in Figure 6, is the tree graph that shows the share price over two trading periods and the related probability.

Utilising Equation 1 at the expiration date, then Equation 7 for one trade period before expiration, then again, for two trade periods before expiration, the premium of the call can be calculated for each case and is plotted into the the tree diagram below.

Here, we denote C_{uu} as the call premium two from now if the price of the underlying share increases twice, in a similar manner, we define C_{dd}. It should be noted that $C_{ud} = C_{du}$. From Equation 1 at the expiration date

$$\begin{cases} C_{uu} = \max\left(0, u^2S - K\right) \\ C_{ud} = \max\left(0, udS - K\right) \\ C_{dd} = \max\left(0, d^2S - K\right) \end{cases}$$

then it follows at one trading period before expiration with Equation 7, that

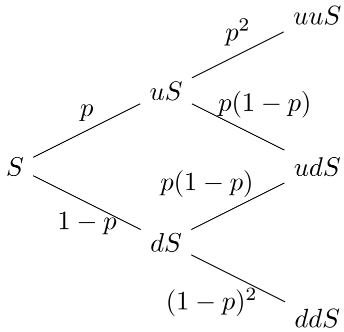

Figure 5: Tree diagram of potential share price and probabilities.

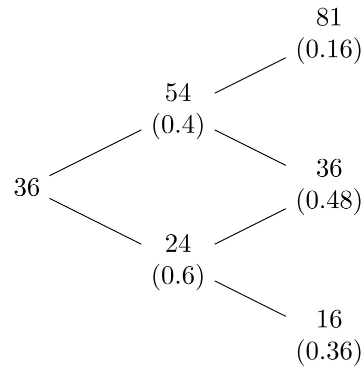

Figure 6: Tree diagram of share price and probabilities, from the worked example.

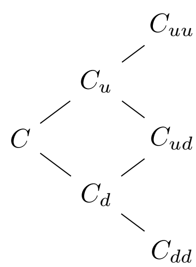

Figure 7: Call option potential 'random walks' tree diagram.

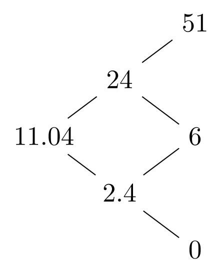

Figure 8: Call option premiums, from worked example.

$$\begin{cases} C_u = \rho^{-1}\big[pC_{uu} + (1-p)C_{ud}\big], \\ C_d = \rho^{-1}\big[pC_{ud} + (1-p)C_{dd}\big]. \end{cases}$$

4.4 Adjusting the value of the call option

For two trading period before expiration, and again using Equation 7, we have

$$\begin{aligned} C &= \rho^{-1}\big[pC_u + (1-p)C_d\big] \\ C &= \rho^{-2}\big[p(pC_{uu} + (1-p)C_{ud}) + (1-p)(pC_{ud} + (1-p)C_{dd})\big]. \end{aligned}$$

Finally,

$$C = \rho^{-2}\big[p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}\big].$$

This aligns with the premium calculated in Figure 8, as

$$\begin{aligned}&C = \rho^{-2}\big[(0.16)(81-30) + (0.48)(36-30) + (0.36)(0)\big].\\ &C = 11.04\end{aligned}$$

4.5 The value of the portfolio before expiration

Using the Equations 4 and 5, then, it follows the theoretical portfolio for one trading period before expiration is

- For position C_u

$$\begin{cases}\Delta_u = 1\\ B_u = -30.\end{cases}$$

  thus, for position C_u, the portfolio equals the premium of the call

$$\begin{aligned}uS\Delta_u + B_u &= C_u\\ \frac{3}{2}\cdot 36\cdot 1 - 30 &= C_u\\ 24 &= C_u.\end{aligned}$$

- For position C_d

$$\begin{cases}\Delta_d = 0.3\\ B_d = -24.5.\end{cases}$$

  again, we observe that the portfolio equals the call

$$dS\Delta_d + B_d = \frac{12}{5} = C_d.$$

Then, for two trading periods before expiration, we have

$$\begin{cases}\Delta = 0.72\\ B_d = -14.88.\end{cases}$$

for the final position, the portfolio equals the call

$$S\Delta + B = 11.4 = C.$$

4.6 Adjusting the value of the portfolio

Suppose that the premium of the call at $n = 2$, is one euro more than what it should be at $C = 12.04$. Acting one this difference, we can make use of it and ensure a riskless profit, no matter if the share price rises or falls over the subsequent trading periods.

At $n = 2$, We begin by selling a call option on the market for a value of 12.04. Taking the true premium of the call 11.04 and investing it into the portfolio of $\Delta = 0.72$ shares. Shorting the remainder $0.72 \cdot 36 - 11.04 = 25.92 - 11.04 = 14.88$, while loaning $12.04 - 11.04 = 1$. In this example we disregard the interest rate r.

- $n = 1u$.
  Now, suppose that the share prices moves up to 54. We now short the difference of Δ, which is $1 - 0.72 = 0.28$, buying at 54 means we owe $0.28 \cdot 54 = 15.12$. This bring our total debt to $14.88 + 15.12 = 30$.
  - $n = 0u$.
    If the share raises again to a price of 81, the call finishes in-the-money, and thus the call can be exercised by the owner, so we can either short the shares or buy back the call for a lost of $81 - 30 = 51$. We sell the shares we own Δ, $1 \cdot 81 = 81$ to make $81 - 51 = 30$. We use the remainder of this profit to pay back the debt $30 - 30 = 0$. Which leaves us with the original 1 euro that we loaned at the start, thus our profit is 1.
  - $n = 0d$,
    Instead, if we consider that the share price goes down to 36, the call finishes in the money, and thus is exercised for our loss of $36 - 30 = 6$. Sell our share $1 \cdot 36 = 36$, thus leaves us 30 to pay of the debt of 30. Which leaves us with the original amount of 1, that we loaned.
- $n = 1d$.
  Alternatively, consider if the share price drop to 24 at the first trading period. If this occurs we sell the difference in Δ, $0.72 - 0.3 = 0.42$ for $0.42 \cdot 24 = 10.08$, use this money to pay off some of the debt $14.88 - 10.08 = 4.8$, which we still owe.
  - $n = 0d$.
    Suppose that the share falls again to a price of 16, here we see that the call finishes out of the money and is not exercised by the owner. Now we sell our remaining shares $0.3 \cdot 16 = 4.8$, which is used to pay off the debt of 4.8, so nothing is outstanding. Which leaves us with the original 1 euro that we loaned at the start, thus our profit is 1.
  - $n = 0u$.
    On the other hand, suppose that the shares are priced at 36, and that the call finishes in the money and is exercised. We short/ buy back the call for a loss of $36 - 30 = 6$. We sell the shares that we own for $0.3 \cdot 36 = 10.8$, which leaves us with $10.8 - 6 = 4.8$ to pay off the debt of 4.8. Which leaves us with the original 1 euro that we loaned at the start, thus our profit is 1.

Thus conclude that in all cases we can always make profit of 1 euro, and there is no risk that we would ever lose money.

4.7 The value of the call option over n trading periods

Extending this idea further, a general discrete binomial equation can be formed. This equation will give the premium of a call at any trading period n, before the call's expiration date.

$$C_n = \rho^{-n} \left[\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{n-j}} \right] \tag{8}$$

This general equation is generated by recursively using Equation 7. Starting at one trading period before expiration and then using these call values with the equation again to obtain the call value two trading periods before expiration. Repeating this method and working backwards from the expiration date, a call premium can be calculated for n many trading periods.

4.8 Separating the binomial formula for the call option

We now define a to be the least positive integer such that

$$u^a d^{n-a} S > K. \tag{9}$$

Since for all $j \geq a$ implies that we have $C_{u^j d^{n-j}} = u^j d^{n-j} S - K$. Substituting this into the Equation 8, results in

$$C = \rho^{-n} \left[\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(u^j d^{n-j} S - K \right) \right].$$

Expanding this expression out gives the following equation

$$C = S \left(\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{\rho^n} \right) - K\rho^{-n} \left(\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \right). \tag{10}$$

The notation of the expression above can be simplified by expressing the bracketed sum terms, with those of a binomial distribution function. Here we denote the binomial distribution function as $\psi_a(n,p)$ with the parameter n for the number of terms and probability of occurring $p \in (0,1)$, then the second sum term can be expressed as

$$\psi_a(n,p) = \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

For the first sum term, we observe that it can be simplified

$$\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{\rho^n} = \sum_{j=a}^{n} \binom{n}{j} \left(\frac{u}{\rho} \cdot p \right)^j \left(\frac{d}{\rho} \cdot (1-p) \right)^{n-j}.$$

Here, we let $p' = (u/\rho)p$ and $(1-p') = (d/\rho)(1-p)$, so that

$$\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{\rho^n} = \sum_{j=a}^{n} \binom{n}{j} p'^j (1-p')^{n-j}.$$

Since p' is bounded by $0 < p' < 1$, we can use it as a probability and express the first sum term as the binomial function

$$\psi_a(n, p') = \sum_{j=a}^{n} \binom{n}{j} p'^j (1-p')^{n-j}.$$

Finally, to conclude this section of the paper by describing the Binomial Option Price Formula as

$$C = S\psi_a(n, p') - K\rho^{-n}\psi_a(n, p). \tag{11}$$

This formula allows us to compute the pricing of a call option in regards to a discrete time model. It achieves this by using the binomial distribution to account for the random walks the stock price may take during n many trading periods.

5 The Black-Scholes formula

In this section, our aim is to develop the model presented in the previous section further inline to that of real stock market behaviour. In order to achieve this, we have to address the assumption we made to simplify the model in section 4, further details can be found in [Lamberton and Lapeyre, 2007]. The first assumption, is that the model is discrete. It becomes quite clear that trading of shares does not occur at defined interval, instead, trading take place on a continuous bases through out the day, weeks and years. Our second assumption, was that the price of the stock can take only two values u and d, at each trading period. It is obvious, on reflecting on the behaviour of an actual market, that the share price can take on any price over the trading period, with the price being independent of it's previous price.

To address these points in a more mathematical manner, we introduce h which represents the elapsed time between trading periods. Now, we let t be the fixed time before the option expiration, and let n be the number of trading periods of length h before the expiration date. Theses terms can be expressed as $h = t/n$. Here, we observe that the more frequently trading occurs n over the fixed time period t, the time between trading periods h approaches zeros; this can be described as $n \to \infty$ then $h \to 0$. With this in mind, we shall consider the affect of $n \to \infty$ has on the variables u,d and ρ.

To show that the variables u, d ρ and a are now dependent on n, we shall add a subscript to them. The increased frequency of trading over a fixed time period gives use a shorter trading period, we observe that the value of the stock is less prone to fluctuations in prices over the shortening trade period. Hence, we assume that as $n \to \infty$ then $u_n \to 1$ and $d_n \to 1$. Moreover, we express this as

$$u_n = e^{\sigma\sqrt{t/n}}, \quad \text{and} \quad d_n = e^{-\sigma\sqrt{t/n}} \tag{12}$$

here, σ is referred to as the annual volatility of the share, being bound by $\sigma > 0$. It should also be noted that $u_n d_n = 1$.

For ρ_n, we let $\hat{r}$ be the interest rate over a single trading period, so that $\rho_n = 1 + \hat{r}$. So then $\rho_n = e^{rt/n}$.

Hence, we can calculate p_n

$$p_n = \frac{\rho_n - d_n}{u_n - d_n} = \frac{e^{rt/n} - e^{-\sigma\sqrt{t/n}}}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}} = \frac{e^{rt/n+\sigma\sqrt{t/n}} - 1}{e^{2\sigma\sqrt{t/n}} - 1} \tag{13}$$

During each trading period, the share price can either rise by a factor of u_n with a probability of q, or the prices can drop by a factor of d_n with a probability of $1 - q$. We assume that this price change is independent of all other previous changes that have occurred.

5.1 Central Limit Theorem and Normal approximation

As described in [van der Vaart et al., 2017], a binomial distribution $\phi_a(n,p)$, with the parameters n denoting the number of trials and p fixed in $(0,1)$ representing success probability, can converge to the standard normal distribution function $\Phi(x)$, as $n \to \infty$, by the Central Limit Theorem. In this section we shall develop theses ideas in order to utilise this convergence of the discrete binomial distribution to that of a continuous standard normal distribution.

Normal approximation implies that the binomial distributions in Equation 10 can be expressed as the probability of a random quantity N_n that has a binomial distribution, being larger than than the variable a_n

$$\psi_a(n,p) = \mathbf{P}\big[N_n \geq a_n\big] = 1 - \mathbf{P}\big[N_n \leq a_n - 1\big] \tag{14}$$

Given that X_n is binomially distributed, with fixed probability p, for a sufficiently large n it will converge to the standard normal distribution $\Phi(x)$ by a special case of the Central Limit theorem. By the definition of Central Limit theorem

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{X_n - \mathbf{E}(X_n)}{\sqrt{\mathbf{V}\text{ar}(X_n)}} \leq x\right] = \Phi(x).$$

Here the expectation of the distribution X_n is denoted as $\mathbf{E}(X_n)$ and the variance is expressed as $\mathbf{V}\text{ar}(X_n)$.Since X_n is binomial distributed, we have $\mathbf{E}(X_n) = np$ and $\mathbf{V}\text{ar}(X_n) = np(1-p)$. Replacing these two terms with the expressions found for a binomial distribution yields

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{X_n - np}{\sqrt{np(1-p)}} \leq x\right] = \Phi(x). \tag{15}$$

In other words, the CLT equation above implies that the standardised sample mean approximately follows the standard normal distribution when the number of observations is large.

5.1.1 Convergent sequence and the CLT

In order to use the Central Limit Theorem in this case we must prove that for any y in $\mathbb{R}$ we have $\mathbf{P}(Y_n \leq y) = \Phi(y)$ and if $x_n \to x$ for some x in $\mathbb{R}$ as $n \to \infty$, then we have also have $\mathbf{P}(Y_n \leq x_n) \to \Phi(x)$.

We want to show that

$$\lim_{n\to\infty} \mathbf{P}(Y_n \leq x_n) = \Phi(x).$$

Let $\epsilon > 0$ be given, we will find a $\delta > 0$ such that $|x_n - x| < \delta$ implies

$$|\mathbf{P}(Y_n \leq x_n) - \Phi(x)| < \epsilon.$$

It is know that the Cumulative Distribution Function of the standard normal distribution Φ, defined as $\Phi : \mathbb{R} \to (0,1)$ maps $\mathbf{P}(Y_n \leq y) \mapsto \Phi(y)$. Since the sequence x_n and its limit x is defined on the interval $\mathbb{R}$, then Φ maps

$$\mathbf{P}(Y_n \leq x_n) = \Phi(x_n).$$

Substituting this result in yields

$$|\mathbf{P}(Y_n \leq x_n) - \Phi(x)| = |\Phi(x_n) - \Phi(x)| < \epsilon.$$

Hence, we need to find a $\delta > 0$ such that $|\Phi(x_n) - \Phi(x)| < \epsilon$. In order to do so, we use the following definition of $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

Without loss of generality, we assume $x_n > x$. The proof works analogously for $x_n < x$.

Therefore, we have

$$\begin{aligned}
|\Phi(x_n) - \Phi(x)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{x_n} e^{-\frac{t^2}{2}} dt - \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \right| \\
&= \frac{1}{\sqrt{2\pi}} \left| \int_{x}^{x_n} e^{-\frac{t^2}{2}} dt \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{x}^{x_n} \left| e^{-\frac{t^2}{2}} \right| dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{x}^{x_n} e^{-\frac{t^2}{2}} dt \\
&\leq \frac{1}{\sqrt{2\pi}} |x_n - x| \max_{t\in[x,x_n]} e^{-\frac{t^2}{2}} \\
&\leq \frac{1}{\sqrt{2\pi}} |x_n - x| \cdot 1 \\
&< |x_n - x| < \delta.
\end{aligned}$$

Therefore, choosing $\delta = \epsilon$ we immediately obtain $|x_n - x| < \delta \implies |\Phi(x_n) - \Phi(x)| < \epsilon$. Using this, we may conclude that

$$\lim_{n\to\infty} \mathbf{P}(Y_n \leq x_n) = \Phi(x).$$

From this working, we can conclude that we are justified using the CLT with a variable a_n that is depend on n and is convergent. In the next section we will find a value for a_n to be used in our model calculations.

5.1.2 Determining the sequence a_n

In order to use this result for N_n, we must find the value of the variable a_n and since p_n is not fixed, we must show that it converges to a limit as $n \to \infty$.

To find the value of a_n, we can makes use of $u_n^{a_n} d_n^{n-a_n} S > K$ to get a lower bound on a_n, by taking natural logs we have

$$a_n \log u_n + (n - a_n) \log d_n > \log \frac{K}{S}.$$

Rearranging, we obtain the following inequality

$$a_n > \frac{\log K/S - n \log d_n}{\log u_n/d_n}$$

Since a_n is defined as the smallest integer for which this inequality holds, it follows that

$$a_n - 1 \leq \frac{\log K/S - n \log d_n}{\log u_n/d_n} < a_n$$

The last pair of inequalities implies that there exists a number $\varepsilon_n \in [0, 1)$ such that

$$a_n - 1 = \frac{\log K/S - n \log d_n}{\log u_n/d_n} - \varepsilon_n \tag{16}$$

We now have an explicit formula for the sequence a_n. With this in hand, we may begin working on applying the CLT to the two binomial distribution.

5.2 First binomial distribution $\psi_a(n, p_n)$

5.2.1 Limits for $\psi_a(n, p_n)$

We must show that the probability of the distribution $\psi_a(n, p_n)$ is bounded by $(0, 1)$, as $n \to \infty$, in order to use the Central Limit Theory. For p_n defined in Equation (13), we can calculate $\lim_{n\to\infty} p_n$ as

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{e^{rt/n+\sigma\sqrt{t/n}} - 1}{e^{2\sigma\sqrt{t/n}} - 1}.$$

Introducing the substitution $y = \sqrt{\frac{t}{n}}$, our limit is identical to the limit $\lim_{y\to 0} q_y$, where

$$q_y = \frac{e^{ry^2+\sigma y}-1}{e^{2\sigma y}-1}$$

This limit is easier to calculate using L'Hopital's rule as it avoids complicated chain rule calculations

$$\begin{aligned}
\lim_{y\to 0} q_y &= \lim_{y\to 0} \frac{e^{ry^2+\sigma y}-1}{e^{2\sigma y}-1} \\
&= \lim_{y\to 0} \frac{(2ry+\sigma)e^{ry^2+\sigma y}}{2\sigma e^{2\sigma y}} \\
&= \lim_{y\to 0} \frac{(2ry+\sigma)e^{ry^2}}{2\sigma} \\
&= \lim_{y\to 0} \left(\frac{rye^{ry^2}}{\sigma} + \frac{1}{2}e^{ry^2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, since $\lim_{n\to\infty} p_n = \lim_{y\to 0} q_y$, we have proven

$$\lim_{n\to\infty} p_n = \frac{1}{2}$$

Similarly, we can calculate $\lim_{n\to\infty} \sqrt{n}(p_n - 1/2)$

$$\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) = \lim_{n\to\infty} \sqrt{n}\left(\frac{e^{rt/n+\sigma\sqrt{t/n}}-1}{e^{2\sigma\sqrt{t/n}}-1} - \frac{1}{2}\right)$$

Again we introduce the substitution $y = \sqrt{\frac{t}{n}}$ as in the last limit. The quantity $\sqrt{n}$ is related to y via $\sqrt{n} = \frac{\sqrt{t}}{y}$. Hence, our limit becomes

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) &= \lim_{y\to 0} \frac{\sqrt{t}}{y}\left(\frac{e^{ry^2+\sigma y}-1}{e^{2\sigma y}-1} - \frac{1}{2}\right) \\
&= \frac{\sqrt{t}}{2} \lim_{y\to 0} \frac{1}{y}\left(\frac{e^{ry^2+\sigma y}-1}{e^{2\sigma y}-1} - \frac{e^{2\sigma y}-1}{2e^{2\sigma y}-2}\right) \\
&= \frac{\sqrt{t}}{2} \lim_{y\to 0} \frac{2e^{ry^2+\sigma y} - e^{2\sigma y} - 1}{y(e^{2\sigma y}-1)}.
\end{aligned}$$

Applying L'Hopital once, we get

$$\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) = \sqrt{t}\lim_{y\to 0} \frac{(2ry+\sigma)e^{ry^2+\sigma y} - \sigma e^{2\sigma y}}{e^{2\sigma y} - 1 + 2\sigma y e^{2\sigma y}}$$

Applying L'Hopital again, we finally get

$$\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) = \sqrt{t} \lim_{y\to 0} \frac{2re^{ry^2+\sigma y} + (2ry+\sigma)^2 e^{ry^2+\sigma y} - 2\sigma^2 e^{2\sigma y}}{4\sigma e^{2\sigma y} + 4\sigma^2 y e^{2\sigma y}}$$

Evaluating the limit we obtain

$$\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) = \sqrt{t}\frac{2r + \sigma^2 - 2\sigma^2}{4\sigma}$$

Finally, after simplifying, we get

$$\lim_{n\to\infty} \sqrt{n}(p_n - 1/2) = \frac{rt - \frac{1}{2}\sigma^2 t}{2\sigma\sqrt{t}} \tag{17}$$

Thus, we have shown that the probability of this distribution converges with large n.

5.2.2 Applying the CLT to $\psi_a(n, p_n)$

As we have shown in the previous section that the probability of $\psi_a(n, p_n)$ converges to $p_n = 1/2$ for a sufficiently large n, we can begin to use the Central Limit Theorem. Following on from Equation (14), we want to find the probability of our random quantity N_n is larger than a_n as n goes to infinity

$$\begin{aligned}\lim_{n\to\infty} \psi_a(n, p_n) &= \lim_{n\to\infty} \big(1 - \mathbf{P}\big[N_n \le a_n - 1\big]\big)\\ &= 1 - \lim_{n\to\infty} \mathbf{P}\left[N_n \le \frac{(a_n - 1) - np_n}{\sqrt{np_n(1-p_n)}}\right].\end{aligned}$$

Now utilising the Central Limit theorem in Equation (15) to obtain

$$\lim_{n\to\infty} \mathbf{P}\left[N_n \le \frac{(a_n - 1) - np_n}{\sqrt{np_n(1-p_n)}}\right] = \Phi\left(\lim_{n\to\infty} \frac{(a_n - 1) - np_n}{\sqrt{np_n(1-p_n)}}\right),$$

Focusing on the right hand side limit expression, we can start by substituting in the expresion we have found for $a_n - 1$ from Equation (16) and with a bit of rearranging it yields

$$\lim_{n\to\infty}\left[\frac{(a_n - 1) - np_n}{\sqrt{np_n(1-p_n)}}\right] = \lim_{n\to\infty}\left[\frac{\log\left(\frac{K}{S}\right) - n(\log(d_n) + p_n \log\left(\frac{u_n}{d_n}\right))}{\log\left(\frac{u_n}{d_n}\right)\sqrt{np_n(1-p_n)}}\right].$$

In order to calculate this limit, we can move the limit into the expression, and consider each of the limits separate for simplicity, then combine the result later on to determine the limit of the expression

$$\frac{\lim_{n\to\infty}\left[\log\left(\frac{K}{S}\right)\right] - \lim_{n\to\infty}\left[n\left(\log(d_n) + p_n\log\left(\frac{u_n}{d_n}\right)\right)\right]}{\lim_{n\to\infty}\left[\log\left(\frac{u_n}{d_n}\right)\sqrt{np_n(1-p_n)}\right]}.$$

We begin with the trivial limit, which is easy to compute since it is not dependent on n, so we have

$$\lim_{n\to\infty} \log\left(\frac{K}{S}\right) = \log\left(\frac{K}{S}\right).$$

Now, consider the key expression $\lim_{n\to\infty}(u_n/d_n)$, which is in most of the terms. To calculate this, we substitute the expression from Equation 12 such that

$$\begin{aligned}\lim_{n\to\infty} \log\left(\frac{u_n}{d_n}\right) &= \lim_{n\to\infty} \log\left(\frac{e^{\sigma\sqrt{t/n}}}{e^{-\sigma\sqrt{t/n}}}\right)\\ &= \lim_{n\to\infty} 2\sigma\sqrt{\frac{t}{n}}\end{aligned}$$

With this result, lets consider the limit of the term in the denominator

$$\begin{aligned}\lim_{n\to\infty} \sqrt{np_n(1-p_n)}\log\left(\frac{u_n}{d_n}\right) &= \lim_{n\to\infty} \sqrt{np_n(1-p_n)}2\sigma\sqrt{\frac{t}{n}},\\ &= \lim_{n\to\infty} 2\sigma\sqrt{tp_n(1-p_n)}.\end{aligned}$$

We have shown that limit of $p_n = 1/2$, so substituting this in yields

$$\lim_{n\to\infty} 2\sigma\sqrt{tp_n(1-p_n)} = 2\sigma\sqrt{t\frac{1}{4}}.$$

Thus, we can show that the limit for this term is

$$\lim_{n\to\infty} \sqrt{np_n(1-p_n)}\log\left(\frac{u_n}{d_n}\right) = \sigma\sqrt{t}.$$

Finally, we can calculate the limit of the term in the numerator, with $\log(d_n) = -\sigma\sqrt{\frac{t}{n}}$, we have

$$\begin{aligned}\lim_{n\to\infty} n\left(p_n\log\left(\frac{u_n}{d_n}\right) + \log(d_n)\right) &= \lim_{n\to\infty} n\left(p_n 2\sigma\sqrt{\frac{t}{n}} - \sigma\sqrt{\frac{t}{n}}\right)\\ &= \lim_{n\to\infty} n\left(2\sigma\sqrt{\frac{t}{n}}\left(p_n - \frac{1}{2}\right)\right)\\ &= \lim_{n\to\infty} 2\sigma\sqrt{tn}\left(p_n - \frac{1}{2}\right).\end{aligned}$$

Noticing that we have already calculated part of this in Equation 17, we substitute this in so that

$$\lim_{n\to\infty} 2\sigma\sqrt{tn}(p_n - 1) = 2\sigma\sqrt{t}\left(\frac{rt - \frac{1}{2}\sigma^2 t}{2\sigma\sqrt{t}}\right).$$

Finally, after simplifying, we get

$$\lim_{n\to\infty} n\left(p_n \log\left(\frac{u_n}{d_n}\right) + \log\left(d_n\right)\right) = t\left(r - \frac{\sigma^2}{2}\right).$$

Now putting together all these limit results back in to the original expression, we can obtain

$$\lim_{n\to\infty}\left(\frac{(a_n - 1) - np_n}{\sqrt{np_n(1-p_n)}}\right) = \left(\frac{\log\left(\frac{K}{S}\right) - tr + t\frac{\sigma^2}{2}}{\sigma\sqrt{t}}\right),$$

this can be further simplified, as we can use $rt = \log(e^{rt})$, so that

$$\lim_{n\to\infty}(1 - \mathbf{P}\big[N_n \leq a_n - 1\big]) = 1 - \Phi\left(\frac{\log\left(\frac{K}{Se^{rt}}\right)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right). \tag{18}$$

Hence we have found an expression, in terms of the standard normal CDF, for the probability that a_n is larger than our binomial distributed $\psi_a(n, p_n)$, random quantity N_n, as $n \to \infty$.

5.3 Second binomial distribution $\psi_a(n, p'_n)$

In a similar manner, we can show that the binomial distribution $\psi_a(n, p'_n)$ converges to a standard normal distribution for large n, as long as we establish that p'_n converges to a limit with in 0 and 1.

5.3.1 Limits for $\psi_a(n, p'_n)$

Recall that we have defined $p'_n = (u_n/\rho_n)p_n$. We may again calculate the quantities $\lim_{n\to\infty} p'_n$ and $\lim_{n\to\infty}\sqrt{n}(p'_n - 1/2)$. We deal with them in the same order as above. Firstly,

$$\lim_{n\to\infty} p'_n = \lim_{n\to\infty}\frac{u_n}{d_n}p_n$$

Since u_n, ρ_n, and p_n are all continuous with the limit as n goes to infinity of u_n/ρ_n being finite, we may simply calculate the limits of u_n/ρ_n and p_n instead, which proves to be a lot easier; particularly because we already know the limit of p_n. Here, we introduce the substitution $y = \sqrt{t/n}$,

$$\begin{aligned}
\lim_{n\to\infty} p'_n &= \lim_{n\to\infty} \frac{u_n}{d_n} \cdot \lim_{n\to\infty} p_n \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{e^{\sigma\sqrt{t/n}}}{e^{rt/n}} \\
&= \frac{1}{2} \lim_{y\to 0} e^{\sigma y - r y^2} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} p'_n = \frac{1}{2}$$

We may also calculate $\lim_{n\to\infty} \sqrt{n}(p'_n - 1/2)$. Using the definitions of u_n, d_n, and p_n, we can easily obtain a familiar expression for p'_n,

$$p'_n = \frac{e^{2\sigma y} - e^{\sigma y - r y^2}}{e^{2\sigma y} - 1}$$

We are now ready to calculate $\lim_{n\to\infty} \sqrt{n}(p'_n - 1/2)$. Just like before, we notice that $\sqrt{n}$ becomes $\sqrt{t}/y$. Hence, we have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}\left(p'_n - \frac{1}{2}\right) &= \lim_{y\to 0} \frac{\sqrt{t}}{y}\left(\frac{e^{2\sigma y} - e^{\sigma y - r y^2}}{e^{2\sigma y} - 1} - \frac{1}{2}\right) \\
&= \frac{\sqrt{t}}{2} \lim_{y\to 0} \frac{1}{y}\left(\frac{e^{2\sigma y} - e^{\sigma y - r y^2}}{e^{2\sigma y} - 1} - \frac{e^{2\sigma y} - 1}{2e^{2\sigma y} - 2}\right) \\
&= \frac{\sqrt{t}}{2} \lim_{y\to 0} \frac{e^{2\sigma y} - 2e^{\sigma y - r y^2} + 1}{y(e^{2\sigma y} - 1)}.
\end{aligned}$$

Just like before, we apply L'Hopital's rule

$$\lim_{n\to\infty} \sqrt{n}\left(p'_n - \frac{1}{2}\right) = \sqrt{t} \lim_{y\to 0} \frac{\sigma e^{2\sigma y} - (\sigma - 2ry)e^{\sigma y - r y^2}}{e^{2\sigma y} - 1 + 2\sigma y e^{2\sigma y}}$$

Applying L'Hopital's rule once again, we get

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}\left(p'_n - \frac{1}{2}\right) &= \sqrt{t} \lim_{y\to 0} \frac{2\sigma^2 e^{2\sigma y} + 2r e^{\sigma y - r y^2} - (\sigma - 2ry)^2 e^{\sigma y - r y^2}}{2e^{2\sigma y} + 2e^{2\sigma y} + 4\sigma^2 y e^{2\sigma y}} \\
&= \sqrt{t}\frac{2\sigma^2 + 2r - \sigma^2}{4\sigma} \\
&= \frac{rt + \frac{1}{2}\sigma^2 t}{2\sigma\sqrt{t}}.
\end{aligned}$$

Therefore, we have solved for the limit. Its value is

$$\lim_{n\to\infty} \sqrt{n}\left(p'_n - \frac{1}{2}\right) = \frac{rt + \frac{1}{2}\sigma^2 t}{2\sigma\sqrt{t}}. \tag{19}$$

Next, we may also wish to find out that $1 - p'_n$ is. Doing so is fairly straightforward;

$$\begin{aligned}
1 - p'_n = 1 - \frac{u_n}{\rho_n} p_n &= \frac{\rho_n(u_n - d_n)}{\rho_n(u_n - d_n)} - \frac{u_n}{\rho_n} \cdot \frac{\rho_n - d_n}{u_n - d_n} \\
&= \frac{\rho_n u_n - \rho_n d_n - \rho_n u_n + u_n d_n}{\rho_n(u_n - d_n)} \\
&= \frac{u_n d_n - \rho_n d_n}{\rho_n(u_n - d_n)} = \frac{d_n}{\rho_n} \cdot \frac{u_n - \rho_n}{u_n - d_n}.
\end{aligned}$$

Noticing that $(u_n - \rho_n)/(u_n - d_n) = 1 - p_n$, we finally obtain

$$1 - p'_n = \frac{d_n}{\rho_n}(1 - p_n).$$

5.3.2 Applying the CLT to $\psi_a(n, p'_n)$

Following the process defined in Applying the CLT to $\psi_a(n, p_n)$, we now use the Central Limit theorem for the binomial distribution function $\psi_a(n, p'_n)$

$$\lim_{n\to\infty} \psi_a(n, p'_n) = 1 - \lim_{n\to\infty} \mathbf{P}\left[N_n \leq \frac{(a_n - 1) - np'_n}{\sqrt{np'_n(1 - p'_n)}}\right] = 1 - \Phi\left(\lim_{n\to\infty} \frac{(a_n - 1) - np'_n}{\sqrt{np'_n(1 - p'_n)}}\right).$$

Furthermore we substitute in for $a - 1$ and do similar rearranging

$$\lim_{n\to\infty} \frac{(a_n - 1) - np'_n}{\sqrt{np'_n(1 - p'_n)}} = \lim_{n\to\infty} \left[\frac{\log\left(\frac{K}{S}\right) - n(\log(d_n) + p'_n \log\left(\frac{u_n}{d_n}\right)}{\log\left(\frac{u_n}{d_n}\right)\sqrt{np'_n(1 - p'_n)}}\right].$$

Observe that these limits are similar to the ones in the previous section, they can be solved easily with replacing p_n with p'_n. Furthermore we arrival at the same results and shall use them here. However the limit for the numerator term slightly differs in this case

$$\begin{aligned}
\lim_{n\to\infty} n\left(p'_n \log\left(\frac{u_n}{d_n}\right) + \log(d_n)\right) &= \lim_{n\to\infty} n\left(p'_n 2\sigma\sqrt{\frac{t}{n}} - \sigma\sqrt{\frac{t}{n}}\right) \\
&= \lim_{n\to\infty} 2\sigma\sqrt{tn}\left(p'_n - \frac{1}{2}\right).
\end{aligned}$$

Since $\sqrt{n}(p'_n - \frac{1}{2})$ has already been computed in Equation 19, then it following that

$$\lim_{n\to\infty} 2\sigma\sqrt{t}\sqrt{n}\left(p'_n - \frac{1}{2}\right) = 2\sigma\sqrt{t}\left(\frac{rt + \frac{1}{2}\sigma^2 t}{2\sigma\sqrt{t}}\right).$$

Therefore, we have solved for the limit. Its value is

$$\lim_{n\to\infty} n\left(p'_n \log\left(\frac{u_n}{d_n}\right) + \log(d_n)\right) = t\left(r + \frac{\sigma^2}{2}\right).$$

Using this results, we can now substitute them in and compute the limit to the original expression in the section as

$$\lim_{n\to\infty}\left[\frac{\log\left(\frac{K}{S}\right) - n(\log(d_n) + p'_n \log\left(\frac{u_n}{d_n}\right)}{\log\left(\frac{u_n}{d_n}\right)\sqrt{np'_n(1-p'_n)}}\right] = \frac{\log\left(\frac{K}{S}\right) - rt - t\frac{\sigma^2}{2}}{\sigma\sqrt{t}}$$

Hence, this gives us

$$\lim_{n\to\infty}\left[1 - \psi_a(n, p'_n)\right] = 1 - \Phi\left(\frac{\log\left(\frac{K}{Se^{rt}}\right)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right). \tag{20}$$

Once again we have found an expression, in terms of the standard normal CDF, for the probability that a_n is larger than our binomial distributed $\psi_a(n, p'_n)$, random quantity N_n, as $n \to \infty$.

5.4 The call and put for continuous

5.4.1 Formula for call

Combining the work done in the previous two sections, we can now compute a general formula for the pricing of call options as n goes to infinity

$$\lim_{n\to\infty} C = \lim_{n\to\infty}\left[S\psi_a(n, p_n) - K\rho_n^{-n}\psi_a(n, p_n)\right].$$

Using Equations 20 and 18 for the results of the limits, with $\rho_n = e^{rt/n}$ and utilising the symmetric property of standard distributions $1 - \Phi(z) = \Phi(-z)$, we can obtain

$$\lim_{n\to\infty} C = S\Phi\left(-\frac{\log\left(\frac{K}{Se^{rt}}\right)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) - Ke^{-rt}\Phi\left(-\frac{\log\left(\frac{K}{Se^{rt}}\right)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right)$$

The logarithms' numerator and denominator now switch due to the negative, hence we now have

$$\lim_{n\to\infty} C = S\Phi\left(\frac{\log\left(\frac{Se^{rt}}{K}\right)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) - Ke^{-srt}\Phi\left(\frac{\log\left(\frac{Se^{rt}}{K}\right)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right)$$

Here, we conclude the European call option premium, of a continuous model is given by

$$C = S\Phi(x) - Ke^{-rt}\Phi(x - \sigma\sqrt{t}), \quad \text{where} \quad x = \frac{\log\left(\frac{Se^{rt}}{K}\right)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}. \tag{21}$$

With this formula, we can now calculate the price of a call option in a continuous model by using the standard normal CDF to account for the random walks the stock price may take during the trading periods. Requiring that we only know the stock price S, the strike price K, the stocks volatility σ, and the time till the expiration date t.

5.4.2 Formula for Put

Furthermore, the European put option premium can be calculated from Equation 21 and the put-call parity, defined in Equation 3, so that $P = C - S + Ke^{-rt}$. Which leads to

$$P = S\Phi(x) - Ke^{-rt}\Phi\left(x - \sigma\sqrt{t}\right) - S + Ke^{-rt}.$$

Factoring similar terms, we can simplify this to

$$P = Ke^{-rt}\left(1 - \Phi(x - \sigma\sqrt{t})\right) - S\left(1 - \Phi(x)\right).$$

Utilising the symmetric property, as described before, gives us the general formula for the European put option defined as

$$P = Ke^{-rt}\Phi(y + \sigma\sqrt{t}) - S\Phi(y), \quad \text{here} \quad y = \frac{\log\left(\frac{K}{Se^{rt}}\right)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}.$$

Once again only requiring that we know the stock price S, the strike price K, the stocks volatility σ, and the time till the expiration date t. The pricing of a put option can be calculated in continuous model by using the standard normal CDF to account for the random walks the stock price may take during the trading periods.

6 Application of the Black-Scholes formula

Here, we assume that the stock prices over a given period is distributed logarithmically. This implies that the relative price denoted as $R_i = S_i/S_{i-1}$ has a normal distribution. This normal distribution has a mean and variance that is proportional to the time period in which it is measured over. This means that each R_i is a random independent sample from a normal distribution, and that the variance of the standard normal is the volatility of the share price. The unbiased estimators for the natural logarithm distribution for the sample mean, is defined as

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log R_i, \quad \text{where} \quad R_i = \frac{S_i}{S_{i-1}} \tag{22}$$

and the variance is

$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n} (\log R_i - \hat{\mu})^2. \tag{23}$$

Here n is the number of relative prices.

Taking an example of share prices on the market for Hagenmeyer in 1996, the data of which is shown in Figure 9.

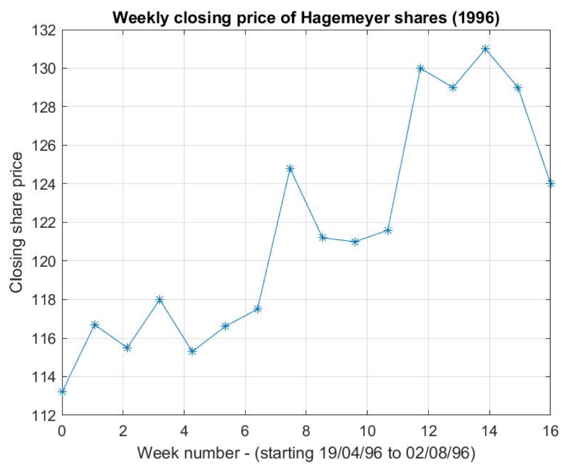

Figure 9: Hagemeyer weekly closing share price.

The first step is to calculating the mean with Equation 22, letting $n = 15$,

$$\hat{\mu} = \frac{1}{15}\sum_{i=1}^{15} \log R_i = \frac{1}{15}(0.091125) = 0.0060750.$$

Furthermore, the variance of is calculated with Equation 23, such that

$$\hat{\sigma}^2 = \frac{1}{14}\sum_{i=1}^{15} (\log R_i - 0.0060750)^2 = \frac{1}{14}(0.0127) = 9.09584\text{e}{-4}$$

Since we are evaluating the closing share prices over a week period, the annual variance can be estimated as $52 * \hat{\sigma}^2$, which, in this case give us an approximate annual variance of 0.0473.

In order to calculate the annual volatility σ, which is required for the Black-Scholes formula, we simply take the square root of the annual variance. Thus, in this example the annual volatility $\sigma = 0.2175$.

Investigating another example, in this case, we observe the daily closing price of Ahold shares over a 40 day trading period, between the 17/11/20 to 15/01/21. The share price can be seen in Figure 10, which was plotted with data collected from Yahoo Fiance website [Yahoo, 2021].

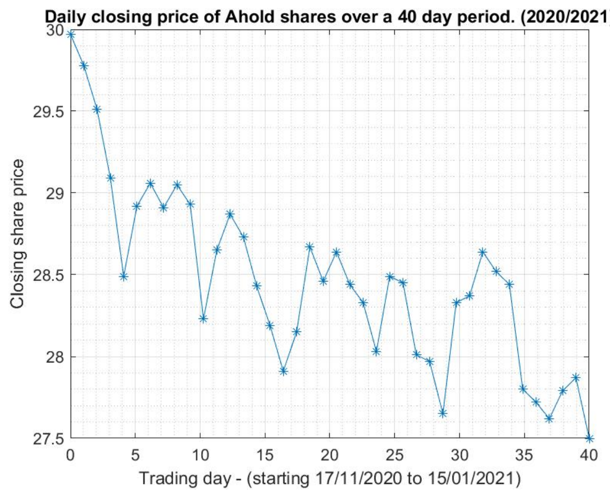

Figure 10: Ahold daily closing share price over 40 day trading period. Source: [Yahoo, 2021].

Following the method describe in the previous part of this section, the mean is once again calculated with Equation 22, so that

$$\hat{\mu} = \frac{1}{39}\sum_{i=1}^{39} \log R_i = \frac{1}{39}(-0.0860) = -0.022,$$

and a variance of

$$\hat{\sigma}^2 = \frac{1}{38}(0.0049) = 0.000128.$$

In this example, we are evaluating the daily closing share price, so the annual variance is calculated as $365 * \hat{\sigma}^2 = 0.0467$. Moreover, this lead us to the annual volatility σ, similar to before, σ is the square root of the annual variance, so $\sigma = 0.2162$.

One of the parameters required to compute option prices via the Black-Scholes formula is the risk-free interest rate r, generally however, such a thing does not exist. We can roughly assimilate r, to the euribor-interest rate (Euro Interbank Offered Rate), this rate is the average interest rates for mutual loans offered between prime banks in the Eurozone. The euribor for a 12-month loan on 15th January is $\hat{r} = -0.508\%$, from data collected from the Nederlandsche Bank [DeNederlandscheBank, 2021]. Due to the current economical climate, we observe that the euribor is negative, which implies the interest rate of mutual loans is negative; instead of the borrower paying money to borrow an amount, it is now the case that money is paid to the borrower. In order to stimulate the financial market, details of this are complex and beyond the scope of this paper.

It should be noted that Black-Scholes model can handle such situations. Since $\hat{r}$ is a percentage, we shall divide it by 100, in order to calculate r. Finally, we observe that $r = log(1 + \hat{r}/100) = -0.0051$.

Using these obtained values, we plot both call option premiums (Figure 11) and put options premiums (Figure 12), with a range of strike prices, defining the expiration date as the 15/01/21. Observe that both plots follow the characteristics behaviour of the underlying share price seen in Figure 10, however, the effect of which changes as the time till expiration date decreases.

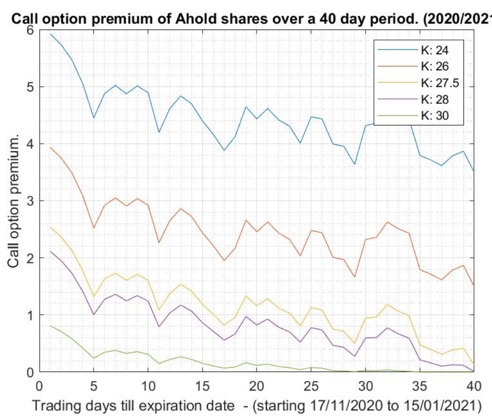

Figure 11: Ahold daily call option premium for expiration date of 15/01/21.

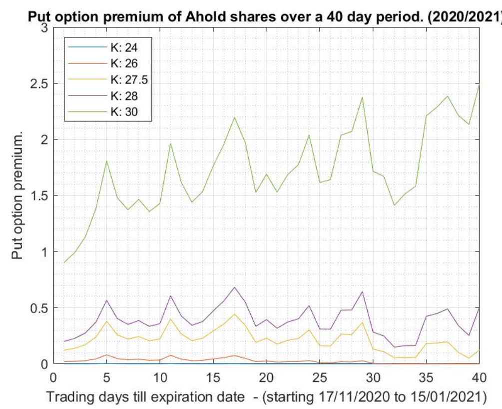

Figure 12: Ahold daily put option premium for expiration date of 15/01/21.

A selection of Ahold options were considered to demonstrates the effect that the time to expiration t has on the value of put and call option. In table 2, we take strike price of $K_1 = 26$ (below current value) and $K_2 = 28$ (above current value).

Here we observe that as the time to expiration date is decreased so is the price of the call and put option, moreover, we can express this as $t_1 < t_2 < t_3$ then it follows $C_{t_1} < C_{t_2} < C_{t_3}$. The option premium is described as being two parts, intrinsic value and extrinsic value. Intrinsic value of premium, is the value of the of the underlying stock against the strike price, $\max(0, S^* - K)$, so in this case for the call option, for K_1 the intrinsic value is 1.5 and for K_2 it is zero, since it finishes out of the money. The extrinsic value part of the premium is made up of the time value and implied volatility of the stock, given by $C - \max(0, S^* - K)$. We calculate with the call option, at a strike price of K_1 at t_1, the extrinsic value is 0.1206, when compared to t_3 extrinsic value 0.3846. The extrinsic value at t_3 is larger than t_1, since the underlying stock has more time, therefore more opportunity to fluctuate in value.

		Expiration date		
		15/02/21	15/03/21	19/04/21
	Trade days	25 (t_1)	45 (t_2)	65 (t_3)
Call	K=26	1.6206	1.7589	1.8846
	K=28	0.4042	0.6068	0.7690
Put	K=26	0.1296	0.1369	0.1442
	K=28	0.9140	2.2765	2.2843

Table 2: Option premiums with a range of different expiration dates.

		Time to Expiration	
	Trade days	3 years	2 years
	Share price	S = 27.50	S= 32.50
Call	K=26	4.5758	7.5092
	K=28	3.7040	6.1701
Put	K=26	3.4761	1.2754
	K=28	4.6351	1.9568

Table 3: Option premiums when the share price increase by 5 Euros over a year period.

Once again, consider the strike prices K_1 and K_2 for both put and call option over a period of three years. Suppose that a year passes, and over that time the share price has moved from 27.5 to 32.50, an increase of 5 euros. The premiums of both the call and put options are calculated and presented in Table 3.

We can now compare this to the percentage increase for the value of the shares and that of the put and call option, which are shown in Table 4. Observe that the share price has only increased by 18.18% in value, while this change in share value has caused the call options premiums to increase by a large amount of approximately 60%, since the call option is now more likely to finish in the money. Similarly, since the share price has gone up, the put premiums are more at risk at finishing out of the money, thus the premiums have decreased, by again approximately 60%.

		Percentage Increase
	Share price	18.18 %
Call	K=26	64.10 %
	K=28	66.58 %
Put	K=26	-63.31 %
	K=28	-57.78 %

Table 4: Percentage increase of the underlying share and option premiums, from data in Table 3.

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