Nonlinear Dynamical Systems
Assignment 3

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Question 1a.

Proof. Give the following set of PDEs;

$$\partial_t v = \partial_x^2 - \gamma v = wv^2,$$
$$\partial_t w = \beta \partial_x w + \alpha - w - wv^2.$$

To find the equilibriums of the given system we set $\partial_t w = 0, \partial_t v = 0$ which yields

$$0 = v\left(-\gamma + wv\right)$$
$$0 = \alpha - w - wv^2.$$

Furthermore, we a bit of work we can compute the 3 equilibriums of the system. The fist case to consider is when $v_1 = 0$ which implies that $w_1 = \alpha$, hence we have $E_1 = (0, \alpha)$. This can be described as the no vegetation state which can always exist in the model.

The other 2 equilibriums arise from when $wv - \gamma = 0$ implies that $v = \gamma/w$, plugging this into the second equations gives us $0 = w^2 - \alpha w + \gamma^2$. Now using the quadratic equation we can solve this to obtain

$$w_{2,3} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\gamma}}{2}.$$

Here we note that if $\alpha < 2\gamma$, the no vegetation state E_1 is the only equilibrium. However if it is the case that $\alpha \geq 2\gamma$ then we have an addition 2 equilibriums, namely

$$E_2 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 - 4\gamma^2}\right)\right), \quad \text{and} \quad E_3 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 - 4\gamma^2}\right)\right).$$

We observe that E_2 is a saddle point and hence is an unstable equilibrium point for the model. Lastly for E_3 we see that the determinate of the Jacobian matrix is give as

$$J_{E_3} = \begin{bmatrix} 2vw - \gamma & v^2 \\ -2vw & -1 - v^2 \end{bmatrix}.$$

Here it can be shown that $\text{Det}(J_{E_3}) > 0$ so then the stability is determined by the trace $\text{Tr}(J_{E_3})$. Moreover, we observe that $\text{Tr}(J_{E_3}) = 0$ along the curve given by $\alpha = \frac{\gamma^2}{\sqrt{\gamma - 1}}$, given $\gamma \neq 1$. The area below this curve will give a stable equilibrium, while the area above will give a unstable equation.

$\square$

Question 2. Linerised

Proof.

$$\partial_t \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = \mathcal{L}\left(v_*, w_*\right) \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$$

Here we define $\mathcal{L} = \left[\frac{\partial f_i}{\partial u_j}\right]$ for i and $j = 1, 2$. Computing this yields the following matrix

$$\mathcal{L}\left(v_*, w_*\right) = \begin{bmatrix} \partial_x^2 - \gamma + 2w_* v_* & {v_*}^2 \\ -2w_* v_* & \beta\partial_x - 1 - {v_*}^2 \end{bmatrix}.$$

□

Question 3. Perturbation of the vegetative equilibrium.

Proof. Using the expression $\gamma = v_* w_*$ which was found in part A we can simplify $\mathcal{L}$ to

$$\mathcal{L} = \begin{bmatrix} \partial_x^2 + \gamma & v_*^2 \\ -2\gamma & \beta\partial_x - 1 - v_*^2 \end{bmatrix}$$

Now using the derivates

$$\partial_x = \frac{d}{dx}(\varphi(x,t)) = \frac{d}{dx}\exp\left(\lambda t + ikx\right) = ik$$

and

$$\partial_x^2 = -k^2$$

yields the following expression

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & v_*^2 \\ -2\gamma & i\beta k - 1 - v_*^2 \end{bmatrix}$$

Using the equilibrium expression for E_3 we have $v_* = \gamma / w_*$, then it follows that

$$\begin{aligned} \frac{\gamma}{w_3} &= \frac{2\gamma}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})} \\ &= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})(\alpha + \sqrt{\alpha^2 - 4\gamma^2})} \\ &= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{4\gamma^2} \\ &= \frac{(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{2\gamma} \\ &= \mu(\alpha, \gamma). \end{aligned}$$

Hence we have

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & \mu^2 \\ -2\gamma & i\beta k - 1 - \mu^2 \end{bmatrix}$$

as required. □

Question 4. Dispersion Relation.

Proof.

```
%clear
clear all, close all, clc
% Question 4.
% function
mu       = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
tau    = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
delta    = @(k,a,b,c) k.^(2) .* (1- i.*b.*k + mu(a,c).^2)
             + c.*(-1+i.*b.*k+mu(a,c).^2);
lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2
             - 4.*delta(k,a,b,c)))/2;
lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2
             - 4.*delta(k,a,b,c)))/2;

rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));

% set parametes
p=[8,20,2];
k= linspace(-2,2,1000);
% Plot dispersion relation (with inset around k=0);
```

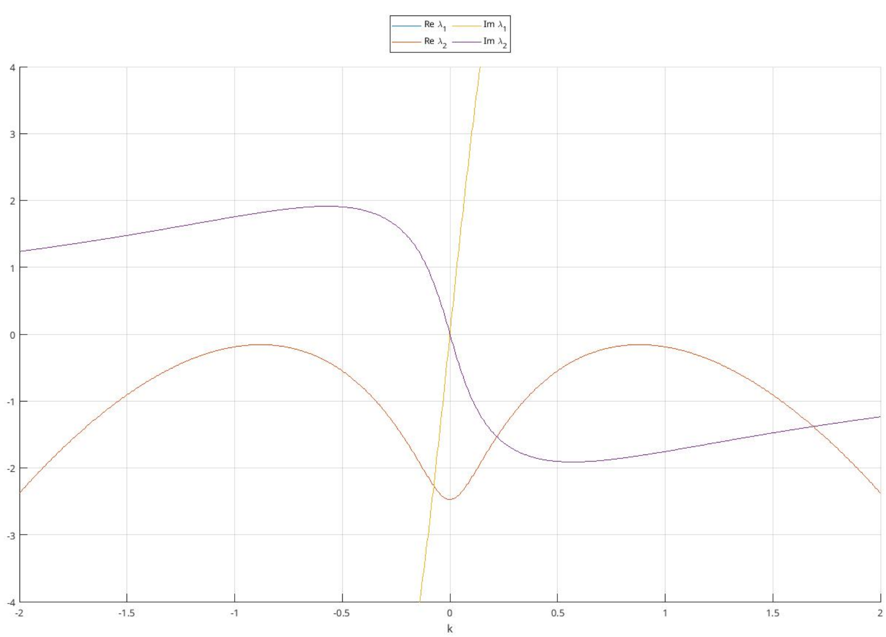

Figure 1: A boat.

```
figure, hold on;
plot(k,rel1(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_1');
plot(k,rel2(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_2');
plot(k,imag1(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_1');
plot(k,imag2(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_2');
hold off; grid on; ylim([-4 4]); xlabel('k');
lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
drawnow;
%
% Fin
```

□

Question 5. Pattern-forming instability.

Proof.

```
%% Question 5 - pattern-forming instability
% Clear
clear all, close all, clc

% functions
mu      = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
tau    = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
delta   = @(k,a,b,c) k.^(2) .* (1- i.*b.*k + mu(a,c).^2)
                            + c.*(-1+i.*b.*k+mu(a,c).^2);
```

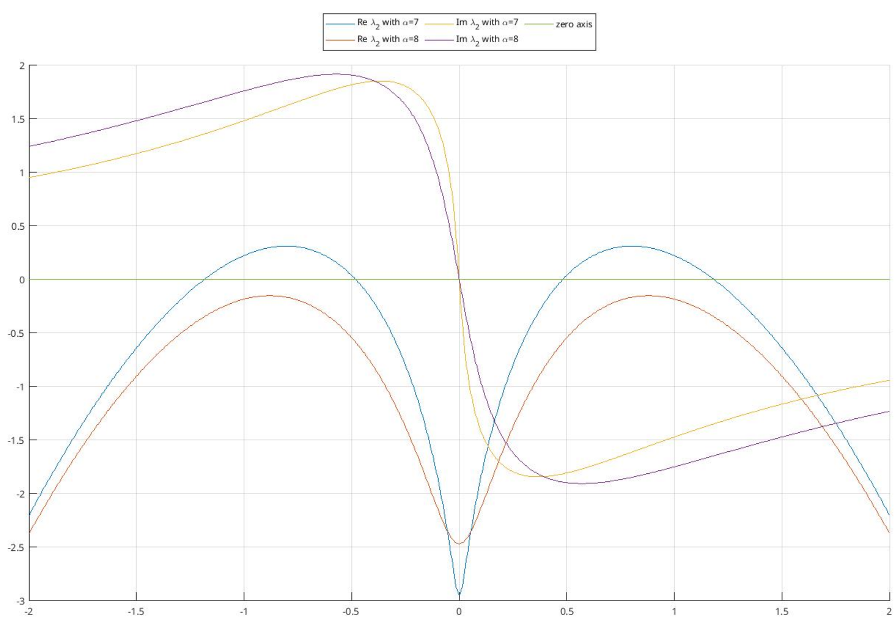

Figure 2: A boat.

```
lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2
                                - 4.*delta(k,a,b,c)))/2;
lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2
                                - 4.*delta(k,a,b,c)))/2;

rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));

% set parameters,
p=[20,2]; % beta , gamma
alpha=[7,8]; %alpha
k=linspace(-2,2,1000);

%
 figure, hold on;
 plot(k,rel2(k,alpha(1),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=7');
 plot(k,rel2(k,alpha(2),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=8');
 plot(k,imag2(k,alpha(1),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=7');
 plot(k,imag2(k,alpha(2),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=8');
 plot(k,zeros(1,1000),'DisplayName','zero axis');
 hold off; grid on; ylim([-3 2]); xlabel('k');
 lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
 drawnow;

% Find when k such that lambda 2 obtains local max on [0,2]
maxrel1 = @(x) rel2(x,alpha(1),p(1),p(2));
[~,kMax] =fminmax(maxrel1 , 0, 2);
lambda2Max = lambda2(kMax, alpha(1),p(1),p(2));

% Wave length
waveLength = 2*pi / kMax
% Wave propagation
wavePropagation = - imag(lambda2Max) / kMax
% function to find max and min of a function
function [min, max] = fminmax(f, lowerbound, upperbound)
    min = fminbnd(f, lowerbound, upperbound);
    max = fminbnd(@(x) -f(x), lowerbound, upperbound);
end
```

□