Nonlinear Dynamical Systems
Assignment 3

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Preface: The matlab files can be accessed and download, along with various other documents, via my Github page, which is linked here.

Question 1.

Proof. Given the following set of PDEs;

$$\partial_t v = \partial_x^2 - \gamma v = wv^2,$$
$$\partial_t w = \beta \partial_x w + \alpha - w - wv^2.$$

To find the equilibriums of the given system we set $\partial_t w = 0, \partial_t v = 0$ which yields

$$0 = v\left(-\gamma + wv\right)$$
$$0 = \alpha - w - wv^2.$$

Furthermore, we a bit of work we can compute the 3 equilibriums of the system. The fist case to consider is when $v_1 = 0$ which implies that $w_1 = \alpha$, hence we have $E_1 = (0, \alpha)$. This can be described as the no vegetation state which can always exist in the model.

The other 2 equilibriums arise from when $wv - \gamma = 0$ implies that $v = \gamma/w$, plugging this into the second equations gives us $0 = w^2 - \alpha w + \gamma^2$. Now using the quadratic equation we can solve this to obtain

$$w_{2,3} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\gamma}}{2}.$$

Here we note that if $\alpha < 2\gamma$, the no vegetation state E_1 is the only equilibrium. However if it is the case that $\alpha \geq 2\gamma$ then we have an addition 2 equilibriums, namely

$$E_2 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 - 4\gamma^2}\right)\right), \quad \text{and} \quad E_3 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 - 4\gamma^2}\right)\right).$$

We observe that E_2 is a saddle point and hence is an unstable equilibrium point for the model. Lastly for E_3 we see that the determinate of the Jacobian matrix is give as

$$J_{E_3} = \begin{bmatrix} 2vw - \gamma & v^2 \\ -2vw & -1 - v^2 \end{bmatrix}.$$

Here it can be shown that $\text{Det}(J_{E_3}) > 0$ so then the stability is determined by the trace $\text{Tr}(J_{E_3})$. Moreover, we observe that $\text{Tr}(J_{E_3}) = 0$ along the curve given by $\alpha = \frac{\gamma^2}{\sqrt{\gamma-1}}$, given $\gamma > 1$. This can be seen in the mathematica note book found on Github. □

Question 2. Linerised

Proof.

$$\partial_t \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = \mathcal{L}\left(v_*, w_*\right) \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$$

Here we define $\mathcal{L} = \left[\frac{\partial f_i}{\partial u_j}\right]$ for i and $j = 1, 2$. Computing this yields the following matrix

$$\mathcal{L}\left(v_*, w_*\right) = \begin{bmatrix} \frac{d}{dv}v & \frac{d}{dw}v \\ \frac{d}{dv}w & \frac{d}{dw}w \end{bmatrix}$$
$$= \begin{bmatrix} \partial_x^2 - \gamma + 2w_* v_* & v_*^2 \\ -2w_* v_* & \beta \partial_x - 1 - v_*^2 \end{bmatrix}.$$

$\square$

Question 3. Perturbation of the vegetative equilibrium.

Proof. Using the expression $\gamma = v_* w_*$ which was found in part A we can simplify $\mathcal{L}$ to

$$\mathcal{L} = \begin{bmatrix} \partial_x^2 + \gamma & v_*^2 \\ -2\gamma & \beta\partial_x - 1 - v_*^2 \end{bmatrix}.$$

Now using the derivates

$$\partial_x = \frac{d}{dx}(\varphi(x,t)) = \frac{d}{dx}\exp(\lambda t + ikx) = ik$$

and

$$\partial_x^2 = -k^2$$

yields the following expression

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & v_*^2 \\ -2\gamma & i\beta k - 1 - v_*^2 \end{bmatrix}.$$

Using the equilibrium expression for E_3 we have $v_* = \gamma/w_*$, then it follows that

$$\begin{aligned}
\frac{\gamma}{w_3} &= \frac{2\gamma}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})} \\
&= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})(\alpha + \sqrt{\alpha^2 - 4\gamma^2})} \\
&= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{4\gamma^2} \\
&= \frac{(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{2\gamma} \\
&= \mu(\alpha, \gamma).
\end{aligned}$$

Hence we have

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & \mu^2 \\ -2\gamma & i\beta k - 1 - \mu^2 \end{bmatrix}$$

as required. $\qquad \square$

Question 4. Dispersion Relation.

Proof. For both λ of the Jacobian matrix we plot the real and imaginary parts for the parameters $(8, 20, 2)$, which can be seen in 1. In which we observe $\Re(\lambda_2) < 0$, implies that the equilibrium is stable. The can also be extended to $\Re(\lambda_1) < 0$ which is not shown in the plot. Moreover we see in both cases $\Im(\lambda_1)$ and $\Im(\lambda_2)$ are not constant functions which implies there is a circle motion in the stream plot. However, we also observe that the they are not the complex conjugate of each other. Hence the vegetative equilibrium is linear stable for $(8, 20, 2)$.
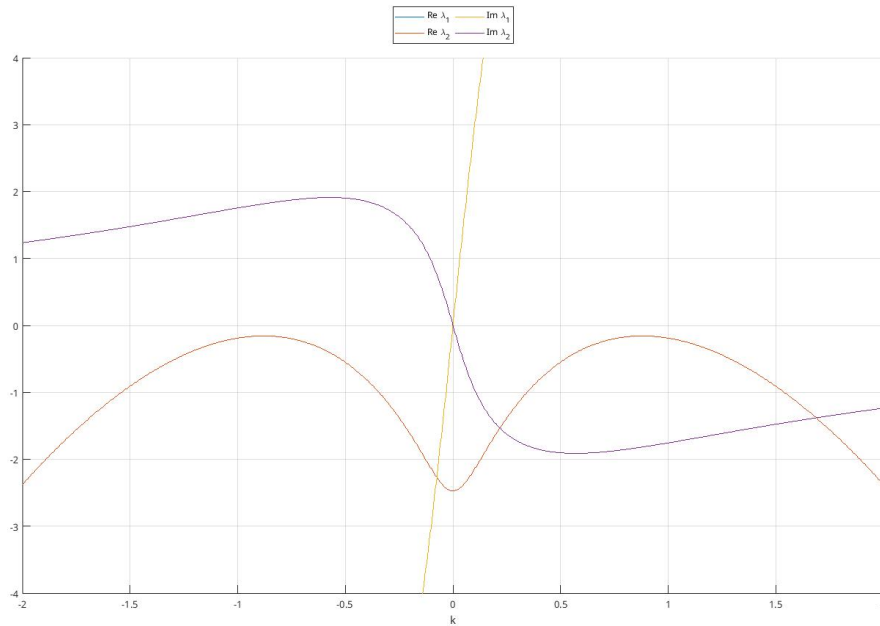

Figure 1: The graph of λ_1 and λ_2 with both the real and imaginary parts plot with the parameters of $\alpha = 8, \beta = 20$ and 2.

```
1   %clear
2   clear all, close all, clc
3   % Question 4.
4   % function
5   mu      = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
6   tau     = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
7   delta   = @(k,a,b,c) k.^(2) .* (1- i.*b.*k + mu(a,c).^2)
8                 + c.*(-1+i.*b.*k+mu(a,c).^2);
9   lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2
10                - 4.*delta(k,a,b,c)))/2;
11  lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2
12                - 4.*delta(k,a,b,c)))/2;
```

```matlab
13
14  rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
15  rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
16  imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
17  imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));
18
19  % set parametes
20  p=[8,20,2];
21  k= linspace(-2,2,1000);
22  % Plot dispersion relation (with inset around k=0);
23   figure, hold on;
24   plot(k,rel1(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_1');
25   plot(k,rel2(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_2');
26   plot(k,imag1(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_1');
27   plot(k,imag2(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_2');
28   hold off; grid on; ylim([-4 4]); xlabel('k');
29   lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
30   drawnow;
31  %
32  % Fin
```

□

Question 5. Pattern-forming instability.

Proof. We can observe numerically from Figure 2 that $\Re(\lambda_2(k,8)) < 0$ so that it is a stable equilibrium and that $\Re(\lambda_2(k,7)) > 0$ which implies that it is an unstable equilibrium.

Moreover, it can be shown that the function $\Re(\lambda_2(k,\alpha))$ is continuous for values of $\alpha = [7,8]$, since λ_2 is a polynomial function. Hence by the Intermediate value theorem there must exits an $\alpha_* \in [7,8]$ and k_* such that $\Re(\lambda_2(k_*,\alpha_*)) = 0$. Namely that α_* is a critical value.

We see that for $\alpha \leq \alpha_*$ there emerges a propagating wave pattern instability. This is since there exist both $\pm k_*$ such that $\Re(\lambda(\pm k_*,\alpha_*)) = 0$ while also we have $\Im(\lambda_2(\pm k_*,\alpha_*)) \neq 0$.

Hence we can compute the wave length by $\Delta = 2\pi/k_*$ so in the case when $\alpha = 7$ we can use matlab to numerically find $k_* = 0.8038$ which yields a wave length of $\Delta = 7.8166$. The speed of the propagating is computed by $c = -\Im(\lambda_*)/k_*$, here λ_* is the maximum value that the λ function obtains. In the case of $\alpha = 7$ we have $\Im(\lambda_*) = -1.6169$, thus we have a travelling wave speed of $c = 2.0115$.
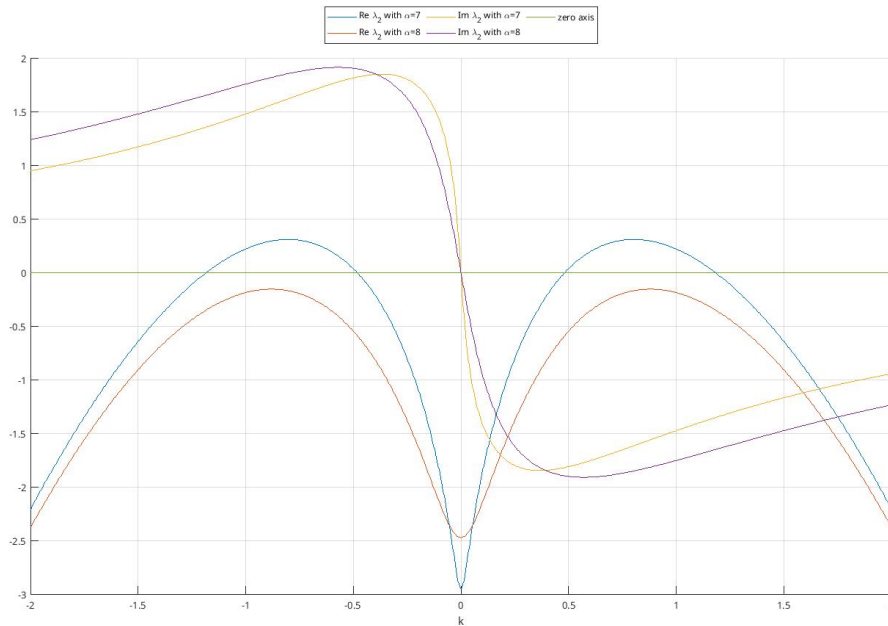

Figure 2: Graphical plots of real and imaginary parts of $\lambda_2(k)$ for parameters $\alpha = [7,8]$, $\beta = 20$ and $\gamma = 2$.

```matlab
1  %% Question 5 - pattern-forming instability
2  % Clear
3  clear all, close all, clc
```

```matlab
4
5  % functions
6  mu      = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
7  tau    = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
8  delta   = @(k,a,b,c) k.^(2).* (1- i.*b.*k + mu(a,c).^2) + c.*(-1+i.*b.*k+mu(a,c).^2
9  lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2 - 4.*delta(k,a,b,c)))/2;
10 lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2 - 4.*delta(k,a,b,c)))/2;
11
12 rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
13 rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
14 imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
15 imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));
16
17 % set parameters,
18 p=[20,2]; % beta , gamma
19 alpha=[7,8]; %alpha
20 k=linspace(-2,2,1000);
21
22 %
23  figure, hold on;
24  plot(k,rel2(k,alpha(1),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=7');
25  plot(k,rel2(k,alpha(2),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=8');
26  plot(k,imag2(k,alpha(1),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=7');
27  plot(k,imag2(k,alpha(2),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=8');
28  plot(k,zeros(1,1000),'DisplayName','zero axis');
29  hold off; grid on; ylim([-3 2]); xlabel('k');
30  lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
31  drawnow;
32
33 % Find when k such that lambda 2 obtains local max on [0,2]
34 maxrel1 = @(x) rel2(x,alpha(1),p(1),p(2));
35 [~,kMax] =fminmax(maxrel1 , 0, 2);
36 lambda2Max = lambda2(kMax, alpha(1),p(1),p(2));
37
38 % Wave length
39 waveLength = 2*pi / kMax
40 % Wave propagation
41 wavePropagation = - imag(lambda2Max) / kMax
42 % function to find max and min of a function
43 function [min, max] = fminmax(f, lowerbound, upperbound)
44     min = fminbnd(f, lowerbound, upperbound);
45     max = fminbnd(@(x) -f(x), lowerbound, upperbound);
46 end
```

□

Question 6.

Proof. For this question we fix the parameters to $(7, 20, 2)$, while we apply the Periodic boundary conditions with setting $L = 15.7$ and the initial conditions of

$$\begin{pmatrix} v(x, 0) \\ w(x, 0) \end{pmatrix} = \begin{pmatrix} v_{*,3} \\ w_{*,3} \end{pmatrix} + \begin{pmatrix} \cos(4\pi x/L) \\ 1 \end{pmatrix}.$$

Using numerically methods and matlab we can obtain the following plots, which shows the existence of a propagating wave. In Figure 4 we see that peaks of the waves occur at $x_1 = 8.52$ and $x_2 = 0.6908$, which gives $\Delta = |x_1 - x_2| = 7.8292$. We see this value is close to the one computed in Question 5, which confirms our findings. Moreover, we also observe that in Figure 3 that we have a dialogical lines, namely that we have periodic in both x and t, this implies that there exists a propagating wave, which we computed as in Question 5.
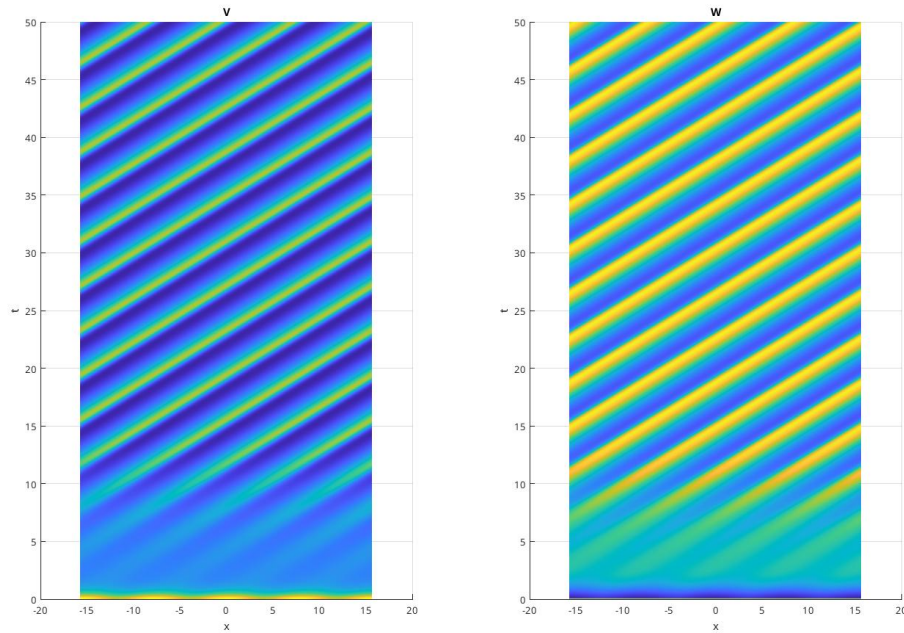

Figure 3: 3D plot of the tiger bush Pattern-forming, in time, space and density. For parameters $(7, 20, 2)$, $L = 15.7$.
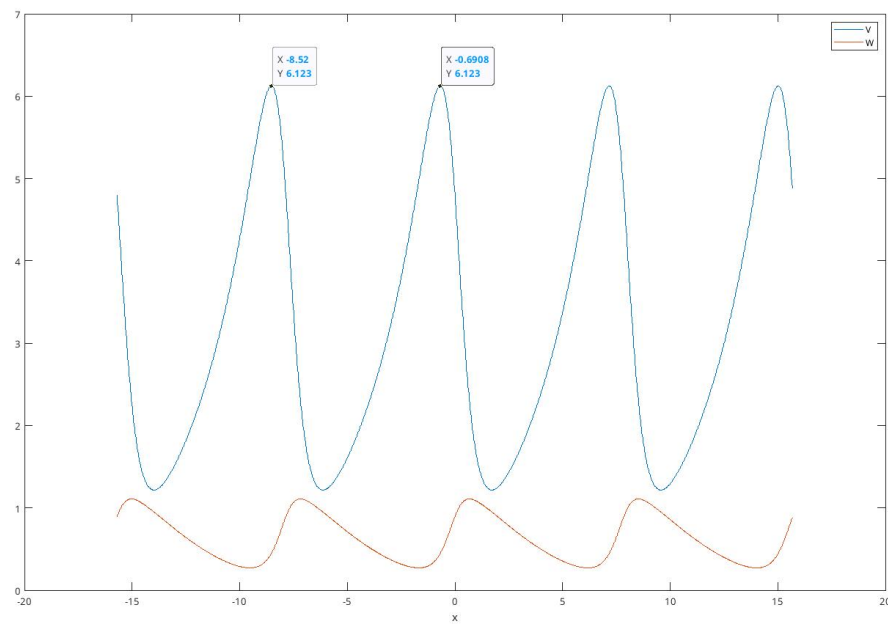
Figure 4: Wave shapes of the Tiger bush pattern. Here we also mark two consecutive peaks in order to measure the wave length. For parameters $(7, 20, 2)$, $L = 15.7$.

```matlab
1  %% Question 6
2  %clear
3  clear all, close all, clc
4
5  % define parameters
6  L=15.7;
7  p=[7,20,2];
8
9  % Instantiating periodic differentiation matrix
10 nx = 1500; [x,Dx,Dxx] = PeriodicDiffMat([-L,L],nx);
11
12 % % Initial condition (steady state + perburbation)
13 e = ones(size(x)); z0 = [p(1)*e; e/p(1)];
14 z0 = z0 + 1*[cos(4*pi/L*x); e];
15
16 % Time step
17 rhs = @(t,z) tigerbush(z,p,Dx,Dxx);
18 jac = @(t,z) tigerbushJacobian(z,p,Dx,Dxx);
19 opts = odeset('Jacobian',jac);
20 tSpan = [0:0.1:50];
21 [t,ZHist] = ode15s(rhs,tSpan,z0,opts);
22
23 % Space-time plot
24  PlotHistory(x,t,ZHist,p,[]);
25 %
26 % Plot final state
27 figure; title('Final state');
28 plot(x,ZHist(end,1:nx),x,ZHist(end,nx+1:2*nx));
29 xlabel('x'); legend({'V','W'});
30
31
32 %% Users functions
33 function [F, DFDZ] = tigerbush(z, p,Dx, Dxx)
34
35      % Rename parameters
36      a =p(1); b=p(2); c=p(3);
37
38      % Ancillary variables and solution split
39      nx=length(z)/2;
40      iU=1:nx;
41      iV=nx+iU;
42      v = z(iU);
43      w = z(iV);
44
45      % Function handles
46      f     = @(v,w) - v.*c + w.*v.^2;
47      dfdv  = @(v,w) -c + 2.* w.*v;
48      dfdw  = @(v,w) v.^2;
49
50      g     =@(v,w) a-w-w.*v.^2;
```

```matlab
51          dgdv   =@(v,w)  -2.*w.*v;
52          dgdw   =@(v,w)  -1-v.^2;
53
54          % Right-hand side
55          F=zeros(size(z));
56          F(iU) = Dxx*v + f(v,w);
57          F(iV) = b*Dx*w + g(v,w);
58
59      if nargout > 1
60         DFDZ = spdiags([],[],2*nx,2*nx);
61         DFDZ(iU,iU) =   Dxx + spdiags(dfdv(v,w),0,nx,nx);
62         DFDZ(iU,iV) =         spdiags(dfdw(v,w),0,nx,nx);
63         DFDZ(iV,iU) =         spdiags(dgdv(v,w),0,nx,nx);
64         DFDZ(iV,iV) = b*Dx + spdiags(dgdw(v,w),0,nx,nx);
65      end
66  end
67
68  function DFDZ = tigerbushJacobian(z,p,Dx, Dxx)
69          [~, DFDZ ] =tigerbush(z,p,Dx,Dxx);
70  end
71  function [x,Dx,Dxx] = PeriodicDiffMat(xSpan,nx)
72
73          % Gridpoints
74          a = xSpan(1); b = xSpan(2);
75          hx = (b-a)/nx;
76          x = a+[0:nx-1]'*hx;
77
78          % Auxiliary vecor
79          e = ones(nx,1);
80
81          % First order differentiation matrix
82          Dx = spdiags([-e e],[-1 1],nx,nx);
83          Dx(1,nx) =   -1; Dx(nx,1) = 1;
84          Dx = Dx/(2*hx);
85
86          % Second order differentiation matrix
87          Dxx = spdiags([e -2*e e],-1:1,nx,nx);
88          Dxx(1,nx) = 1; Dxx(nx,1) = 1;
89          Dxx = Dxx/(hx^2);
90
91  end
92
93  function plotHandle = PlotHistory(x,t,U,p,parentHandle)
94
95      numComp = 2;
96      nx = size(U,2)/2;
97
98      %% Assign solution label
99      solLabel(1).name = "V";
100     solLabel(2).name = "W";
```

```matlab
101
102      %% Position and eventually grab figure
103      if isempty(parentHandle)
104        %scrsz = get(0,'ScreenSize');
105        % plotHandle = figure('Position',[scrsz(3)/2 scrsz(4)/2 scrsz(3)/4 scrsz(4)/4]
106        plotHandle = figure();
107        parentHandle = plotHandle;
108       else
109         plotHandle = parentHandle;
110      end
111      figure(parentHandle);
112
113      %% Grid
114      [T,X] = meshgrid(t,x);
115
116      %% Plots
117      for k = 1:numComp
118        subplot(1,numComp,k)
119        % pcolor(X,T,U(:,idx(:,k))'); shading interp; view([0 90]);
120        surf(X,T,U(:,nx*(k-1)+[1:nx])'); shading interp; view([0 90]);
121        title(solLabel(k).name);
122        xlabel('x'); ylabel('t');
123      end
124
125      %% Save
126      % print -dtiff history.tiff
127
128  end
```

□