Nonlinear Dynamical Systems
Assignment 3

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Question 1.

Proof. Given the following set of PDEs;

$$\partial_t v = \partial_x^2 - \gamma v = w v^2,$$
$$\partial_t w = \beta \partial_x w + \alpha - w - w v^2.$$

To find the equilibriums of the given system we set $\partial_t w = 0, \partial_t v = 0$ which yields

$$0 = v\left(-\gamma + wv\right)$$
$$0 = \alpha - w - wv^2.$$

Furthermore, we a bit of work we can compute the 3 equilibriums of the system. The fist case to consider is when $v_1 = 0$ which implies that $w_1 = \alpha$, hence we have $E_1 = (0, \alpha)$. This can be described as the no vegetation state which can always exist in the model.

The other 2 equilibriums arise from when $wv - \gamma = 0$ implies that $v = \gamma / w$, plugging this into the second equations gives us $0 = w^2 - \alpha w + \gamma^2$. Now using the quadratic equation we can solve this to obtain

$$w_{2,3} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\gamma}}{2}.$$

Here we note that if $\alpha < 2\gamma$, the no vegetation state E_1 is the only equilibrium. However if it is the case that $\alpha \geq 2\gamma$ then we have an addition 2 equilibriums, namely

$$E_2 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 - 4\gamma^2}\right)\right), \quad \text{and} \quad E_3 = \left(\frac{\gamma}{w}, \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 - 4\gamma^2}\right)\right).$$

We observe that E_2 is a saddle point and hence is an unstable equilibrium point for the model. Lastly for E_3 we see that the determinate of the Jacobian matrix is give as

$$J_{E_3} = \begin{bmatrix} 2vw - \gamma & v^2 \\ -2vw & -1 - v^2 \end{bmatrix}.$$

Here it can be shown that $\mathrm{Det}(J_{E_3}) > 0$ so then the stability is determined by the trace $\mathrm{Tr}(J_{E_3})$. Moreover, we observe that $\mathrm{Tr}(J_{E_3}) = 0$ along the curve given by $\alpha = \frac{\gamma^2}{\sqrt{\gamma - 1}}$, given $\gamma > 1$. This can be seen in [edit]. $\square$

Question 2. Linerised

Proof.

$$\partial_t \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = \mathcal{L}\left(v_*, w_*\right) \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$$

Here we define $\mathcal{L} = \left[\frac{\partial f_i}{\partial u_j}\right]$ for i and $j = 1, 2$. Computing this yields the following matrix

$$\begin{aligned} \mathcal{L}\left(v_*, w_*\right) &= \begin{bmatrix} \frac{d}{dv} v & \frac{d}{dw} v \\ \frac{d}{dv} w & \frac{d}{dw} w \end{bmatrix} \\ &= \begin{bmatrix} \partial_x^2 - \gamma + 2w_* v_* & {v_*}^2 \\ -2w_* v_* & \beta \partial_x - 1 - {v_*}^2 \end{bmatrix}. \end{aligned}$$

□

Question 3. Perturbation of the vegetative equilibrium.

Proof. Using the expression $\gamma = v_* w_*$ which was found in part A we can simplify $\mathcal{L}$ to

$$\mathcal{L} = \begin{bmatrix} \partial_x^2 + \gamma & v_*^2 \\ -2\gamma & \beta\partial_x - 1 - v_*^2 \end{bmatrix}.$$

Now using the derivates

$$\partial_x = \frac{d}{dx}(\varphi(x,t)) = \frac{d}{dx}\exp\left(\lambda t + ikx\right) = ik$$

and

$$\partial_x^2 = -k^2$$

yields the following expression

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & v_*^2 \\ -2\gamma & i\beta k - 1 - v_*^2 \end{bmatrix}.$$

Using the equilibrium expression for E_3 we have $v_* = \gamma / w_*$, then it follows that

$$\begin{aligned} \frac{\gamma}{w_3} &= \frac{2\gamma}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})} \\ &= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{(\alpha - \sqrt{\alpha^2 - 4\gamma^2})(\alpha + \sqrt{\alpha^2 - 4\gamma^2})} \\ &= \frac{2\gamma(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{4\gamma^2} \\ &= \frac{(\alpha + \sqrt{\alpha^2 - 4\gamma^2})}{2\gamma} \\ &= \mu(\alpha, \gamma). \end{aligned}$$

Hence we have

$$\mathcal{L} = \begin{bmatrix} -k^2 + \gamma & \mu^2 \\ -2\gamma & i\beta k - 1 - \mu^2 \end{bmatrix}$$

as required. □

Question 4. Dispersion Relation.

Proof. For both λ of the Jacobian matrix we plot the real and imaginary parts for the parameters $(8, 20, 2)$, which can be seen in 1. In which we observe $\Re(\lambda_2) < 0$, implies that the equilibrium is stable. The can also be extended to $\Re(\lambda_1) < 0$ which is not shown in the plot. Moreover we see in both cases $\Im(\lambda_1)$ and $\Im(\lambda_2)$ are not constant functions which implies there is a circle motion in the stream plot. However, we also observe that the they are not the complex conjugate of each other. Hence the vegetative equilibrium is linear stable for $(8, 20, 2)$.

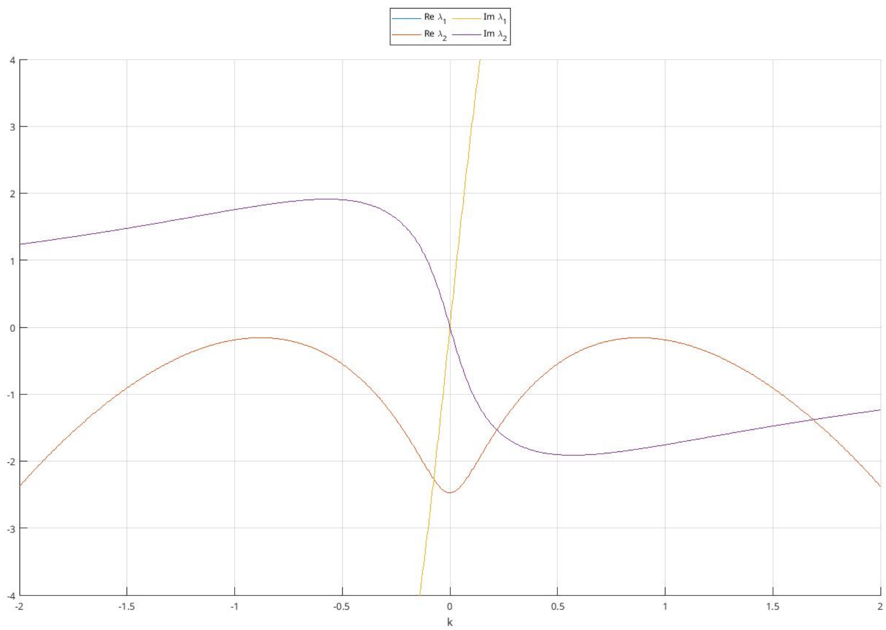

Figure 1: The graph of λ_1 and λ_2 with both the real and imaginary parts plot with the parameters of $\alpha = 8, \beta = 20$ and 2.

```
%clear
clear all, close all, clc
% Question 4.
% function
mu      = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
tau    = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
delta   = @(k,a,b,c) k.^(2) .* (1- i.*b.*k + mu(a,c).^2)
              + c.*(-1+i.*b.*k+mu(a,c).^2);
lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2
              - 4.*delta(k,a,b,c)))/2;
lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2
              - 4.*delta(k,a,b,c)))/2;
```

```matlab

rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));

% set parametes
p=[8,20,2];
k= linspace(-2,2,1000);
% Plot dispersion relation (with inset around k=0);
 figure, hold on;
 plot(k,rel1(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_1');
 plot(k,rel2(k,p(1),p(2),p(3)),'DisplayName','Re \lambda_2');
 plot(k,imag1(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_1');
 plot(k,imag2(k,p(1),p(2),p(3)),'DisplayName','Im \lambda_2');
 hold off; grid on; ylim([-4 4]); xlabel('k');
 lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
 drawnow;
%
% Fin
```

□

Question 5. Pattern-forming instability.

Proof. We can observe numerically from Figure 2 that $\Re(\lambda_2(k, 8)) < 0$ so that it is a stable equilibrium and that $\Re(\lambda_2(k, 7)) > 0$ which implies that it is an unstable equilibrium.

Moreover, it can be shown that the function $\Re(\lambda_2(k, \alpha))$ is continuous for values of $\alpha = [7, 8]$, since λ_2 is a polynomial function. Hence by the Intermediate value theorem there must exits an $\alpha_* \in [7, 8]$ and k_* such that $\Re(\lambda_2(k_*, \alpha_*)) = 0$. Namely that α_* is a critical value.

We see that for $\alpha \leq \alpha_*$ there emerges a propagating wave pattern instability. This is since there exist both $\pm k_*$ such that $\Re(\lambda(\pm k_*, \alpha_*)) = 0$ while also we have $\Im(\lambda_2(\pm k_*, \alpha_*)) \neq 0$.

Hence we can compute the wave length by $\Delta = 2\pi/k_*$ so in the case when $\alpha = 7$ we can use matlab to numerically find $k_* = 0.8038$ which yields a wave length of $\Delta = 7.8166$. The speed of the propagating is computed by $c = -\Im(\lambda_*)/k_*$, here λ_* is the maximum value that the λ function obtains. In the case of $\alpha = 7$ we have $\Im(\lambda_*) = -1.6169$, thus we have a travelling wave speed of $c = 2.0115$.

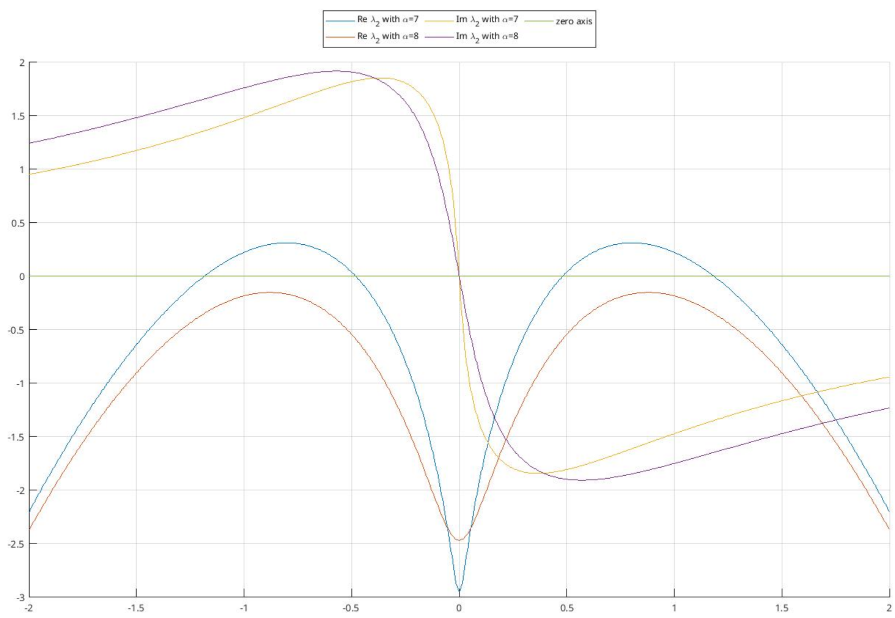

Figure 2: Graphical plots of real and imaginary parts of $\lambda_2(k)$ for parameters $\alpha = [7, 8]$, $\beta = 20$ and $\gamma = 2$.

```
%% Question 5 - pattern-forming instability
% Clear
clear all, close all, clc
```

```matlab

% functions
mu      = @(a,c) (a+sqrt(a^2-4.*c^2))/(2.*c);
tau    = @(k,a,b,c) -1 +c + i.*b.*k-k.^2-mu(a,c).^2;
delta   = @(k,a,b,c) k.^(2).* (1- i.*b.*k + mu(a,c).^2) + c.*(-1+i.*b.*k+mu(a,c).^2
lambda1 = @(k,a,b,c) (tau(k,a,b,c)-sqrt(tau(k,a,b,c).^2 - 4.*delta(k,a,b,c)))/2;
lambda2 = @(k,a,b,c) (tau(k,a,b,c)+sqrt(tau(k,a,b,c).^2 - 4.*delta(k,a,b,c)))/2;

rel1 = @(k,a,b,c) real(lambda1(k,a,b,c));
rel2 = @(k,a,b,c) real(lambda2(k,a,b,c));
imag1 = @(k,a,b,c) imag(lambda1(k,a,b,c));
imag2 = @(k,a,b,c) imag(lambda2(k,a,b,c));

% set parameters,
p=[20,2]; % beta , gamma
alpha=[7,8]; %alpha
k=linspace(-2,2,1000);

%
 figure, hold on;
 plot(k,rel2(k,alpha(1),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=7');
 plot(k,rel2(k,alpha(2),p(1),p(2)),'DisplayName','Re \lambda_2 with \alpha=8');
 plot(k,imag2(k,alpha(1),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=7');
 plot(k,imag2(k,alpha(2),p(1),p(2)),'DisplayName','Im \lambda_2 with \alpha=8');
 plot(k,zeros(1,1000),'DisplayName','zero axis');
 hold off; grid on; ylim([-3 2]); xlabel('k');
 lgd = legend; lgd.Location = 'northoutside'; lgd.NumColumns = 3;
 drawnow;

% Find when k such that lambda 2 obtains local max on [0,2]
maxrel1 = @(x) rel2(x,alpha(1),p(1),p(2));
[~,kMax] =fminmax(maxrel1 , 0, 2);
lambda2Max = lambda2(kMax, alpha(1),p(1),p(2));

% Wave length
waveLength = 2*pi / kMax
% Wave propagation
wavePropagation = - imag(lambda2Max) / kMax
% function to find max and min of a function
function [min, max] = fminmax(f, lowerbound, upperbound)
    min = fminbnd(f, lowerbound, upperbound);
    max = fminbnd(@(x) -f(x), lowerbound, upperbound);
end
```

□

Question 6.

Proof. For this question we fix the parameters to $(7, 20, 2)$, while we apply the Periodic boundary conditions with setting $L = 15.7$ and the initial conditions of

$$\begin{pmatrix} v(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} v_{*,3} \\ w_{*,3} \end{pmatrix} + \begin{pmatrix} \cos(4\pi x/L) \\ 1 \end{pmatrix}.$$

Using numerically methods and matlab we can obtain the following plots, which shows the existence of a propagating wave. In Figure 4 we see that peaks of the waves occur at $x_1 = 8.52$ and $x_2 = 0.6908$, which gives $\Delta = |x_1 - x_2| = 7.8292$. We see this value is close to the one computed in Question 5, which confirms our findings. Moreover, we also observe that in Figure 3 that we have a dialogical lines, namely that we have periodic in both x and t, this implies that there exists a propagating wave, which we computed as in Question 5.

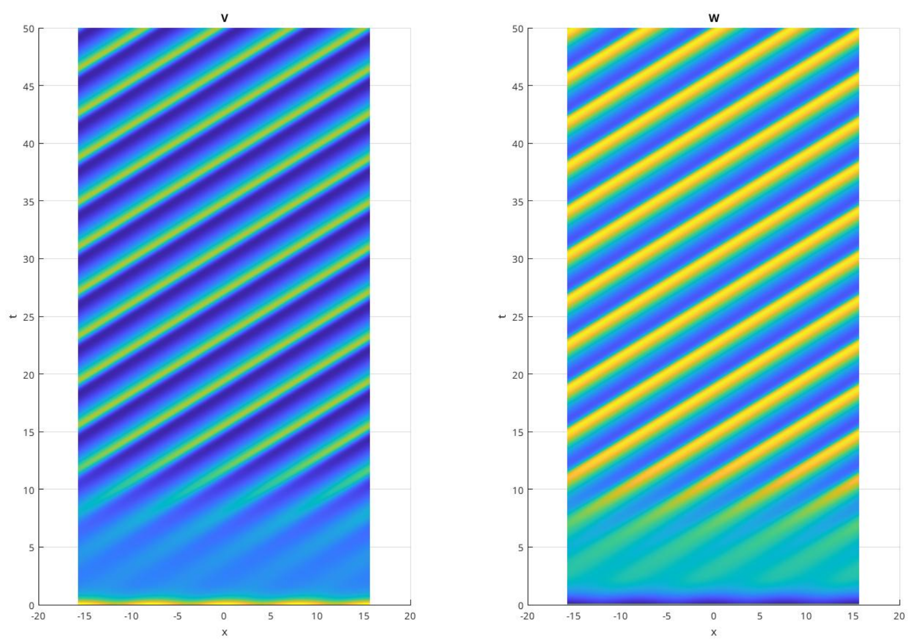

Figure 3: 3D plot of the tiger bush Pattern-forming, in time, space and density. For parameters $(7, 20, 2)$, $L = 15.7$.

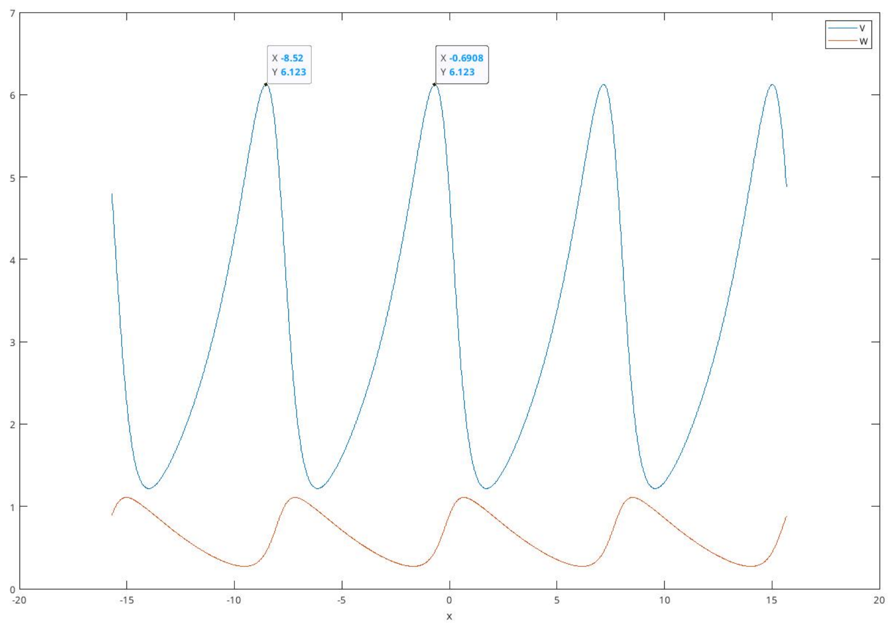

Figure 4: Wave shapes of the Tiger bush pattern. Here we also mark two consecutive peaks in order to measure the wave length. For parameters $(7, 20, 2)$, $L = 15.7$.

```
%% Question 6
%clear
clear all, close all, clc

% define parameters
L=15.7;
p=[7,20,2];

% Instantiating periodic differentiation matrix
nx = 1500; [x,Dx,Dxx] = PeriodicDiffMat([-L,L],nx);

% % Initial condition (steady state + perburbation)
e = ones(size(x)); z0 = [p(1)*e; e/p(1)];
z0 = z0 + 1*[cos(4*pi/L*x); e];

% Time step
rhs = @(t,z) tigerbush(z,p,Dx,Dxx);
jac = @(t,z) tigerbushJacobian(z,p,Dx,Dxx);
opts = odeset('Jacobian',jac);
tSpan = [0:0.1:50];
[t,ZHist] = ode15s(rhs,tSpan,z0,opts);

% Space-time plot
 PlotHistory(x,t,ZHist,p,[]);
%
% Plot final state
figure; title('Final state');
plot(x,ZHist(end,1:nx),x,ZHist(end,nx+1:2*nx));
xlabel('x'); legend({'V','W'});


%% Users functions
function [F, DFDZ] = tigerbush(z, p,Dx, Dxx)

      % Rename parameters
      a =p(1); b=p(2); c=p(3);

      % Ancillary variables and solution split
      nx=length(z)/2;
      iU=1:nx;
      iV=nx+iU;
      v = z(iU);
      w = z(iV);

      % Function handles
      f     = @(v,w) - v.*c + w.*v.^2;
      dfdv  = @(v,w) -c + 2.* w.*v;
      dfdw  = @(v,w) v.^2;

      g     =@(v,w) a-w-w.*v.^2;
```

```
      dgdv  =@(v,w) -2.*w.*v;
      dgdw  =@(v,w) -1-v.^2;

      % Right-hand side
      F=zeros(size(z));
      F(iU) = Dxx*v + f(v,w);
      F(iV) = b*Dx*w + g(v,w);

  if nargout > 1
    DFDZ = spdiags([],[],2*nx,2*nx);
    DFDZ(iU,iU) =   Dxx + spdiags(dfdv(v,w),0,nx,nx);
    DFDZ(iU,iV) =         spdiags(dfdw(v,w),0,nx,nx);
    DFDZ(iV,iU) =         spdiags(dgdv(v,w),0,nx,nx);
    DFDZ(iV,iV) = b*Dx + spdiags(dgdw(v,w),0,nx,nx);
  end
end

function DFDZ = tigerbushJacobian(z,p,Dx, Dxx)
      [~, DFDZ ] =tigerbush(z,p,Dx,Dxx);
end
function [x,Dx,Dxx] = PeriodicDiffMat(xSpan,nx)

      % Gridpoints
      a = xSpan(1); b = xSpan(2);
      hx = (b-a)/nx;
      x = a+[0:nx-1]'*hx;

      % Auxiliary vecor
      e = ones(nx,1);

      % First order differentiation matrix
      Dx = spdiags([-e e],[-1 1],nx,nx);
      Dx(1,nx) =  -1; Dx(nx,1) = 1;
      Dx = Dx/(2*hx);

      % Second order differentiation matrix
      Dxx = spdiags([e -2*e e],-1:1,nx,nx);
      Dxx(1,nx) = 1; Dxx(nx,1) = 1;
      Dxx = Dxx/(hx^2);

end

function plotHandle = PlotHistory(x,t,U,p,parentHandle)

  numComp = 2;
  nx = size(U,2)/2;

  %% Assign solution label
  solLabel(1).name = "V";
  solLabel(2).name = "W";
```

```
    %% Position and eventually grab figure
    if isempty(parentHandle)
      %scrsz = get(0,'ScreenSize');
      % plotHandle = figure('Position',[scrsz(3)/2 scrsz(4)/2 scrsz(3)/4 scrsz(4)/4]
      plotHandle = figure();
      parentHandle = plotHandle;
     else
       plotHandle = parentHandle;
    end
    figure(parentHandle);

    %% Grid
    [T,X] = meshgrid(t,x);

    %% Plots
    for k = 1:numComp
      subplot(1,numComp,k)
      % pcolor(X,T,U(:,idx(:,k))'); shading interp; view([0 90]);
      surf(X,T,U(:,nx*(k-1)+[1:nx])'); shading interp; view([0 90]);
      title(solLabel(k).name);
      xlabel('x'); ylabel('t');
    end

    %% Save
    % print -dtiff history.tiff

end
```

□