Ramsey Theory

An introduction to classic results

Carmen Oliver
2661615

James Zoryk
2663347

Supervisor: Dr.Nirvana Coppola.

Bachelor Mathematics
Project Pure Mathematics

Department of Mathematics
Faculty of Sciences
Vrije Universiteit
Amsterdam, Netherlands

29 June 2021

Abstract

The purpose of this paper is to give an introduction to Ramsey theory results, of which we state the classic theorems, give suitable proofs for such and explicit examples where possible.

The first section of this paper describes a well know problem which is refereed to as the Party Problem, which naturally leads our discussion into the following section that provides details about the fundamentals of Ramsey Theorem. In the subsequent sections we extended the fundamental ideas of Ramsey theory to that of hypergraphs and for the finite and infinite version of the Ramsey theorem. We also explore the results of Schür, with his proof of Fermat's last Theorem in the case of finite groups. Then extending the ideas of Ramsey theory to arithmetical progressions, by stating the work of B.L van der Waerden. Lastly, with the ideas of topology we show the compactness Theorem.

Ramsey Theory is a subfield of combinatorics, named in honor of Frank Ramsey (1903 to 1930), who published a paper [Ramsey, 1930] on set theory that generalised the Pigeon Hole Principle (PHP). In the years since the publication the whole subject formed one of the most developed combinatorial theories which transcendents by far the original motivations.

Contents

1 Introduction

Ramsey theory, named after the British mathematician and philosopher Frank P. Ramsey, studies the appearance of order in a substructure given a structure of a known size. The question might be "how big must some substructure be to guarantee that a particular property holds".

1.1 Pigeonhole Principle

Theorem 1.1. (The Generalised Pigeonhole Principle) If N objects are placed into k boxes then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof. Suppose that none of the boxes contain more than than $\lceil N/k \rceil - 1$ objects. Using then inequality $\lceil N/k \rceil < (N/k) + 1$, it then follows that the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} - 1 \right\rceil\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N.$$

This is a contradiction since we have a total of N objects. Hence the statement holds true. □

In the next example, we shall utilise the Generalise Pigeonhole Principle and apply it to an important part of combinatorics which is referred to as Ramsey Theory.

1.2 Party Problem

Let us examine the famous problem referred to as the Party Problem or sometimes called the Friends and Strangers Theorem.

Example 1.1. Assume that in a group of six people, each pair of individuals consists of two friends or two strangers. Show that there are either three mutual friends or three mutual strangers in the group.

Proof. Let A be an arbitrary person of the group of six people. Of the remaining five other people in the group, there are either three or more who are friends of A, or three or more who are strangers of A. This follows from the The Generalised Pigeonhole Principle [1.1], as when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements.

In the former case, let us suppose that B, C, D are friends of A, this is illustrated in Figure [1]. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise B, C, D form a set of three mutual strangers. The proof in the latter cases, when there are three or more strangers of A, follows the same logic.

Therefore in a group of six people, there will be at least either a group of three mutual friends or a group of three mutual strangers. □

Now the question arises is this property particular to a group of six people or can it hold in the cases when the group size is less than six.

Proposition 1.1. A group of six people is the minimum amount needed for Example [1.1] to hold true.

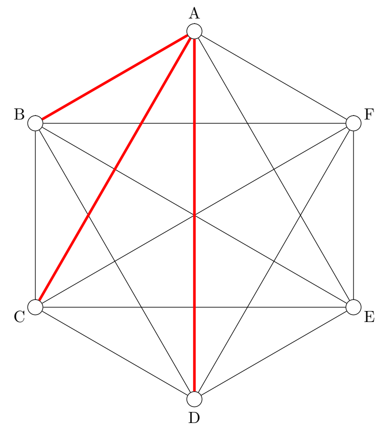

Figure 1: Diagram of K_6, with 3 edges coloured red with respect to the vertex A. Showing that we will have a monochromatic clique K_3.

Proof. This can been shown by counter example with five people, consider the example when (A, B), (B, C), (C, D), (D, E) and (A, E) are friends, with the remainder strangers. This is shown in Figure [2]. Then it follows that there is no mutual connection of either three friends or three strangers.

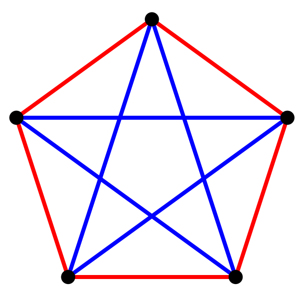

Figure 2: Diagram of K_5, showing a colouring that does not hve monochromatic clique K_3. Hence this is a counter example for $R(3, 3) = 5$.

Hence, six people is the smallest group for which a subgroup of three people are either friends or strangers. □

The Ramsey number $R(s, t)$, where s and t are positive integers that are greater or equal to 2, denotes the minimum number of people at a 'party' such that there are either s 'mutual friends' or t 'strangers', here assuming that every pair of 'people' at the party are fiends or strangers.

These seems like a reasonable statements, but we want to prove them in a more mathematical manner, in order to explore the properties in further details.

1.3 Analysis of the problem

When we consider the party problem in more mathematical sense, we observe that the problem of the six people and their knowledge (or their lack of knowledge) of each other is similar to a complete graph K_6, where the people represent the vertices of the graph while the edges denote the relationship between the given people. Since we are dealing with a complete graph, it is know that the number of edges is given by

$$\#\ \text{edges}\,(K_n) = \frac{n(n-1)}{2}.$$

In our party problem example we see that there is a total of 15 edges. These edges represents the relationship of a pair of vertices either being a friends or strangers, which we can denote with a specific colour. Then with the assistance of combinatorics, there are two possible choice for each edge in the graph, we have that the total number of possible colourings are

$$\#\ \text{possible graphs}\,(K_6) = 2^{15} = 32,768$$

In general, if the number of colours is denoted by k and number of edges by e, then the total number of possible number of configurations of the graph is k^k.

We quickly see that even in the simple example of K_6 the number of configurations is large and will grow at alarming rate as the number of vertices is increased. However, we would like to be able to say something about the generalisation of these complete graphs and whether they contain monochromatic subgraph. As we have seen in the example, K_6 with a two-colouring, will contain at least a K_3 subgraph of either colour.

At this point I hope that the reader begins to understand the complexity that is involved in answering such a question for larger graphs and with an increasing number of colouring choices. In the next section we shall begin to generalise this concept, from which we can introduce new concepts.

2 Ramsey Numbers (Two-colours)

2.1 Introduction

In the language of graph theory the definition of the Ramsey Number that was described in the Party Problem can be given as the following.

Definition 2.1. **(Ramsey Property)** Let s and t be integers with $s, t > 1$. Then a positive integer r has the (s, t) Ramsey property, if for any the colouring of the edges of K_r, then the graph contains a monochromatic clique H of either colour, i.e: red $H = K_s$ or blue $H = K_t$.

Definition 2.2. **(Ramsey Number)** The Ramsey Number r is the minimum number of vertices $r = R(s, t)$ such that K_r contains a monochromatic clique of order s, or of order t.

More specifically the Ramsey Number for a two-colour problem, written as $r = R(s, t)$, is the smallest integer n such that the 2-coloured graph K_r, using the colours green and black for edges, contains a blue monochromatic subgraph K_s or a red monochromatic subgraph K_t.

We will later on prove that indeed this number exists and it is a natural finite number. That is, we will prove Theorem [2.1] (Ramsey's Theorem). But first, we need to describe this number and its properties further.

2.2 Basic Properties of Ramsey Numbers.

Proposition 2.1.

(1) For all positive integers $t > 1$, then $R(2, t) = t$.

(2) If there exists $R(s, t)$, then there also exists $R(s, t)$, and $R(s, t) = R(s, t)$.

(3) If $m = R(s, t)$ then every integer $n > m$ has the Ramsey (s, t)-property.

(4) Similarly, if an positive integer m does not have the Ramsey (s, t)-property, then neither does any integer $n < m$.

(5) For all positive integers s_1, s_2 with $s_1 \geq s_2 > 1$, we have that $R(s_1, t) \geq R(s_2, t)$.

Proof. (1) Consider $K_{(t-1)}$ in which every edge is coloured red. In this case there is neither a blue edge nor a complete red K_t. Observe that the the complete graph K_2 consist of two vertices and one edge. This implies that $R(2, t) > t - 1$. Now consider any graph with t vertices.

We can have two scenarios; There is at least one edge coloured blue, which means we have a red pair of vertices.Else, all edges are red which means we have a complete red K_t.

(2) Consider a K_2 graph say G, its inversely 2-coloured complete graph is $G^{'}$ such that any red edge in G will be coloured blue in $G^{'}$ and vice versa.

We know that $R(s, t)$ requires that any edge colouration of $K_{R(s,t)}$ will have a blue monochromatic subgraph K_s or a red monochromatic subgraph K_t. Similarly the inversely 2-gon coloured graph will have a blue monochromatic subgraph K_t or a red monochromatic subgraph K_s.

Since the inverses of all edge colouring are just all edge colouring, we have the equivalent conditions for R(t, s).

□

The proof for the remaining $3 - 5$ properties following directly from the definition and theorem for Ramsey Numbers and will be omitted here.

2.3 Existence and Bounds of Ramsey Numbers.

2.3.1 Existence

In the examples presented until now the computation of the Ramsey number was not difficult. Let us now consider the case for large values of s and t. A natural question is whether there actually exists an r in $\mathbb{R}$ such that $r = R(s, t)$. As we observe that r becomes more challenging to obtain as s and t are increased. That is why we make use of upper and lower bounds, some of which are in terms of other Ramsey numbers. The first upper bound we will introduce was proved in 1955 by Greenwood and Gleason, for more details see [Greenwood and Gleason, 1955].

Lemma 2.1. Let s and t be integers with $s, t > 1$ then it follows that

$$R(s,t) \leq R(s-1,t) + R(s,t-1). \tag{1}$$

Proof. Let $m = R(s-1,t) + R(s,t-1)$ and consider the complete graph K_m, which has m vertices with edges two-coloured. Now pick an arbitrary vertex $v \in K_m$,then partition the remaining vertices into two sets , as follows

$$\begin{aligned} A &:= \{w \mid \forall w \in V(K_m) \setminus \{v\}, \text{ edge } (w,v) \text{ is blue.}\}, \\ B &:= \{z \mid \forall z \in V(K_m) \setminus \{v\}, \text{ edge } (z,v) \text{ is red.}\}. \end{aligned}$$

Furthermore, as all the vertices other than v belong to either A or B, we obtain

$$|A| + |B| + 1 = R(s-1,t) + R(s,t-1) = m.$$

It follow that we either have

$$\begin{cases} |A| \geq R(s-1,t), \text{ then } |B| < R(s,t-1), \\ |B| \geq R(s,t-1), \text{ then } |A| < R(s-1,t). \end{cases}$$

Otherwise we would have $|A \cup B| + |\{v\}| < m$, which is clearly a contradiction.

In the first case, $|A| \geq R(s-1,t)$, then $|B| < R(s,t-1)$, we consider A. In A there is either a red monochromatic clique with $s-1$ vertices or a blue monochromatic clique with t vertices. In the form case consider the set $\{K_{s-1} + v\}$ which is red monochromatic clique with s vertices.

Now, if $|B| \geq R(s,t-1)$, then $|A| < R(s-1,t)$, we observe that B contains either a red monochromatic clique with s vertices or a blue monochromatic clique of $t-1$ vertices. In the former case consider the set $\{K_{t-1} + v\}$, which is a blue monochromatic clique of t vertices.

Hence the inequality holds in both cases and thus proving the Lemma.

□

This now leads us to show the existence of Ramsey Numbers for two-coloured completed graphs.

Theorem 2.1. Let s and t be integers with $s, t > 1$, then there exists a positive integer r with $R(s,t) = r < \infty$.

Proof. In order to prove this statement we will use double mathematical induction on s and t. First, we see that this theorem is satisfied in the base case $s = t = 2$ as $R(2,2) = 2$.

Now, assume that $R(s-1,t)$ and $R(s,t-1)$ exists, then we must have that $R(s-1,t) + R(s,t-1)$ exists. Thus, by lemma 1 we can bound $R(s,t)$ such that $R(s,t) \leq R(s-1,t) + R(s,t-1)$ which implies that $R(s,t)$ exists.

Thus by induction, $R(s,t) < \infty$ for all $s, t \in \mathbb{N}$.

□

Form the results in the proof of Theorem [2.1], it can be extend to from an alternative upper bound for $R(s,t)$. Here we use $\binom{N}{k}$ to denote the binomial coefficient. Furthermore in the induction step, we show that $R(s-1,t) \leq \binom{s+t-3}{s-2}$ and similarly $R(s,t) \leq \binom{s+t-3}{s-1}$. Then adding these with Pascal's Triangle Rule, we have

$$\binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$

Therefore, we can then conclude by induction that $R(s,t) \leq \binom{s+t-2}{s-1}$ is an upper bound for all s and t.

Corollary 2.1. Let s and t be integers with $s, t > 1$, then

$$R(s,t) \leq \binom{s+t-2}{s-t}.$$

2.3.2 Other bounds for Ramsey Numbers.

Again the computation of the Ramsey Numbers is usually almost impossible. In the case of the diagonal Ramsey number of the form $R(k,k)$, we have that $R(k,k) \leq \binom{2k-2}{k-1}$. This bound is only useful if k is not large. We can define different bounds for specific cases that can help us compute certain Ramsey numbers that would otherwise be very difficult.

Theorem 2.2. $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$ whenever $R(s-1,t)$ and $R(s,t-1)$ are both even.

Proof. Take K_n. Fix a vertex v, $R(s-1,t) = n_1$ and $R(s,t-1) = n_2$ where $n_1, n_2 \in \mathbb{N}$ are even. Set $n = n_1 + n_2 - 1$. So v is connected to exactly $n - 1 = n_1 + n_2 - 2$ edges. There are three different cases to consider:

1. n_1 or more edges end at v
2. n_2 or more edges end at v
3. $n_1 - 1$ blue edges and $n_2 - 1$ red edges end at v.

First case, we consider the set T_1 of the vertices connected to v by the n_1 or more edges. So there must be either a blue K_{s-1} or a red K_t. But if there is a blue K_{s-1} then we can form K_s with the vertex v. So the theorem holds.

In the second case a similar argument holds.

In the third case, by contradiction we assume that all vertices of K_n have $n_1 - 1(blue) + n_2 - 1(red)$ vertices that are end points with multiplicity, that is a total number of $n(n_1 - 1)$ but this number is odd, so we have reached a contradiction. So at least one does not have this property, so this case is not possible and the statement holds. □

2.3.3 Lower bounds.

Erdős' lower bound proof for two colour Ramsey Number has the interesting feature that it never presents a specific colouring. This has been noted as one of the first occurrence to incorporate the Probabilistic Method in the field of combinatorics. Simply stated the Probabilistic Method says; 'If, in a given set of objects, the probability that an object does not have a certain property is less than 1, then there must exist an object with this property.' See [Aigner et al., 2010]. Namely, this work is about showing that for an as large as possible $n < R(s, s)$ that there exists a colouring of the edges of K_n such that no red or blue cliques of order s are present.

Theorem 2.3. (Erdös-1947) For all natural numbers $s \geq 3$ the following inequality holds for the lower bound

$$2^{(s/2)} < R(s, s).$$

Proof. In order to show that $n < R(s, s)$, it is sufficient to show that there exists a colouring of the edges of K_n that contains no monochromatic K_s.

Note that $R(3, 3) = 6 \geq 2^{3/2}$. Let us define $n \leq 2^{(s/2)}$ and consider all the two colouring's possible on the edges of K_n. Assuming that the choice of colour for each edges is independent, so that

$$\mathbb{P}(\text{edge is red}) = \mathbb{P}(\text{edge is blue}) = \frac{1}{2}.$$

Moreover, the probability that all the edges of the vertices in A are monochromatic is $2/2^{\binom{s}{2}}$. Then it follows that the probability that K_n has a monochromatic s clique is bounded by

$$\mathbb{P}\left(\cup_s A\right) \leq \sum_s \mathbb{P}\left(A \,\text{is monochromatic}\right) = \binom{n}{s} \frac{2}{2^{\binom{s}{2}}}.$$

Using $n \leq 2^{s/2}$ and the Binomial coefficient bound

$$\binom{n}{s} \leq \frac{n^s}{s!},$$

yields the following results for all $s \geq 3$,

$$\begin{aligned}
\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} &\leq \frac{n^s}{s!} 2^{1-s(s-1)/2}, \\
&\leq \frac{\left(2^{s/2}\right)^s}{s!} 2^{1+s/2-s^2/2}, \\
&= \frac{2^{1+s/2}}{s!}, \\
&< 1.
\end{aligned}$$

Therefore, we have shown that probability of s vertices having monochromatic edges is $p < 1$. Thus there must be a colouring of K_n with no monochromatic clique of order s. Namely that K_n does not have the Ramsey property (s, s). Therefore we conclude that $R(s, s) > 2^{2/s}$. □

With the use of the Sterling's Formula, Erdős' lower bound for $R(s, s)$ can be improved to the lower bound of $s2^{s/2}$.

2.4 Examples

At this point in the paper we can begin to compute certain examples. We look at cases of the type $R(3,t)$, Many mathematicians have been working in Ramsey numbers of that type but have only reached bounds. See Table 1 below.

However there are some examples that we can indeed compute; such as $R(3,4)$ and $R(3,5)$.

Theorem 2.4. R(3,4) = 9 and R(3,5) = 14

Proof. We have that

$$R(3,4) \leq R(2,4) + R(3,3) - 1 \leq 9$$

as R(2,4)=4 (by property 2) and R(3,3)=6 (by the theorem of Friends and Strangers).

We also have that $R(3,5) \leq R(2,5) + R(3,4) \leq 14$.

Need to prove $R(3,5) \geq 14$. We claim that $R(3,5) > 13$. Now it is left to prove that $R(3,5) \neq 13$. This is done with a counterexample that we will leave for the reader to compute.

From this we conclude that indeed $R(3,5) \geq 14$ and also $R(3,5) \leq 14$ so it must be equal to 14, then we must have $R(3,4) = 9$ too. □

r/l	2	3	4	5	6	7	8	9	10
2	2								
3	3	6							
4	4	9	18						
5	5	14	25	43-48					
6	6	18	36-41	58-87	102-165				
7	7	23	49-61	80-143	115-298	205-540			
8	8	28	59-84	101-216	134-495	217-1031	282-1870		
9	9	36	73-115	133-316	183-780	252-1713	329-3583	565-6588	
10	10	40-42	92-149	149-442	204-1171	292-2826	343-6090	581-12677	798-23556

Table 1: Two-colour Ramsey Numbers and best estimate for bounds of small values of r and l. Note that the table is symmetrical across the diagonal. For further details see [N.J.A.Sloane, 2001] and [Radziszowski, 2011].

2.5 Ramsey Number Generalised. (k-colouring).

2.5.1 Introduction

Now that we have a general understanding of Ramsey numbers and its bounds we want to make sure that indeed this number exists and its defined in all cases.

We define $R(s_1, \ldots, s_n)$ as the minimum $r \in \mathbb{N}$ such that for each n-colouring of the complete graph K_r, it contains, for at least one $i \in \{1, \ldots, n\}$, a K_{s_i} isomorphic subgraph completely coloured with the i-th colour.

Theorem 2.5. (Ramsey's theorem) We claim that $R(s_1, \ldots, s_n)$ is well defined for every n and for every $s_1, \ldots, s_n$.

Proof. To prove this theorem we use induction on the number of colours k. The base case is when $n = 2$, which is simply Theorem 2.1.

For the inductive step, we observe that from a n-colouring we can obtain a (n - 1)-colouring by unifying the two colours c_1 and c_2, i.e. assigning a new colour c_0 to all the arcs that were previously coloured with c_1 or c_2 and leaving the others unchanged.

By inductive hypothesis, there exists $R(v)$, where v is any $(n-1)$- tuple of a natural number (greater than or equal to 2).

In particular, there exists

$$r = R\left(R(s_1, s_2), s_3, \ldots, s_n\right)$$

that is, K_r contains either a monochromatic subgraph K_{s_i} all coloured with c_i, for some $i = 3, ..., k$, or it contains a K_m, where $m = R(s_1, s_2)$, monochromatic with c_0.

Now, by breaking c_0 again into the two components c_1 and c_2, we have a 2-colouring of K_m and m is precisely the Ramsey number relative to s_1 and s_2: we can find inside it a monochromatic K_{s_1} of colour c_1 or K_{s_2} of colour c_2. In any case, we have shown that r matches the condition of the definition of a Ramsey number; this proves that $R(s_1, \ldots, s_n)$ exists (and is less than or equal to r). □

Here we will show one example of a non-trivial k-colour Ramsey Number. Until 2016 the only prove k-colouring Ramsey Number was $R(3, 3, 3) = 17$. However, it has been recently shown that $R(3, 3, 4) = 30$, accomplished by the efforts of Codish, Frank, Itzhakov and Miller, as for some time it has been bound by $30 \leq R(3, 3, 4) \leq 31$, see [Codish et al., 2016].

Theorem 2.6. $R(3, 3, 3) = 17$

Proof. Let K_n be a complete graph of order n. Then suppose that you colour the edges of K_n with three colours, namely red, yellow and green, such that there does not exist a monochromatic clique of order 3 of any colour.

Now, choosing a vertex $v \in K_n$, we shall consider the following three sets of vertices defined as

$$\begin{cases} A := \{u \mid \forall u \in V(K_n) \setminus \{v\}, \text{ edge}\,(u, v) \text{ is red.}\}, \\ B := \{w \mid \forall w \in V(K_n) \setminus \{v\}, \text{ edge}\,(w, v) \text{ is yellow.}\}, \\ C := \{z \mid \forall z \in V(K_n) \setminus \{v\}, \text{ edge}\,(z, v) \text{ is green.}\}. \end{cases}$$

Observe that the set A does not contain any red edges between the vertices, otherwise we would have monochromatic clique of order 3. Therefor A contains vertices with edges of colour yellow and green. Thus, A must contains at most 5 vertices, otherwise 6 vertices implies that $R(3, 3)$. A similar argument can be used for the sets B and C. Therefore, K_n has a most

$$|\{v\}| + |A| + |B| + |C| = 1 + 5 + 5 + 5 = 16$$

vertices. From which we can conclude that $R(3, 3, 3) \leq 17$.

In order to show that $R(3, 3, 3) = 17$, it is left to prove that $R(3, 3, 3) \geq 17$. This is achieved by a counter example, namely showing that the graph K_{16} can be coloured with 3 colours such that there are no monochromatic cliques of order 3. There are exactly 2 possible colouring of K_{16}, in which this can occurs. We will omit the drawings of these two graphs due to the complex nature of them, for detailed draws please see [Radziszowski, 2011].

Thus, we can finally conclude that $R(3, 3, 3) = 17$. □

The reader should note that there are $3^{\frac{16(15)}{2}} = 1.79 \times 10^{57}$ possible permutation of K_{16}, with 3 colours. In this example the probability that choosing a colouring of K_{16} at random that would produce a counterexample is $2/3^{\frac{16(15)}{2}}$. This shows how difficult it is to find counterexamples for larger graphs.

3 Hypergraphs

3.1 Introduction

Until now we have been working with graphs given by a collection of pairs i.e. edges connect exactly two edges. In this section we introduce the notion of a hypergraph, which is a more generalised version of a graph where edges can connect any number of vertices. In particular, if all hyperedges connect the same number of vertices, we call this a r-uniform hypergraph.

Here we use the notation of $\mathcal{P}_r(V)$, which mean we only consider sets of length r. We denote $\mathcal{P}_2(V)/\{0\}$ to be $\mathcal{P}_2(V) = \{\{i, j\} | i \neq j \in V\}$ a subset of $\mathcal{P}(V)$ with cardinality two.

Definition 3.1. A hypergraph is a pair (V, E) in which V is an arbitrary set of vertices and $E \subset \mathcal{P}/\{\emptyset\}$ is called the set of hyperedges. When all elements of E have the same order we call this an r-uniform hypergraph. (As we have mentioned before).

A complete r-uniform hypergraph is $K_n^{(r)} = (V, \mathcal{P}_r(V))$ with $|V| = n$. In the previous presented cases we have defined the complete graph as $K_n^{(2)}$.

Definition 3.2. The hypergraph Ramsey number $R^{(r)}(s, t)$ is the minimum number n such that any r-uniform hypergraph on n vertices contains a clique of size s or a clique of size t.

In a similar manner as presented in the previous section, the definition of a hypergraph Ramsey Number can be extended beyond just two colours, which yields the following generalised Ramsey Number definition of k-coloured hypergraphs.

Definition 3.3. The Ramsey number $R^{(r)}(s_1, \ldots, s_k)$ is the minimum number n such that any coloring of the edges of the complete hypergraph $K_n^{(r)}$ with k colours contains a clique of size s_i whose edges all have colour i, for some $i \in \{1, \ldots, k\}$.

The existence of such a number is not obvious. Next we shall introduce two theorems that implies the existence of Ramsey Number in both a finite and infinite perspective. Firstly we state the Finite Ramsey Theorem.

3.1.1 Finite Ramsey theorem

Theorem 3.1. For all $m, r, k \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that for all k-colouring of $\mathcal{P}_r(\{1, \ldots, n\})$ there exists $H \subset \{1, \ldots, n\}$ such that $|H| = m$ with $\mathcal{P}_r(H)$ monochromatic i.e. $R^{(r)}(m, \ldots, m)$ exists.

Now we can introduce the Infinite Ramsey Theorem, which we will also prove. This theorem will tell us something about the nature of infinite sets, and how we can also understand them through Ramsey Theory with the help of this following theorem.

3.1.2 Infinite Ramsey theorem

Theorem 3.2. For all k-colouring on $\mathcal{P}_r(A)$ there exists $X \subset A$ where X is infinite such that $\mathcal{P}_r(X)$ is monochromatic i.e. $R^{(r)}(m, \ldots, m)$ exists.

Proof. We prove this theorem by induction on r. Let $r = 1$ and A be a countable set, so $\mathcal{P}_1(A) = A$ as they are the sigletons of A. Now consider the map

$$\mathbf{C} : \mathcal{P}_1(A) \to \{1, 2, \ldots, k\}$$

which is bijective. Since $\mathcal{P}_1(A) = A$ is infinite, any k-partition contains at least an infinite set.

Now we want to prove that if the theorem is true for $(r-1)$-uniform hypergraphs it is then also true for r-uniform hypergraphs.

Fix c a k-colouring on $\mathcal{P}_r(A)$. Take $A_0 = A$ then fix an element $x_0 \in A$. Call $B_1 = A_0 \setminus \{x_0\}$. Now take a hyperedge $\tau \in \mathcal{P}_{r-1}(B_1)$, then $|\tau \cup \{x_0\}| = r$. So we define a colouring c_1 on $\mathcal{P}_{r-1}(B)$ as $\mathbf{C}_1(\tau) = \mathbf{C}(\tau \cup \{x_0\})$.

By induction B_1 has a monochromatic infinite subset with respect to c_1 namely A_1 which means $\mathcal{P}_{(r-1)}(A_1)$ is monochromatic. We can do this again taking A_0 to be A_1. So we obtain $A = A_1 \supset A_1 \supset A_2 \supset \ldots$, which are all infinite such that $P_{(r-1)}(A_i)$ is monochromatic with respect to c_i and $Y = \{x_i\}_{i \in \mathbb{N}}$.

In particular $x_i \in A_{i-1} \setminus A_i$. We have that all x_i are all different and there are infinitely many of them, thus implying that Y is infinite. Since

$$\{x_{i_1}, \ldots, \{x_{i_r}\}\} = \{x_{i_1} \cup \{\{x_{i_2}, \ldots, \{x_{i_r}\}\}\}\},$$

then by the definition of $\mathbf{C}_{i_1}$ we can obtain the following expression

$$\mathbf{C}(\{x_{i_1}, \ldots, \{x_{i_r}\}\}) = \mathbf{C}_{i_1}(\{x_{i_2}, \ldots, \{x_{i_r}\}\}) = d_{i_1}.$$

From which we observe that $\{d_i \,|\, i \in \mathbb{N}\}$ is finite as it is contained in $\{1, \ldots, k\}$.

In particular, let d be one colour such that there exists infinitely many i with $d_i = d$. So now we know that this colour does indeed exists as we have an infinite set Y to which we can associate a $k-$colouring. Let $X = \{x_i \,|\, d_i = d\}$ (which is infinite) thus $\mathcal{P}_r(X)$ is monochromatic, precisely with colour d. □

4 Schur's Theorem.

4.1 Introduction

Ramsey theory for integers is about finding monochromatic subsets with a certain arithmetic structure. It starts with the following theorem of Schur, which turns out to be a simple application of Ramsey's Theorem for graphs.

In 1916 Schur considered the following problem, given a set of natural numbers X, is it possible to colour this set with k colours such that there exist monochromatic elements with $x + y = z$.

As an example lets consider a simple case $X_4 := \{1, 2, 3, 4\}$ with $k = 2$. We want to find monochromatic $x, y, z \in X$ such that $x + y = z$. We start by colouring 1 as blue, assuming that x, y are not distinct then we must colour 2 as red otherwise $1 + 1 = 2$. Furthermore, we then must colour 4 as blue to avoid $2 + 2 = 4$. Now we are left with 3 which can be coloured blue. Here we conclude there doesn't exist a colouring such that $x + y = z$ is monochromatic.

Lets now increase the order of the set so that we now have $X_5 := \{1, 2, 3, 4, 5\}$, still with $k = 2$. We begin the same, colouring 1 as blue, so then 2 is coloured red, it follows 4 is coloured blue. Here, we must colour 5 to red, to avoid a monochromatic subset $1 + 4 = 5$. However this leaves us to colour 3. We quickly observe that no matter what colour we choose for 3, we will end up with a monochromatic subset $x + y = z$.

Generalising these examples using Ramsey theory, is there an order $|X| = n$ that ensure that for some colouring k we have a monochromatic subset which $x + y = z$ holds.

Theorem 4.1. (Schür's Theorem) For any $k \geq 2$, there is $n > 3$ such that for any colouring of $\{1, 2, \ldots, n\}$, there are three integers x, y, z of the same colour such that $x+y = z$.

Proof. Let $R_k(3, 3, \ldots, 3) = n$, be the Ramsey Number such that for any k-colouring of complete graph K_n contains a monochromatic clique.

Given a colouring $\mathbf{C} : [n] \to [k]$, we can define an edge colouring of K_n: the colour of the edge (i, j) will be $\mathbf{C}'((i, j)) = \mathbf{C}(|j - i|)$.

By Theorem [2.5], Ramsey Theorem for graphs, there is a monochromatic clique (i, j, k). Assuming that $i < j < k$.

Then we set $x = j - i$, $y = k - j$ and $z = k - i$. So we have $\mathbf{C}(x) = \mathbf{C}(y) = \mathbf{C}(z)$ and $x + y = z$. □

4.2 Fermat's Last Theorem.

The purpose of such a problem purposed by Schur was to investigate Fermat's Last Theorem which states; for any natural number $k \geq 3$, the equation $x^k + y^k = z^k$ does not have any solutions in the natural numbers. A theorem that was conjectured by Fermat in 1637, with it only been proven in 1995 by Andrew Wiles. In Schur's 1916 work, he was able to prove that Fermat's Last Theorem does not hold for finite field $\mathbb{Z}_p$ for any sufficiently large prime number p, by using the Pigeon Hole Principle [1.1]. This can be considered one of the first results of Ramsey Theory. [Jukna, 2011].

Theorem 4.2. Schur 1916. For every $k \geq 1$ there exists a n such that for any prime $p > n$ the equation

$$x^k + y^k = z^k \pmod{n}$$

has a solution.

Proof. As p is prime it follows that the multiplication group $\mathbb{Z}_p^* = \{1, 2, \ldots, p - 1\}$ is cyclic of order $p - 1$, moreover it has a generation g. Therefor each element in $\mathbb{Z}_p^*$ can be expressed as $x = g^{mj+i}$ with $0 \leq i \leq m$.

Now we can colour all the elements of $\mathbb{Z}_p^*$ by k colours, where we set $c(x) = i$ if $x = g^{kj+i}$. Now by Schur's Theorem [4.1] for p sufficiently large, then there is a subset of elements x, y, z in $\mathbb{Z}_p^*$ such that $\mathbf{C}(x') = \mathbf{C}(y') = \mathbf{C}(z')$. Therefore, we have

$$x' = g^{kj_x+i}, \quad y' = g^{kj_y+i} \quad \text{and } z' = g^{kj_z+i}.$$

Which then yields the following expression

$$g^{kj_x+i} + g^{kj_y+i} = g^{kj_z+i}.$$

Now dividing this expression by g^i, lead us to set $x = g^{j_x}$, $y = g^{j_y}$ and $z = g^{j_z}$ to give the desired solution to $x^k + y^k = z^k \pmod{n}$. □

5 Van der Waerden Theorem

Here we introduce an important classical result, that has greatly assisted in the development of the subfield of Ramsey Theory. It was a conjecture originally purposed by the mathematician Schur, and was eventually proven a few years later by the Dutch mathematician Van Der Warden in 1927. The original statement has been modified to the following:

Theorem 5.1. (van der Warden 1927) Let $l, k \in \mathbb{N}$. There exists a number $w \in \mathbb{N}$ which depends on l and k such that for every k-colouring of $\{1, \ldots, w\}$, it contains a monochromatic Arithmetic Progression (AP) of l terms.

The minimum w that satisfies this theorem is called the Van der Waerden number associated with l and k and is denoted by $w = W(l, k)$.

Before proving such a statement, let's observe $W(l, k)$ for small values of l and k in order to better understand the intuition required for the proof.

The first is the trivial case of $l = 2$, which for any k in $\mathbb{N}$, by the Pigeon Hole Principle [1.1] it holds that $W(2, k) = k + 1$.

Theorem 5.2. $W(3, 2) \leq 325$.

Proof. Here, we claim that the upper bound is $W(3, 2) \leq 325$. Denote the natural number $1, 2, \ldots 325$ as $[1, 325]$, which we can partition into 65 equal blocks of length 5, namely $[1, 325] = \bigcup_{i=0}^{64} B_i$. Each block is the set $B_i = \{b_i + 1, b_i + 2, \ldots, b_i + 5\}$. As $k = 2$, each block B_i has $2^5 = 32$ possible colourings.

Considering the set of blocks, The Pigeon Hole Principle [1.1] implies that if we take 33 blocks then at least two blocks will be coloured the same i.e $c(B_s) = c(B_t)$, with $s < t$. Again, by The Pigeon Hole Principle, if we consider the first three elements of B_s it implies that at least two are monochromatic. If all three are monochromatic then we are done. So instead lets assume that only two are monochromatic, denoting these elements as $b_s + a_1$ and $b_s + a_2$ with $a_1 < a_2$. Let $a_3 = 2a_2 - a_1$, if $c(b_s + a_1) = c(b_s + a_2) = c(b_s + a_3)$, this would complete the proof as we have a monochromatic AP of length 3, else there are two possible cases to consider; As $c(B_s) = c(B_t)$ this implies that $c(b_s + a_3) = c(b_t + a_3)$, now setting $B_m = 2B_t - B_s$. Then if $c(b_s + a_3) = c(b_t + a_3) = c(b_m + a_3)$, then we are done. Else, if $c(b_s + a_3) = c(b_t + a_3) \neq c(b_m + a_3)$, then $c(b_s + a_1) = c(b_t + a_2) = c(b_m + a_3)$ holds.

Therefore we can conclude that $W(3, 2) \leq 5 \cdot 65 = 325$. □

The bound of 325 is an extremely bad upper bound for $W(3,2)$, with a little bit of work it can be shown that $W(3,2) = 9$. The proof above gives us a foundation to the generalise proof of the van der Waerden as we observe that the choice of block size being 5 was due $2\cdot W(3-1,2)-1 = 2\cdot W(2,2)-1 = 5$, hence setting each block as $\{1,2,\ldots,2W(3-1,2)-1\}$ ensures that there exist monochromatic AP of length $3-1=2$. In a similar manner, we see that the choice of number of blocks arises from $2\cdot W(3-1,2^5)-1 = 65$, which ensures that at least $3-1=2$ blocks have the same colouring, which in turn implies that we have an monochromatic AP of at least length 2.

To prove the general case of the van der Warden Theorem [5.1], we shall use double induction on the length l of the AP required, and also the number of colours k. We have to use that $W(l,k-1)$ exists and also $W(l-1,r')$ exist for all r' in $\mathbb{N}$.

Proof - (Theorem 5.1): Let $X_{l,m}$ be the set of all sequences $x \in \{0,1,2,\ldots,l\}^m$, with $x = (x_1,x_2,\ldots x_m)$, such that if $x_i = l$, then $x_j = l$ for all $j \leq i$.

We say two sequences x and x' in $X_{l,m}$ are l-equivalent if there exists i in $\{0,1,\ldots m\}$ such that $x_j \neq l$ and $x'_j \neq l$ for all $j < i$, and $x_j = x'_j = l$ for all $j \geq i$. Here, $X_{l,m}$ is partitioned into $m+1$ l-equivalent classes.

In order to simplify the proof of the van der Warden Theorem, we shall introduce a stronger claim that implies the van der Warden Theorem.

Claim $S(l,m)$: For $l,m \geq 1$ and k, there exists $n = \mathbf{N}(l,m,k)$ such that for every function $\mathbf{C} : [0,,\mathbf{N}(l,m,k)] \to [1,k]$, there exist positive integers $a, d_1, d_2, \ldots, d_m$ such that

$$\mathbf{C}\left(a + \sum_{i=1}^{m} x_i\, d_i\right)$$

is constant on all sequences in each l-equivalent classes of $X_{l,m}$.

This claim is stronger than van der Waerden Theorem and hence we will show that it implies the van der Waerden's theorem for l term AP.

Let $m = 1$, then there are two l-equivalent classes on $X_{l,1}$, namely

$$X_{l,1} = \begin{cases} (0),(1),(2),(3),\ldots,(l-1) \\ (l). \end{cases}$$

This implies that $\mathbf{C}(a) = \mathbf{C}(a+d_1) = \mathbf{C}(a+2d_1) = \ldots = \mathbf{C}(a+(l-1)d_1)$, which are constant.

Hence this claim implies van der Waerden theorem.

We will now use mathematical double induction on m and l to prove $S(l,m)$ holds for all $m,l > 1$ and therefore the van der Waerden Theorem.

For the base case, let $m = 1$ and $l = 1$. The equivalence class on $[0,1]$ is the trivial, as each is formed with only one element. Hence the colour mapping is constant on all sequences in each equivalent classes.

1. $S(l,m) \Rightarrow S(l,m+1)$.
   For a fixed value of k, let $M = \mathbf{N}(l,m,k)$ and $M' = \mathbf{N}(l,1,k^M)$, with setting $N = M\,M'$. Now suppose that $\mathbf{C} : ([1,N]) \to [1,k]$, and define $\mathbf{C}' : ([1,M']) \to [1,k^M$ with $\mathbf{C}'(x) = \mathbf{C}'(x')$ if and only if $\mathbf{C}(xM-j) = \mathbf{C}(x'M-j$ for all j in $\{0,1,\ldots M\}$.

   Now by the induction hypothesis there exists $a', d' > 0$ such that $\mathbf{C}'(a'+xd')$ is constant for all x in $[0,l-1]$. Define an interval $I = [a'M-(M-1), a'M]$. As $S(l,m)$ can be applied to the interval I then by the choice of M, there exist $a, d_2, \dots d_{m+1}$ with all sum $a + \sum_{i=2}^{m+1} x_i d_i$ for each x_i in $[0, l]$, in I with $\mathbf{C}(a + \sum_{i=2}^{m+1} x_i d_i)$ constant on l-equivalence classes. Now setting $d'_i = d_i$ for all i in the range $\{2, 3, \dots m+1\}$ and $d'_1 = d'M$. Therefore $S(l, m+1)$ holds.

2. $S(l,m)$ for all $m \geq 1 \Rightarrow S(l+1,m)$.
For a fixed value of k, let $N = \mathbf{N}(l,k,k)$, suppose that $\mathbf{C} : [1,N] \to [1,k]$. Then there exist by the induction hypothesis $a, d_1, d_2, \dots, d_K$, such that $a + \sum x_i d_i \leq N$ and $\mathbf{C}\left(a + \sum x_i d_i\right)$ is constant for x in each l-equivalent classes, i.e $x \in \{1, 2, \dots l\}^k$.

Lets consider the following

$$k + 1\text{values} \begin{cases} \mathbf{C}\left(a + l(d_1 + d_2 + d_3 \dots + d_k)\right), \\ \mathbf{C}\left(a + l(d_2 + d_3 + \dots + d_k)\right), \\ \mathbf{C}\left(a + l(d_3 + \dots + d_k)\right), \\ \qquad \vdots \\ \mathbf{C}\left(a + l d_k\right), \\ \mathbf{C}\left(a\right). \end{cases}$$

So by the Pigeon Hole Principle, at least two of them must be equal. There is $1 \leq u < v \leq k+1$ such that

$$\mathbf{C}\left(a + l\sum_{i=u}^{k} d_i\right) = \mathbf{C}\left(a + l\sum_{i=v}^{k} d_i\right).$$

Therefore

$$\mathbf{C}\left(a + l\sum_{i=v}^{k} d_i + x_i\sum_{i=u}^{v-1} d_i\right).$$

is constant for all i in $\{1, 2, \dots, l\}$.

Therefore we have shown $S(l+1, 1)$ is true.

Hence by $S(l,m) \Rightarrow S(l,m+1)$ and $S(l,m)$ for all $m \geq 1 \Rightarrow S(l+1,m)$, we can conclude that Theorem 5.1 is true.

□

6 Compactness Principle

In this section we take a look at Ramsey Theory through another mathematical perspective, that is topology. We introduce the compactness principle which is a key tool when we work with infinite structures, for example. This will allow us to start with an infinite structure, that we know has some property about some finite substructure, than you can get rid of the infinite part of the structure we started with, and work with something finite.

We use some concepts such as the definition of compactness and its properties and the Tychonoff principle of topology. (The reader should be familiar with them).

Before we investigate further on this interesting view of Ramsey Theory through topology let us define some notation for this paper.

Suppose that Z is a structure on which we can define a k-colouring, that is Z can be a graph on n vertices such that we colour the edges with k different colours. Moreover, Z can be a set (finite or infinite), for which we colour its subsets of cardinality r (i.e. a hypergraph). In particular, if Z is finite of size n, and $r = 2$, we obtain colourings on the edges of a finite graph again.

We define the X_i's to be finite substructures of Z. For example, if Z is a complete graph on n vertices, one can take X_i to be the complete graph on a certain (smaller) number of vertices.

We write

$$Z \to (X_1, \dots, X_k)$$

if for any k-colouring of Z there is a substructure isomorphic to X_i (for some i), monochromatic and of the i-th colour.

You can use this notation to denote the existence of "classical" Ramsey numbers, writing $K_n \to (K_s, K_t)$ to mean that n is at least equal to the Ramsey number $R(s,t)$.

Finally, we write $Z \to (X)_k$ if all the X_i's are isomorphic. Let us now state the theorem.

Theorem 6.1. Let M be infinite. Then $M \to (\mathcal{P})_k$ if and only if there exists $X \subset M$ finite such that $X \to (\mathcal{P})_k$.

Proof. The proof from right to left is immediate as if $P \subseteq X$ such that P in $\mathcal{P}$ and is monochromatic, then it follows that P is also a subset of M and hence has the same properties.

Now, let us look at the proof from left to right. A k-colouring of M is a colouring $c : M \to \{1, \dots, k\}$, which is an element of the product space $\mathcal{F} := \{1, 2, \dots, k\}^M$. We give the space $\{1, 2, \dots, k\}$ the discrete topology. Then the fact that this set is finite implies that it is compact, and by Tychonoff's Theorem. We have that the product over compact spaces is compact. So $\{1, \dots, k\}^M$ is compact.

Now, for any $Y \subset M$ finite and nonempty. Consider

$$N(Y) = \{c \in \mathcal{F} \,:\, c|_Y \nexists,\, P \subset Y \text{ monochromatic } P \in \mathcal{P}\}$$

We observe that, $c, d \in \mathcal{F}$ where if $c|_Y = d|_Y$ then $c \in N(Y)$ if and only if $d \in N(Y)$. Now we define the following set and it complement as

$$G = \{f \in \{1, \dots, k\}^Y \,:\, \exists P \subseteq Y,\, P \in \mathcal{P} \text{ Monochromatic respect to } f\},$$
$$\overline{G} = \{f \in \{1, \dots, k\}^Y \,:\, \nexists P \subseteq Y,\, P \in \mathcal{P} \text{ Monochromatic respect to } f\}.$$

Here we observe that we have the following expressions

$$N(Y) = \cup_{f \in \overline{G}} \{c \in \mathcal{F} \,:\, c|_Y = f\}.$$
$$\mathcal{F} \setminus N(Y) = \cup_{f \in G} \{c \in \mathcal{F} \,:\, c|_Y = f\}.$$

Where $\{c \in \mathcal{F} \,:\, c|_Y = f\}$ is given by

$$\prod_{y \in Y} \{f(y)\} \times \prod_{y \notin Y} \{1, \dots, k\}$$

so it follows that $N(Y)$ is both open and closed in $\mathcal{F}$.

We use contradiction to finish this proof. Assume there does not exist a finite X such that $X \to (\mathcal{P})_k$. Then $N(X) \neq \emptyset$ for all finite X.

Moreover for all finite $X, Z \subset M$ we have that

$$N(X \cup Z) \subseteq N(X) \cap N(Z).$$

Therefore the family of closed nonempty sets $\{N(X) : X \subset M,\ X \text{ is finite}\}$ has the Finite Intersection Property (FIP). Which means each intersection finite elements of the family contains another elements of that same family. As $\mathcal{F}$ is compact, it follows that

$$\bigcap_{\substack{X \subset M \\ X \text{finite}}} N(X) \neq \emptyset.$$

As this intersection is nonempty, let c be an element in the intersection defined above. Now observe that for all finite x, c in $N(X)$, which means that for all finite X, there is no monochromatic set P that is isomorphic to $\mathcal{P}$. Since this holds for all X, this implies that M does not have a monochromatic P that is isomorphic to $\mathcal{P}$ for the colouring c, hence we reach a contraction of the stated hypothesis that $M \to (\mathcal{P})_k$.

Thus proving this statement.

$\square$

7 Conclusion

In conclusion, in this paper we have stated the most important ideas of classic Ramsey Theory furthered explained with interesting examples. We have discussed the numerical difficulty of computing non-trivial Ramsey Numbers and have shown the best efforts of forming bounds for Ramsey Number. Furthermore, we have highlight the pioneering work of Schür, as he attempted to prove Fermat's last Theorem and also the work of B.L Van der Waerden on Arithmetic Progression.

Following on from the intuitive introduction to the results of Ramsey theory, many researchers in the field have expressed their beliefs that graph theory may not be the best method to processed with future explorations of Ramsey theory, see [Graham and Butler, 2015]. This can been seen with the lengthy enumeration of all possible cases, i.e $R(3, 3, 3) \neq 16$. Numerous alternatives have been suggested, however it is still unclear on which method will be the dominate in future research in this field.

References

[Aigner et al., 2010] Aigner, M., Ziegler, G. M., Hofmann, K. H., and Erdos, P. (2010). *Proofs from the Book*, volume 274. Springer.

[Codish et al., 2016] Codish, M., Frank, M., Itzhakov, A., and Miller, A. (2016). Computing the ramsey number r (4, 3, 3) using abstraction and symmetry breaking. *Constraints*, 21(3):375–393.

[Graham and Butler, 2015] Graham, R. and Butler, S. (2015). *Rudiments of Ramsey theory*, volume 123. American Mathematical Soc.

[Greenwood and Gleason, 1955] Greenwood, R. E. and Gleason, A. M. (1955). Combinatorial relations and chromatic graphs. *Canadian Journal of Mathematics*, 7:1–7.

[Jukna, 2011] Jukna, S. (2011). *Extremal combinatorics: with applications in computer science*. Springer Science & Business Media.

[N.J.A.Sloane, 2001] N.J.A.Sloane (2001). The On-Line Encyclopedia Of Integer Sequences. http://oeis.org/A059442. [Online; accessed 09-June-2021].

[Radziszowski, 2011] Radziszowski, S. (2011). Small ramsey numbers. *The electronic journal of combinatorics*, 1000:DS1–Aug.

[Ramsey, 1930] Ramsey, F. P. (1930). On a problem of formal logic. *Proceedings of the London Mathematical Society*, 2(1):264–286.