Classic Theorems of Ramsey Theory

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Table of Contents

Introduction Ramsey Theory

Ramsey Theory - Basics

Ramsey Theory is a subfield of combinatorics, named in honor of Frank Ramsey (1903 to 1930), who published a paper in 1930 on set theory that generalised the Pigeon Hole Principle (PHP).

Ramsey Theory - Basics

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Ramsey Theory in essence tries to answer: "How big must some structure be to guarantee that a particular property holds?"

Ramsey Property

Definition (Ramsey Property)
Let s and t be integers with $s, t > 1$. Then a positive integer r has the (s, t) Ramsey property, if K_r contains a monochromatic clique H of either colour, i.e: red $H = K_s$ or blue $H = K_t$, no matter the colouring of the edges of K_r.

The minimum r that satisfies this (s, t)-Ramsey property is called the Ramsey number is denoted by $r = R(s, t)$.

Ramsey Theory - Party Problem

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The stated problem can be translated into a classic Ramsey Theory result.

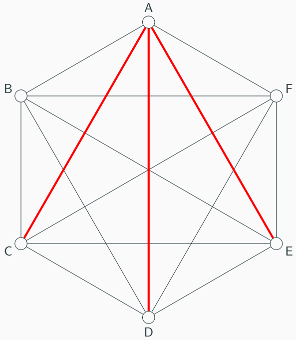
Theorem
$R(3,3) = 6$.

A sketch of the proof will be illustrated to show the intuition and the use of the PHP.

Example - R(3,3) $\leq$ 6

Consider the set of edges of the vertex A, by PHP we have '5 pigeons and 2 pigeon holes', hence at least 3 edges are either red or blue.

Observe if any of the edges $\{(C,D),(C,E),(E,D)\}$ are red, we would have a red K_3, otherwise we have a blue K_3.



R(3,3) $\neq$ 5 - Counter Example

To show that 6 is the minimum and hence 6 is the associated Ramsey number, a counter example of K_5 is used to show that $R(3,3) \neq 5$.

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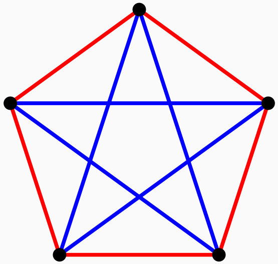

$R(3,3) \neq 5$ - Counter Example

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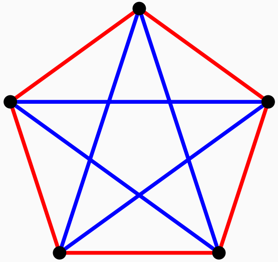

Given that 5 does not have the (s,t)-Ramsey property, then all $r < 5$ will also not have the associated (s,t)-Ramsey property.

Calculated values of two colour Ramsey numbers

‘Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshall all our computer and all our mathematicians and attempt to find the value. But suppose, instead they ask for $R(6,6)$. In that case, he believes, we we should attempt to destroy the aliens’ [Spencer, 1994].

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s/t	2	3	4	5	6	7	8	9	10
2	2								
3	3	6							
4	4	9	18						
5	5	14	25	43-48					
6	6	18	36-41	58-87	102-165				
7	7	23	49-61	80-143	115-298	205-540			
8	8	28	59-84	101-216	134-495	217-1031	282-1870		
9	9	36	73-115	133-316	183-780	252-1713	329-3583	565-6588	
10	10	40-42	92-149	149-442	204-1171	292-2826	343-6090	581-12677	798-23556

Table 1: Two colour Ramsey Numbers and best estimate for bounds of small values of s and t. Note that the table is symmetrical across the diagonal. For further details see [N.J.A.Sloane, 2001] and [Radziszowski, 2011].

Ramsey Theory - Existence I

The Ramsey Number $R(s,t)$ is the smallest positive integer r that has the the (s,t)-Ramsey property. We have shown that it exists for $s = t = 3$, but we can generalise and say that for all $s, t > 1$ we can find an r.

To answer this with the help of the following upper bound;

Lemma (1)
Let s and t be integers with $s, t > 1$. Then it follows that

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Ramsey Theory - Existence II

Using double mathematical induction, with the Lemma [1], we can show the existence of r for all $s, t > 1$.

In the base case we use the property that $R(2, t) = t$.

1. The base case follows trivially for $s = t = 2$ $R(2, 2) = 2 < \infty$.
2. Next we use double induction on s and t and the Lemma. We assume that $R(s-1, t)$ and $R(s, t-1)$ exists.
3. Thus, by lemma 1 we can bound $R(s, t)$ such that $R(s, t) \leq R(s-1, t) + R(s, t-1)$ which implies that $R(s, t)$ exists.
4. Thus by induction, $R(s, t) < \infty$ for all $s, t \in \mathbb{N}$.

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4. Thus by induction, $R(s, t) < \infty$ for all $s, t \in \mathbb{N}$.

Theorem (Ramsey's Theorem)
Let s and t be integers with $s, t > 1$. Then $R(s, t)$ exists.

Ramsey Theory - Erdös' lower bound

The diagonal Ramsey numbers are of the form $R(s,s)$. The lower bound for these diagonals is given as the following;

Theorem (Erdös - 1947)
For all positive integers numbers $s \geq 3$ the following inequality holds for the lower bound

$$2^{(s/2)} \leq R(s,s).$$

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Erdős ' proof of this theorem was the first to incorporate the Probabilistic Method.

The Probabilistic Method says; 'If, in a given set of objects, the probability that an object does not have a certain property is less than 1, then there must exist an object with this property.'

Ramsey Theory - Generalised

Ramsey Theory - k colouring

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Definition
Suppose that $s_1, s_2, \ldots, s_k$ are positive integers, where each $s_i > 1$. A positive integer r has the $(s_1, s_2, \ldots, s_k)$-Ramsey property if, given k colours $1, \ldots, k$, K_r has a monochromatic clique of the i-th colour, no matter how the edges are coloured with k colours.

The smallest r with the $(s_1, s_2, \ldots, s_k)$-Ramsey property is called the Ramsey number $R(s_1, s_2, \ldots, s_k)$.

Note: If $k = 2$, then these are the Ramsey numbers we have seen in the slides before.

k colouring - Example

The only two proven non-trivial examples are $R(3,3,3) = 17$ [Greenwood and Gleason, 1955] and recently $R(3,3,4) = 40$. [Codish et al., 2016].

An interesting remark in regards to the proof of $R(3,3,3) = 17$, is that one key step is to find a 3-colouring of K_{16} that is a counter example. Out of the

$$3^{\frac{16(15)}{2}} = 1.79 \times 10^{57}$$

possible 3-colourings, only two do not have a monochromatic clique.

Hypergraphs

Ramsey Theory - Hypergraphs Basics

Until now a graph has been given by a collection of pairs on a set of vertices, which form edges. We can extended Ramsey Numbers to a more generalised version of a graph where edges can connect any number of vertices.

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Hypergraphs of interest are r-uniform hypergraphs, which have hyperedges of the same size r.

Definition
An r-uniform hypergraphs is a pair $H = (V, E)$ where V is a finite set of vertices and $E \subseteq \mathcal{P}_r(V)$ is a set of hyperedges.

A complete r-uniform hypergraph is $K_n^{(r)} = (V, \mathcal{P}_r(V))$ with $|V| = n$.

Note: In the previous presented cases we have defined the complete graph as $K_n^{(2)}$.

Hypergraphs- Ramsey Number

Naturally we can apply the previous definition of Ramsey Numbers to the case for k-colouring complete r-uniform hypergraph as:

Definition

The Ramsey number $R^{(r)}(s_1, \ldots, s_k)$ is the minimum number n such that any coloring of the edges of the complete hypergraph $K_n^{(r)}$ with k colours contains a clique of size s_i whose edges all have colour i, for some $i \in \{1, \ldots, k\}$.

Van der Waerden Theorem

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Theorem (van der Waerden 1927)

Let $l, k \in \mathbb{N}$. There exists a number $w \in \mathbb{N}$ which depends on l and k such that for every k-coloring of $\{1, \ldots, w\}$, it contains a monochromatic Arithmetic Progression (AP) of l terms.

The minimum w that satisfies this theorem is called the van der Waerden number associated with l and k and is denoted by $w = W(l, k)$.

Recall: An Arithmetic Progression (AP) is a sequence of numbers such that the difference between the consecutive terms is constant. In general for $n, m \in \mathbb{N}$ with $n > m$, the following expression holds

$$a_n = a_m + (n - m)d.$$

Van der Waerden Theorem - Example

In order to gain intuition about $W(l, k)$, let us consider the cases for small values of l and k.

The first is the trivial case of $l = 2$, for any k in $\mathbb{N}$, by the PHP, it holds that $W(2, k) = k + 1$.

Van der Waerden Theorem - $W(3,2) \leq 325$

The second case to consider is for an AP of length $l = 3$, with two colours $k = 2$, which leads to the following statement.

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Theorem
$W(3,2) \leq 325$.

The reason for the specified upper bound is as follows.

Suppose we partition $\{1, 2, \ldots, 325\} = [325]$ by the disjointed union $[325] = \bigcup_{i=0}^{64} B_i$, with $|B_i| = 5$ for all $i \in \{0, \ldots 64\}$.

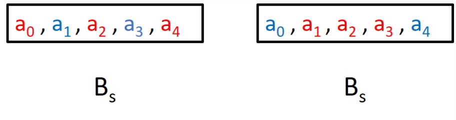
Then it follows that each block B_i has $2^5 = 32$ possible colourings.

Consider the set of blocks, $\{B_0, B_1, \ldots, B_{64}\}$. Then by the PHP, if we take the first 33 blocks, then at least two blocks will be coloured the same. Say $\mathrm{C}(B_s) = \mathrm{C}(B_t)$, with $0 \leq s < t < 34$.

Van der Waerden Theorem - $W(3,2) \leq 325$ - pt ii

Now, let us consider the first three elements of B_s. If all three elements are the same colour this would complete the proof. However by the PHP at least two are same colour, denote these elements of B_s as α_1 and α_2, with $\alpha_1 < \alpha_2$.

Let $\alpha_3 \in B_s$ such that $\alpha_3 = 2\alpha_2 - \alpha_1$, if $\mathrm{C}(\alpha_1) = \mathrm{C}(\alpha_2) = \mathrm{C}(\alpha_3)$, we are done.



Two possible colourings of the block B_S that give an AP of length 3. Here a_0, a_2, a_4 represents $\alpha_1, \alpha_2, \alpha_3$ respectfully. Left: Red AP with $d = 2$. Right: Red AP with $d = 1$

Van der Waerden Theorem - $W(3,2) \leq 325$ - pt iii

Now suppose that $C(\alpha_1) = C(\alpha_2) \neq C(\alpha_3)$. Then since $C(B_s) = C(B_t)$, we set $B_m = 2B_t - B_s$.

51 , 52 , 53 , 54 , 55	·····	126 , 127 , 128, 129 , 130	·····	201 ,202 , 203,204 , 205
B_{10}		B_{25}		B_{40}

Possible colouring of [325]. Here $s = 10$, $t = 25$, which yields $m = 40$. The green coloured 205 indicates, in this case the element γ_3, as $\alpha_3 = 54$ implies that $\beta_3 = 129$.

Observe in the example above that if 205 is coloured red, it yields an AP $\{51, 128, 205\}$. This is equivalent to $C(\alpha_1) = C(\beta_2) = C(\gamma_3)$.

Else if 205 is coloured blue, this results in the AP $\{55, 130, 205\}$. In this case it is equivalent to $C(\alpha_3) = C(\beta_3) = C(\gamma_3)$.

Van der Waerden Theorem - Proof Sketch

The choice of block size being 5 was due $2 \cdot W(3-1, 2) - 1 = 5$.

Hence setting each block as $\{1, 2, \ldots, 2W(3-1, 2) - 1\}$ ensures that there exist monochromatic AP of length $3 - 1 = 2$.

In a similar manner, we see that the choice of number of blocks arises from $2 \cdot W(3-1, 2^5) - 1 = 65$.

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To prove the general case of the van der Waerden Theorem , we use double induction on the length l of the AP required, and also the number of colours k.

In the proof we use that $W(l,k-1)$ exists and also $W(l-1,r')$ exist for all r' in $\mathbb{N}$.

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Thank you for your attention!

Questions?