Rings and Fields
Assignment 3

James Zoryk.
Student Number: 2663347.

Question 1. Show that $\mathbb{Z}[\sqrt{2}]$ is Euclidean domain.

Proof. Define the norm as $N(a + b\sqrt{2}) = |a^2 - 2b^2|$. Then for a nonzero element $\beta = b + \hat{b}\sqrt{2} \in \mathbb{Z}\sqrt{2}$, we have $N(\beta) = |b^2 - 2\hat{b}^2| \geq 1$. Then for all nonzero α and $\beta \in \mathbb{Z}[\sqrt{2}]$, $N(\alpha) \leq N(\alpha)N(\beta) = N(\alpha\beta)$. This proves one of the the condition for a Euclidean domain.
To prove the division algorithm part, let α and β be in $\mathbb{Z}[\sqrt{2}]$, with β being a nonzero element, such that $\alpha = a + \hat{a}\sqrt{2}$ and $\beta = b + \hat{b}\sqrt{2}$.
It must be shown that

$$\alpha = \beta\sigma + \rho,$$

where either $\rho = 0$ or that $N(\rho) < N(\beta)$. where σ is the quotient and ρ is the remainder term.
Now, we define the quotient as $\sigma = q + \hat{q}\sqrt{2}$, and the remainder as $\rho = \alpha - \beta\sigma$.

$$N(\frac{\alpha}{\beta} - \sigma) = N\big((c + \hat{c}\sqrt{2}) - (q + \hat{q}\sqrt{2})\big).$$

Where we let

$$\frac{\alpha}{\beta} = c + \hat{c}\sqrt{2},$$

with c and $\hat{c}$ in $\mathbb{Q}$. Then, there exists integers q and $\hat{q}$ in $\mathbb{Z}$, which are as close as possible to c and $\hat{c}$ respectively. If $\rho = 0$, then it would complete the proof, if $\rho \neq 0$ we need to show $N(\rho) < N\beta)$ holds.
Note that the closest possible integers can be bounded such that $|c - q| \leq 1/2$ and also $|\hat{c} - \hat{q}| \leq 1/2$. This implies

$$N\big((c - q) + (\hat{c} - \hat{q})\sqrt{2})\big) \leq |(\frac{1}{2})^2 - 2(\frac{1}{2})^2| = \frac{1}{4}.$$

Hence, we can obtain

$$N(\rho) = N(\alpha - \beta\sigma) = N(\beta(\frac{\alpha}{\beta} - \sigma)) = N(\beta)N(\frac{\alpha}{\beta} - \sigma) \leq N(\beta)\frac{1}{4}.$$

Furthermore we see that

$$N(\rho) \leq N(\beta)\frac{1}{4} < N(\beta),$$

which shows that $\mathbb{Z}[\sqrt{2}]$ is Euclidean domain. □

Question 2. Determine all principal ideals of $\mathbb{Z}[\sqrt{-7}]$ that contain the ideal $(4, 1+\sqrt{-7})$.

Proof. We need to show that there exists a principal ideal(α) in $\mathbb{Z}[\sqrt{-7}]$, such that

$$(4, 1+\sqrt{-7}) \subseteq (\alpha).$$

Define the Norm as $N(a + \hat{a}\sqrt{-7}) = a^2 + 7\hat{a}^2$.

Note: Here, we see that if we try to take the $\gcd(4, 1+\sqrt{-7})$, our choices of the quotient from $\mathbb{C}[\sqrt{-7}] \to \mathbb{Z}[\sqrt{-7}]$ results in the remainder not being less than quotient, $N(r) \nless N(q)$.

The ideal $(4, 1+\sqrt{-7})$ consists of the set of possible linear combination of 4 and $1+\sqrt{-7}$, such that

$$(\alpha) = x(4) + y(1+\sqrt{-7})$$

with x and y in $\mathbb{Z}[\sqrt{-7}]$. Furthermore, we apply the Norm so that

$$N(\alpha) = N(x)16 + N(y)8.$$

This implies that $N(\alpha)|8$, meaning that $N(\alpha) \in \{1, 2, 3, 8\}$. Using

$$a^2 + 7\hat{a}^2 = N(\alpha) \in \{1, 2, 4, 8\},$$

we see that $N(\alpha) = 2$ has no solution in $\mathbb{Z}$ for a and $\hat{a}$, while $N(\alpha) = 1$ results in $a = 1$, $N(\alpha) = 4$ gives $a = 2$ and finally $N(\alpha) = 8$ results in $a = \{\pm 1\}$ and $\hat{a} = \{\pm 1\}$.
Now suppose that $\alpha = 1 + \sqrt{-7}$, then there must exists a r in $\mathbb{Z}[\sqrt{-7}]$ such that

$$4 = r(1+\sqrt{-7}).$$

However, this equation has no solution for r, a similar argument can be made for $(1-\sqrt{-7})$. We can also rule out $N(\alpha) = 2$ since

$$1 + \sqrt{-7} = (2)s,$$

has no solution for some s in $\mathbb{Z}[\sqrt{-7}]$.
This only leaves the the trivially ideal (1) as our principal ideal that contains $(4, 1+\sqrt{-7})$. □