Assignment 1

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1 Question 1

Given that $\rho = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$, and $R = \{a + b\rho | a, b \in \mathbb{Z}\}$.

1.1 A)

Show that R is a subring in $\mathbb{R}$, and that $\mathbb{Z}[\sqrt{5}] \subset R$.

i) Let $a, b = 0$ then $0a + 0\rho = 0$ which shows that addition identity 0_R is in R and thus $R \neq \emptyset$.

ii)Let $x, y \in R$ such that $x = a + b\rho$ and $y = c + d\rho$, with $a, b, c, d \in \mathbb{Z}$ then

$$\begin{aligned} x - y &= (a + b\rho) - (c + d\rho) \\ &= (a - c) + (b - d)\rho \end{aligned}$$

Since a, b, c, d are in $\mathbb{Z}$ which is closed under addition and inverse, then it follows that $(a - c)$ and $(b - d)$ are in $\mathbb{Z}$.
Hence $(x - y)$ is in R, which show R is closed under addition and inverse.

iii)Let $x, y \in R$ such that $x = a + b\rho$ and $y = c + d\rho$, with $a, b, c, dinZ$ then

$$\begin{aligned} x * y &= (a + b\rho) * (c + d\rho) \\ &= ac + (ad + bc)\rho + bd\rho^2 \end{aligned}$$

Since ρ^2 is always an integer, then it follows that $x * y \in$ R, which implies that R is is closed under multiplication.
Therefore we have shown that R is a subring of $\mathbb{R}$.

let $x = a + b(\frac{1+\sqrt{5}}{2})$, with $a, b \in \mathbb{Z}$, such that $x = a + \frac{b}{2} + \frac{b\sqrt{5}}{2}$. Now we can write $x = a + c + c(\sqrt{5})$, where $c = 2b$, with $c \in \mathbb{Z}$. This shows that $\mathbb{Z}[\sqrt{5}] \subset R$. However $\mathbb{Z}[\sqrt{5}] \not\supset R$, as let $a, b = 1$ such that $1 + (\frac{1+\sqrt{5}}{2}) \notin \mathbb{Z}[\sqrt{5}]$.

1.2 B)

Given that $N(\alpha) = |\alpha\bar{\alpha}|$, where $\bar{\alpha} = \overline{a + b\rho} = (a + b) - b\rho$.
Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all α, β in R, but that in general we do not have that $N(\alpha + \beta) = N(\alpha) + N(\beta)$.

Let $\alpha = a + b\rho$ and $\beta = c + d\rho$.

Then

$$\begin{aligned} \alpha * \beta &= (a + b\rho)(c + d\rho) \\ &= ac + (ad + bc)\rho + bd\rho^2. \end{aligned}$$

Applying the norm function, $N(\alpha * \beta) =$

$$\begin{aligned} &[ac + (ad + bc)(\frac{1 + \sqrt{5}}{2}) + bd(\frac{1 + \sqrt{5}}{2})^2][ac + (ad + bc)\frac{1 - \sqrt{5}}{2} + bd(\frac{1 - \sqrt{5}}{2})^2] \\ &= (a^2c^2 + a^2cd - a^2d^2) + (abc^2 + abcd - abd^2) + (-b^2c^2 - b^2cd + b^2d^2) \end{aligned}$$

Now, finding the norm of α,

$$\begin{aligned} N(\alpha) &= (a + b\rho)\overline{(a + b\rho)} \\ &= [a + b(\frac{1 + \sqrt{5}}{2})][a + b(\frac{1 - \sqrt{5}}{2})] \\ &= (a^2 + ab - b^2). \end{aligned}$$

The steps above can be repeated for β, which give us $N(\beta) = (c^2 + cd - d^2)$. Using both equation found for the norm of α and β, the product of the two can be calculated as

$$N(\alpha)N(\beta) = (a^2c^2 + a^2cd - a^2d^2) + (abc^2 + abcd - abd^2) + (-b^2c^2 - b^2cd + b^2d^2).$$

Hence $N(\alpha\beta) = N(\alpha)N(\beta)$ for all α, β in R.

Now, the addition of α and β is given by $(a + c) + (b + d)\rho$, applying the norm function to this gives us the follow results

$$N(\alpha + \beta) = a^2 + ab + 2ac + ad - b^2 + bc - 2bd + c^2 + cd - d^2.$$

while,

$$N(\alpha) + N(\beta) = a^2 + c^2 + ab + cd - b^2 - d^2$$

This show that $N(\alpha + \beta) \neq N(\alpha) + N(\beta)$ in generally.

1.3 C)

Consider the infinite set $\{u^n | n \in \mathbb{N}\}$, where u is a unit in R and u has infinite order. The each element of the set is a unit of R.
Now assume $v \in$R is the inverse of u such that $uv = 1_R$ then,

$$\begin{aligned} u^n v^n = 1_R = uv \\ = u^{n-1}(uv)v^{n-1} \\ = u^{n-1}(1_R)v^{n-1} \\ = u^{n-1}v^{n-1} \\ = 1_R \end{aligned}$$

$N(\alpha) = \pm 1$, using Pell's equation $a^2 + ab - b^2 = \pm 1$, which can be arranged into $(2a+b)^2 - 5b^2 = \pm 4$.
Since $(\frac{-3+\sqrt{5}}{2})(\frac{3+\sqrt{5}}{2}) = 1_R$, then $(\frac{3+\sqrt{5}}{2})$ is a unit in R with infinite order, as $(\frac{-3+\sqrt{5}}{2})^n(\frac{3+\sqrt{5}}{2})^n = 1_R$.

2 Question 2

Given $S = \mathbb{Z} * \mathbb{C}$ with addition and multiplication given by

$$\begin{aligned} (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \\ (a_1, b_1) * (a_2, b_2) = (a_1 * a_2, a_1 * b_2 + b_1 * a_2) \end{aligned}$$

with $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{C}$, and S being a commutative ring.

2.1 a)

Determine all zero divisors

A nonzero element a of S is a zero divisor if there is a nonzero element b in S such that either $ab = 0$ or $ba = 0$.
Using $(a_1*a_2, a_1*b_2+b_1*a_2)$, we see that $a_1*a_2 = 0$ and $a_1*b_2+b_1*a_2 = 0$, has to be satisfied in order for it to be a zero divisor. Solving these set of equations give us a zero divisor of the form $(0, b)$ with $b \in \mathbb{C}$. Since

$$\begin{aligned} (0, b_1) * (0, b_2) = (0 * 0, 0 * b_2 + b_1 * 0) \\ = (0, 0)_S. \end{aligned}$$

Therefor we have shown that S has a zero divisor, given by $\{(0, b) | b \in \mathbb{C}\}$.

2.2 b)

Show that this ring has an identity $1_s \neq 0_s$, and determine s^*.

The 0_S is define by the addition identity such that,

$$\begin{aligned}(0,0)_S + (a,b) = (0+a, 0+b) = (a,b)\\ (a,b) + (0,0)_S = (a,b).\end{aligned}$$

While the multiplication identity of S is denoted by 1_S is given by

$$(a_1, b_1)_S * (a_2, b_2) = (a_2, b_2).$$

So we have

$$\begin{cases} a_1 * a_2 = a_2 \\ a_1 * b_2 + b_1 * a_2 = b_2 \end{cases}$$

Which implies the multiplication identity of S is $1_s = (1,0) \neq (0,0) = 0_S$. Hence, for the ring S, $1_s \neq 0_s$.

The units of the ring S, which form the set S^* are given as $(a_1, b_1) * (a_2, b_2) = (1,0)$, this implies that $a_1 * a_2 = 1$ and $a_1 * b_2 + b_1 * a_2 = 0$. Then it follows that $a_1, a_2 = \pm 1$, so that $b_2 + b_1 = 0$, then clearly $b_1 = -b_2$. Since $b_1, b_2 \in \mathbb{C}$ they can take the form of $\{x + yi | x, y \in \mathbb{R}\}$.
Hence, S^* is the set $\{(\pm 1, b) | b \in \mathbb{C}\}$, such that $(1,b) * (1,-b) = (1,0) = 1_S$.