Rings and Fields
Assignment 6

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Question 1.

Part a. Show that $f(x) = x^6 + 3$ is irreducible in $\mathbb{Q}[x]$.

Proof. Consider $f(x) = x^6 + 3$ in $\mathbb{Z}[x]$, applying Eisensetien's criterion with $p = 3$, observe that $3|0, 3$ and $3^2 \not| 3$, thus $f(x)$ is irreducible in $\mathbb{Z}[x]$. Moreover, since the polynomial $f(x)$ is monic and irreducible over $\mathbb{Z}[x]$, by Gauss's lemma, this implies that $f(x)$ is irreducible in $\mathbb{Q}[x]$, as required. □

Now let α be a root $\mathbb{C}$ of $f(x)$, so $\alpha^6 = -3$, and let $F = \mathbb{Q}[x]$.

Part. Prove that $\mathbb{Q}(\sqrt[3]{3}, \rho) \subseteq F$, where $\rho = e^{2\pi i/3}$.

Proof. A polynomial $f(x)$ in $\mathbb{Q}[x]$ has α as a root if and only if $m_\beta(x)$ divides $f(x)$ in $\mathbb{Q}[x]$. The minimal polynomial for the element $\sqrt[3]{3}$ is over $\mathbb{Q}$ is $x^3 - 3$ and $\sqrt[3]{3}$ is of degree 3 over $\mathbb{Q}$. The minimal polynomial for the element ρ over $\mathbb{Q}$ is $x^2 + x + 1$ and then $[\mathbb{Q}(\rho) : \mathbb{Q}] = 2$

□

Part c. Show that $F = \mathbb{Q}(\sqrt[3]{3}, \rho)$.

Proof. this is the text for the proof □

Part d. Show that $f(x)$ splits completely in $F[x]$.

Proof. Let α be the root of the polynomial $f(X) = x^3 + 3$ in $\mathbb{Q}[x]$. Observe that $f(x)$ has 6 complex roots of unity, with the cyclotomic equation we have $r^6 e^{6ia} = -3 = 3e^{i\pi}$. then This gives $r = \sqrt[6]{3}$ and $a = \frac{\pi}{6} + \frac{\pi}{3}k$ □