The Characterisation and Manipulation of Novel Topological Phases of Matter

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This thesis is dedicated to...

Acknowledgements

My thanks to...

Abstract

My abstract in here...

Abbreviations

k_B	Boltzmann's constant
k_BT	Thermal energy
...	...

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Chapter 1

Introduction

1.1 Introduction

Defining and distinguishing phases of matter has been a continuing effort by physicists for many years.

1.2 1D Topological Superconductor

In order to illustrate the essential elements of a topological condensed matter system, we now present a construction and analysis of the simplest superconducting lattice model, the Kitaev wire [1].

1.2.1 Dirac fermions

Given $N \in \mathbb{N}^+$ Dirac fermions, hereby denoted simply as fermions, they can be represented by a set of second quantised fermionic field operators $\{a_j\}$ and their conjugate partners $\{a_j^\dagger\}$, where $j = 1, ..., N$. They obey the following commutation relations

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad\qquad \{a_i^{(\dagger)}, a_j^{(\dagger)}\} = 0 \tag{1.1}$$

where δ_{ij} Kronercker delta function. These operators act on a tensor product of

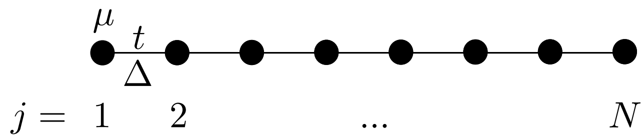

Figure 1.1: A schematic representation of the Kitaev 1D wire. A set of N sites denoted by the black dots and indexed by $j = 1, .., N$ are connected by black lines. To each site we associate a fermion a_j to which we associate a chemical potential $\mu \in \mathbb{R}$. We allow for fermions to tunnel to adjacent sites with amplitude $t \in \mathbb{R}$ and pair with adjacent fermions with amplitude $\Delta \in \mathbb{R}$

Fock states. A general state of the fermionic system, $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{n_i=0,1} \left(\alpha_{n_1,\dots,n_N} \bigotimes_{i=1}^{N} |n_i\rangle \right), \tag{1.2}$$

where $\alpha_{n_1,\dots,n_N} \in \mathbb{C}$ and

$$\bigotimes_{j=1}^{N} |n_j\rangle = \left(\bigotimes_{j=1}^{N} (a_j^\dagger)^{n_j} \right) \left(\bigotimes_{j=1}^{N} |0\rangle \right). \tag{1.3}$$

1.2.2 Real space tight-binding model

We take a chain of N sites indexed by $j = 1, ..., N$ and to each site we associate a fermion a_j. To each fermion we associate the same chemical potential μ and we allow for nearest neighbour tunnelling and pairing with amplitudes t and Δ respectively, with $\mu, t, \Delta \in \mathbb{R}$. This arrangement is shown in fig. 1.1. With this information we can write down a tight binding Hamiltonian

$$H = \sum_{j=1}^{N} \left(\mu a_j^\dagger a_j - \frac{1}{2} + t a_j^\dagger a_{j+1} + \Delta a_j a_{j+1} \right) + h.c., \tag{1.4}$$

where $h.c.$ denotes the Hermitian conjugate. We have chosen periodic boundary conditions such that $N + 1$ is 1.

1.2.3 Momentum space and the Fourier tansform

Because (1.4) is translationally invariant and has preiodic boundary conditions, we can transform it into momentum space via the Fourier transform. The transformation is defined as

$$a_j = \sum_p e^{ipj} a_p \qquad\qquad a_j^\dagger = \sum_p e^{-ipj} a_p^\dagger, \tag{1.5}$$

where $p \in [-\pi, \pi)$, also called the Brillouin zone (BZ). The transformed Hamiltonian is written as

$$H = \sum_p (\mu + t\cos(p)) \left(a_p^\dagger a_p - a_{-p} a_{-p}^\dagger\right) + i\Delta \sin(p) \left(a_p a_{-p} - a_{-p}^\dagger a_p^\dagger\right). \tag{1.6}$$

We can now write the Hamiltonian in Bogoliubov-de Gennes form

$$H = \sum_p \boldsymbol{\psi}_p^\dagger h(\boldsymbol{\lambda}, p) \boldsymbol{\psi}_p, \tag{1.7}$$

where $\boldsymbol{\psi}_p = \begin{pmatrix} a_p & a_{-p}^\dagger \end{pmatrix}^{\mathrm{T}}$, $\boldsymbol{\lambda} = \begin{pmatrix} \mu & \Delta & t \end{pmatrix}$, and $h(p)$ is a 2×2 Hermitian matrix given by

$$h(p) = \begin{pmatrix} \epsilon(p) & \Xi(p) \\ \Xi^*(p) & -\epsilon(p) \end{pmatrix} \tag{1.8}$$

where $\epsilon(\mu, t, p) = \mu + t\cos(p)$ and $\Xi(\Delta, p) = i\Delta \sin(p)$. We shall call $h(\boldsymbol{\lambda}, p)$ the *kernel Hamiltonian*. From the kernal Hamiltonian we can extract many useful quantities such as the energy spectrum and the model's topological invariant, the winding number.

1.2.4 Symmetries

Free fermionic models can be classified by the symmetries of the kernel Hamiltonian [2, 3, 4]. The presence (or not) of time-reversal (TR), particle-hole (PH) and sublattice (S) symmetries determines which of 10 classes a given Hamiltonian is in. More shall be said of the so called 10-fold way later in this document. The model as it stands obeys both TR and PH symmetries and by implication S symmetry.

1.2.5 Energy Spectrum and ground state

The model supports a pair of eigenvalues and eigenvectors, $E^\pm(\boldsymbol{\lambda}, p)$ and $|\psi_\pm(\boldsymbol{\lambda}, p)\rangle$. As the model is PH symmetric the spectrum will be symmetric about zero energy.

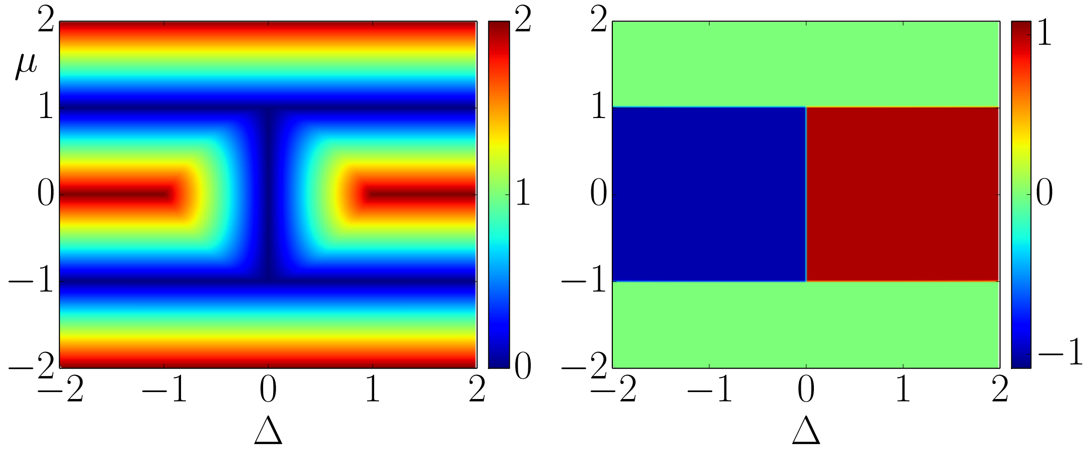

Figure 1.2: (Left) The energy gap, ΔE of the Kitaev wire as a function of the chemical potential, μ, and the superconducting order parameter Δ with $t = 1$. The data was obtained via exact diagonalisation of (1.8). (Right) The winding number, ν_{1D}, of the Kitaev wire. Each gapped phase is separated by a gappless line as depicted in the diagram of the gap on the left. The data was obtained via numerical evaluation of (1.12).

This is confirmed when we look at the analytic expression for the eigenvalues of (1.8),

$$E^{\pm}(\boldsymbol{\lambda}, p) = \pm\sqrt{|\epsilon(\mu, t, p)|^2 + |\Xi(\Delta, p)|^2}. \tag{1.9}$$

We define the *energy gap*, denoted $\Delta E(\boldsymbol{\lambda})$, as

$$\Delta E(\boldsymbol{\lambda}) = 2 \cdot \min_p |E^+(\boldsymbol{\lambda}, p)|. \tag{1.10}$$

In fig. 1.2 (left) we plot $\Delta E(\boldsymbol{\lambda})$ as a function of μ and Δ and for $t = 1$. The diagram is separated into four regions where $\Delta E(\boldsymbol{\lambda}) \neq 0$ which are separated by lines where $\Delta E(\boldsymbol{\lambda}) = 0$.

We take the system to be at half filling. This means that all of the negative energy states are occupied and the ground state is state associated with $E^-(p)$, $|\psi_{gs}(p)\rangle = |\psi_-(p)\rangle$.

1.2.6 Winding number

The topological phase of the system is determined by the winding number, ν_{1D}. In order to define ν_{1D} we must first define a unit vector $\hat{\boldsymbol{s}}(p)$ which parametrises the kernel Hamiltonian

$$h(\boldsymbol{\lambda}, p) = \boldsymbol{s}(\boldsymbol{\lambda}, p) \cdot \boldsymbol{\sigma} = |\boldsymbol{s}(\boldsymbol{\lambda}, p)| \hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p) \cdot \boldsymbol{\sigma} \tag{1.11}$$

where $\boldsymbol{\sigma} = \begin{pmatrix} \sigma^x & \sigma^y & \sigma^z \end{pmatrix}^{\mathrm{T}}$ and $\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p) : T^1 \to S^2$. The vector $\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p)$ is a map between the 1D-torus that is the Brillouin zone and the unit two sphere. We define $\nu_{\mathrm{1D}} : \mathbb{R}^3 \to \mathbb{Z}$ as

$$\nu_{\mathrm{1D}}(\boldsymbol{\lambda}) = \int_{\mathrm{BZ}} dp \;\; \boldsymbol{\theta}(\boldsymbol{\lambda}, p) \cdot \hat{\boldsymbol{u}}_{\perp}(\boldsymbol{\lambda}, p), \tag{1.12}$$

where $\boldsymbol{\theta}(\boldsymbol{\lambda}, p) = \big(\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p) \times \partial_p \hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p)\big)$ and $(\hat{\boldsymbol{u}}_{\perp}(\boldsymbol{\lambda}, p))_i = |\theta_i(\boldsymbol{\lambda}, p)|/|\boldsymbol{\theta}(\boldsymbol{\lambda}, p)|$. The integral (1.12) counts the number of times the vector $\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p)$ winds around S^2. Fig. 1.2 (right) depicts the winding number for the Kitaev wire as a function of μ and Δ with $t = 1$. There are two distinct topological phases with $\nu = \pm 1$ and two separated trivial phases with $\nu = 0$. The winding number is invariant within each gapped phase, only changing value when $\Delta E(\boldsymbol{\lambda}) \to 0$. This can be understood by looking at the definition of $\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p)$. It is easy to show that $|\boldsymbol{s}(\boldsymbol{\lambda}, p)| = E^{\pm}(\boldsymbol{\lambda}, p)$. Therefore when $\Delta E(\boldsymbol{\lambda}) \to 0$, $\hat{\boldsymbol{s}}(\boldsymbol{\lambda}, p)$ becomes undefined. This discontinuity in $\nu_{\mathrm{1D}}(\boldsymbol{\lambda})$ allows the invariant to change value.

1.2.7 Majorana fermions and edge states

A natural basis for describing topological superconductors is the Majorana basis. It is related to the Dirac fermion basis in the following way

$$a_j = \frac{\gamma_{1,j} + i\gamma_{2,j}}{2} \qquad\qquad a_j^{\dagger} = \frac{\gamma_{1,j} - i\gamma_{2,j}}{2} \tag{1.13}$$

where $\gamma_{\alpha,j}$ are the Majorana operators. They have the singular property that they are self dual, i.e. $\gamma_{\alpha,j} = \gamma_{\alpha,j}^{\dagger}$, and they obey the following commutation relations

$$\big\{\gamma_{\alpha,i}, \gamma_{\beta,j}\big\} = 2\delta_{\alpha\beta}\delta_{ij}, \tag{1.14}$$

implying that $\gamma_{\alpha,j}^2 = 1$. We can rewrite (1.4) in the Majorana basis to get

$$H = \sum_{j=1}^{N} \mu\gamma_{1,j}\gamma_{2,j} + \big(t + \Delta\big)\gamma_{2,j}\gamma_{1,j+1} + \big(-t + \Delta\big)\gamma_{1,j}\gamma_{2,j+1} \tag{1.15}$$

Chapter 2

Decomposition of the Chern Number

2.1 Introduction

2.1.1 The Chern Number

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