Multiuser Downlink Beamforming: Rank-Constrained SDP

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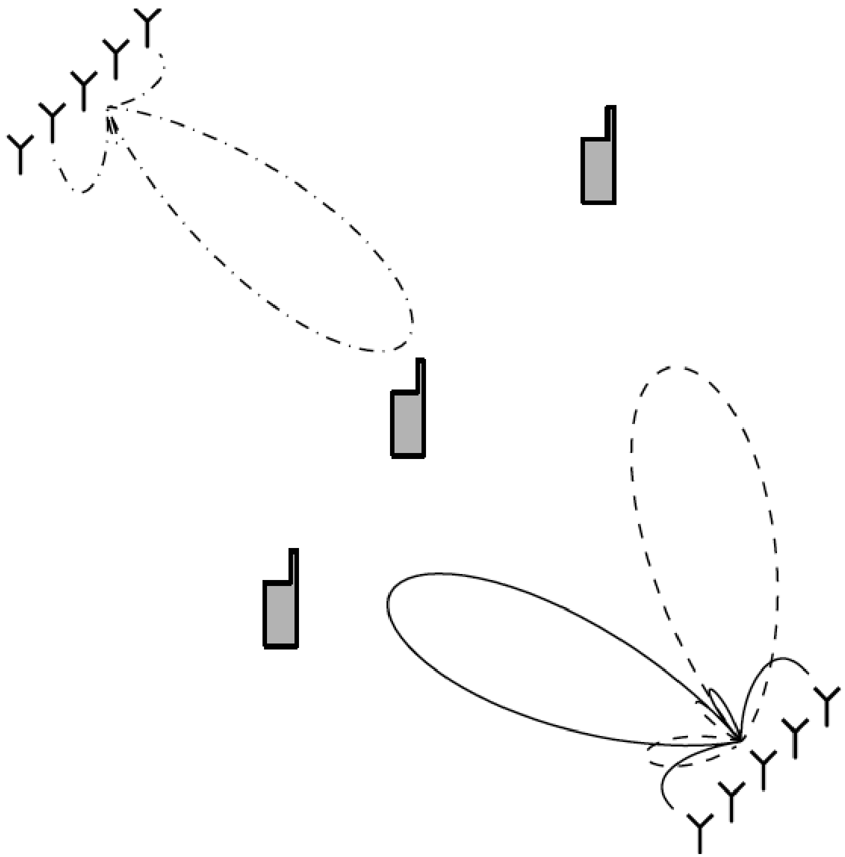
Outline of Lecture

- Problem formulation: downlink beamforming
- SOCP formulation, SDP relaxation, and equivalent reformulation
- General framework: rank-constrained SDP
- Numerical results
- Summary

Problem Formulation: Downlink Beamforming

Downlink Beamforming

- Consider several multi-antenna base stations serving single-antenna users:



Signal Model

- Signal transmitted by the kth base station:

$$\mathbf{x}_k(t) = \sum_{m \in \mathcal{I}_k} \mathbf{w}_m s_m(t)$$

where $\mathcal{I}_k$ are the users assigned to the kth base station.

- Signal received by mth user:

$$r_m(t) = \mathbf{h}_{m,\kappa_m}^H \mathbf{x}_{\kappa_m} + \sum_{k \neq \kappa_m} \mathbf{h}_{m,k}^H \mathbf{x}_k + \mathbf{n}_m(t)$$

where κ_m is the base station assigned to user m, $\mathbf{h}_{m,k}$ is the channel vector from base station k to user m with correlation matrix $\mathbf{R}_{m,k}$.

SINR Constraints

- SINR of the mth user is

$$\mathsf{SINR}_m = \frac{\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m}{\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2}.$$

- We can similarly consider an instantaneous version of the SINR, as well as the SINR for a dirty-paper coding type of transmission.

- Optimization variables: beamvectors $\mathbf{w}_l$ for $l = 1, \ldots, L$.

- The SINR constraints are

$$\mathsf{SINR}_m \geq \rho_m \qquad \text{for} \quad m = 1, \ldots, L.$$

Formulation of Optimal Beamforming Problem

- Minimization of the overall transmitted power subject to SINR constraints:

$$\begin{array}{ll}\underset{\{\mathbf{w}_l\}}{\text{minimize}} & \sum_{l=1}^{L}\left\|\mathbf{w}_l\right\|^2 \\ \text{subject to} & \frac{\mathbf{w}_m^H\mathbf{R}_{mm}\mathbf{w}_m}{\sum_{l\neq m}\mathbf{w}_l^H\mathbf{R}_{ml}\mathbf{w}_l+\sigma_m^2}\geq\rho_m, \qquad m=1,\ldots,L.\end{array}$$

- This beamforming optimization is nonconvex!!

SOCP Formulation

- Assume rank-one direct channels: $\mathbf{R}_{mm} = \mathbf{h}_m \mathbf{h}_m^H$.

- By adding, w.l.o.g., the constraint $\mathbf{w}_m^H \mathbf{h}_m \geq 0$, we can write the beamforming problem as

$$\begin{array}{ll} \underset{\{\mathbf{w}_l\}}{\text{minimize}} & \sum_{l=1}^{L} \left\|\mathbf{w}_l\right\|^2 \\ \text{subject to} & \left(\mathbf{w}_m^H \mathbf{h}_m\right)^2 \geq \rho_m \left(\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2\right), \quad \forall m \\ & \mathbf{w}_m^H \mathbf{h}_m \geq 0. \end{array}$$

- Observe we can write

$$\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2 = \left\|\widetilde{\mathbf{R}}_m^{1/2} \widetilde{\mathbf{w}}_m\right\|^2$$

where

$$\widetilde{\mathbf{w}}_m = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_L \\ 1 \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{R}}_m = \text{diag}\begin{pmatrix} \mathbf{R}_{m1} \\ \vdots \\ \mathbf{0} \\ \vdots \\ \mathbf{R}_{mL} \\ \sigma_m^2 \end{pmatrix}.$$

- The beamforming problem can be finally written as the SOCP:

$$\begin{array}{ll} \underset{\{\mathbf{w}_l\}}{\text{minimize}} & \sum_{l=1}^{L} \|\mathbf{w}_l\|^2 \\ \text{subject to} & \mathbf{w}_m^H \mathbf{h}_m \geq \rho_m \left\| \widetilde{\mathbf{R}}_m^{1/2} \widetilde{\mathbf{w}}_m \right\|, \quad \forall m. \end{array}$$

SDP Relaxation

- Let's start by rewriting the SINR constraint

$$\frac{\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m}{\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2} \geq \rho_m$$

as

$$\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m - \rho_m \sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l \geq \rho_m \sigma_m^2.$$

- Then, define the rank-one matrix: $\mathbf{X}_l = \mathbf{w}_l \mathbf{w}_l^H$.

- Notation: observe that

$$\mathbf{w}_l^H \mathbf{A} \mathbf{w}_l = \mathsf{Tr}\left(\mathbf{A}\mathbf{X}_l\right) \triangleq \mathbf{A} \bullet \mathbf{X}_l.$$

- The beamforming optimization can be rewritten as

$$
\begin{array}{ll}
\underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \mathbf{I} \bullet \mathbf{X}_l \\
\text{subject to} & \mathbf{R}_{mm} \bullet \mathbf{X}_m - \rho_m \sum_{l \neq m} \mathbf{R}_{ml} \bullet \mathbf{X}_l \geq \rho_m \sigma_m^2, \quad \forall m \\
& \mathbf{X}_l \succeq \mathbf{0} \\
& \text{rank}\left(\mathbf{X}_l\right) = 1
\end{array}
$$

- This is still a nonconvex problem, but if we remove the rank-one constraint, it becomes an SDP!

- Thus, a relaxation of the beamforming problem is an SDP which can be solved in polynomial time.

Equivalent Reformulation

- An equivalent reformulation was obtained in the seminal work:
  - M. Bengtsson and B. Ottersten, “Optimal and suboptimal trasmit beamforming,” in *Handbook of Antennas in Wireless Communications*, ch. 18, L.C. Godara, Ed., Boca Raton, FL: CRC Press, Aug. 2001.
  - M. Bengtsson and B. Ottersten, "Optimal transmit beamforming using semidefinite optimization," in *Proc. of 37th Annual Allerton Conference on Communication, Control, and Computing*, Sept. 1999.

- **Theorem**: If the SDP relaxation is feasible, then it has at least one rank-one solution.

- The proof is quite involved as it requires tools from Lagrange duality and Perron-Frobenius theory on nonnegative matrices.

- **Lemma**: The following two problems have the same unique optimal solution λ:

$$\begin{array}{ll}\underset{\lambda,\{\mathbf{u}_m\}}{\text{minimize}} & \sum_{l=1}^{L} \lambda_l \rho_l \sigma_l^2 \\ \text{subject to} & \mathbf{u}_m^H \left(\mathbf{I} - \lambda_m \mathbf{R}_{mm} + \sum_{l \neq m} \lambda_l \rho_l \mathbf{R}_{ml} \right) \mathbf{u}_i = 0, \quad \forall m \\ & \|\mathbf{u}_m\| = 1 \\ & \lambda_m \geq 0 \end{array}$$

and

$$\begin{array}{ll}\underset{\lambda}{\text{maximize}} & \sum_{l=1}^{L} \lambda_l \rho_l \sigma_l^2 \\ \text{subject to} & \mathbf{I} - \lambda_m \mathbf{R}_{mm} + \sum_{l \neq m} \lambda_l \rho_l \mathbf{R}_{ml} \succeq \mathbf{0}, \quad \forall m \\ & \lambda_m \geq 0. \end{array}$$

- The next two steps of the proof as the following:
  1. show that the dual problem of the beamforming problem is exactly the maximization in the previous lemma (this is quite simple)
  2. show that the beamforming problem can be rewritten as the minimization in the previous lemma (this is quite challenging and requires Perron-Frobenius theory).

- At this point, one just needs to realize that what has been proved is that strong duality holds. This implies that an optimal solution of the original problem is also an optimal solution of the relaxed problem; hence, the relaxed problem must have a rank-one solution.

More General Problem Formulation: Downlink Beamforming

Soft-Shaping Interference Constraints

- Constraint to control the amount of interference generated along some particular direction:

$$\sum_{k=1}^{N} \mathrm{E}\left[\left|\mathbf{h}_{m,k}^{H}\mathbf{x}_{k}\left(t\right)\right|^{2}\right] \leq \tau_{m} \qquad \text{for} \quad m = L+1, \ldots, M.$$

- Defining $\mathbf{S}_{ml} = \mathbf{h}_{m,\kappa_l}\mathbf{h}_{m,\kappa_l}^{H}$, we can rewrite it as

$$\sum_{l=1}^{L} \mathbf{w}_{l}^{H}\mathbf{S}_{ml}\mathbf{w}_{l} \leq \tau_{m} \qquad \text{for} \quad m = L+1, \ldots, M.$$

- We can similarly consider long-term constraints, high rank matrices $\mathbf{S}_{ml}$, total power transmitted from one particular base station to one external user.

Individual Shaping Interference Constraints

- Some interference constraints only affect one beamvector and we call them individual constraints.

- We consider the following two types of constraints:

$$\begin{aligned} \mathbf{w}_l^H \mathbf{B}_l \mathbf{w}_l &= 0 \qquad \text{for} \quad l \in \mathcal{E} \\ \mathbf{w}_l^H \mathbf{B}_l \mathbf{w}_l &\geq 0 \qquad \text{for} \quad l \notin \mathcal{E}. \end{aligned}$$

- The following global null shaping constraint is equivalent to individual constraints (assuming $\mathbf{S}_{ml} \succeq \mathbf{0}$):

$$\sum_{l=1}^{L} \mathbf{w}_l^H \mathbf{S}_{ml} \mathbf{w}_l \leq 0.$$

Formulation of Optimal Beamforming Problem

- Minimization of the overall transmitted power subject to all the previous SINR and interference constraints:

$$
\begin{array}{lll}
\underset{\{\mathbf{w}_l\}}{\text{minimize}} & \sum_{l=1}^{L} \left\|\mathbf{w}_l\right\|^2 & \\
\text{subject to} & \frac{\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m}{\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2} \geq \rho_m, & m = 1, \ldots, L \\
& \sum_{l=1}^{L} \mathbf{w}_l^H \mathbf{S}_{ml} \mathbf{w}_l \leq \tau_m, & m = L+1, \ldots, M \\
& \mathbf{w}_l^H \mathbf{B}_l \mathbf{w}_l = (\geq)\, 0, & l \in (\notin)\, \mathcal{E}_1 \\
& \mathbf{w}_l^H \mathbf{D}_l \mathbf{w}_l = (\geq)\, 0, & l \in (\notin)\, \mathcal{E}_2.
\end{array}
$$

- This beamforming optimization is nonconvex!!

SDP Relaxation

- Let's start by rewriting the SINR constraint

$$\frac{\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m}{\sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l + \sigma_m^2} \geq \rho_m$$

as

$$\mathbf{w}_m^H \mathbf{R}_{mm} \mathbf{w}_m - \rho_m \sum_{l \neq m} \mathbf{w}_l^H \mathbf{R}_{ml} \mathbf{w}_l \geq \rho_m \sigma_m^2.$$

- Then, define the rank-one matrix: $\mathbf{X}_l = \mathbf{w}_l \mathbf{w}_l^H$.

- Notation: observe that

$$\mathbf{w}_l^H \mathbf{A} \mathbf{w}_l = \mathsf{Tr}\left(\mathbf{A}\mathbf{X}_l\right) \triangleq \mathbf{A} \bullet \mathbf{X}_l.$$

- The beamforming optimization can be rewritten as

$$\begin{array}{ll}
\underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \mathbf{I} \bullet \mathbf{X}_l \\
\text{subject to} & \mathbf{R}_{mm} \bullet \mathbf{X}_m - \rho_m \sum_{l \neq m} \mathbf{R}_{ml} \bullet \mathbf{X}_l \geq \rho_m \sigma_m^2, \;\; m = 1, \ldots, L \\
& \sum_{l=1}^{L} \mathbf{S}_{ml} \bullet \mathbf{X}_l \leq \tau_m, \qquad m = L+1, \ldots, M \\
& \mathbf{B}_l \bullet \mathbf{X}_l = (\geq)\, 0, \qquad l \in (\notin)\, \mathcal{E}_1 \\
& \mathbf{D}_l \bullet \mathbf{X}_l = (\geq)\, 0, \qquad l \in (\notin)\, \mathcal{E}_2 \\
& \mathbf{X}_l \succeq \mathbf{0}, \;\; \text{rank}\,(\mathbf{X}_l) = 1 \qquad l = 1, \ldots, L.
\end{array}$$

- This is still a nonconvex problem, but if we remove the rank-one constraint, it becomes an SDP!

- Thus, a relaxation of the beamforming problem is an SDP which can be solved in polynomial time.

Existing Literature

- M. Bengtsson and B. Ottersten, "Optimal and suboptimal trasmit beamforming," in *Handbook of Antennas in Wireless Communications*, ch. 18, L.C. Godara, Ed., Boca Raton, FL: CRC Press, Aug. 2001.

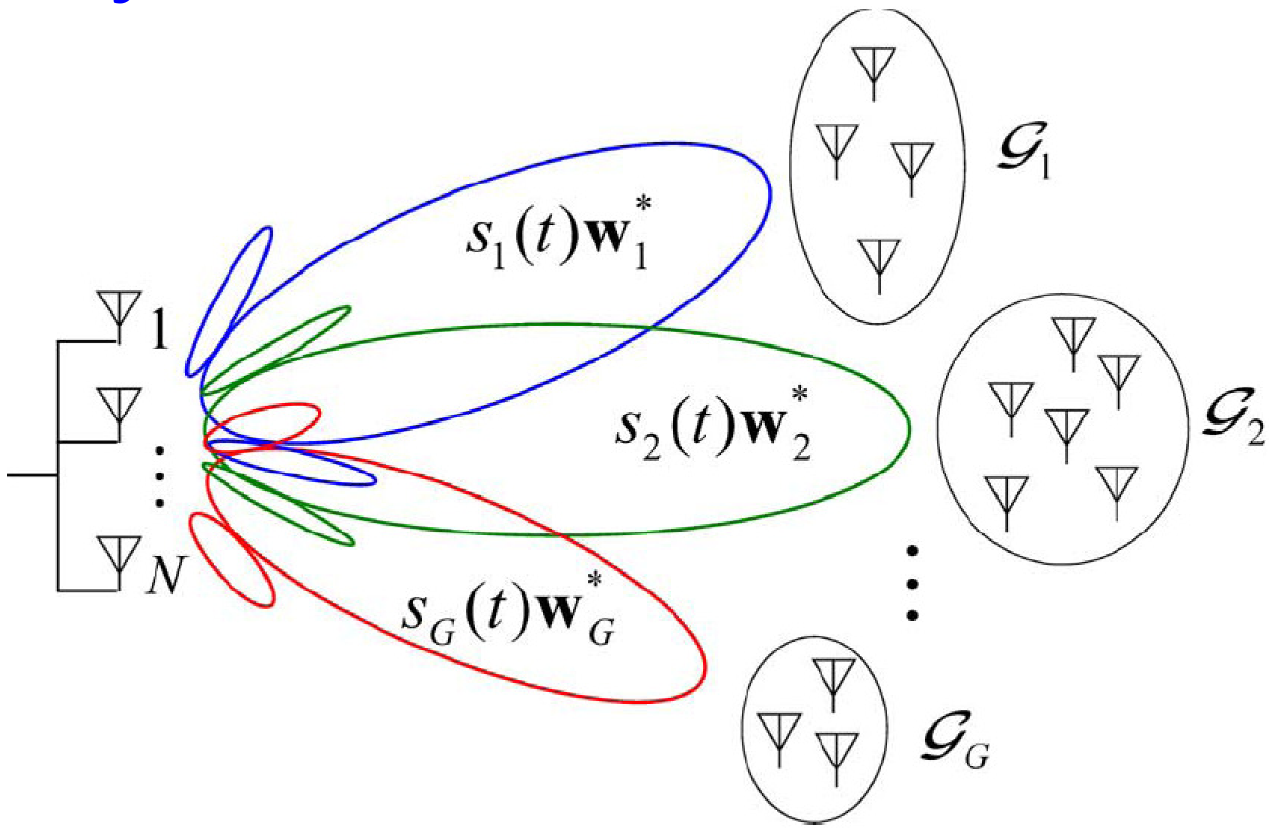
- M. Bengtsson and B. Ottersten, "Optimal transmit beamforming using semidefinite optimization," in *Proc. of 37th Annual Allerton Conference on Communication, Control, and Computing*, Sept. 1999.

- D. Hammarwall, M. Bengtsson, and B. Ottersten, "On downlink beamforming with indefinite shaping constraints," *IEEE Trans. on Signal Processing*, vol. 54, no. 9, Sept. 2006.

- G. Scutari, D. P. Palomar, and S. Barbarossa, "Cognitive MIMO Radio," *IEEE Signal Processing Magazine*, vol. 25, no. 6, Nov. 2008.

- Z.-Q. Luo and T.-H. Chang, "SDP relaxation of homogeneous quadratic optimization: approximation bounds and applications," *Convex Optimization in Signal Processing and Communications*, D.P. Palomar and Y. Eldar, Eds., Cambridge University Press, to appear in 2009.

Curiosity: Multicast Downlink Beamforming



- This small variation was proven to be NP-hard:
  - N. D. Sidiropoulos, T. N. Davidson, Z.-Q. (Tom) Luo, "Transmit Beamforming for Physical Layer Multicasting", *IEEE Trans. on Signal Processing*, vol. 54, no. 6, June 2006.

Rank-Constrained SDP

Separable SDP

- Consider the following separable SDP:

$$\begin{array}{lll} \underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \mathbf{C}_l \bullet \mathbf{X}_l & \\ \text{subject to} & \sum_{l=1}^{L} \mathbf{A}_{ml} \bullet \mathbf{X}_l = b_m, & m = 1, \ldots, M \\ & \mathbf{X}_l \succeq \mathbf{0}, & l = 1, \ldots, L. \end{array}$$

- If we solve this SDP, we will obtain an optimal solution $\mathbf{X}_1^\star, \ldots, \mathbf{X}_L^\star$ with an arbitrary rank profile.

- Can achieve a low-rank optimal solution? (recall that for the beamforming problem we want a rank-one solution)

Rank-Constrained Solutions

- A general framework for rank-constrained SDP was developed in
  - Yongwei Huang and Daniel P. Palomar, "Rank-Constrained Separable Semidefinite Programming for Optimal Beamforming Design," in *Proc. IEEE International Symposium on Information Theory (ISIT'09)*, Seoul, Korea, June 28 - July 3, 2009.
  - Yongwei Huang and Daniel P. Palomar, "Rank-Constrained Separable Semidefinite Programming with Applications to Optimal Beamforming," *IEEE Trans. on Signal Processing*, vol. 58, no. 2, pp. 664-678, Feb. 2010.

- **Theorem**: Suppose the SDP is solvable and strictly feasible. Then, it has always an optimal solution such that

$$\sum_{l=1}^{L} \mathrm{rank}^2 \left(\mathbf{X}_l\right) \leq M.$$

Purification-Type Algorithm

Algorithm: Rank-constrained solution procedure for separable SDP

1. Solve the SDP finding $\mathbf{X}_1, \ldots, \mathbf{X}_L$ with arbitrary rank.

2. while $\sum_{l=1}^{L} \text{rank}^2\left(\mathbf{X}_l\right) > M$, then

3. decompose $\mathbf{X}_l = \mathbf{V}_l \mathbf{V}_l^H$ and find a nonzero solution $\{\boldsymbol{\Delta}_l\}$ to
$$\sum_{l=1}^{L} \mathbf{V}_l^H \mathbf{A}_{ml} \mathbf{V}_l \bullet \boldsymbol{\Delta}_l = 0 \qquad m = 1, \ldots, M.$$

4. choose maximum eigenvalue $\lambda_{\max}$ of $\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_L$

5. compute $\mathbf{X}_l = \mathbf{V}_l \left(\mathbf{I} - \frac{1}{\lambda_{\max}} \boldsymbol{\Delta}_l\right) \mathbf{V}_l^H$ for $l = 1, \ldots, L$.

6. end while

A Simple Consequence

- Observe that if $L = 1$, then $\text{rank}(\mathbf{X}) \leq \sqrt{M}$ (real-valued case by [Pataki'98]).

- Furthermore, if $M \leq 3$, then $\text{rank}(\mathbf{X}) \leq 1$:

$$\begin{array}{ll} \underset{\mathbf{X} \succeq \mathbf{0}}{\text{minimize}} & \mathbf{C} \bullet \mathbf{X} \\ \text{subject to} & \mathbf{A}_m \bullet \mathbf{X} = b_m, \qquad m = 1, \ldots, M. \end{array}$$

- **Corollary**: The following QCQP has no duality gap:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^H \mathbf{C} \mathbf{x} \\ \text{subject to} & \mathbf{x}^H \mathbf{A}_m \mathbf{x} = b_m, \qquad m = 1, 2, 3. \end{array}$$

Rank-One Solutions for Beamforming

- Our result allows us to find rank-constrained solutions satisfying $\sum_{l=1}^{L} \mathsf{rank}^2 (\mathbf{X}_l) \leq M$.

- But, what about rank-one solutions for beamforming?

- Recall that to write the beamforming problem as an SDP we used $\mathbf{X}_l = \mathbf{w}_l \mathbf{w}_l^H$.

- **Corollary**: Suppose the SDP is solvable and strictly feasible, and that any optimal primal solution has no zero matrix component. Then, if $M \leq L + 2$, the SDP has always a rank-one optimal solution.

- The following conditions guarantee that any optimal solution has no zero component:

$$\begin{array}{rcll} L & \leq & M & \\ -\mathbf{A}_{ml} & \succeq & \mathbf{0}, & \forall l \neq m,\ m,l = 1,\ldots,L \\ b_m & > & 0, & m = 1,\ldots,L. \end{array}$$

- These conditions are satisfied by the beamforming problem:

$$\begin{array}{ll} \underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \mathbf{I} \bullet \mathbf{X}_l \\ \text{subject to} & \mathbf{R}_{mm} \bullet \mathbf{X}_m - \rho_m \sum_{l \neq m} \mathbf{R}_{ml} \bullet \mathbf{X}_l \geq \rho_m \sigma_m^2,\ m = 1,\ldots,L \\ & \sum_{l=1}^{L} \mathbf{S}_{ml} \bullet \mathbf{X}_l \leq \tau_m, \qquad m = L+1,\ldots,M \\ & \mathbf{X}_l \succeq \mathbf{0}, \qquad l = 1,\ldots,L. \end{array}$$

Contribution Thus Far

- Original result:
  - original seminal 2001 paper by Bengtsson and Ottersten showed strong duality for the rank-one beamforming problem with $M = L$
  - the proof relied on Perron-Frobenius nonnegative matrix theory and on the uplink-downlink beamforming duality
  - it seemed that strong duality was specific to that scenario.

- Current result:
  - applies to a much wide problem formulation of separable SDP
  - in the context of beamforming, it allows for external interference constraints
  - proof based purely on optimization concepts (no Perron-Frobenius, no uplink-downlink duality).

Rank-Constrained SDP with Additional Individual Constraints

Separable SDP with Individual Constraints

- Consider the following separable SDP with individual constrains:

$$
\begin{array}{lll}
\underset{\mathbf{X}_1,\ldots,\mathbf{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \mathbf{C}_l \bullet \mathbf{X}_l & \\
\text{subject to} & \sum_{l=1}^{L} \mathbf{A}_{ml} \bullet \mathbf{X}_l = b_m, & m = 1, \ldots, M \\
& \mathbf{B}_l \bullet \mathbf{X}_l = (\geq)\, 0, & l \in (\notin)\, \mathcal{E}_1 \\
& \mathbf{D}_l \bullet \mathbf{X}_l = (\geq)\, 0, & l \in (\notin)\, \mathcal{E}_2 \\
& \mathbf{X}_l \succeq \mathbf{0}, & l = 1, \ldots, L.
\end{array}
$$

- By directly invoking our previous result we get:

$$
\sum_{l=1}^{L} \text{rank}^2 (\mathbf{X}_l) \leq M + 2L.
$$

- Can we do better?

Low-Rank Solutions

- **Theorem**: Suppose the SDP is solvable and strictly feasible. Then, it has always an optimal solution such that

$$\sum_{l=1}^{L} \mathsf{rank}\left(\mathbf{X}_l\right) \leq M.$$

- Observe that the main difference is the loss of the square in the ranks. This implies that solutions will have higher ranks.

- Can we do better?

- We can sharpen the previous result is we assume that some of the matrices $(\mathbf{B}_l, \mathbf{D}_l)$ are semidefinite.

- **Theorem**: Suppose the SDP is solvable and strictly feasible, and that $(\mathbf{B}_l, \mathbf{D}_l)$ for $l = 1, \ldots, L'$ are semidefinite. Then, the SDP has always an optimal solution such that

$$\sum_{l=1}^{L'} \mathsf{rank}^2\left(\mathbf{X}_l\right) + \sum_{l=L'+1}^{L} \mathsf{rank}\left(\mathbf{X}_l\right) \leq M.$$

- Observation: individual constraints defined by semidefinite matrices do not contribute to the loss of the square in the rank.

A Simple Consequence

- **Corollary**: The following QCQP has no duality gap (assuming $\mathbf{B}$ and $\mathbf{D}$ semidefinite):

$$\begin{array}{lll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^H\mathbf{C}\mathbf{x} & \\ \text{subject to} & \mathbf{x}^H\mathbf{A}_m\mathbf{x} = (\geq)\, b_m, & m = 1, 2, 3 \\ & \mathbf{x}^H\mathbf{B}\mathbf{x} = (\geq)\, 0 & \\ & \mathbf{x}^H\mathbf{D}\mathbf{x} = (\geq)\, 0. & \end{array}$$

Rank-One Solutions for Beamforming

- What about rank-one solutions for beamforming?

- Recall that to write the beamforming problem as an SDP we used $\mathbf{X}_l = \mathbf{w}_l \mathbf{w}_l^H$.

- **Corollary**: Suppose the SDP is solvable and strictly feasible, and that any optimal primal solution has no zero matrix component. Then, the SDP has a rank-one solution if one of the following hold:

  – $M = L$
  – $M \leq L + 2$ and $(\mathbf{B}_l, \mathbf{D}_l)$ are semidefinite for $l = 1, \ldots, L$.

Numerical Simulations

Simulation Setup

- Scenario: single base station with $K = 8$ antennas serving 3 users at $-5°$, $10°$, and $25°$ with an angular spread of $\sigma_\theta = 2°$.

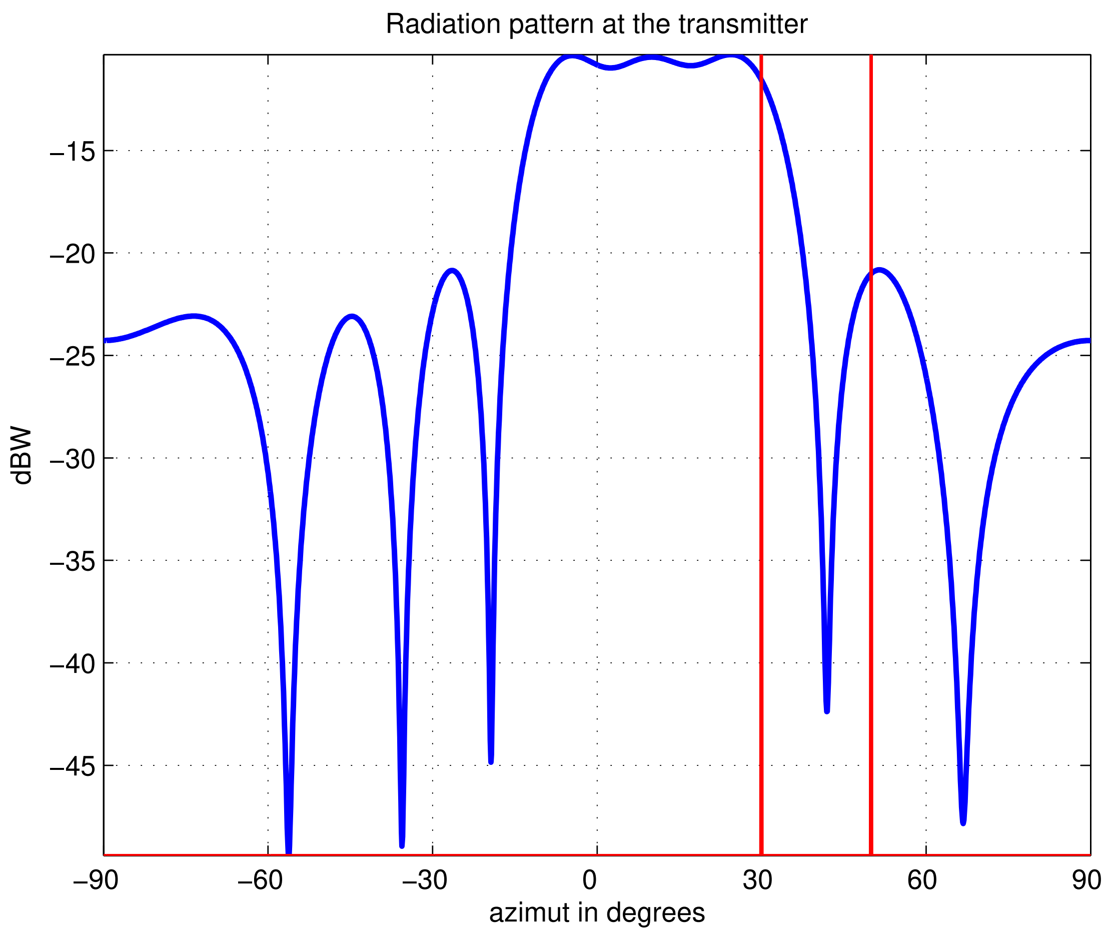
- SINR constraints set to: $\mathsf{SINR}_l \geq 1$ for $l = 1, 2, 3$.

- External users from another coexisting wireless system at $30°$ and $50°$ with no angular spread.

- Soft-shaping interference constraints for external users set to: $\tau_1 = 0.001$ and $\tau_2 = 0.0001$.

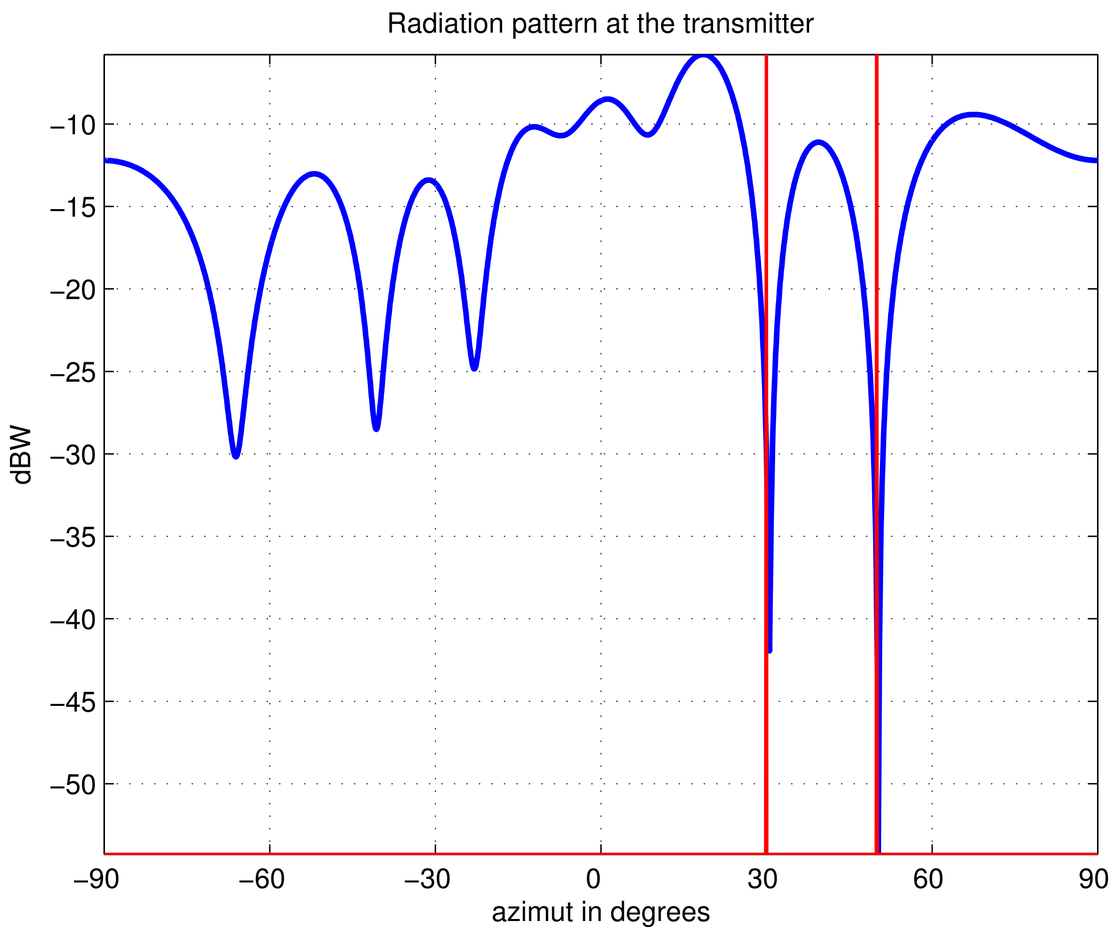
Simulation #1

- With no soft-shaping interference constraints (15.36 dBm):



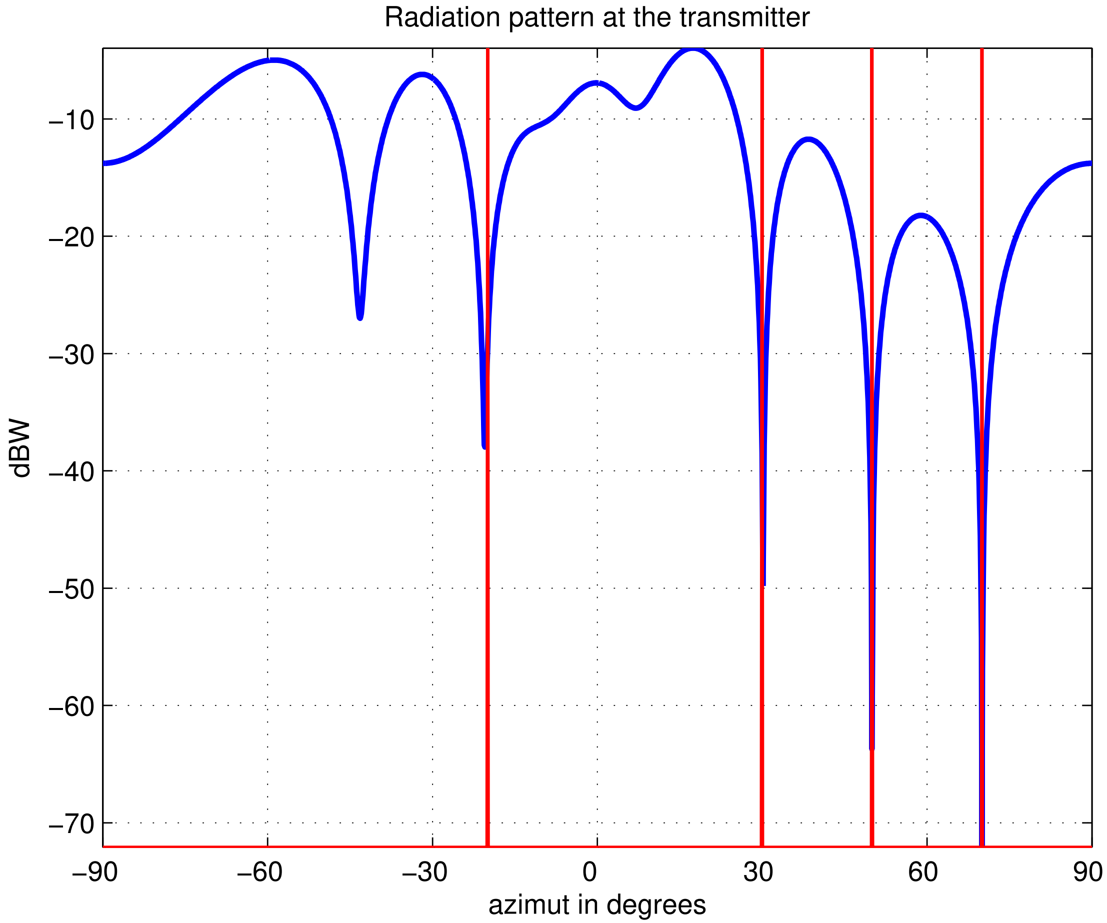
Simulation #2

- With soft-shaping interference constraints (18.54 dBm):



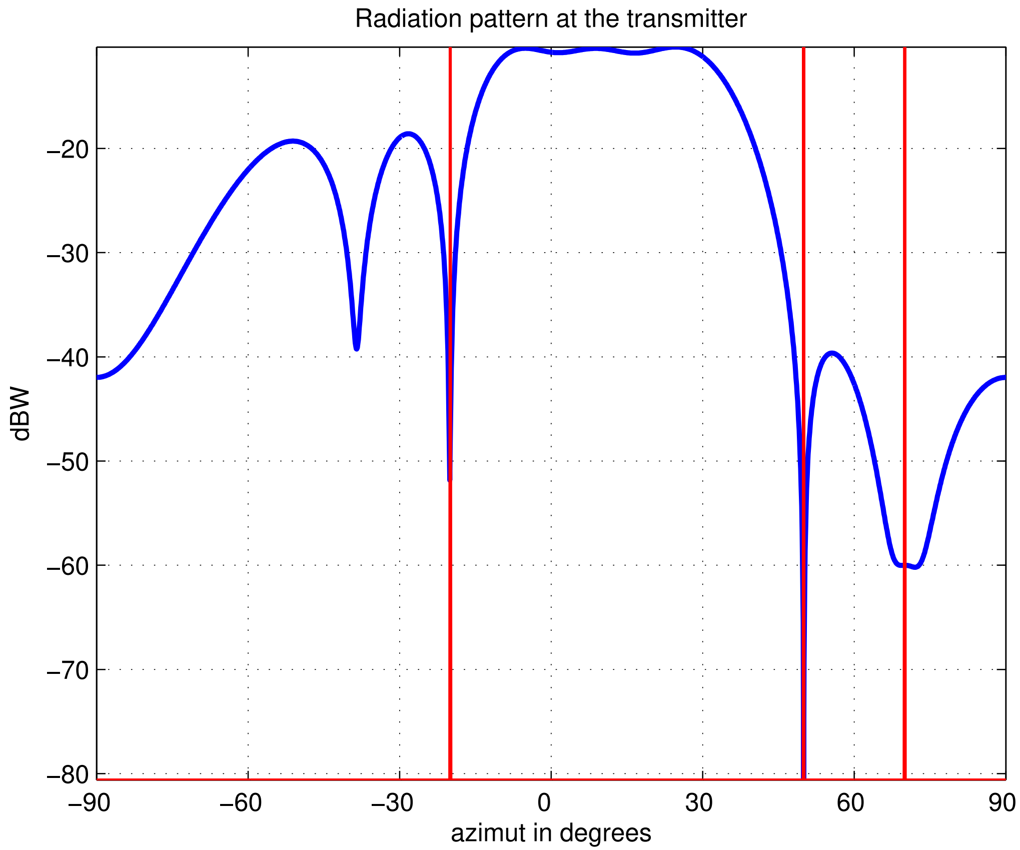
Simulation #3

- With soft- and null-shaping interference constraints (at $-20°$, $30°$ and $50°$, $70°$) (20.88 dBm):



Simulation #4

- With soft- and null-shaping interference constraints (at $-20°$, $50°$, and $70°$ with $\tau_1 = 0.00001$, $\tau_2 = 0$, $\tau_3 = 0.000001$) as well as a derivative null constraint at $70°$ (15.66 dBm):



Summary

- We have considered the optimal downlink beamforming problem that minimizes the transmission power subject to:
  - SINR constraints for users within the system
  - soft-shaping interference constraints to protect users from coexisting systems
  - individual null-shaping interference constraints
  - derivative interference constraints.
- The problem belongs to the class of rank-constrained separable SDP.
- We have proposed conditions under which strong duality holds as well as rank reduction procedures.