ML Detection via SDP Relaxation

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ELEC 5470 - Convex Optimization
Fall 2017-18, HKUST, Hong Kong

Outline of Lecture

1. BPSK Signal Detection
   - Problem Statement
   - Convex Relaxations
   - Comparison of Various Relaxations
   - Reconstruct Binary Solutions

2. Extension to Different Constellations
   - QPSK Constellation
   - M-PSK Constellations
   - 16-QAM Constellation
   - Universal Binary SDP Relaxation

3. Efficient Implementation
   - Dual Barrier Method for SDP

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   - Universal Binary SDP Relaxation
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BPSK Signal Detection

- Given the linear observation model

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w},$$

  where $\mathbf{y} \in \mathbf{R}^m$ is the received signal, $\mathbf{H} \in \mathbf{R}^{m \times n}$ is the known channel matrix, $\mathbf{s} \in \{\pm 1\}^n$ is the vector of transmitted BPSK symbols, and $\mathbf{w}$ is the Gaussian noise vector $\mathbf{w} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right)$.
- The maximum likelihood (ML) detection problem is

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2 \\ \text{subject to} & \mathbf{s} \in \{\pm 1\}^n. \end{array}$$

How Hard the Problem is?

- It is a ***nonconvex*** problem, due to the binary constraint set.
- We could solve it by evaluating all possible combinations; i.e., brute-force search.
- The complexity of a brute-force search is $\mathcal{O}(2^n)$, not okay at all for large n!
- NP-hard in general — no polynomial-time algorithm exists.

- What can we do?

- One possibility is to relax the problem.

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Unconstrained and Box Relaxation

- Unconstrained relaxation:

$$\underset{\mathbf{s}\in\mathbf{R}^n}{\text{minimize}} \quad \|\mathbf{y}-\mathbf{H}\mathbf{s}\|_2^2$$

  The result is zero forcing (ZF) and the problem has a closed-form solution $\mathbf{s}^\star = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$.

- Box relaxation:

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \|\mathbf{y}-\mathbf{H}\mathbf{s}\|_2^2 \\ \text{subject to} & -1 \leq s_i \leq 1, \quad i = 1, \ldots, n. \end{array}$$

SDP Relaxation I

- Since

$$\begin{aligned}\|\mathbf{y}-\mathbf{H}\mathbf{s}\|_2^2 &= \mathbf{y}^T\mathbf{y}+\mathbf{s}^T\mathbf{H}^T\mathbf{H}\mathbf{s}-2\mathbf{y}^T\mathbf{H}\mathbf{s}\\ &= \mathbf{x}^T\mathbf{L}\mathbf{x},\end{aligned}$$

where

$$\mathbf{L}=\begin{bmatrix}\mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{y}\\ -\mathbf{y}^T\mathbf{H} & \mathbf{y}^T\mathbf{y}\end{bmatrix} \quad \text{and} \quad \mathbf{x}=\begin{bmatrix}\mathbf{s}\\ 1\end{bmatrix}.$$

- The problem can be rewritten in homogeneous form as

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^T\mathbf{L}\mathbf{x}\\ \text{subject to} & \mathbf{x}\in\{\pm 1\}^{n+1}\\ & x_{n+1}=1.\end{array}$$

SDP Relaxation II

- Note that the last constraint $x_{n+1} = 1$ is unnecessary, since if $\mathbf{x}$ is a solution, so is $-\mathbf{x}$. Thus, we can simply remove the constraint.
- Defining $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, the problem is equivalent to

$$\begin{array}{ll} \underset{\mathbf{X},\mathbf{x}}{\text{minimize}} & \mathrm{Tr}(\mathbf{L}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{n+1} \\ & \mathbf{X} = \mathbf{x}\mathbf{x}^T. \end{array}$$

SDP Relaxation III

- The constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ is equivalent to $\mathbf{X} \succeq \mathbf{0}$ and $\mathrm{rank}(\mathbf{X}) = 1$.
- The key idea in SDP relaxation is to relax the problem by removing the rank constraint, and the problem becomes

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{minimize}} & \mathrm{Tr}(\mathbf{L}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{n+1} \\ & \mathbf{X} \succeq \mathbf{0}, \end{array}$$

which is an SDP.

An Alternative Way to Derive SDR I

- Recalling the original problem (nonhomogeneous form)

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \mathbf{s}^T\mathbf{H}^T\mathbf{H}\mathbf{s} - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\ \text{subject to} & \mathbf{s} \in \{\pm 1\}^n. \end{array}$$

- By letting $\mathbf{S} = \mathbf{s}\mathbf{s}^T$, we get

$$\begin{array}{ll} \underset{\mathbf{S},\mathbf{s}}{\text{minimize}} & \mathrm{Tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\ \text{subject to} & \mathrm{diag}(\mathbf{S}) = \mathbf{1}_n \\ & \mathbf{S} = \mathbf{s}\mathbf{s}^T. \end{array}$$

An Alternative Way to Derive SDR II

- Relaxing $\mathbf{S} = \mathbf{s}\mathbf{s}^T$ to $\mathbf{S} \succeq \mathbf{s}\mathbf{s}^T$, we can derive an SDR

$$\begin{array}{ll} \underset{\mathbf{S},\mathbf{s}}{\text{minimize}} & \mathrm{Tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\ \text{subject to} & \mathrm{diag}(\mathbf{S}) = \mathbf{1}_n \\ & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T. \end{array}$$

- This nonhomogeneous SDR is equivalent to the previously derived SDR, by Schur complement:

$$\mathbf{S} \succeq \mathbf{s}\mathbf{s}^T \Longleftrightarrow \mathbf{X} = \begin{bmatrix} \mathbf{S} & \mathbf{s} \\ \mathbf{s}^T & 1 \end{bmatrix} \succeq \mathbf{0}$$

Schur Complement

- Consider a matrix $\mathbf{X} \in \mathbf{S}^n$ partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A} \in \mathbf{S}^k$. If $\mathbf{A}$ is nonsingular, the matrix

$$\mathbf{S} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$$

is called the ***Schur complement*** of $\mathbf{A}$ in $\mathbf{X}$.

- Important characterizations:
  - $\mathbf{X} \succ \mathbf{0}$ if and only if $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{S} \succ \mathbf{0}$.
  - If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{S} \succeq \mathbf{0}$.

SDR as the Dual of the Dual I

- Recalling the original problem in homogeneous form

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^T\mathbf{L}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in \{\pm 1\}^{n+1} \end{array}$$

- The Lagrangian is

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\lambda}) &= \mathbf{x}^T\mathbf{L}\mathbf{x} + \sum_{i=1}^{n+1} \lambda_i (x_i^2 - 1) \\ &= -\boldsymbol{\lambda}^T\mathbf{1} + \mathbf{x}^T(\mathbf{L} + \mathrm{Diag}(\boldsymbol{\lambda}))\mathbf{x} \end{aligned}$$

- The dual problem is

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & -\boldsymbol{\lambda}^T\mathbf{1} \\ \text{subject to} & \mathbf{L} + \mathrm{Diag}(\boldsymbol{\lambda}) \succeq \mathbf{0}. \end{array}$$

SDR as the Dual of the Dual II

- Recalling the SDP relaxation

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{minimize}} & \mathrm{Tr}(\mathbf{LX}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{n+1} \\ & \mathbf{X} \succeq \mathbf{0}, \end{array}$$

- The dual function is

$$\begin{aligned} g(\mathbf{Z}, \boldsymbol{\lambda}) &= \inf_{\mathbf{X}} \ \mathrm{Tr}\left(\mathbf{LX}\right) - \mathrm{Tr}\left(\mathbf{ZX}\right) + \boldsymbol{\lambda}^T (\mathrm{diag}(\mathbf{X}) - \mathbf{1}) \\ &= \inf_{\mathbf{X}} \ \mathrm{Tr}\left((\mathbf{L} - \mathbf{Z} + \mathrm{Diag}(\boldsymbol{\lambda}))\mathbf{X}\right) - \boldsymbol{\lambda}^T \mathbf{1} \\ &= \begin{cases} -\boldsymbol{\lambda}^T \mathbf{1} & \mathbf{L} + \mathrm{Diag}(\boldsymbol{\lambda}) = \mathbf{Z} \succeq \mathbf{0} \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

SDR as the Dual of the Dual III

- Thus, the dual problem of the SDR is

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & -\boldsymbol{\lambda}^T \mathbf{1} \\ \text{subject to} & \mathbf{L} + \mathrm{Diag}(\boldsymbol{\lambda}) \succeq \mathbf{0}, \end{array}$$

  which is exactly the dual problem of the original problem.
- Since strong duality holds for the SDP, the SDP relaxation can be considered as the dual of the dual of the original problem.

- Which relaxation is better?

Comparison via Simulation

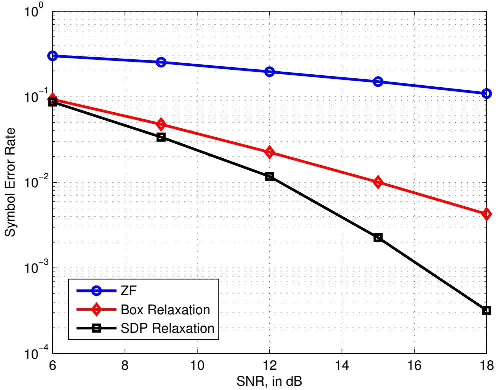

Figure: SER performance using quantization, $n = m = 8$, BPSK.

Comparison of Various Relaxations

- From the figure, we can see that the performance of the unconstrained relaxation (ZF) is the worst.
- This is easy to understand, since the other two relaxations are much tighter than the unconstrained relaxation.
- From the figure, we can also see that the SDP relaxation performs better than the box relaxation.

- *Is the SDP relaxation tighter than the box relaxation?*

Comparison of Various Relaxations

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- This is easy to understand, since the other two relaxations are much tighter than the unconstrained relaxation.
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- ***Is the SDP relaxation tighter than the box relaxation?***

Box Relaxation Vs SDP Relaxation I

- Recalling the box relaxation and the SDP relaxation

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \|\mathbf{y}-\mathbf{H}\mathbf{s}\|^2 \\ \text{subject to} & s_i^2 \leq 1, \quad i=1,\ldots,n. \end{array}$$

$$\begin{array}{ll} \underset{\mathbf{S},\mathbf{s}}{\text{minimize}} & \mathrm{Tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\ \text{subject to} & \mathrm{diag}(\mathbf{S}) = \mathbf{1}_n \\ & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T. \end{array}$$

- It seems hard to compare these two relaxations from the primal perspective.

Box Relaxation Vs SDP Relaxation II

- But we know that the optimal value of the SDP relaxation is equal to that of the dual of the original problem, thus

$$p^\star_{\mathrm{SDR}} = \max_{\boldsymbol{\lambda}} \min_{\mathbf{s}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 + \sum_{i=1}^{n} \lambda_i (s_i^2 - 1)$$

- For the box relaxation, by strong duality

$$p^\star_{\mathrm{Box}} = \max_{\boldsymbol{\lambda} \geq \mathbf{0}} \min_{\mathbf{s}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 + \sum_{i=1}^{n} \lambda_i (s_i^2 - 1)$$

- The feasible set of $\boldsymbol{\lambda}$ in SDP relaxation contains that in box relaxation, i.e., the box relaxation may be seen as further relaxation of the SDP relaxation, thus $p^\star_{\mathrm{Box}} \leq p^\star_{\mathrm{SDR}} \leq p^\star_{\mathrm{ML}}$.

- The original problem requires the solution to be binary.
- If the optimal solutions of the unconstrained relaxation and box relaxation are binary, or the optimal solution of the SDP relaxation is rank 1, then clearly we have solved the original problem.
- But usually the optimal solution of the relaxed problems will not be binary.

- *How can we reconstruct binary solutions?*

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Simple Quantization

- For unconstrained relaxation and box relaxation, a natural approach is to quantize the solution by looking at the sign, i.e.,

$$\hat{\mathbf{s}} = \mathrm{sign}(\mathbf{s}^\star_{\mathrm{relax}}).$$

- For SDP relaxation, based on the fact that, when the solution $\hat{\mathbf{X}}$ is rank 1, it will have the structure $\hat{\mathbf{X}} = \begin{bmatrix} \mathbf{s}\mathbf{s}^T & \mathbf{s} \\ \mathbf{s}^T & 1 \end{bmatrix}$, one simple approach is to quantize the last column of $\hat{\mathbf{X}}$, i.e.,

$$\hat{\mathbf{s}} = \mathrm{sign}(\hat{\mathbf{X}}_{1:n,n+1}).$$

- This is the quantization method we have used in the previous figure.

Eigenvalue Decomposition

- There are better approaches for SDP relaxation, since we have the whole matrix $\hat{\mathbf{X}}$.
- Suppose λ is the largest eigenvalue of $\hat{\mathbf{X}}$ and $\mathbf{u}$ is the corresponding eigenvector. Like before, based on the fact that, when $\hat{\mathbf{X}}$ is rank 1, then

$$\hat{\mathbf{X}} = \begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}^T & 1 \end{bmatrix} = \lambda \mathbf{u}\mathbf{u}^T,$$

i.e., $\mathbf{u}$ is a scaled version of $\begin{bmatrix} \mathbf{s} \\ 1 \end{bmatrix}$. Thus, we can take

$$\hat{\mathbf{s}} = \operatorname{sign}\left(\frac{\mathbf{u}_{1:n}}{u_{n+1}}\right).$$

Randomization

- The idea of randomization is to generate random points using $\hat{\mathbf{X}}$ as the covariance matrix, and then quantize and keep the best of the points.
- A convenient way to implement this is to factorize $\hat{\mathbf{X}}$ as $\hat{\mathbf{X}} = \mathbf{V}\mathbf{V}^T$, multiply $\mathbf{V}$ by random vectors $\mathbf{r}^{(i)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and then quantize

$$\hat{\mathbf{x}}^{(i)} = \operatorname{sign}\left(\frac{\mathbf{V}\mathbf{r}^{(i)}}{\mathbf{V}_{n+1,:}\mathbf{r}^{(i)}}\right) \qquad i = 1, \ldots, N.$$

  Then just keep the point $\hat{\mathbf{x}}^{\star}$ with minimum objective value.
- Randomization usually can provides better performance compared with the previous two methods.

Approximation Bound of Randomization

- When we generate candidate solutions using the randomization approach, the expected value of the objective can be computed explicitly:

$$\mathbb{E}(\hat{\mathbf{x}}^T \mathbf{L} \hat{\mathbf{x}}) = \mathrm{Tr}(\mathbf{L}\mathbb{E}(\hat{\mathbf{x}}\hat{\mathbf{x}}^T)) = \frac{2}{\pi}\mathrm{Tr}(\mathbf{L}\arcsin(\hat{\mathbf{X}})).$$

- We are guaranteed to reach this expected value after sampling a few points $\hat{\mathbf{x}}$, hence we know that the optimal value $p^\star_{\mathrm{ML}}$ of the ML detection problem is bounded by

$$\mathrm{Tr}(\mathbf{L}\hat{\mathbf{X}}) \leq p^\star_{\mathrm{ML}} \leq \hat{\mathbf{x}}^{\star T} \mathbf{L} \hat{\mathbf{x}}^\star \leq \frac{2}{\pi}\mathrm{Tr}(\mathbf{L}\arcsin(\hat{\mathbf{X}})).$$

SER Performance

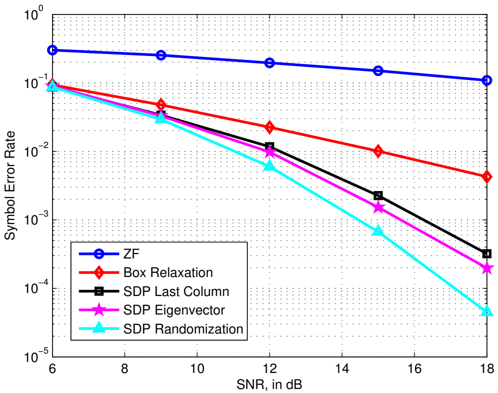

Figure: SER performance, $n = m = 8$, BPSK.

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- So far we have considered the SDP relaxation for the case with BPSK symbols and real-valued channels.
- Can we handle more general complex-valued channels and constellations, such as QPSK, M-PSK and 16-QAM?
- **Yes!** There are different papers dealing with different constellations.
  - BPSK and QPSK [Tan and Rasmussen, 2001]
  - MPSK [Ma et al., 2004]
  - 16-QAM [Wiesel et al., 2005], [Sidiropoulos and Luo, 2006]

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General Channel Model

- From now on, we will consider the following complex-valued MIMO model

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{w}},$$

  where $\tilde{\mathbf{y}} \in \mathbf{C}^m$ is the received signal, $\tilde{\mathbf{H}} \in \mathbf{C}^{m \times n}$ is the channel matrix, $\tilde{\mathbf{s}} \in \mathcal{A}^n$ is the vector of transmitted symbols, with $\mathcal{A}$ being the constellation set, and $\tilde{\mathbf{w}}$ is the complex Gaussian noise vector $\tilde{\mathbf{w}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$.
- Constellations:
  - QPSK: $\mathcal{A} = \{\pm 1 \pm j\}$
  - M-PSK: $\mathcal{A} = \{s = e^{\frac{\pi}{M}(2k-1)j} | k = 1, \ldots, M\}$
  - 16-QAM: $\mathcal{A} = \{s = s_1 + js_2 | s_1, s_2 \in \{\pm 1, \pm 3\}\}$

ML Detection with QPSK Constellation

- The signal model is

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{w}},$$

  where $\tilde{s}_i \in \{\pm 1 \pm j\}$.

- The ML detection problem in the QPSK case:

$$\begin{array}{ll} \underset{\tilde{\mathbf{s}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|^2 \\ \text{subject to} & \tilde{\mathbf{s}} \in \{\pm 1 \pm j\}^n. \end{array}$$

SDP Relaxation for QPSK

- The complex-valued problem can be reformulated as a real-valued one if we define

$$\mathbf{y} = \begin{bmatrix} \mathrm{Re}\{\tilde{\mathbf{y}}\} \\ \mathrm{Im}\{\tilde{\mathbf{y}}\} \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} \mathrm{Re}\{\tilde{\mathbf{H}}\} & -\mathrm{Im}\{\tilde{\mathbf{H}}\} \\ \mathrm{Im}\{\tilde{\mathbf{H}}\} & \mathrm{Re}\{\tilde{\mathbf{H}}\} \end{bmatrix};$$

$$\mathbf{s} = \begin{bmatrix} \mathrm{Re}\{\tilde{\mathbf{s}}\} \\ \mathrm{Im}\{\tilde{\mathbf{s}}\} \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} \mathrm{Re}\{\tilde{\mathbf{w}}\} \\ \mathrm{Im}\{\tilde{\mathbf{w}}\} \end{bmatrix}.$$

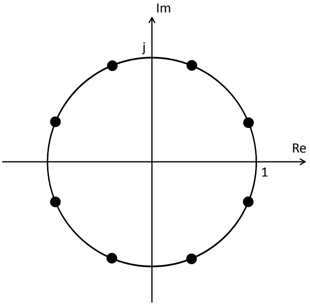
- The problem reduces to a BPSK one with twice the dimension:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w}, \quad \mathbf{s} \in \{\pm 1\}^{2n},$$

and thus can be solved via SDP relaxation.

M-PSK Constellations

- M-PSK: $\mathcal{A} = \{s = e^{\frac{\pi}{M}(2k-1)j} | k = 1, \ldots, M\}$
- The complex constellation points are equally spaced on the unit-circle.
- For example, the 8-PSK constellation is as follows:



8-PSK

ML Detection with M-PSK Constellations

- The signal model is
$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{w}},$$
where $\tilde{s}_i \in \{s = e^{\frac{\pi}{M}(2k-1)j} | k = 1, \ldots, M\}$.
- The ML detection problem in the M-PSK case:
$$\begin{array}{ll} \underset{\tilde{\mathbf{s}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|^2 \\ \text{subject to} & \tilde{\mathbf{s}} \in \{s = e^{\frac{\pi}{M}(2k-1)j} | k = 1, \ldots, M\}^n. \end{array}$$

SDP Relaxation for M-PSK I

- Idea: relax the constellation constraints to $|\tilde{s}_i| = 1$, i.e., the whole unit-circle, and then apply SDP relaxation.
- Following the idea, we first relax the problem to be

$$\begin{array}{ll} \underset{\tilde{\mathbf{s}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|^2 \\ \text{subject to} & |\tilde{s}_i| = 1, \quad i = 1, \ldots, n. \end{array}$$

By letting $\tilde{\mathbf{S}} = \tilde{\mathbf{s}}\tilde{\mathbf{s}}^H$, we get

$$\begin{array}{ll} \underset{\tilde{\mathbf{S}}, \tilde{\mathbf{s}}}{\text{minimize}} & \mathrm{Tr}(\tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \tilde{\mathbf{S}}) - 2\mathrm{Re}\{\tilde{\mathbf{y}}^H \tilde{\mathbf{H}} \tilde{\mathbf{s}}\} + \tilde{\mathbf{y}}^H \tilde{\mathbf{y}} \\ \text{subject to} & \mathrm{diag}(\tilde{\mathbf{S}}) = \mathbf{1}_n \\ & \tilde{\mathbf{S}} = \tilde{\mathbf{s}}\tilde{\mathbf{s}}^H. \end{array}$$

SDP Relaxation for M-PSK II

- As before, by further relaxing $\tilde{\mathbf{S}} = \tilde{\mathbf{s}}\tilde{\mathbf{s}}^H$ to $\tilde{\mathbf{S}} \succeq \tilde{\mathbf{s}}\tilde{\mathbf{s}}^H$, we can derive a complex-valued SDR

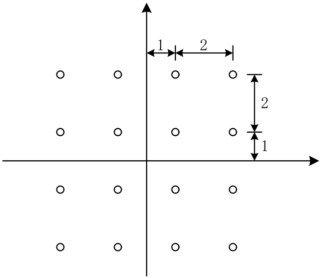
$$\begin{array}{ll}\underset{\tilde{\mathbf{S}},\tilde{\mathbf{s}}}{\text{minimize}} & \mathrm{Tr}(\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\tilde{\mathbf{S}}) - 2\mathrm{Re}\{\tilde{\mathbf{y}}^H\tilde{\mathbf{H}}\tilde{\mathbf{s}}\} + \tilde{\mathbf{y}}^H\tilde{\mathbf{y}} \\ \text{subject to} & \mathrm{diag}(\tilde{\mathbf{S}}) = \mathbf{1}_n \\ & \tilde{\mathbf{S}} \succeq \tilde{\mathbf{s}}\tilde{\mathbf{s}}^H. \end{array}$$

- By Schur complement, we can rewrite the complex-valued SDR in homogeneous form:

$$\begin{array}{ll}\underset{\mathbf{X}\in\mathbf{C}^{n+1\times n+1}}{\text{minimize}} & \mathrm{Tr}(\tilde{\mathbf{L}}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{n+1} \\ & \mathbf{X} \succeq \mathbf{0}, \end{array} \quad \text{where } \tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{H}}^H\tilde{\mathbf{H}} & -\tilde{\mathbf{H}}^H\tilde{\mathbf{y}} \\ -\tilde{\mathbf{y}}^H\tilde{\mathbf{H}} & \tilde{\mathbf{y}}^H\tilde{\mathbf{y}} \end{bmatrix}.$$

16-QAM Constellation

- 16-QAM: $\mathcal{A} = \{s = s_1 + js_2 | s_1, s_2 \in \{\pm 1, \pm 3\}\}$



ML Detection with 16-QAM Constellation

- The signal model is
$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{w}},$$
where $\tilde{s}_i \in \{s = s_1 + js_2 | s_1, s_2 \in \{\pm 1, \pm 3\}\}$.
- The ML detection problem in the 16-QAM case:
$$\begin{array}{ll} \underset{\tilde{\mathbf{s}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|^2 \\ \text{subject to} & \tilde{\mathbf{s}} \in \{s = s_1 + js_2 | s_1, s_2 \in \{\pm 1, \pm 3\}\}^n. \end{array}$$
- As in the QPSK case, we can rewrite the problem as a real-valued one
$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 \\ \text{subject to} & \mathbf{s} \in \{\pm 1, \pm 3\}^{2n}. \end{array}$$

SDP Relaxation with Polynomial Constraints I

- The constellation constraints can be written in a polynomial form $(s_i^2-1)(s_i^2-9)=0$ $(i=1,\cdots,2n)$ to obtain the following equivalent formulation:

$$\begin{array}{ll} \underset{\mathbf{s},\mathbf{t}}{\text{minimize}} & \|\mathbf{y}-\mathbf{H}\mathbf{s}\|^2 \\ \text{subject to} & s_i^2 = t_i, \quad i=1,\cdots,2n \\ & t_i^2 - 10t_i + 9 = 0, \quad i=1,\cdots,2n. \end{array}$$

- Since there are quadratic equality constraints, the problem is nonconvex.

SDP Relaxation with Polynomial Constraints II

Defining a new variable

$$\mathbf{X} = \begin{bmatrix} \mathbf{s} \\ \mathbf{t} \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{s}^T & \mathbf{t}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{s}\mathbf{s}^T & \mathbf{s}\mathbf{t}^T & \mathbf{s} \\ \mathbf{t}\mathbf{s}^T & \mathbf{t}\mathbf{t}^T & \mathbf{t} \\ \mathbf{s}^T & \mathbf{t}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{1,1} & \mathbf{X}_{1,2} & \mathbf{X}_{1,3} \\ \mathbf{X}_{2,1} & \mathbf{X}_{2,2} & \mathbf{X}_{2,3} \\ \mathbf{X}_{3,1} & \mathbf{X}_{3,2} & \mathbf{X}_{3,3} \end{bmatrix}$$

then the quadratic equality constraints become

$$\begin{aligned} \operatorname{diag}(\mathbf{X}_{1,1}) - \mathbf{X}_{2,3} &= \mathbf{0} \\ \operatorname{diag}(\mathbf{X}_{2,2}) - 10\mathbf{X}_{2,3} + 9 &= \mathbf{0} \end{aligned}$$

and the objective function can be written as

$$\|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 = \operatorname{Tr}\left(\begin{bmatrix} \mathbf{H}^T\mathbf{H} & \mathbf{0} & -\mathbf{H}^T\mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{y}^T\mathbf{H} & \mathbf{0} & \mathbf{y}^T\mathbf{y} \end{bmatrix} \mathbf{X} \right).$$

SDP Relaxation with Polynomial Constraints III

- The problem can be rewritten as

$$\begin{array}{ll}\underset{\mathbf{X}}{\text{minimize}} & \mathrm{Tr}\left(\begin{bmatrix} \mathbf{H}^T\mathbf{H} & \mathbf{0} & -\mathbf{H}^T\mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{y}^T\mathbf{H} & \mathbf{0} & \mathbf{y}^T\mathbf{y} \end{bmatrix}\mathbf{X}\right) \\ \text{subject to} & \begin{cases} \mathrm{diag}(\mathbf{X}_{1,1}) - \mathbf{X}_{2,3} = \mathbf{0} \\ \mathrm{diag}(\mathbf{X}_{2,2}) - 10\mathbf{X}_{2,3} + 9 = \mathbf{0} \\ \mathbf{X} \succeq \mathbf{0} \\ \mathbf{X}_{3,3} = 1 \\ \mathrm{rank}(\mathbf{X}) = 1 \end{cases}\end{array}$$

- Finally, the problem can be relaxed to an SDP by dropping the rank-one constraint.

SDP Relaxation with Bound Constraints I

- Still consider the equivalent real-valued problem

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 \\ \text{subject to} & \mathbf{s} \in \{\pm 1, \pm 3\}^{2n}. \end{array}$$

- By letting $\mathbf{S} = \mathbf{s}\mathbf{s}^T$, we get

$$\begin{array}{ll} \underset{\mathbf{S}, \mathbf{s}}{\text{minimize}} & \mathrm{Tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\ \text{subject to} & S_{ii} \in \{1, 9\}, \quad i = 1, \ldots, 2n \\ & \mathbf{S} = \mathbf{s}\mathbf{s}^T. \end{array}$$

SDP Relaxation with Bound Constraints II

- Relaxing $\mathbf{S} = \mathbf{s}\mathbf{s}^T$ to $\mathbf{S} \succeq \mathbf{s}\mathbf{s}^T$ is not enough to yield a convex relaxation.
- We also relaxes $\{1, 9\}$ to the interval $[1, 9]$, leading to

$$
\begin{array}{ll}
\underset{\mathbf{S},\mathbf{s}}{\text{minimize}} & \mathrm{Tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{y}^T\mathbf{H}\mathbf{s} + \mathbf{y}^T\mathbf{y} \\
\text{subject to} & 1 \leq S_{ii} \leq 9, \quad i = 1, \ldots, 2n \\
& \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T.
\end{array}
$$

- Rather unexpectedly, it is shown in [Ma et al., 2009] that the two SDP relaxations are equivalent.

- So far, we have seen different specialized SDP relaxations for different constellations.
- Can we come up with a universal SDP relaxation to deal with all the constellations?
- **Yes!** Next we are going to talk about a universal binary SDP relaxation [Fan et al., 2013], which can handle any constellations.

- So far, we have seen different specialized SDP relaxations for different constellations.
- Can we come up with a universal SDP relaxation to deal with all the constellations?

- **Yes!** Next we are going to talk about a universal binary SDP relaxation [Fan et al., 2013], which can handle any constellations.

Binarization of Arbitrary Constellations I

- The starting point of the universal binary SDR is the existence of a binary space covering the original non-binary signal space.
- For any constellation $\mathcal{A} \subseteq \mathbf{C}$ of size $|\mathcal{A}| = M$, there always exists a covering binary space $\mathcal{B}$ defined by a vector $\boldsymbol{\alpha} \in \mathbf{C}^q$, such that

$$\mathcal{A} \subseteq \mathcal{B} = \{s | s = \boldsymbol{\alpha}^H \mathbf{b}, \quad \mathbf{b} \in \{\pm 1\}^q\},$$

with q satisfying $\log_2(M) \leq q \leq M + 1$.

Binarization of Arbitrary Constellations II

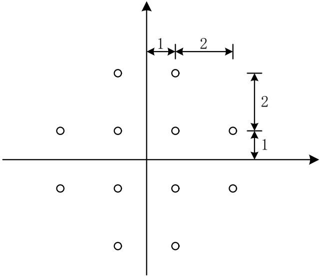
- For example, if $\mathcal{A} = \{s_1, s_2, ..., s_M\}$, we can construct a covering binary space $\mathcal{B}$ by setting

$$\boldsymbol{\alpha} = \left[\frac{1}{2}s_1, \frac{1}{2}s_2, \cdots, \frac{1}{2}s_M, \frac{1}{2}\sum_i s_i\right]^H,$$

  which is not very desirable as it achieves the most expensive binary expansion with $q = M + 1$.
- But the binary mapping (i.e., the choice of $\boldsymbol{\alpha}$) is not unique for a given constellation $\mathcal{A}$, and for most constellations we can have tighter binary mappings with smaller q.
- For example, the 16-QAM constellation, we can construct $\mathcal{B}$ by using $\boldsymbol{\alpha} = [2, 2j, 1, j]^H$ and thus $q = \log_2 16 = 4$.

Binarization of Arbitrary Constellations III

- In the case of the 16-QAM constellation, it just so happens that $\mathcal{B} = \mathcal{A}$.
- In other cases, however, like the 12-QAM constellation shown below, $\mathcal{B}$ will contain more points than $\mathcal{A}$, i.e., $\mathcal{B} \supset \mathcal{A}$ and we need to add some constraints to exclude all the undesired points.



Binarization of Arbitrary Constellations IV

- In the following lemma, we derive linear constraints to remove all the undesired points.

Lemma

For any constellation $\mathcal{A} \subseteq \mathbf{C}$ of size $|\mathcal{A}| = M$, there always exists a vector $\boldsymbol{\alpha} \in \mathbf{C}^q$ (with q satisfying $\log_2(M) \leq q \leq M+1$) and a matrix $\mathbf{D}$ such that

$$\mathcal{A} = \{s | s = \boldsymbol{\alpha}^H \mathbf{b},\ \mathbf{b}^T \mathbf{D} \leq (q-2)\mathbf{1}^T,\ \mathbf{b} \in \{\pm 1\}^q\}.$$

- One possible choice for $\mathbf{D}$ is $\mathbf{D} = [\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_l]$, where $\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_l$ are the binary vectors that generate the difference $\mathcal{B} - \mathcal{A}$, i.e., $\boldsymbol{\alpha}^H \mathbf{d}_i \in \mathcal{B} - \mathcal{A}$, $i = 1, \ldots, l$.

Table: Binarization for different constellations

Constellation	$\mathcal{A}$	$\boldsymbol{\alpha}$	$\mathbf{D}$
BPSK	$\mathcal{A}=\{-1,1\}$	1	$\varnothing$
QPSK	$\mathcal{A}=\{\pm 1\pm j\}$	$[1,j]^H$	$\varnothing$
16-QAM	$\mathcal{A}=\{s \mid s=s_1+js_2, s_1\in\mathcal{S},\ s_2\in\mathcal{S}\}$ $\mathcal{S}=\{-3,-1,1,3\}$	$[2,2j,1,j]^H$	$\varnothing$
64-QAM	$\mathcal{A}=\{s \mid s=s_1+js_2, s_1\in\mathcal{S},\ s_2\in\mathcal{S}\}$ $\mathcal{S}=\{-7,-5,-3,-1,1,3,5,7\}$	$[4,4j,2,2j,1,j]^H$	$\varnothing$
256-QAM	$\mathcal{A}=\{s \mid s=s_1+js_2, s_1\in\mathcal{S},\ s_2\in\mathcal{S}\}$ $\mathcal{S}=\{\pm(2i+1),\ i=0,1,...,7\}$	$[8,8j,4,4j,2,2j,1,j]^H$	$\varnothing$
12-QAM	$\mathcal{A}=\{s \mid s\in\mathcal{A}_{16-\mathrm{QAM}}, \lvert s\rvert^2<18\}$	$[2,2j,1,j]^H$	$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$
32-QAM	$\mathcal{A}=\{s \mid s\in\mathcal{A}_{64-\mathrm{QAM}}, \lvert s\rvert^2<50\}$	$[4,4j,2,2j,1,j]^H$	$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 4 & -4 \\ 1 & -1 & 1 & -1 & 4 & -4 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 4 & -4 \\ 1 & -1 & 1 & -1 & 4 & -4 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 4 & -4 \\ -1 & 1 & -1 & 1 & 4 & -4 & 0 & 0 \end{bmatrix}$
8-PSK	$\mathcal{A}=\{s \mid s=e^{j\frac{(2i+1)\pi}{8}}, i=0,1,...,7\}$	$[a,-b,aj,bj]^H$ $(a=\frac{\sqrt{2}}{2}cos(\frac{\pi}{8})$ $b=\frac{\sqrt{2}}{2}sin(\frac{\pi}{8}))$	$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}$

Universal Binary SDP Relaxation I

- With the binary representation of arbitrary constellations in the previous lemma, the signal space $\mathcal{A}^n$ can be expressed as:

$$\mathcal{A}^n = \{\tilde{\mathbf{s}} | \tilde{\mathbf{s}} = \tilde{\boldsymbol{\alpha}}^H \tilde{\mathbf{b}},\ \tilde{\mathbf{b}}^T \tilde{\mathbf{D}} \leq (q-2)\mathbf{1}^T,\ \tilde{\mathbf{b}} \in \{\pm 1\}^{qn}\},$$

  where $\tilde{\boldsymbol{\alpha}}^H = \boldsymbol{\alpha}^H \otimes \mathbf{I}_{n\times n}$, $\tilde{\mathbf{b}} = [\mathbf{b}_1^T, ..., \mathbf{b}_q^T]^T$ (with the n-dimensional vector $\mathbf{b}_q$ containing the q-th bit of the n transmitted symbols) and $\tilde{\mathbf{D}} = \mathbf{D} \otimes \mathbf{I}_{n\times n}$.
- The implication is that any n-dimensional signal space $\mathcal{A}^n$ can be expressed by all qn-dimensional binary vectors $\tilde{\mathbf{b}}$, that satisfy some linear inequalities constraints $\tilde{\mathbf{b}}^T \tilde{\mathbf{D}} \leq (q-2)\mathbf{1}^T$.

Universal Binary SDP Relaxation II

- Recalling the general ML detection problem

$$\begin{array}{ll} \underset{\tilde{\mathbf{s}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\mathbf{s}}\|^2 \\ \text{subject to} & \tilde{\mathbf{s}} \in \mathcal{A}^n. \end{array}$$

- By applying the binary representation of $\mathcal{A}^n$, the problem becomes

$$\begin{array}{ll} \underset{\tilde{\mathbf{b}}}{\text{minimize}} & \|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\tilde{\boldsymbol{\alpha}}^H\tilde{\mathbf{b}}\|^2 \\ \text{subject to} & \tilde{\mathbf{b}}^T\tilde{\mathbf{D}} \leq (q-2)\mathbf{1}^T \\ & \tilde{\mathbf{b}} \in \{\pm 1\}^{qn}, \end{array}$$

which is now a qn-dimensional binary ML detection problem.

Universal Binary SDP Relaxation III

- The problem reduces to the BPSK case, but with additional linear inequality constraints.
- By letting $\mathbf{x} = [\tilde{\mathbf{b}}^T \ 1]^T$ and $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, the problem becomes

$$\begin{array}{ll} \underset{\mathbf{X},\mathbf{x}}{\text{minimize}} & \mathrm{Tr}(\mathbf{L}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{qn+1} \\ & \mathbf{X} = \mathbf{x}\mathbf{x}^T \\ & \mathbf{X}_{qn+1,1:qn}\tilde{\mathbf{D}} \leq (q-2)\mathbf{1}^T \end{array}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{H}^H\mathbf{H} & -\mathbf{H}^H\tilde{\mathbf{y}} \\ -\tilde{\mathbf{y}}^H\mathbf{H} & \tilde{\mathbf{y}}^H\tilde{\mathbf{y}} \end{bmatrix}, \quad \text{with } \mathbf{H} = \tilde{\mathbf{H}}\tilde{\boldsymbol{\alpha}}^H.$$

Universal Binary SDP Relaxation IV

- By relaxing $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ to $\mathbf{X} \succeq \mathbf{0}$, finally we get the SDP relaxation

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{minimize}} & \mathrm{Tr}(\mathbf{L}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{qn+1} \\ & \mathbf{X} \succeq \mathbf{0} \\ & \mathbf{X}_{qn+1,1:qn}\tilde{\mathbf{D}} \leq (q-2)\mathbf{1}^T. \end{array}$$

- To sum up, given a constellation, we first find $\boldsymbol{\alpha}$ and $\mathbf{D}$, then let $\tilde{\boldsymbol{\alpha}}^H = \boldsymbol{\alpha}^H \otimes \mathbf{I}_{n\times n}$, $\tilde{\mathbf{D}} = \mathbf{D} \otimes \mathbf{I}_{n\times n}$ and construct $\mathbf{L}$ as in the previous page. With $\tilde{\mathbf{D}}$ and $\mathbf{L}$, we then solve the above SDP relaxation.

Outline of Lecture

1. BPSK Signal Detection
   - Problem Statement
   - Convex Relaxations
   - Comparison of Various Relaxations
   - Reconstruct Binary Solutions
2. Extension to Different Constellations
   - QPSK Constellation
   - M-PSK Constellations
   - 16-QAM Constellation
   - Universal Binary SDP Relaxation
3. Efficient Implementation
   - Dual Barrier Method for SDP

Solving the SDP Relaxation

- Having the SDR formulation, a general purpose software, such as CVX, can be used to solve the SDP conveniently.
- But to improve the computational efficiency, we can write our own interior point method (IPM) to explicitly exploit the problem structure.
- For ease of illustration, we take the SDP relaxation in the BPSK case as an example, to show the derivation of a specialized (and fast) IPM.

A First Look at the Problem

- Recalling the problem formulation

$$\begin{array}{ll} \underset{\mathbf{X}}{\text{minimize}} & \mathrm{Tr}(\mathbf{L}\mathbf{X}) \\ \text{subject to} & \mathrm{diag}(\mathbf{X}) = \mathbf{1}_{n+1} \\ & \mathbf{X} \succeq \mathbf{0}. \end{array}$$

- We know that in the barrier method we need to compute the gradient and Hessian. But in the above problem, the variable is an $(n+1) \times (n+1)$ matrix, and the Hessian will be of size $(n+1)^2 \times (n+1)^2$, which is hard to handle.
- Thus, instead of directly dealing with the primal problem, we resort to the dual problem.

Dual Barrier Method for SDP I

- The dual problem is

$$\begin{array}{ll} \underset{\mathbf{v}}{\text{maximize}} & -\mathbf{1}^T\mathbf{v} \\ \text{subject to} & \mathbf{L} + \mathrm{Diag}(\mathbf{v}) \succeq \mathbf{0}. \end{array}$$

- The logarithmic barrier of $\mathbf{L} + \mathrm{Diag}(\mathbf{v}) \succeq \mathbf{0}$ is $-\text{logdet}\left(\mathbf{L} + \mathrm{Diag}(\mathbf{v})\right)$.
- The dual barrier method will solve the following problem based on Newton's method:

$$\underset{\mathbf{v}}{\text{minimize}} \quad f\left(\mathbf{v}\right) = t\left(\mathbf{1}^T\mathbf{v}\right) - \text{logdet}\left(\mathbf{L} + \mathrm{Diag}(\mathbf{v})\right),$$

where t is the barrier parameter.

Dual Barrier Method for SDP II

- Denoting $\mathbf{v}^\star(t)$ the optimal solution of the problem for a given t, the gradient will be zero

$$\nabla f(\mathbf{v}^\star(t)) = t\mathbf{1} - \mathrm{diag}\left((\mathbf{L} + \mathrm{Diag}(\mathbf{v}^\star(t)))^{-1}\right) = \mathbf{0}.$$

- Defining

$$\mathbf{X}^\star(t) = \frac{1}{t}(\mathbf{L} + \mathrm{Diag}(\mathbf{v}^\star(t)))^{-1},$$

then

$$\mathrm{diag}(\mathbf{X}^\star(t)) = \mathbf{1}_{n+1},$$

so it's a feasible point for the primal.

Dual Barrier Method for SDP III

- The duality gap is

$$\begin{aligned}
\mathrm{Tr}(\mathbf{L}\mathbf{X}^\star(t)) - (-\mathbf{v}^\star(t)^T\mathbf{1}) &= \mathrm{Tr}(\mathbf{L}\mathbf{X}^\star(t)) + \mathbf{v}^\star(t)^T \mathsf{diag}\left(\mathbf{X}^\star(t)\right) \\
&= \mathrm{Tr}(\mathbf{L}\mathbf{X}^\star(t)) + \mathrm{Tr}(\mathsf{Diag}\left(\mathbf{v}^\star(t)\right)\mathbf{X}^\star(t)) \\
&= \mathrm{Tr}((\mathbf{L} + \mathsf{Diag}\left(\mathbf{v}^\star(t)\right))\mathbf{X}^\star(t)) \\
&= \frac{1}{t}\mathrm{Tr}(\mathbf{I}_{n+1}) \\
&= \frac{n+1}{t}
\end{aligned}$$

which means $\mathrm{Tr}(\mathbf{L}\mathbf{X}^\star(t))$ is not far away from the desired optimal value $p^\star$ of the original problem by more than $\frac{n+1}{t}$.

Dual Barrier Method for SDP IV

The algorithm for solving the SDP is summarized as follows:

Algorithm: dual barrier method for SDP

Input: $\mathbf{L}$, $\mu > 1$, $t > 0$, tolerance $\epsilon > 0$

Repeat

1. Centering step. Compute $\mathbf{v}^{\star}(t)$ by solving

$$\underset{\mathbf{v}}{\text{minimize}} \quad t\left(\mathbf{1}^{T}\mathbf{v}\right) - \text{logdet}\left(\mathbf{L} + \mathrm{Diag}(\mathbf{v})\right)$$

   using Newton's method.
2. Stopping criterion. **Quit** if $(n+1)/t < \epsilon$
3. Increase t. $t = \mu t$.

Output: $\mathbf{X} = (1/t)\left(\mathbf{L} + \text{diag}\left(\mathbf{v}^{\star}(t)\right)\right)^{-1}$

Computational Complexity

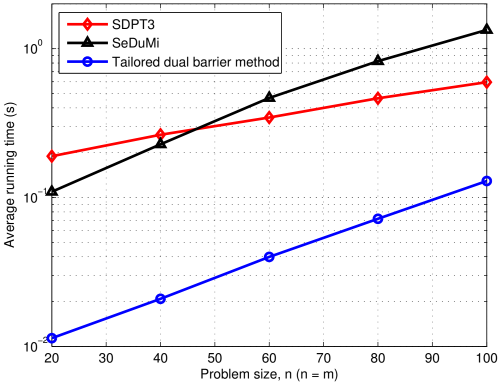


Figure: Average running time vs problem size.

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Thanks

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