Index Tracking in Finance

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Outline

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Investment Strategies

Fund managers follow two basic investment strategies:

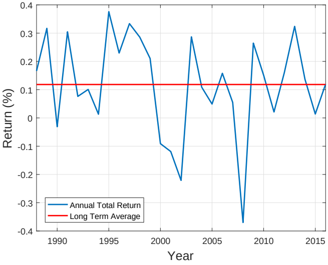
Active

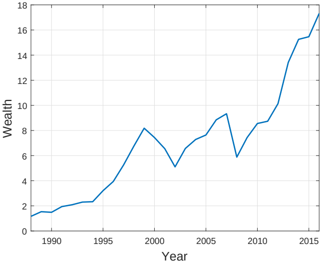
- Assumption: markets are not perfectly efficient.
- Through expertise add value by choosing high performing assets.

Passive

- Assumption: market cannot be beaten in the long run.
- Conform to a defined set of criteria (e.g. achieve same return as an index).

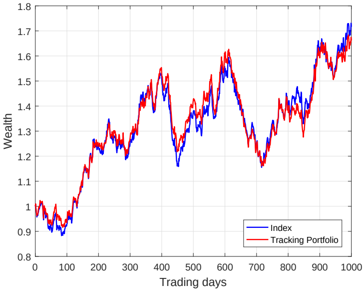
The stock markets have historically risen, e.g. S&P 500.





- Partly misleading: e.g. inflation.
- Still, reasonable returns can be obtained without the active management's risk.
- Makes passive investment more attractive.

Index Tracking



- Index tracking is a popular passive portfolio management strategy.
- **Goal:** construct a portfolio that replicates the performance of a financial index.

Definitions

- Price (at time t) of an asset or an index: p_t
- Net return: $r_t = \frac{p_t - p_{t-1}}{p_{t-1}}$
- Returns of an index in a period of T days: $\mathbf{r}^b = [r_1^b, \dots, r_T^b]^\top \in \mathbb{R}^T$
- Returns of N assets in a period of T days: $\mathbf{X} = [\mathbf{r}_1, \dots, \mathbf{r}_T]^\top \in \mathbb{R}^{T \times N}$
  - $\mathbf{r}_t \in \mathbb{R}^N$ with $t \in [1, T]$

Definitions

- Assume that an index is composed by a weighted collection of N assets.
- $\mathbf{b} \in \mathbb{R}_+^N$: normalized index weights
  - $\mathbf{b} > \mathbf{0}$
  - $\mathbf{b}^T\mathbf{1} = 1$
  - $\mathbf{X}\mathbf{b} = \mathbf{r}^b$
- A portfolio $\mathbf{w} \in \mathbb{R}_+^N$ is defined to be the proportion of the money we allocate in each asset.
- $\mathbf{w} \in \mathbb{R}_+^N$: tracking portfolio we wish to design
  - $\mathbf{w} \geq \mathbf{0}$
  - $\mathbf{w}^T\mathbf{1} = 1$

Full Replication

- How should we select $\mathbf{w}$?

$\Longrightarrow$ Straightforward solution: Full replication.

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  - Buy appropriate quantities of all the assets.
  - Perfect tracking.

Full Replication

- How should we select $\mathbf{w}$?

  $\Longrightarrow$ Straightforward solution: Full replication.

- Full replication: $\mathbf{w} = \mathbf{b}$
  - Buy appropriate quantities of all the assets.
  - Perfect tracking.

- Limitations:
  - Transaction costs: increase as $\text{card}(\mathbf{w})$ increases.
  - Illiquid assets: cannot buy/sell an asset easily or in market price.

Sparse Index Tracking

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- Use a small number of assets: $\mathsf{card}(\mathbf{w}) < N$
  - Reduce transaction costs.
  - Avoid illiquid assets.
  - Tradeoff: imperfect tracking.

Sparse Index Tracking

- How can we overcome there limitations?

  $\Longrightarrow$ Sparse index tracking.

- Use a small number of assets: card$(\mathbf{w}) < N$
  - Reduce transaction costs.
  - Avoid illiquid assets.
  - Tradeoff: imperfect tracking.

- Challenges:
  - Which assets should we select?
  - What should be their relative weight?

Existing Methods

- Two step approach:
  - Stock selection:
    - Largest market capital.
    - Most correlated to the index.
    - A combination cointegrated well with the index.
  - Capital allocation:
    - Naive allocation: proportional to the original weights.
    - Optimized allocation: usually a convex problem.
- Mixed Integer Programming
  - Practical only for small dimensions, e.g. $\binom{100}{20} > 10^{20}$.
- Genetic Algorithms
  - Solve the MIP problems in reasonable time.
  - Worse performance, cannot prove optimality.

Outline

Tracking Error Measure

A common tracking measure is the **empirical tracking error (ETE)**:

$$\text{ETE}(\mathbf{w}) = \frac{1}{T}\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2$$

- $\text{ETE}(\mathbf{w}) = \begin{cases} 0, & \text{if } \mathbf{w} = \mathbf{b}, \\ +, & \text{otherwise.} \end{cases}$

Goal: Construct a sparse tracking portfolio with minimum ETE.

Problem Formulation

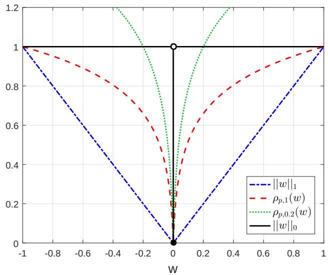
$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2 + \lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array} \tag{1}$$

- $\mathcal{W}$: A set of general **convex** constraints.
  - We assume that $\{\mathbf{w}|\mathbf{w} \geq \mathbf{0}, \mathbf{w}^\top\mathbf{1} = 1\} \subseteq \mathcal{W}$.
  - We will state separately any non-convex constraints.
- The optimization problem (1) is too difficult to deal with directly:
  - Discontinuous, non-differentiable, non-convex objective function.

- Approximation of the ℓ_0-norm (indicator function):

$$\rho_{p,\gamma}(w) = \frac{\log(1 + |w|/p)}{\log(1 + \gamma/p)}.$$

- Good approximation in the interval $[-\gamma, \gamma]$.



- For our problem we set $\gamma = u$, where $u \leq 1$ is an upperbound of the weights.
  - If $\{\mathbf{w} | \mathbf{w} \leq u\mathbf{1}\} \nsubseteq \mathcal{W}$ (no upperbound constraint) then implicitly $u = 1$.

Approximate Formulation

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array} \quad (2)$$

- $\boldsymbol{\rho}_{p,u}(\mathbf{w}) = [\rho_{p,u}(w_1), \ldots, \rho_{p,u}(w_N)]^\top$.
- Problem (2) is continuous and differentiable for $\mathbf{w} \geq \mathbf{0}$.
- Still non-convex: $\boldsymbol{\rho}_{p,u}(\mathbf{w})$ is concave for $\mathbf{w} \geq \mathbf{0}$.
- We will use MM to deal with the non-convex part.

Majorization of $\rho_{p,\gamma}$

Lemma 1

The function $\rho_{p,\gamma}(w)$, with $w \geq 0$, is upperbounded at $w^{(k)}$ by the surrogate function

$$h_{p,\gamma}(w, w^{(k)}) = d_{p,\gamma}(w^{(k)})w + c_{p,\gamma}(w^{(k)}),$$

where

$$d_{p,\gamma}(w^{(k)}) = \frac{1}{\log(1+\gamma/p)(p+w^{(k)})},$$

$$c_{p,\gamma}(w^{(k)}) = \frac{\log\left(1+w^{(k)}/p\right)}{\log(1+\gamma/p)} - \frac{w^{(k)}}{\log(1+\gamma/p)(p+w^{(k)})},$$

are constants.

Proof of Lemma 1

- The function $\rho_{p,\gamma}(w)$ is concave for $w \geq 0$.
- An upper bound is its first-order Taylor approximation at any point $w_0 \in \mathbb{R}_+$.

$$\begin{aligned}
\rho_{p,\gamma}(w) &= \frac{\log(1+w/p)}{\log(1+\gamma/p)} \\
&\leq \frac{1}{\log(1+\gamma/p)} \left[\log\left(1+w_0/p\right) + \frac{1}{p+w_0}(w-w_0)\right] \\
&= \underbrace{\frac{1}{\log(1+\gamma/p)(p+w_0)}}_{d_{p,\gamma}} w + \underbrace{\frac{\log\left(1+w_0/p\right)}{\log(1+\gamma/p)} - \frac{w_0}{\log(1+\gamma/p)(p+w_0)}}_{b_{p,\gamma}}.
\end{aligned}$$

Iterative Formulation via MM

- Now in every iteration we need to solve the following problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w} \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array} \tag{3}$$

- $\mathbf{d}_{p,u}^{(k)}=\left[d_{p,u}(w_1^{(k)}),\ldots,d_{p,u}(w_N^{(k)})\right]^\top$.
- Problem (3) is convex (QP).
- Requires a solver in each iteration.

LAIT

Algorithm 1: Linear Approximation for the Index Tracking problem (LAIT)

Set $k = 0$, choose $\mathbf{w}^{(0)} \in \mathcal{W}$
repeat
    Compute $\mathbf{d}_{p,u}^{(k)}$
    Solve (3) with a solver and set the optimal solution as $\mathbf{w}^{(k+1)}$
    $k \leftarrow k + 1$
until convergence
return $\mathbf{w}^{(k)}$

The Big Picture

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

The Big Picture

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array}$$

ℓ_0-norm approximation

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{1}^\top\rho_{p,u}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array}$$

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MM

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

Should we stop here?

- Advantages:
  - ✓ The problem is convex.
  - ✓ Can be solved efficiently by an off-the-shelf solver.
- Disadvantages:
  - × Needs to be solved many times (one for each iteration).
  - × Calling a solver many times increases significantly the running time.
- Can we do something better?
  - ✓ For specific constraint sets we can derive closed-form update algorithms!

- We consider the following convex set parametrized by u:

$$\mathcal{W}_u = \left\{ \mathbf{w} \middle| \mathbf{w}^\top \mathbf{1} = 1, \mathbf{0} \leq \mathbf{w} \leq u\mathbf{1} \right\}.$$

- Expand the objective:

$$\frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda \mathbf{d}_{p,u}^{(k)\top}\mathbf{w} = \frac{1}{T}\mathbf{w}^\top\mathbf{X}^\top\mathbf{X}\mathbf{w}+\left(\lambda \mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b\right)^\top \mathbf{w}+const.$$

- The problem becomes:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\mathbf{w}^\top\mathbf{X}^\top\mathbf{X}\mathbf{w} + \left(\lambda \mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b\right)^\top \mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W}_u \end{array}$$

Lemma 2

Let $\mathbf{L}$ and $\mathbf{M}$ be real symmetric matrices such that $\mathbf{M} \succeq \mathbf{L}$. Then, for any point $\mathbf{w}^{(k)} \in \mathbb{R}^N$ the following inequality holds:

$$\mathbf{w}^\top \mathbf{L} \mathbf{w} \le \mathbf{w}^\top \mathbf{M} \mathbf{w} + 2\mathbf{w}^{(k)\top} (\mathbf{L} - \mathbf{M})\, \mathbf{w} - \mathbf{w}^{(k)\top} (\mathbf{L} - \mathbf{M})\, \mathbf{w}^{(k)}.$$

Equality is achieved when $\mathbf{w} = \mathbf{w}^{(k)}$.

Proof:

$$\begin{aligned}
\mathbf{w}^\top \mathbf{L} \mathbf{w} &= \mathbf{w}^{(k)\top} \mathbf{L} \mathbf{w}^{(k)} + 2\left(\mathbf{w} - \mathbf{w}^{(k)}\right)^\top \mathbf{L} \mathbf{w}^{(k)} + \left(\mathbf{w} - \mathbf{w}^{(k)}\right)^\top \mathbf{L} \left(\mathbf{w} - \mathbf{w}^{(k)}\right) \\
&\le \mathbf{w}^{(k)\top} \mathbf{L} \mathbf{w}^{(k)} + 2\left(\mathbf{w} - \mathbf{w}^{(k)}\right)^\top \mathbf{L} \mathbf{w}^{(k)} + \left(\mathbf{w} - \mathbf{w}^{(k)}\right)^\top \mathbf{M} \left(\mathbf{w} - \mathbf{w}^{(k)}\right) \\
&= \mathbf{w}^\top \mathbf{M} \mathbf{w} + 2\mathbf{w}^{(k)\top} (\mathbf{L} - \mathbf{M})\, \mathbf{w} - \mathbf{w}^{(k)\top} (\mathbf{L} - \mathbf{M})\, \mathbf{w}^{(k)}
\end{aligned}$$

- Based on Lemma 2:
  - Majorize the quadratic term $\frac{1}{T}\mathbf{w}^\top\mathbf{X}^\top\mathbf{X}\mathbf{w}$.
  - In our case $\mathbf{L}_1 = \frac{1}{T}\mathbf{X}^\top\mathbf{X}$.
  - We set $\mathbf{M}_1 = \lambda_{\max}^{(\mathbf{L}_1)}\mathbf{I}$ so that $\mathbf{M}_1 \succeq \mathbf{L}_1$ holds.

- The objective becomes:

$$
\begin{aligned}
&\mathbf{w}^\top\mathbf{L}_1\mathbf{w} + \left(\lambda\mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b\right)^\top \mathbf{w} \\
\leq &\mathbf{w}^\top\mathbf{M}_1\mathbf{w} + 2\mathbf{w}^{(k)^\top}(\mathbf{L}_1 - \mathbf{M}_1)\,\mathbf{w} - \underbrace{\mathbf{w}^{(k)^\top}(\mathbf{L}_1 - \mathbf{M}_1)\,\mathbf{w}^{(k)}}_{\text{const.}} + \left(\lambda\mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b\right)^\top \mathbf{w} \\
= &\lambda_{\max}^{(\mathbf{L}_1)}\mathbf{w}^\top\mathbf{w} + \left(2\left(\mathbf{L}_1 - \lambda_{\max}^{(\mathbf{L}_1)}\mathbf{I}\right)\mathbf{w}^{(k)} + \lambda\mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b\right)^\top \mathbf{w} + const.
\end{aligned}
$$

Specialized Iterative Formulation

The new optimization problem at the $(k+1)$-th iteration becomes:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top \mathbf{w} + \mathbf{q}_1^{(k)\top} \mathbf{w} \\ \text{subject to} & \left.\begin{array}{l} \mathbf{w}^\top \mathbf{1} = 1, \\ \mathbf{0} \leq \mathbf{w} \leq u\mathbf{1}, \end{array}\right\} \mathcal{W}_u \end{array} \tag{4}$$

where

$$\mathbf{q}_1^{(k)} = \frac{1}{\lambda_{\max}^{(\mathbf{L}_1)}} \left(2 \left(\mathbf{L}_1 - \lambda_{\max}^{(\mathbf{L}_1)} \mathbf{I} \right) \mathbf{w}^{(k)} + \lambda \mathbf{d}_{p,u}^{(k)} - \frac{2}{T} \mathbf{X}^\top \mathbf{r}^b \right).$$

- Problem (4) can be solved with a closed-form update algorithm.
- Consider two cases of $\mathcal{W}_u$:
  1. $u = 1$ (we can discard the upper bound - easier KKT)
  2. $u < 1$

AS_1: Active Set for $u = 1$

Proposition 1

The optimal solution of the optimization problem (4) *with* $u = 1$ *is:*

$$w_i^\star = \begin{cases} -\frac{\mu+q_i}{2}, & i \in \mathcal{A}, \\ 0, & i \notin \mathcal{A}, \end{cases}$$

with

$$\mu = -\frac{\sum_{i\in\mathcal{A}} q_i + 2}{\textit{card}(\mathcal{A})},$$

and

$$\mathcal{A} = \{i \big| \mu + q_i < 0\},$$

where $\mathcal{A}$ *can be determined in* $O(\log(N))$ *steps.*

AS$_u$: Active Set for $u < 1$

Proposition 2

The optimal solution of the optimization problem (4) *with* $u < 1$ *is:*

$$w_i^\star = \begin{cases} u, & i \in \mathcal{B}_1, \\ -\frac{\mu+q_i}{2}, & i \in \mathcal{B}_2, \\ 0, & i \notin \mathcal{B}_1 \cup \mathcal{B}_2, \end{cases}$$

with

$$\mu = -\frac{\sum_{i\in\mathcal{B}_2} q_i + 2 - \textit{card}(\mathcal{B}_1)2u}{\textit{card}(\mathcal{B}_2)},$$

$$\mathcal{B}_1 = \{i | \mu + q_i \leq -2u\},$$

$$\mathcal{B}_2 = \{i | -2u < \mu + q_i < 0\},$$

where $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *can be determined in* $O(N\log(N))$ *steps.*

SLAIT

Algorithm 2: Specialized Linear Approximation for the Index Tracking problem (SLAIT)

Set $k = 0$, choose $\mathbf{w}^{(0)} \in \mathcal{W}_u$
Compute $\lambda_{\text{max}}^{(\mathbf{L}_1)}$
repeat
  Compute $\mathbf{q}_1^{(k)}$
  Solve (4) with AS_1 or AS_u and set the optimal solution as $\mathbf{w}^{(k+1)}$
  $k \leftarrow k + 1$
until convergence
return $\mathbf{w}^{(k)}$

The Big Picture

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

ℓ_0-norm approximation

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\mathbf{1}^\top \boldsymbol{\rho}_{p,u}(\mathbf{w}) \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

MM

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

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MM (for the set $\mathcal{W}_u$)

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top\mathbf{w} + \mathbf{q}_1^{(k)\top}\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W}_u \end{array}$$

Outline

Holding Constraints

- In practice, the constraints that are usually considered in the index tracking problem can be written in a convex form.
- Exception: holding constraints to avoid extreme positions or brokerage fees for very small orders.

$$\mathbf{l} \odot \mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}} \leq \mathbf{w} \leq \mathbf{u} \odot \mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}$$

- Active constraints only for the selected assets ($w_i > 0$).
- $\mathbf{w} \leq \mathbf{u} \odot \mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}} \iff \mathbf{w} \leq \mathbf{u}$ (convex).
  - Can be included in $\mathcal{W}$.

Problem Formulation

The problem formulation with holding constraints becomes (after the ℓ_0-"norm" approximation):

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W}, \\ & \mathbf{l}\odot\mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}\leq\mathbf{w}. \end{array} \tag{5}$$

- How should we deal with the non-convex constraint?

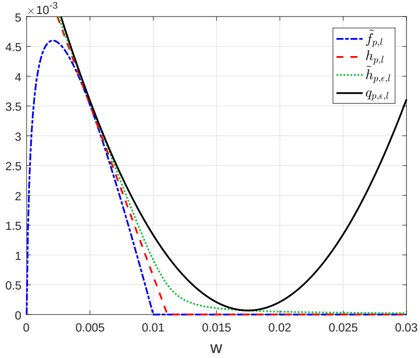
Penalization of Violations

- Hard constraint $\Longrightarrow$ Soft constraint.
  - Penalize violations in the objective.
- A suitable penalty function for a general entry w is (since the constraints are separable):

$$f_l(w) = \left(\mathcal{I}_{\{0<w<l\}} \cdot l - w\right)^+ .$$

- Approximate the indicator function with $\rho_{p,\gamma}(w)$. Since we are interested for the interval $[0, l]$ we select $\gamma = l$.

$$\tilde{f}_{p,l}(w) = \left(\rho_{p,l}(w) \cdot l - w\right)^+ .$$



- Penalty function $\tilde{f}_{p,l}(w)$ for $l = 0.01$, $p = 10^{-4}$.
- $h_{p,l}(w)$, $\tilde{h}_{p,\epsilon,l}(w)$, $q_{p,\epsilon,l}(w)$?

The optimization problem becomes:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^{b}\right\|_2^2+\lambda\mathbf{1}^{\top}\boldsymbol{\rho}_{p,u}(\mathbf{w})+\boldsymbol{\nu}^{\top}\tilde{\mathbf{f}}_{p,l}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W}, \end{array} \tag{6}$$

- $\boldsymbol{\nu}$ is a parameter vector that controls the penalization.
- $\tilde{\mathbf{f}}_{p,l}(\mathbf{w}) = [\tilde{f}_{p,l}(w_1), \ldots, \tilde{f}_{p,l}(w_N)]^{\top}$.
- Problem (6) is not convex:
  - $\rho_{p,u}(w)$ is concave $\Longrightarrow$ Linear upperbound [Lemma 1].
  - $\tilde{f}_{p,l}(w)$ is neither convex nor concave.

Lemma 3

The function $\tilde{f}_{p,l}(w)$ *is majorized at* $w^{(k)} \in [0, u]$ *by the convex function*

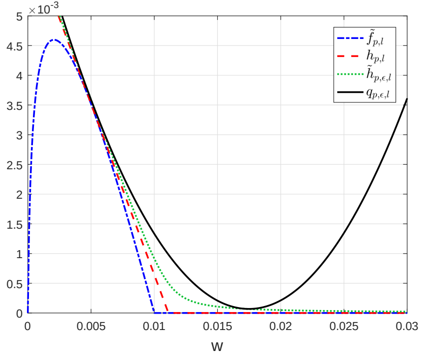
$$h_{p,l}(w, w^{(k)}) = \left(\left(d_{p,l}(w^{(k)}) \cdot l - 1\right) w + c_{p,l}(w^{(k)}) \cdot l\right)^{+},$$

where $d_{p,l}(w^{(k)})$ *and* $c_{p,l}(w^{(k)})$ *are given in Lemma 1.*

Proof: $\rho_{p,l}(w) \leq d_{p,l}(w^{(k)})w + c_{p,l}(w^{(k)})$ for $w \geq 0$ [Lemma 1]. Thus:

$$\begin{aligned} \tilde{f}_{p,l}(w) &= \max\left(\rho_{p,l}(w) \cdot l - w, 0\right) \\ &\leq \max\left((d_{p,l}(w^{(k)})w + c_{p,l}(w^{(k)})) \cdot l - w, 0\right) \\ &= \max\left(\left(d_{p,l}(w^{(k)}) \cdot l - 1\right) w + c_{p,l}(w^{(k)}) \cdot l, 0\right). \end{aligned}$$

$h_{p,l}(w, w^{(k)})$ is convex as the maximum of two convex functions.



- Penalty function $\tilde{f}_{p,l}(w)$ for $l = 0.01$, $p = 10^{-4}$.
- $h_{p,l}(w)$: linear upperbound of $\tilde{f}_{p,l}(w)$.
- $\tilde{h}_{p,\epsilon,l}(w)$, $q_{p,\epsilon,l}(w)$?

Reminder: $$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w})+\boldsymbol{\nu}^\top\tilde{\mathbf{f}}_{p,l}(\mathbf{w})\\ \text{subject to} & \mathbf{w}\in\mathcal{W}.\end{array}$$

The optimization problem at the $(k+1)$-th iteration becomes:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\left\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\right\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w}+\boldsymbol{\nu}^\top\mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)})\\ \text{subject to} & \mathbf{w}\in\mathcal{W}.\end{array} \tag{7}$$

- $\boldsymbol{\rho}_{p,u}(\mathbf{w})\leq\mathbf{d}_{p,u}^{(k)\top}\mathbf{w}+const.$ [Lemma 1]
- $\tilde{\mathbf{f}}_{p,l}(\mathbf{w})=(\boldsymbol{\rho}_{p,l}(\mathbf{w})\cdot\mathbf{l}-\mathbf{w})^+\leq\left(\text{Diag}\left(\mathbf{d}_{p,l}^{(k)}\odot\mathbf{l}-\mathbf{1}\right)\mathbf{w}+\mathbf{c}_{p,l}^{(k)}\odot\mathbf{l}\right)^+ = \mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)})$ [Lemma 3]
- Problem (7) is convex.

LAITH

Algorithm 3: Linear Approximation for the Index Tracking problem with Holding constraints (LAITH)

Set $k = 0$, choose $\mathbf{w}^{(0)} \in \mathcal{W}_u$

repeat

Compute $\mathbf{d}_{p,l}^{(k)}, \mathbf{d}_{p,u}^{(k)}$

Compute $\mathbf{c}_{p,l}^{(k)}$

Solve (7) with a solver and set the optimal solution as $\mathbf{w}^{(k+1)}$

$k \leftarrow k + 1$

until convergence

return $\mathbf{w}^{(k)}$

The Big Picture

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\|_2^2 + \lambda\|\mathbf{w}\|_0 \\
\text{subject to} & \mathbf{w} \in \mathcal{W}, \\
& \mathbf{l} \odot \mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}} \leq \mathbf{w}
\end{array}
$$

The Big Picture

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w}\in\mathcal{W}, \\ & \mathbf{l}\odot\mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}\leq\mathbf{w} \end{array}$$

ℓ_0-norm approximation / soft constraint

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w}) \\ & +\boldsymbol{\nu}^\top\tilde{\mathbf{f}}_{p,l}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array}$$

The Big Picture

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\|\mathbf{w}\|_0 \\ \text{subject to} & \mathbf{w}\in\mathcal{W}, \\ & \mathbf{l}\odot\mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}\leq\mathbf{w} \end{array}$$

ℓ_0-norm approximation / soft constraint

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w}) \\ & +\boldsymbol{\nu}^\top\tilde{\mathbf{f}}_{p,l}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array}$$

MM

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w} \\ & +\boldsymbol{\nu}^\top\mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W} \end{array}$$

Should we stop here?

✓ Again, for specific constraint sets we can derive closed-form update algorithms!

Smooth Approximation of $(\cdot)^+$ Operator

- To get a closed-form update algorithm we need to majorize again the objective.
- Let us begin with the majorization of the third term, i.e.,

$$\mathbf{h}_{p,l}(\mathbf{w}, \mathbf{w}^{(k)}) = \left(\text{Diag}\left(\mathbf{d}_{p,l}^{(k)} \odot \mathbf{l} - \mathbf{1}\right)\mathbf{w} + \mathbf{c}_{p,l}^{(k)} \odot \mathbf{l}\right)^+.$$

✓ Separable: focus only in the univariate case, i.e., $h_{p,l}(w, w^{(k)})$.

× Not smooth: cannot define majorization function at the non-differentiable point.

Smooth Approximation of $(\cdot)^+$ Operator

- Use a smooth approximation of the $(\cdot)^+$ operator:

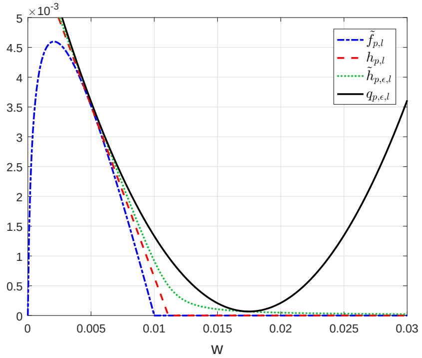
$$(x)^+ \approx \frac{x + \sqrt{x^2 + \epsilon^2}}{2},$$

where $0 < \epsilon \ll 1$ controls the approximation.

- Apply this to $h_{p,l}(w, w^{(k)}) = \left(\left(d_{p,l}(w^{(k)}) \cdot l - 1\right) w + c_{p,l}(w^{(k)}) \cdot l\right)^+$:

$$\tilde{h}_{p,\epsilon,l}(w, w^{(k)}) = \frac{\alpha^{(k)} w + \beta^{(k)} + \sqrt{(\alpha^{(k)} w + \beta^{(k)})^2 + \epsilon^2}}{2},$$

where $\alpha^{(k)} = d_{p,l}(w^{(k)}) \cdot l - 1$, and $\beta^{(k)} = c_{p,l}(w^{(k)}) \cdot l$.



- Penalty function $\tilde{f}_{p,l}(w)$ for $l = 0.01$, $p = 10^{-4}$.
- $h_{p,l}(w)$: linear upperbound of $\tilde{f}_{p,l}(w)$.
- $\tilde{h}_{p,\epsilon,l}(w)$: smooth approximation of $h_{p,l}(w)$ for $\epsilon = 10^{-3}$.
- $q_{p,\epsilon,l}(w)$?

- Now that the function is smooth we can derive a majorizer.

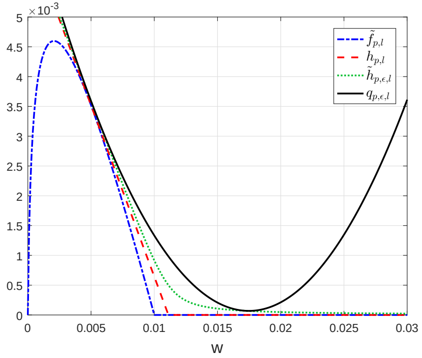
Lemma 4

The function $\tilde{h}_{p,\epsilon,l}(w, w^{(k)})$ *is majorized at* $w^{(k)}$ *by the quadratic convex function* $q_{p,\epsilon,l}(w, w^{(k)}) = a_{p,\epsilon,l}(w^{(k)})w^2 + b_{p,\epsilon,l}(w^{(k)})w + c_{p,\epsilon,l}(w^{(k)})$, *where*

$$\begin{aligned} a_{p,\epsilon,l}(w^{(k)}) &= \frac{(\alpha^{(k)})^2}{2\kappa}, \\ b_{p,\epsilon,l}(w^{(k)}) &= \frac{\alpha^{(k)}\beta^{(k)}}{\kappa} + \frac{\alpha^{(k)}}{2}, \end{aligned}$$

and $c_{p,\epsilon,l}(w^{(k)}) = \frac{(\alpha^{(k)}w^{(k)})(\alpha^{(k)}w^{(k)}+2\beta^{(k)})+2(\beta^{(k)2}+\epsilon^2)}{2\kappa} + \frac{\beta^{(k)}}{2}$ *is an optimization irrelevant constant, with* $\kappa = 2\sqrt{(\alpha^{(k)}w^{(k)} + \beta^{(k)})^2 + \epsilon^2}$.

Proof: Majorize the square root term of $\tilde{h}_{p,\epsilon,l}(w, w^{(k)})$ (concave) with its first-order Taylor approximation.



- Penalty function $\tilde{f}_{p,l}(w)$ for $l = 0.01$, $p = 10^{-4}$.
- $h_{p,l}(w)$: linear upperbound of $\tilde{f}_{p,l}(w)$.
- $\tilde{h}_{p,\epsilon,l}(w)$: smooth approximation of $h_{p,l}(w)$ for $\epsilon = 10^{-3}$.
- $q_{p,\epsilon,l}(w)$: quadratic majorizer of $\tilde{h}_{p,\epsilon,l}(w)$.

Reminder: $$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w}+\boldsymbol{\nu}^\top\mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)}) \\ \text{subject to} & \mathbf{w}\in\mathcal{W}. \end{array}$$

The optimization problem at the $(k+1)$-th iteration becomes:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top\left(\frac{1}{T}\mathbf{X}^\top\mathbf{X}+\text{Diag}\left(\mathbf{a}_{p,\epsilon,l}^{(k)}\odot\boldsymbol{\nu}\right)\right)\mathbf{w} \\ & +\left(\lambda\mathbf{d}_{p,u}^{(k)}-\frac{2}{T}\mathbf{X}^\top\mathbf{r}^b+\mathbf{b}_{p,\epsilon,l}^{(k)}\odot\boldsymbol{\nu}\right)^\top\mathbf{w} \\ \text{subject to} & \mathbf{w}\in\mathcal{W}_u. \end{array} \quad (8)$$

- $\tilde{\mathbf{h}}_{p,\epsilon,l}(\mathbf{w})\leq\mathbf{w}^\top\text{Diag}\left(\mathbf{a}_{p,\epsilon,l}^{(k)}\odot\boldsymbol{\nu}\right)\mathbf{w}+\mathbf{b}_{p,\epsilon,l}^{(k)}\odot\boldsymbol{\nu}^\top\mathbf{w}+const.$ [Lemma 4]

- Problem (8) is a QP.
- Use Lemma 2 to majorize the quadratic part:
  - $\mathbf{L}_2 = \frac{1}{T}\mathbf{X}^\top\mathbf{X} + \mathrm{Diag}\left(\mathbf{a}_{p,\epsilon,l}^{(k)} \odot \boldsymbol{\nu}\right)$.
  - $\mathbf{M}_2 = \lambda_{\max}^{(\mathbf{L}_2)}\mathbf{I}$.

The new optimization problem at the $(k+1)$-th iteration becomes:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top\mathbf{w} + \mathbf{q}_2^{(k)\top}\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W}_u, \end{array} \tag{9}$$

where

$$\mathbf{q}_2^{(k)} = \frac{1}{\lambda_{\max}^{(\mathbf{L}_2)}}\left(2\left(\mathbf{L}_2 - \lambda_{\max}^{(\mathbf{L}_2)}\mathbf{I}\right)\mathbf{w}^{(k)} + \lambda\mathbf{d}_{p,u}^{(k)} - \frac{2}{T}\mathbf{X}^\top\mathbf{r}^b + \mathbf{b}_{p,\epsilon,l}^{(k)} \odot \boldsymbol{\nu}\right).$$

SLAITH

Algorithm 4: Specialized Linear Approximation for the Index Tracking problem with Holding constraints (SLAITH)

Set $k = 0$, choose $\mathbf{w}^{(0)} \in \mathcal{W}_u$
Compute $\lambda_{\text{max}}^{(\mathbf{L}_2)}$
repeat
  Compute $\mathbf{q}_2^{(k)}$
  Solve (9) with AS$_1$ or AS$_u$ and set the optimal solution as $\mathbf{w}^{(k+1)}$
  $k \leftarrow k + 1$
until convergence
return $\mathbf{w}^{(k)}$

The Big Picture

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\|\mathbf{w}\|_0\\ \text{subject to} & \mathbf{w}\in\mathcal{W},\\ & \mathbf{l}\odot\mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}\leq\mathbf{w}\end{array}$$

ℓ_0-norm approximation / soft constraint

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w})\\ & +\boldsymbol{\nu}^\top\tilde{\mathbf{f}}_{p,l}(\mathbf{w})\\ \text{subject to} & \mathbf{w}\in\mathcal{W}\end{array}$$

MM

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w}\\ & +\boldsymbol{\nu}^\top\mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)})\\ \text{subject to} & \mathbf{w}\in\mathcal{W}\end{array}$$

The Big Picture

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\|\mathbf{w}\|_0 \\
\text{subject to} & \mathbf{w}\in\mathcal{W}, \\
 & \mathbf{l}\odot\mathcal{I}_{\{\mathbf{w}>\mathbf{0}\}}\leq\mathbf{w}
\end{array}
$$

ℓ_0-norm approximation / soft constraint

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{1}^\top\boldsymbol{\rho}_{p,u}(\mathbf{w}) \\
 & +\boldsymbol{\nu}^\top\tilde{\mathbf{f}}_{p,l}(\mathbf{w}) \\
\text{subject to} & \mathbf{w}\in\mathcal{W}
\end{array}
$$

MM

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{minimize}} & \frac{1}{T}\|\mathbf{X}\mathbf{w}-\mathbf{r}^b\|_2^2+\lambda\mathbf{d}_{p,u}^{(k)\top}\mathbf{w} \\
 & +\boldsymbol{\nu}^\top\mathbf{h}_{p,l}(\mathbf{w},\mathbf{w}^{(k)}) \\
\text{subject to} & \mathbf{w}\in\mathcal{W}
\end{array}
$$

MM (for the set $\mathcal{W}_u$) / smooth approximation

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top\mathbf{w}+\mathbf{q}_2^{(k)\top}\mathbf{w} \\
\text{subject to} & \mathbf{w}\in\mathcal{W}_u
\end{array}
$$

Extension to Other Tracking Error Measures

In all the previous formulations we used the **empirical tracking error (ETE)**:

$$\text{ETE}(\mathbf{w}) = \frac{1}{T}\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2.$$

However, we can use other tracking error measures such as:

- Downside risk: $\text{DR}(\mathbf{w}) = \frac{1}{T}\left\|(\mathbf{r}^b - \mathbf{X}\mathbf{w})^+\right\|_2^2$.
- Value-at-Risk (VaR) relative to an index.
- Conditional VaR (CVaR) relative to an index.

Extension to Downside Risk

- $\mathrm{DR}(\mathbf{w})$ is convex: can be used directly without any manipulation.
- Interestingly, if we consider the set $\mathcal{W}_u$, specialized algorithms can be derived for the DR too.

Lemma 5

The function $DR(\mathbf{w}) = \frac{1}{T}\left\|(\mathbf{r}^b - \mathbf{X}\mathbf{w})^+\right\|_2^2$ *is majorized at* $\mathbf{w}^{(k)}$ *by the quadratic convex function* $\frac{1}{T}\left\|\mathbf{r}^b - \mathbf{X}\mathbf{w} - \mathbf{y}^{(k)}\right\|_2^2$*, where*

$$\mathbf{y}^{(k)} = -\left(\mathbf{X}\mathbf{w}^{(k)} - \mathbf{r}^b\right)^+.$$

Proof of Lemma 5 (1/4)

For convenience set $\mathbf{z} = \mathbf{r}^b - \mathbf{X}\mathbf{w}$. Then:

$$\mathsf{DR}(\mathbf{w}) = \frac{1}{T}\left\|(\mathbf{z})^+\right\|_2^2 = \frac{1}{T}\sum_{i=1}^{T}\tilde{z}_i^2,$$

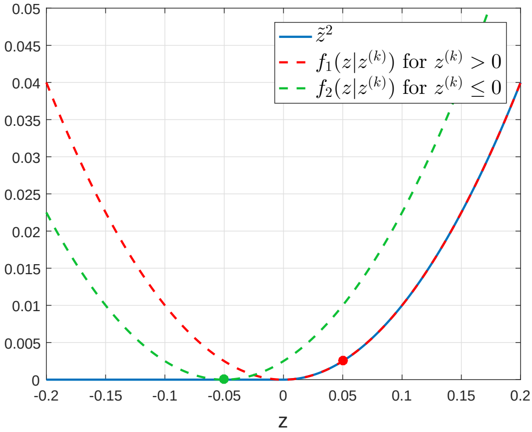
where

$$\tilde{z}_i = \begin{cases} z_i, & \text{if } z_i > 0, \\ 0, & \text{if } z_i \leq 0. \end{cases}$$

- Majorize each $\tilde{z}_i^2$. Two cases:
  - For a point $z_i^{(k)} > 0$, $f_1(z_i|z_i^{(k)}) = z_i^2$ is an upper bound of $\tilde{z}_i^2$, with $f_1(z_i^{(k)}|z_i^{(k)}) = \left(z_i^{(k)}\right)^2 = \left(\tilde{z}_i^{(k)}\right)^2$.
  - For a point $z_i^{(k)} \leq 0$, $f_2(z_i|z_i^{(k)}) = \left(z_i - z_i^{(k)}\right)^2$ is an upper bound of $\tilde{z}_i^2$, with $f_2(z_i^{(k)}|z_i^{(k)}) = \left(z_i^{(k)} - z_i^{(k)}\right)^2 = 0 = \left(\tilde{z}_i^{(k)}\right)^2$.

Proof of Lemma 5 (2/4)

For both cases the proofs are straightforward and they are easily shown pictorially:



Proof of Lemma 5 (3/4)

Combining the two cases:

$$
\begin{aligned}
\tilde{z}_i^2 \leq & \begin{cases} f_1(z_i|z_i^{(k)}), & \text{if } z_i^{(k)} > 0, \\ f_2(z_i|z_i^{(k)}), & \text{if } z_i^{(k)} \leq 0, \end{cases} \\
= & \begin{cases} (z_i - 0)^2, & \text{if } z_i^{(k)} > 0, \\ (z_i - z_i^{(k)})^2, & \text{if } z_i^{(k)} \leq 0, \end{cases} \\
= & (z_i - y_i^{(k)})^2,
\end{aligned}
$$

where

$$
\begin{aligned}
y_i^{(k)} = & \begin{cases} 0, & \text{if } z_i^{(k)} > 0, \\ z_i^{(k)}, & \text{if } z_i^{(k)} \leq 0, \end{cases} \\
= & -(-z_i^{(k)})^+.
\end{aligned}
$$

Proof of Lemma 5 (4/4)

Thus, $\mathrm{DR}(\mathbf{z})$ is majorized as follows:

$$\mathrm{DR}(\mathbf{w}) = \frac{1}{T}\sum_{i=1}^{T}\tilde{z}_i^2 \leq \frac{1}{T}\sum_{i=1}^{T}(z_i - y_i^{(k)})^2 = \frac{1}{T}\left\|\mathbf{z} - \mathbf{y}^{(k)}\right\|_2^2.$$

Substituting back $\mathbf{z} = \mathbf{r}^b - \mathbf{X}\mathbf{w}$, we get

$$\mathrm{DR}(\mathbf{w}) \leq \frac{1}{T}\left\|\mathbf{r}^b - \mathbf{X}\mathbf{w} - \mathbf{y}^{(k)}\right\|_2^2.$$

where $\mathbf{y}^{(k)} = -(-\mathbf{z}^{(k)})^+ = -(\mathbf{X}\mathbf{w}^{(k)} - \mathbf{r}^b)^+$.

Extension to Other Penalty Functions

- Apart from the various performance measures, we can select a different penalty function.
- We have used only the ℓ_2-norm to penalize the differences in the portfolio and the index.
- We can use the Huber penalty function for robustness:

$$\phi(x) = \begin{cases} x^2, & |x| \leq M, \\ M(2|x| - M), & |x| > M. \end{cases}$$

- The ℓ_1-norm.
- Many more...

Huber

Lemma 6

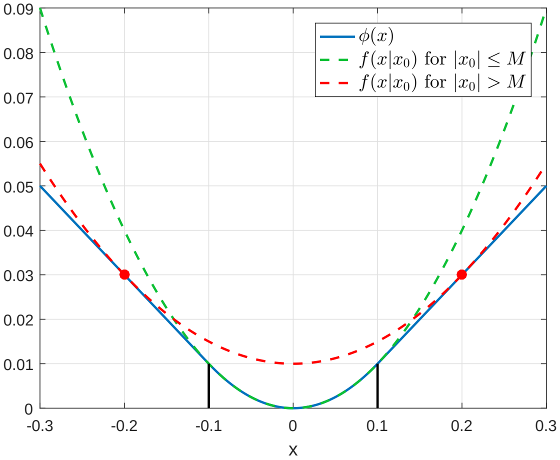
The function $\phi(x)$ *is majorized at* $x^{(k)}$ *by the quadratic convex function* $f(x|x^{(k)}) = a^{(k)}x^2 + b^{(k)}$, *where*

$$a^{(k)} = \begin{cases} 1, & |x^{(k)}| \leq M, \\ \frac{M}{|x^{(k)}|}, & |x^{(k)}| > M, \end{cases}$$

and

$$b^{(k)} = \begin{cases} 0, & |x^{(k)}| \leq M, \\ M(|x^{(k)}| - M), & |x^{(k)}| > M. \end{cases}$$

Huber


$\phi(x)$
$f(x|x_0)$ for $|x_0| \leq M$
$f(x|x_0)$ for $|x_0| > M$
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
x

Outline

For the numerical experiments we use historical data of two indices:

- S&P 500
- Russell 2000

Index	Data Period	T_{tr}	T_{tst}
S&P 500	01/01/10 - 31/12/15	252	252
Russell 2000	01/06/06 - 31/12/15	1000	252

Table 1: Index information.

- Rolling window.
- Performance measure: magnitude of daily tracking error (MDTE)

$$\text{MDTE} = \frac{1}{T - T_{\text{tr}}} \left\| \text{diag}(\mathbf{X}\mathbf{W}) - \mathbf{r}^b \right\|_2,$$

where $\mathbf{X} \in \mathbb{R}^{(T-T_{\text{tr}})\times N}$ and $\mathbf{r}^b \in \mathbb{R}^{T-T_{\text{tr}}}$.

Benchmarks

- MIP solution by Gurobi solver (MIP_{Gur}).
- Diversity Method [Jansen et al., 2002] where the $\ell_{1/2}$-norm approximation is used ($\text{DM}_{1/2}$).
- Hybrid Half Thresholding (HHT) algorithm [F. Xu et al., 2015].

S&P 500 - w/o holding constraints

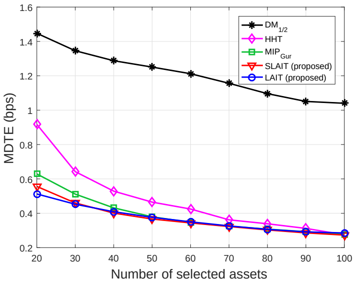


Figure 1: Magnitude of daily tracking error.

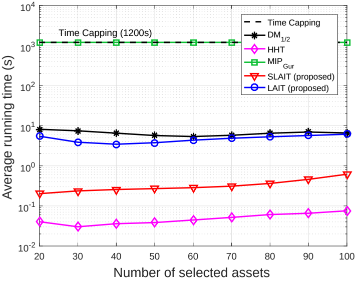


Figure 2: Average running time.

Russell 2000 - w/o holding constraints

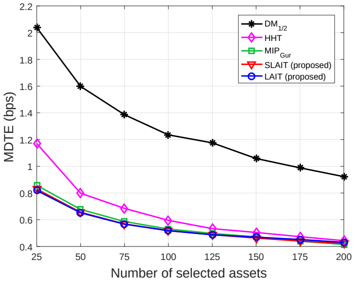


Figure 3: Magnitude of daily tracking error.

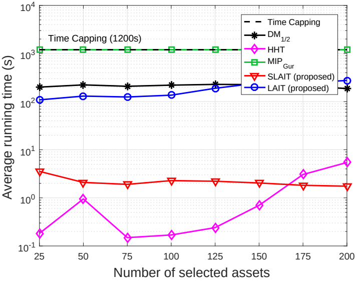


Figure 4: Average running time.

S&P 500 - w/ holding constraints

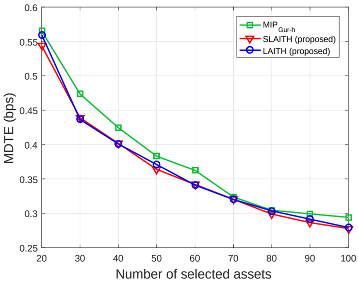


Figure 5: Magnitude of daily tracking error.

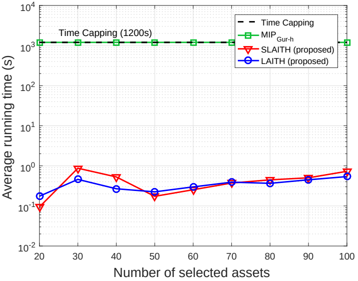


Figure 6: Average running time.

Russell 2000 - w/ holding constraints

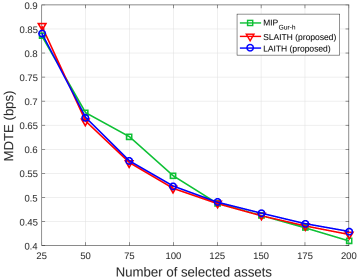


Figure 7: Magnitude of daily tracking error.

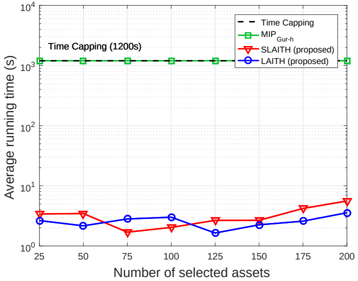


Figure 8: Average running time.

Wealth

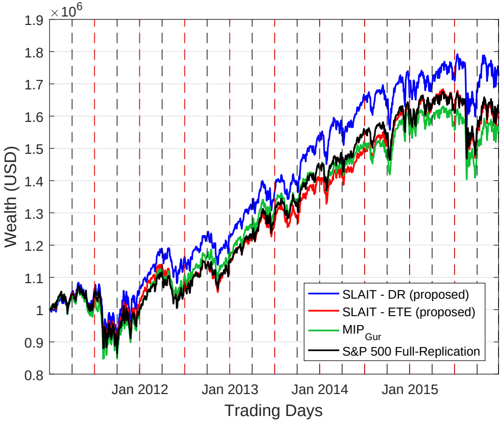


Figure 9: S&P 500 and out-of-sample tracking portfolio wealth with $\text{card}(\mathbf{w}) = 40$ (apart from the full-replication portfolio). Vertical black dashed lines: redesign. Vertical red dashed lines: rebalancing.

Average Running Time of AS_1 - AS_u

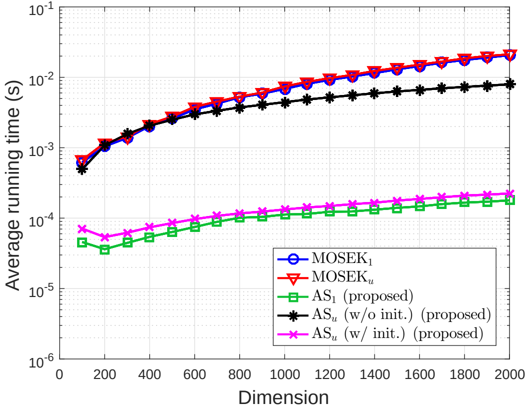


Figure 10: Comparison of AS_1 and AS_u. The algorithms $MOSEK_1$ and $MOSEK_u$ correspond to the solution using the MOSEK solver.

Conclusions

- We have developed efficient algorithms that promote sparsity for the index tracking problem.
- The algorithms are derived based on the MM framework.
  - Derivation of surrogate functions.
  - Majorization of convex problems for closed-form solutions.
- Many possible extensions.

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