STRUCTURAL DYNAMICS

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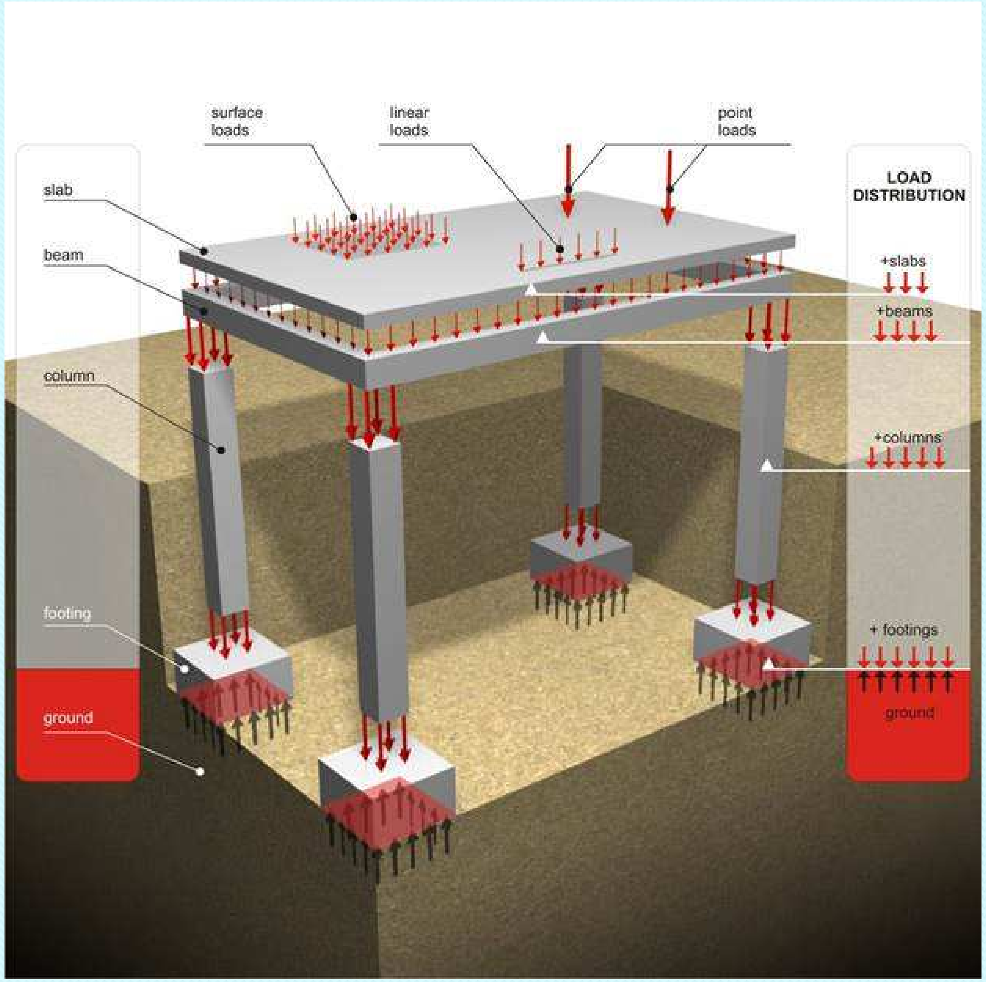
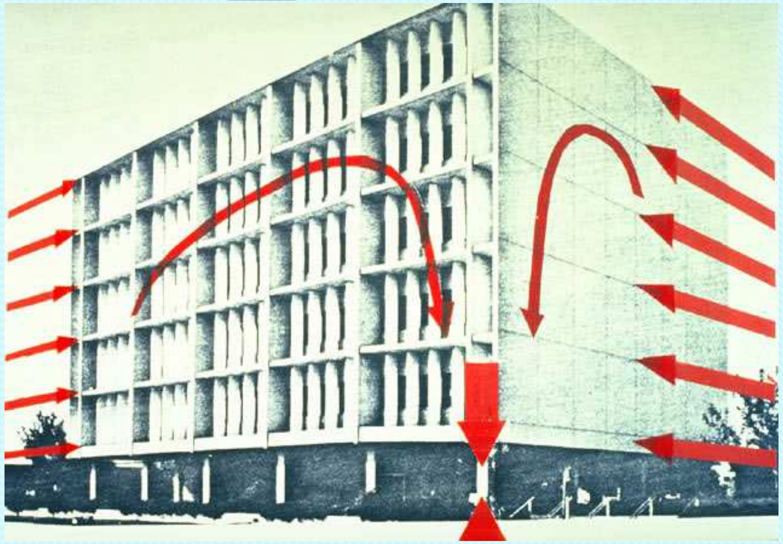
Overview of Structural Dynamics

- Structure – Members, joints, strength, stiffness, ductility
- Structure–Action (Forces)–Response (Stresses, Displacements)
- Forces :Static – Gravity, Dynamic – Time Varying, Lateral (Wind, seismic)
  (Harmonic, Random)

- Structural Dynamics – Response of structure (deflection, drift, stresses) due to application of dynamic forces.

Force	↔	(Stiffness,k)	↔	Displacements
↕		↕		↕
c/s prop. (A,I)		(EI/L, AE/L)		(Variation)
Stresses	↔	(Elastic constants, E,μ)	↔	Strains


surface loads
linear loads
point loads
slab
beam
column
footing
ground
LOAD DISTRIBUTION
+slabs
+beams
+columns
+ footings
ground



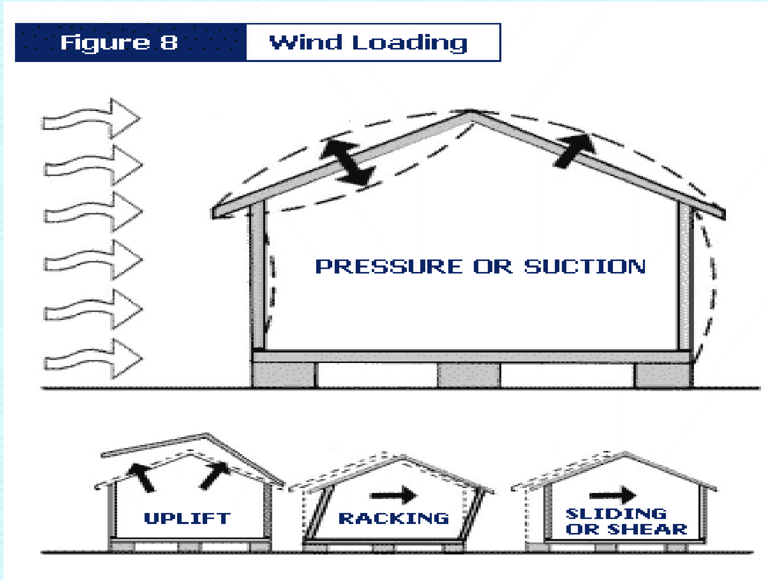
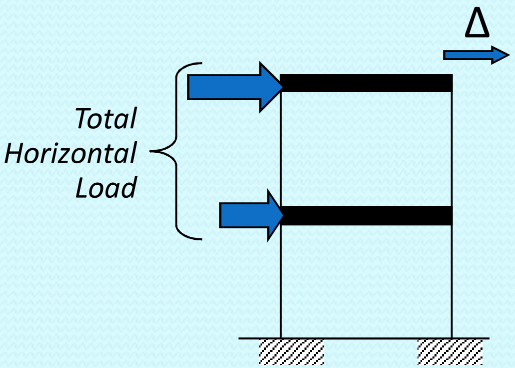
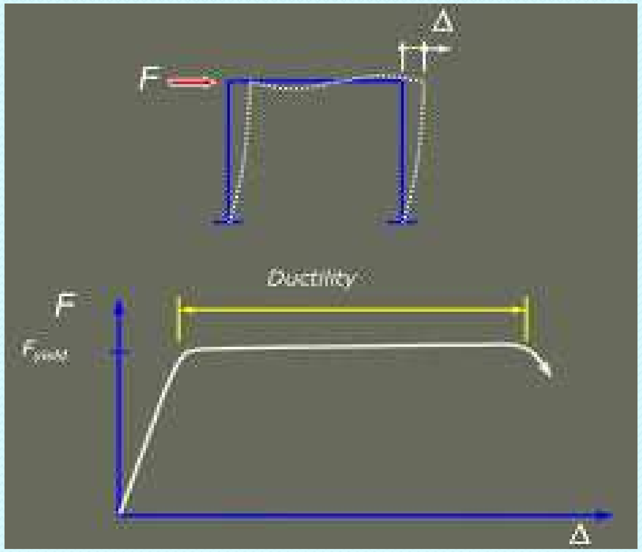
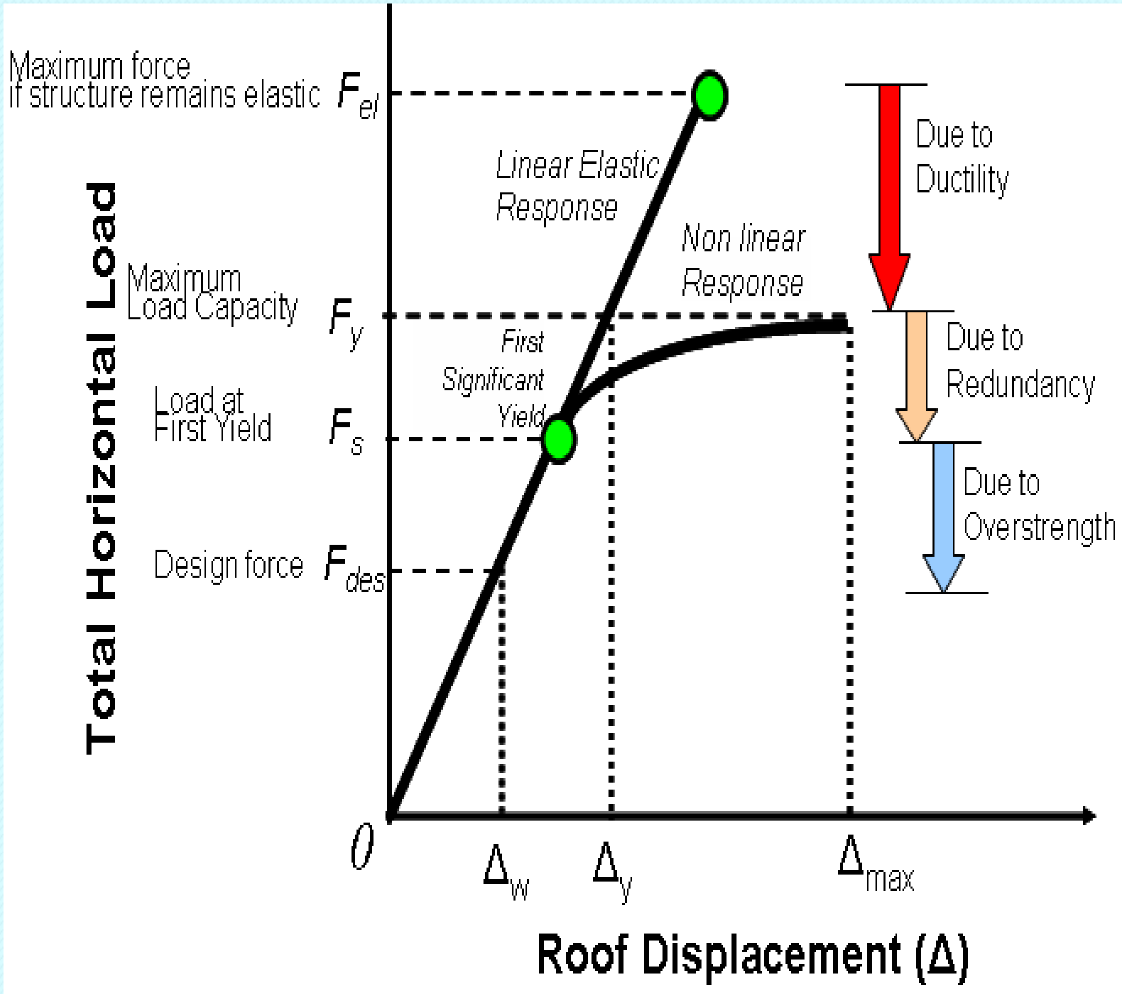


Figure 8
Wind Loading
PRESSURE OR SUCTION
UPLIFT
RACKING
SLIDING OR SHEAR


Δ
Total
Horizontal
Load


F
Δ
Ductility
F
Δ


Maximum force
if structure remains elastic
F_{el}
Linear Elastic
Response
Non linear
Response
Due to
Ductility
Maximum
Load Capacity
F_y
Due to
Redundancy
First
Significant
Yield
Load at
First Yield
F_s
Due to
Overstrength
Design force
F_{des}
Total Horizontal Load
0
Δ_w
Δ_y
Δ_{max}
Roof Displacement (Δ)

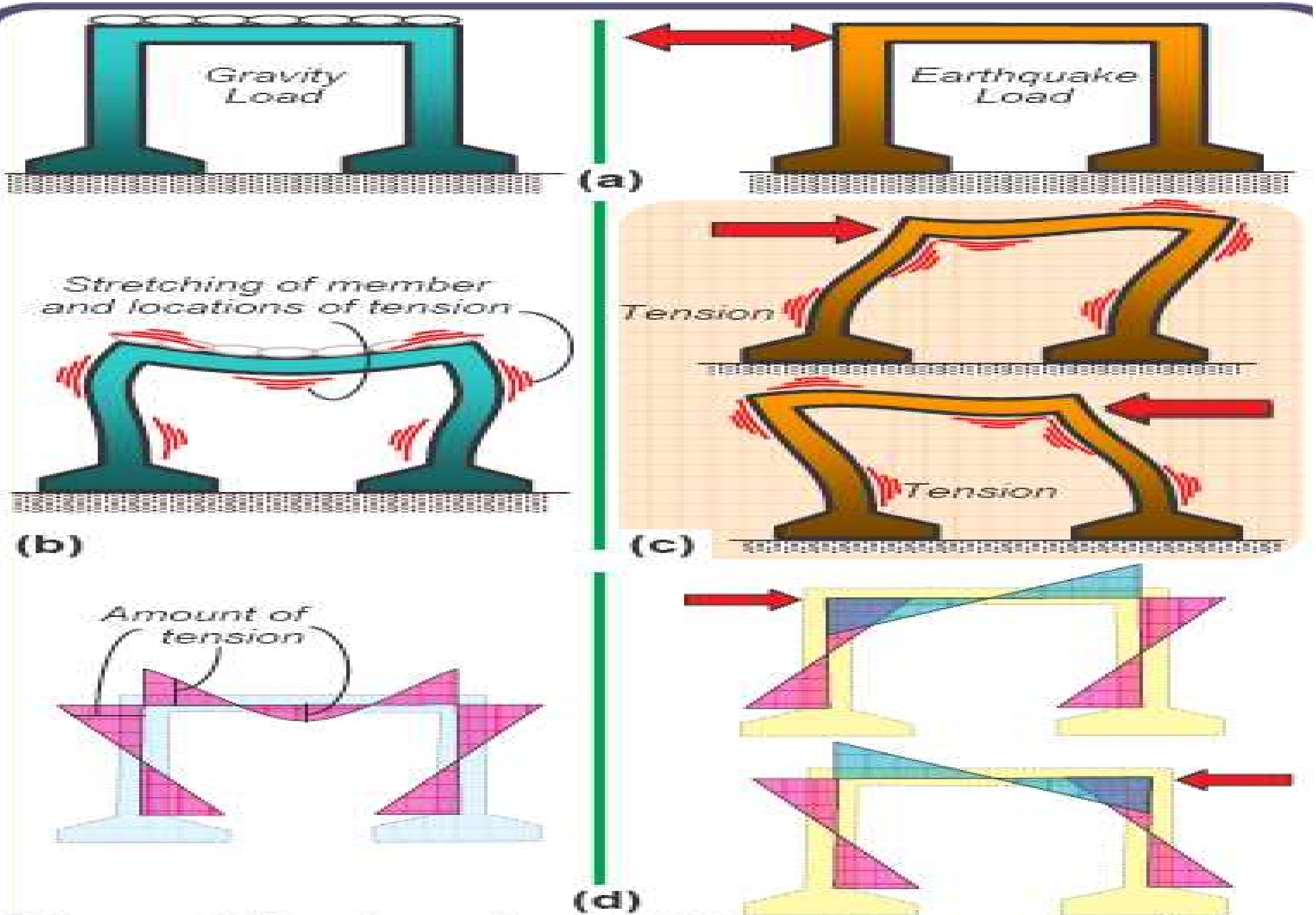
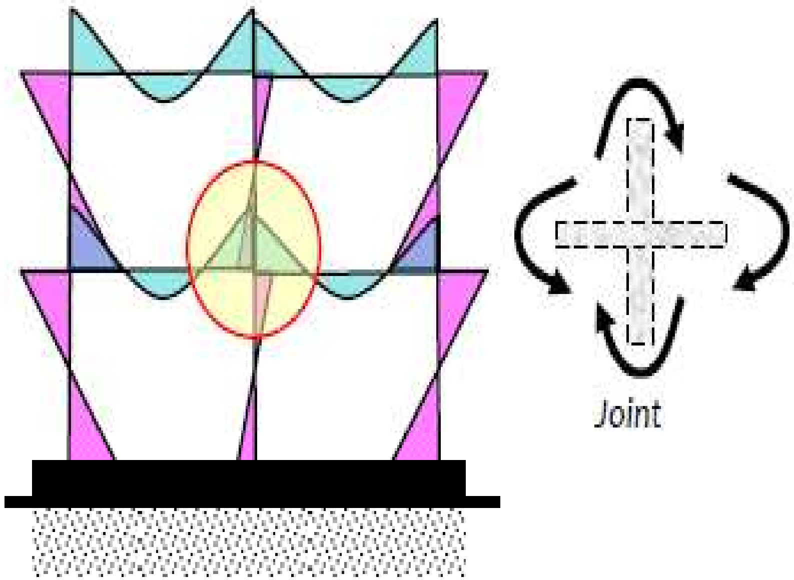
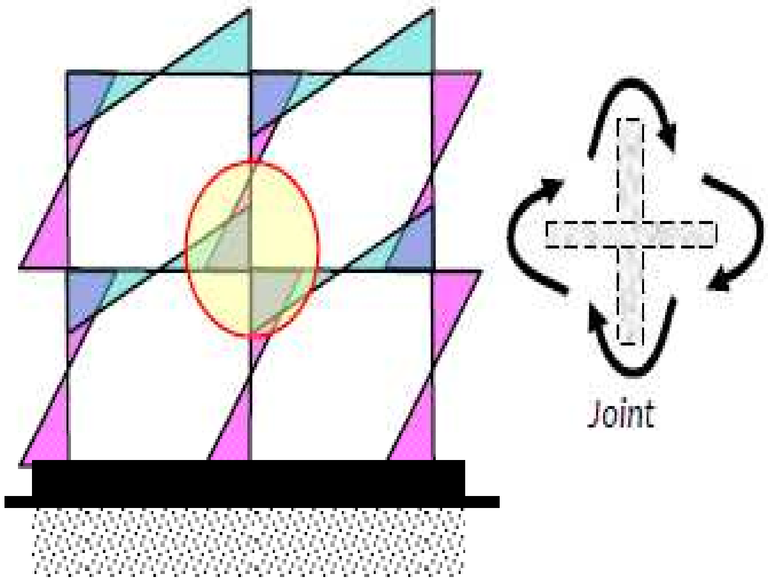


Figure 4: Earthquake shaking reverses tension and compression in members – *reinforcement is required on both faces of members.*





- ➢Forces : Non deterministic , lateral , reversible
- ➢Displacements : Large, inelastic, fatigue
- ➢Stresses : inelastic, stress reversal,
  Large SF and BM in columns and beam-col. joints

Influence of Relative Stiffness on the Seismic Response of Framed Structures

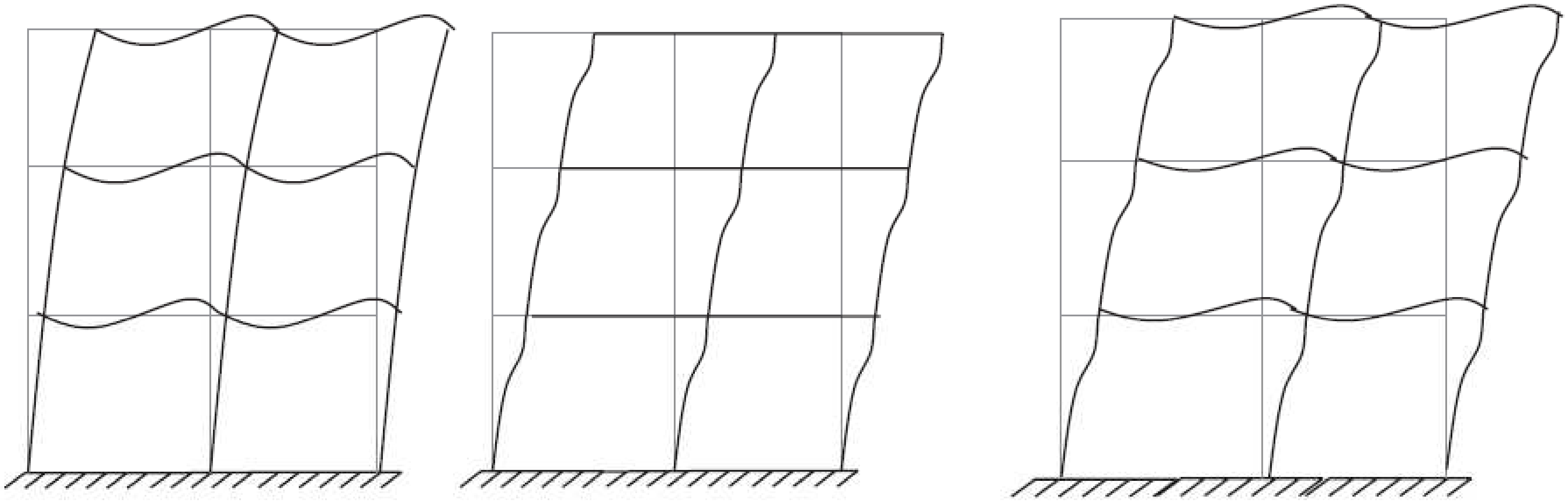


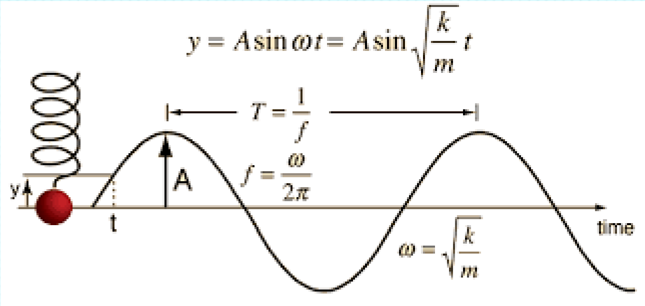
Figure 5 (a) Structure having $\lambda = 0$. (b) Structure having $\lambda = \infty$. (c) Structure having small λ.

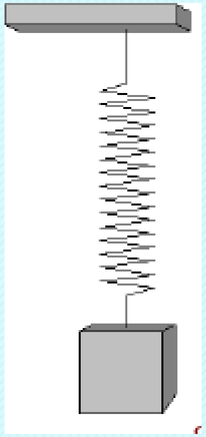
$$\lambda = \frac{\sum_{\text{Beams}} \left(\frac{EI_B}{L_B} \right)}{\sum_{\text{Columns}} \left(\frac{EI_C}{L_C} \right)}$$

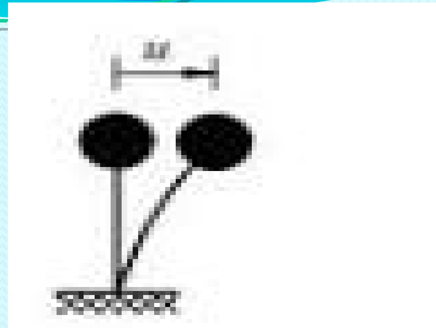
- Dynamic Response – Natural frequencies, modes
- Structural parameters affecting Dynamic Response:
  - Natural frequency (ω_n)
  - Excitation frequency (ω)
  - Damping (ξ)
  - Frequency ratio ($\eta = \omega / \omega_n$)

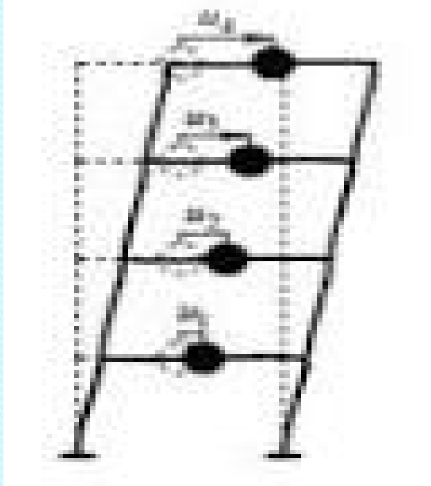
$$\omega_n = \sqrt{\frac{K_e}{m}}$$

HARMONIC VIBRATION



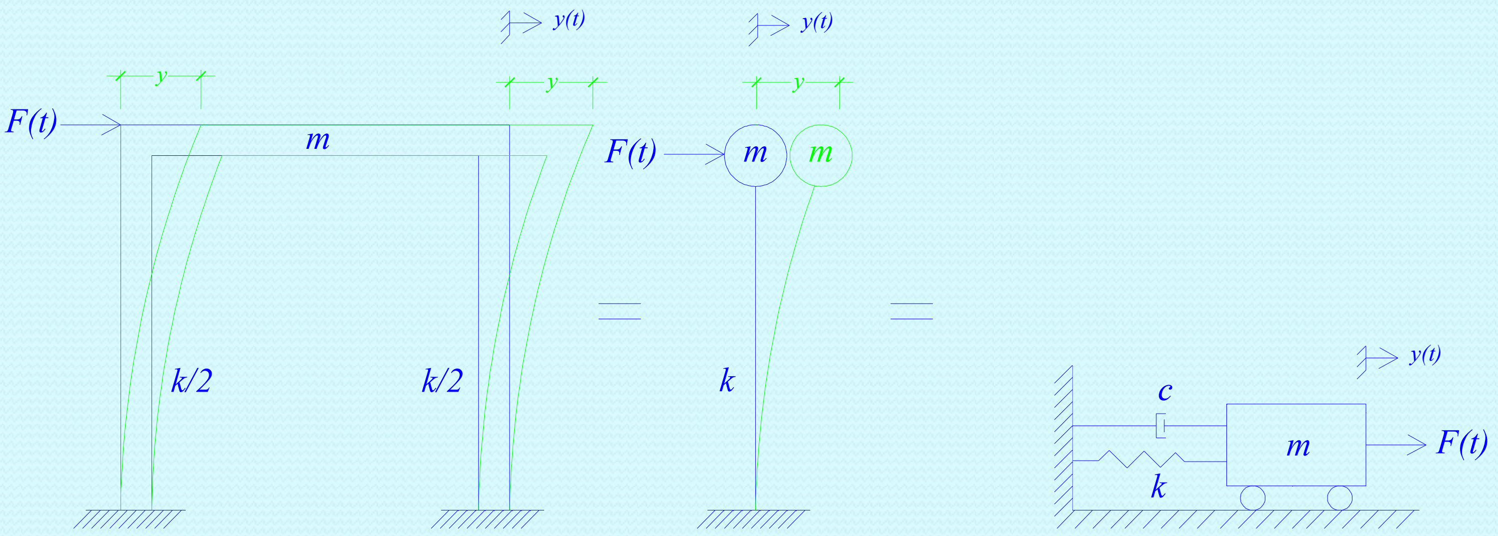






D.O.F : The number of independent coordinates required to specify configuration of vibrating system in space at any instant of time.

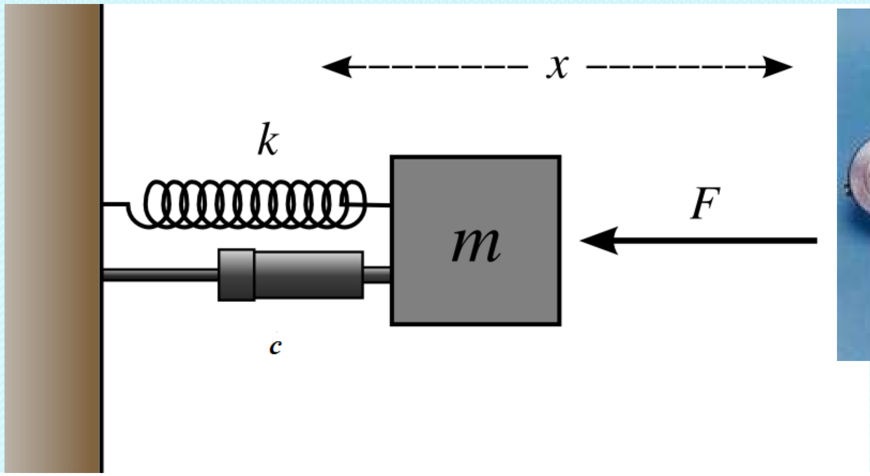
Structure - Model



Conventional Single
Story Structure SDOF

SDOF Lumped
Mass System

Mathematical Model
for SDOF



The physical system needs to be represented as an analytical or mathematical model with conceptual idealisation and simplifying assumptions.

The idealisations can be broadly classified as,

(i) Idealisation of material such as homogeneity, isotropy and idealisation of structural behaviour such as linearly elastic, elastic and nonlinear.
(ii) Idealisation of mass to be concentrated at the geometric centre and idealisation of loading to be constant, impulsive or periodic.
(iii) Idealisation of the geometry of the structure such as idealisation of continuous system as equivalent discrete systems, idealisation of one dimensional structural elements into beams, bars and two dimensional structural elements into plates, shells.

Often it becomes necessary to reduce the infinite number of degrees of freedom of a physical continuous system to a discrete number through the idealisation.

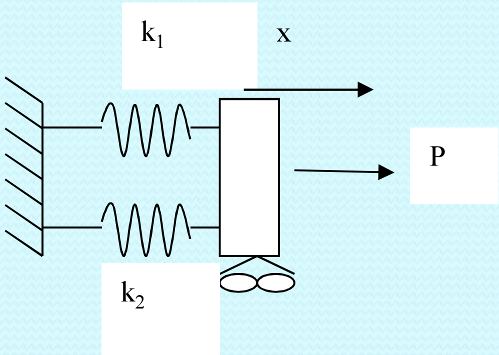
ANALYTICAL MODEL OF DYNAMIC SYSTEM

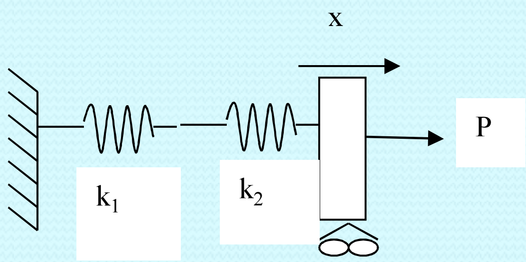
The model consists of three elements, namely the mass element (m), spring element (k) and the damping element(c). The degree of freedom is the independent displacement component (generalised co-ordinate) describing the position of the mass at any instant of time.

i. The mass element signifies the mass and the inertial characteristics of the structural mass (inertia force, m $\ddot{x}$).
ii. The spring element signifies the potential energy of and the stiffness characteristics (elastic restoring force, k x) of the structural system.
iii. The damping element signifies the energy dissipation characteristics (damping force, $c\,\dot{x}$) of the structural system. The damping in the system is generally represented as the equivalent viscous damping.
iv. The external force is represented by the excitation force, F(t) which is obviously a function of time.

Where, x is the displacement, $\dot{x}$ or (dx/dt) the velocity and $\ddot{x}$ or (d^2x/dt^2) the acceleration of mass.

Concept of equivalent spring (stiffness, k_e): Generally in the realistic structure, multiple stiffness elements exist. When such multiple stiffness elements are to be combined, the equivalent spring (stiffness, k_e) is computed as,





$$P = P_1 + P_2, \quad x_1 = x_2 = x$$

$$(k_e \times x) = (k_1 \times x) + (k_2 \times x),$$

$$k_e = k_1 + k_2$$

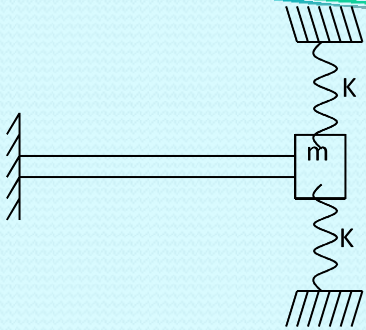
$$x = x_1 + x_2 \qquad P_1 = P_2 = P$$

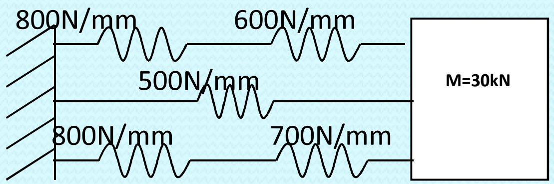
$$\frac{P}{k_e} = \frac{P}{k_1} + \frac{P}{k_2}$$

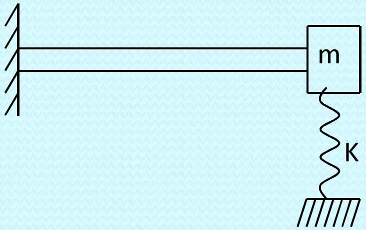
$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

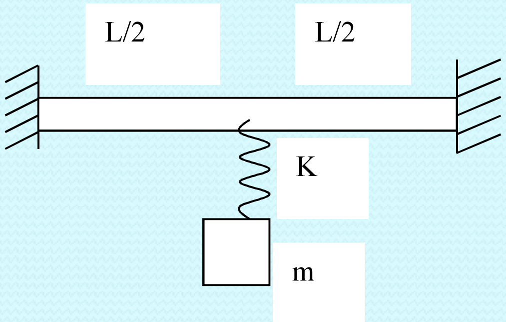
The mass may be imagined to be given a displacement to know whether the springs are in series or in parallel. If all the springs displace by the same amount, then the springs are in parallel if not, they are in series.

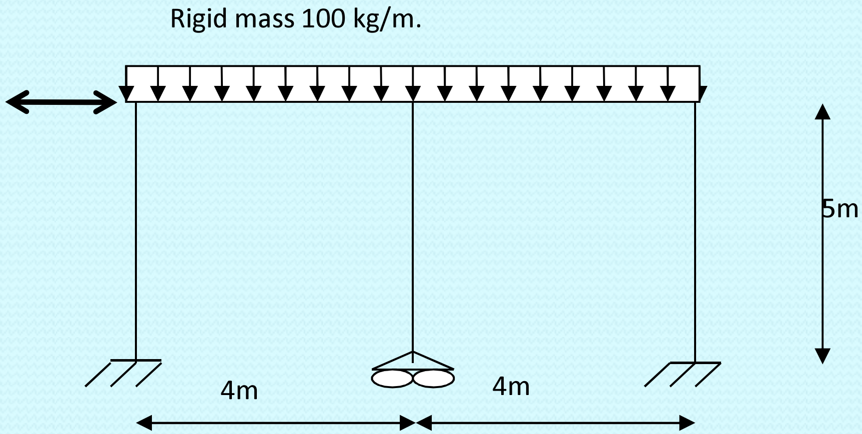

K
m

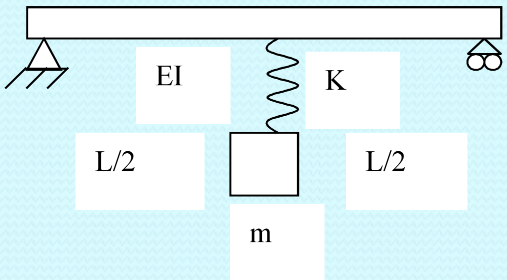

K
m
K


800N/mm
600N/mm
M=30kN
500N/mm
800N/mm
700N/mm

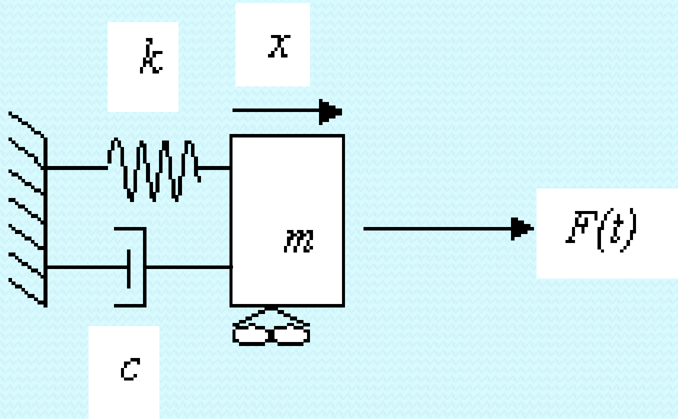

m
K


L/2
L/2
K
m

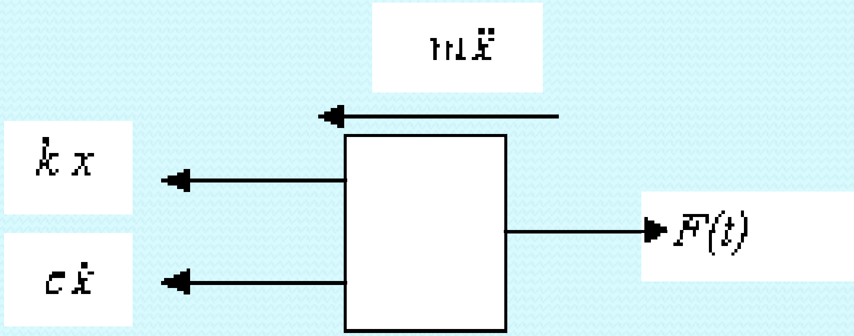

Rigid mass 100 kg/m.
5m
4m
4m


EI
K
L/2
L/2
m

For equilibrium (using D'Alembert's principle),



Idealisation



Free body diagram

$$m\ddot{x} + c\dot{x} + kx = F(t),$$

$$m\ddot{x} + c\dot{x} + kx = 0,$$

$$m\ddot{x} + kx = 0,$$

Displacement of the un-damped single degree of freedom (sdof) system

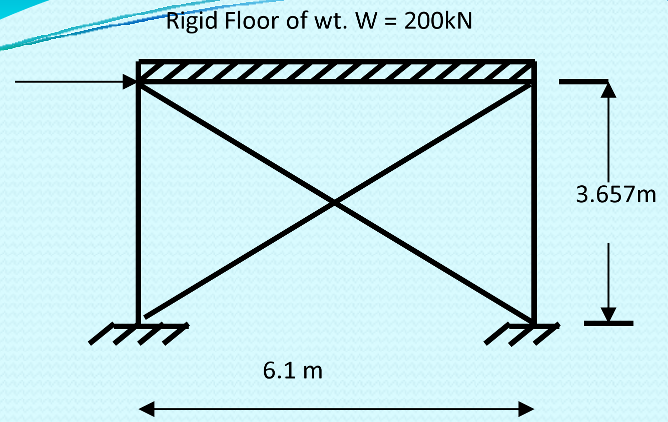
Equation of motion, $m\ddot{x} + kx = 0$

$\omega_n^2 = k/m$, $\ddot{x} + \omega_n^2 x = 0$

The general solution, $x = A\cos\omega_n t + B\sin\omega_n t$

$$x = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

$Since, \omega_n T = 2\pi$, $T = \frac{2\pi}{\omega_n}$ $A = \sqrt{x_0{}^2 + \frac{\dot{x}_0}{\omega_n}^2}$



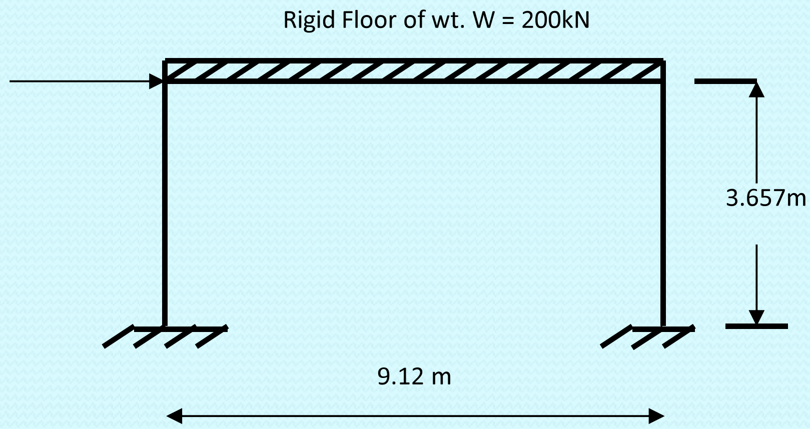
$E = 2.0 \times 10^5\ N/mm^2$

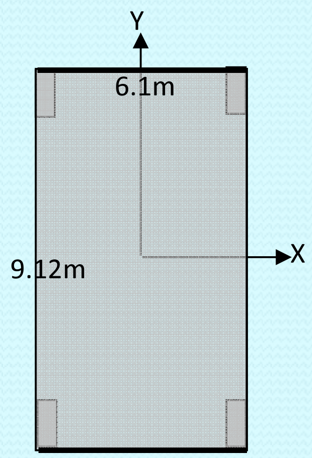
$I_{Y(col)} = 7.617 \times 10^6\ mm^4$

$I_{X(col)} = 34.16 \times 10^6\ mm^4$

$A_{\ brace} = 506.5\ mm^2$

$L_{\ brace} = 7.11m$





In X-Direction:

$$K_{brace} = \frac{A\,E}{L} \cos^2\theta \qquad K_{col} = 2 \times \frac{12\,EI}{L^3}$$

$$K = 2\times(K_{brace} + K_{col}) = 22.486 \times 10^3\ N/mm$$

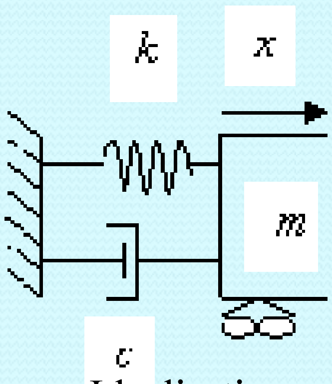
$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{22.486 \times 10^3}{(200 \times 10^3/9810)}} = 33.20\ \text{rad/sec}$$

In Y-Direction

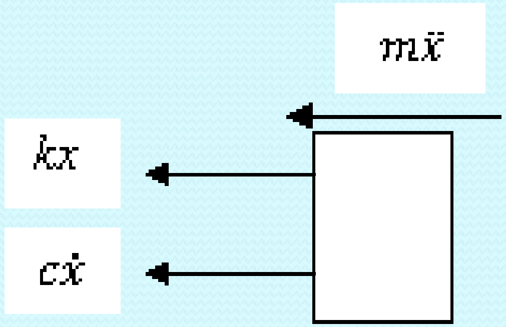
$$K = 2\times K_{col} = 6.944 \times 10^3\ N/mm$$

Natural frequency, $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{6.944 \times 10^3}{(200 \times 10^3/9810)}} = 18.455\ \text{rad/sec}$

Free damped single degree of freedom (sdof) system



a. Idealisation



b. Free body diagram

The equation of motion is, $m\ddot{x} + c\dot{x} + kx = 0$

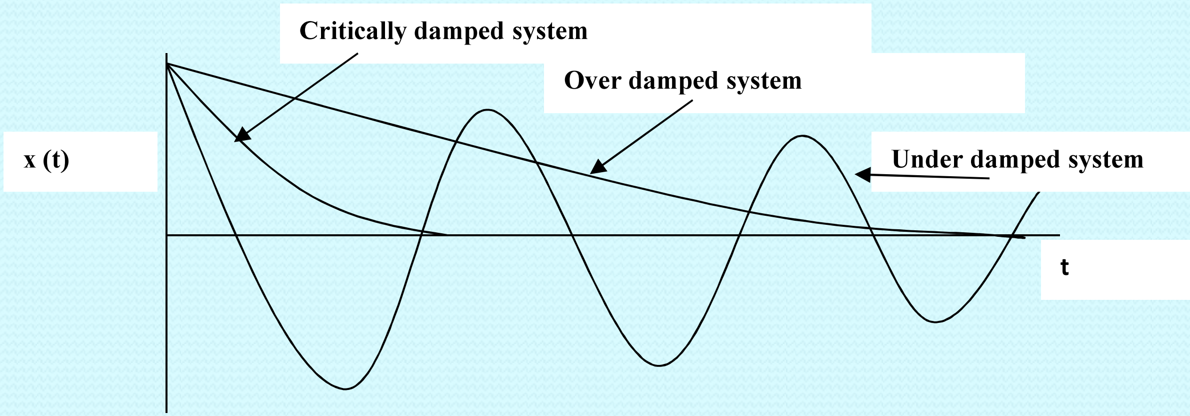
The solution $x = A\,e^{\lambda t}$

Characteristic Eqn $\lambda^2 + 2\dfrac{c}{2m}\lambda + \omega_n^2 = 0$

Solution to Characteristic Eqn $\lambda_{1,2} = -\left(\dfrac{c}{2m}\right) \pm \sqrt{\left(\dfrac{c}{2m}\right)^2 - \omega_n^2}$

The general solution, $x = e^{-\omega_n \zeta t}\left[x_0 \cos \omega_d t + \frac{\dot{x_0} + \omega_n \zeta x_0}{\omega_d} \sin \omega_d t\right]$

Amplitude, $A = \sqrt{(x_0)^2 + \left(\frac{\dot{x_0} + \omega_n \zeta x_0}{\omega_d}\right)^2}$



$$\zeta = \frac{c}{c_{cr}} = \frac{c}{(2m\omega_n)};$$

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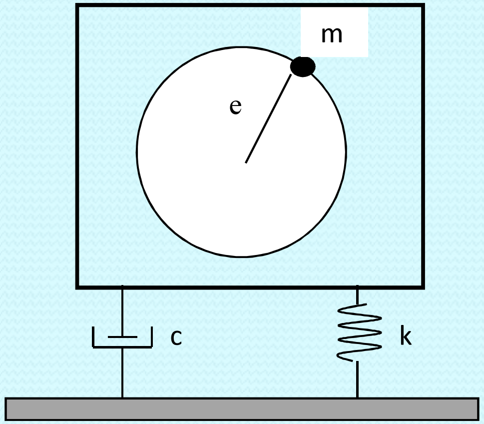
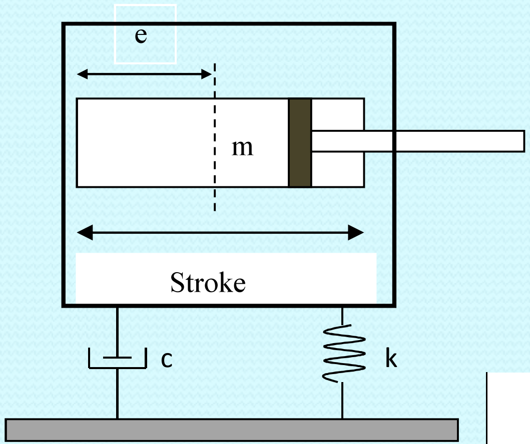
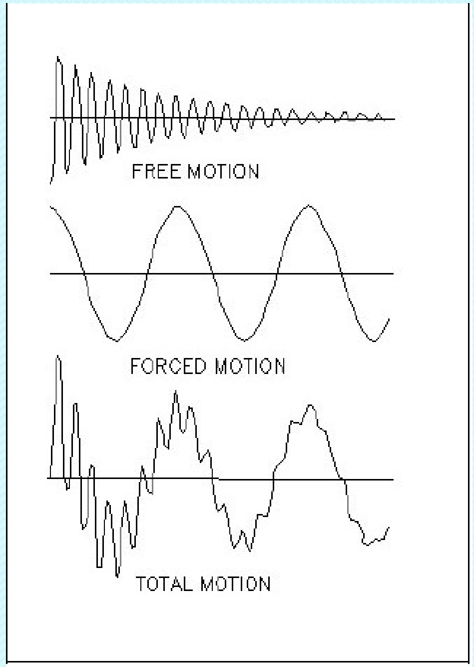
RESPONSE OF A SINGLE DEGREE OF FREEDOM (SDOF) SYSTEM TO HARMONIC LOADING





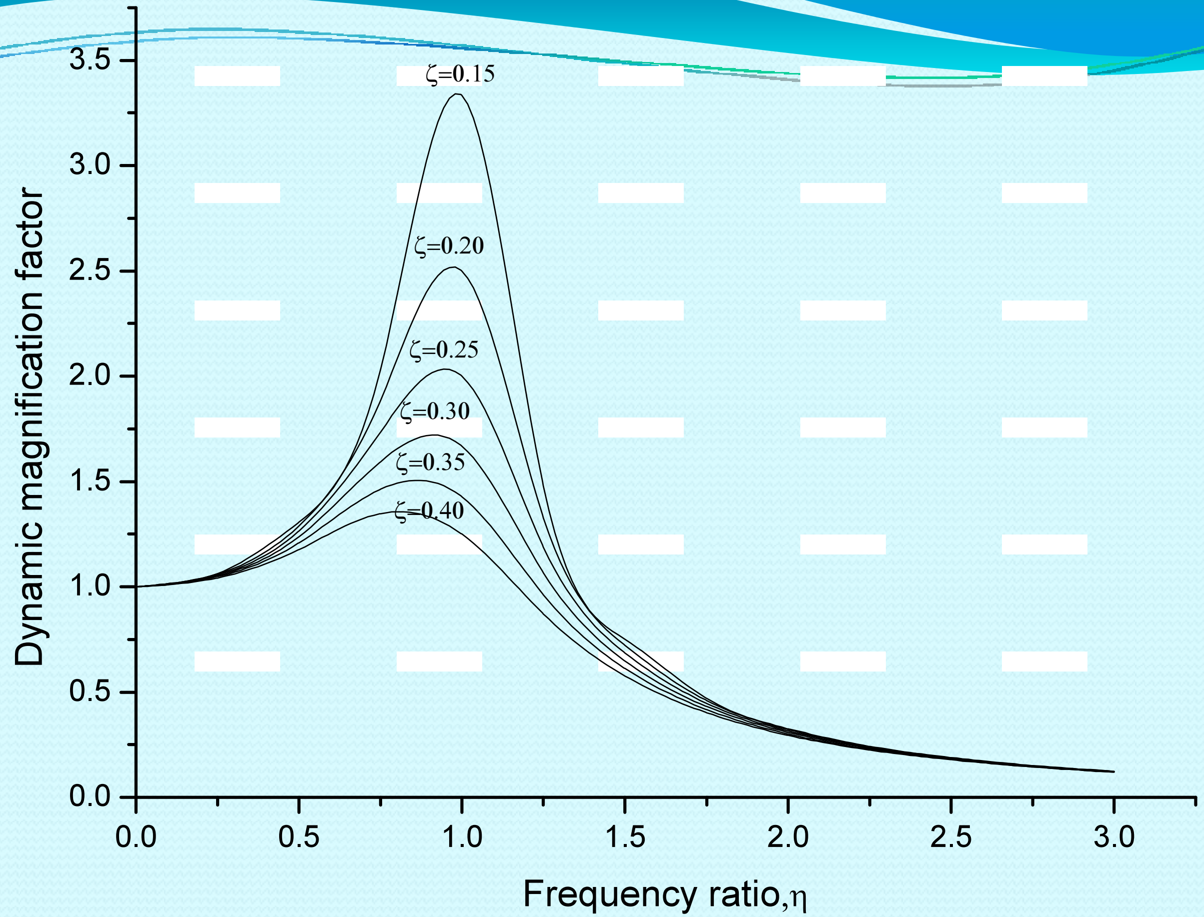


$$F_0 = m\, e\, \omega^2$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

$$x = e^{-\omega_n \zeta t}\left[x_0 \cos \omega_d t + \frac{\dot{x}_0 + \omega_n \zeta x_0}{\omega_d} \sin \omega_d t\right]$$

$$+ \frac{\frac{F_0}{K}}{\sqrt{[(1-\eta^2)^2 + (2\,\eta\,\zeta)^2]}} \sin(\omega t - \varphi),$$

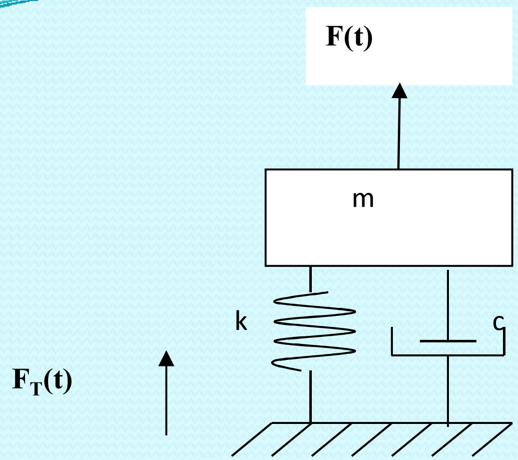


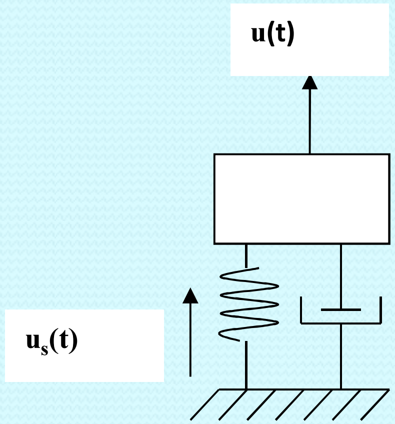
$$x_{max} = \frac{\frac{F_0}{K}}{\sqrt{[(1-\eta^2)^2 + (2\,\eta\,\zeta)^2]}}$$

η (resonant frequency ratio) = ω / ω_n ,

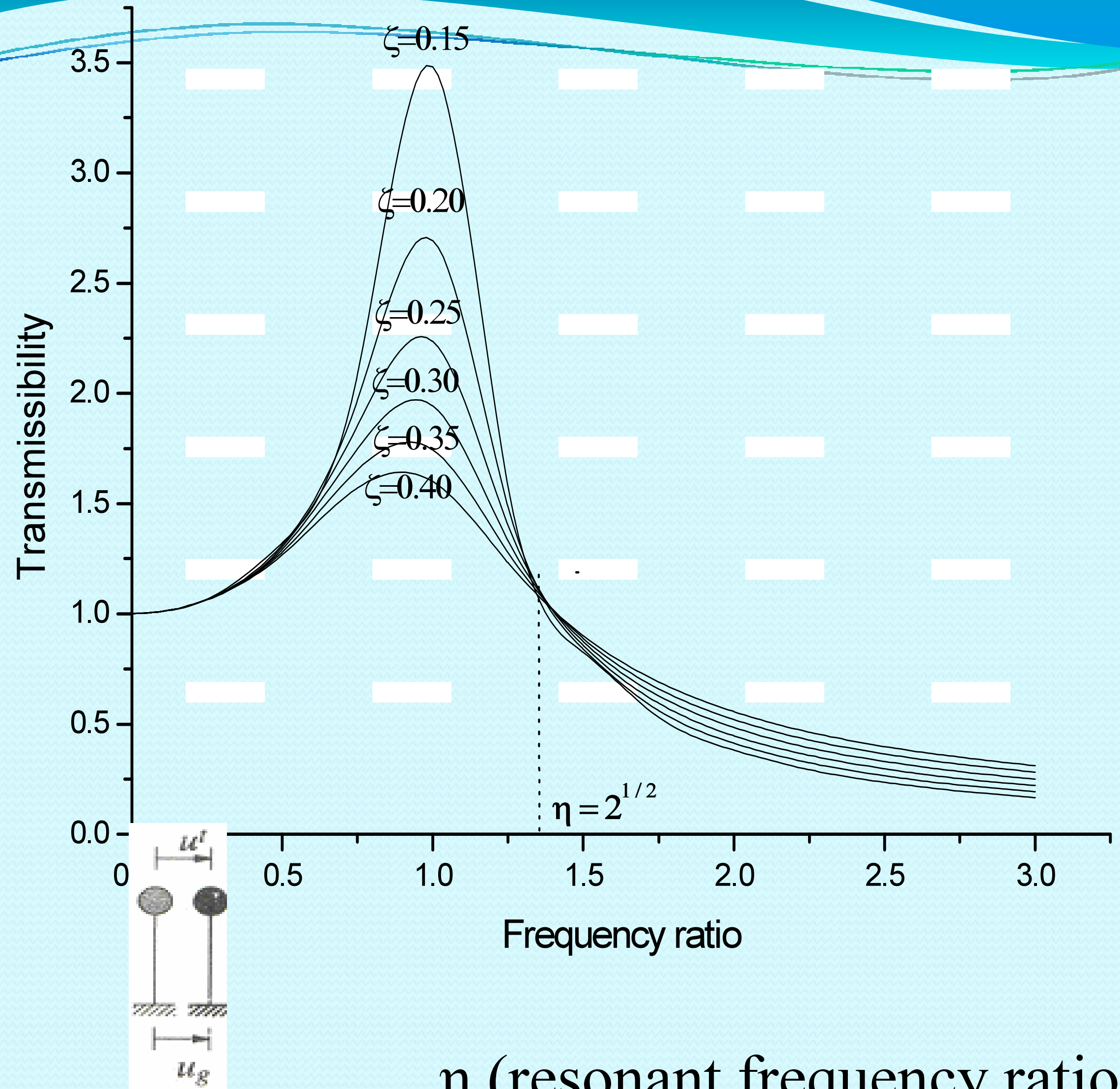
ζ (damping ratio) = $c / 2m\omega_n$

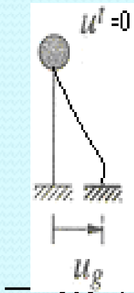
The transmissibility





$$T_r = \frac{F_{T_0}}{F_0} = \frac{\sqrt{1 + (2\zeta\eta)^2}}{\sqrt{[(1-\eta^2)^2 + (2\,\eta\,\zeta)^2]}} \qquad T_r = \frac{U}{u_{s0}} = \frac{\sqrt{1 + (2\zeta\eta)^2}}{\sqrt{[(1-\eta^2)^2 + (2\,\eta\,\zeta)^2]}}$$





$$T_r = \frac{F_{T_0}}{F_0} = \frac{\sqrt{1 + (2\zeta\eta)^2}}{\sqrt{[(1-\eta^2)^2 + (2\,\eta\,\zeta)^2]}}$$

η (resonant frequency ratio) = ω / ω_n

,

ζ (damping ratio) = $c / 2m\omega_n$

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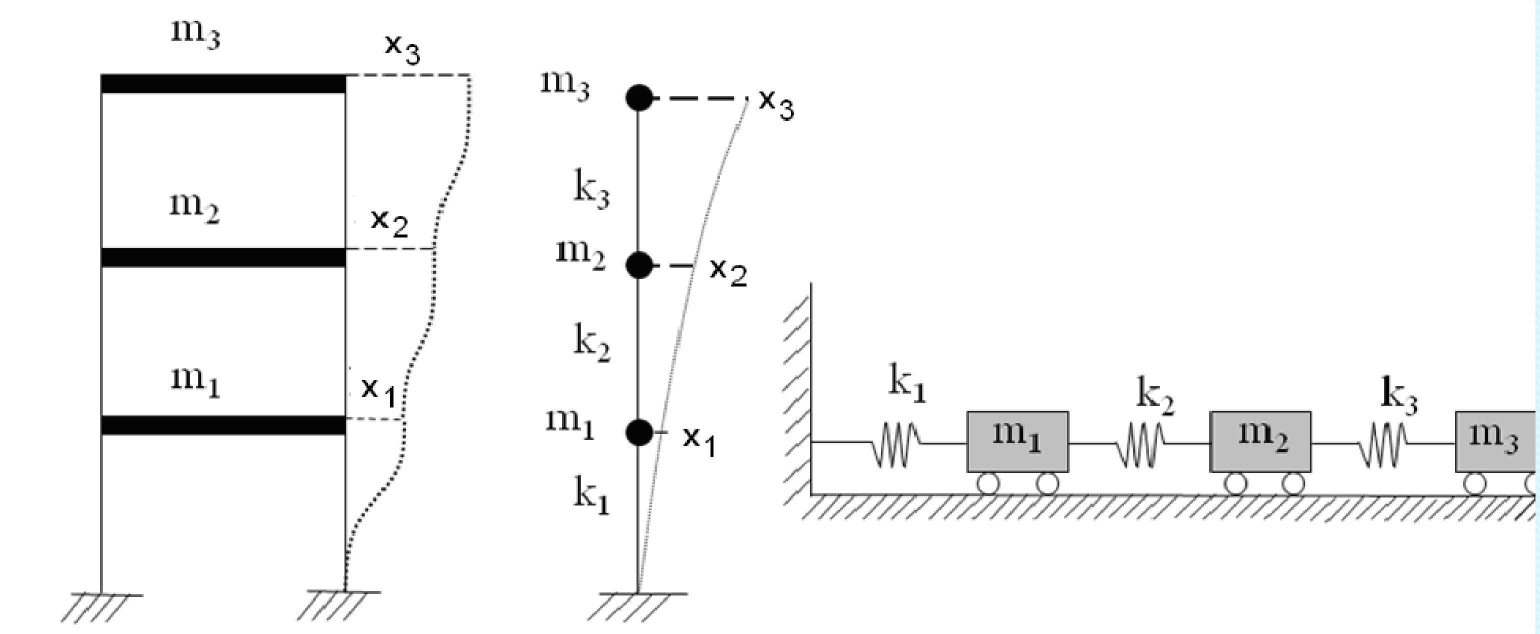
MULTI DEGREE OF FREEDOM (MDOF) SYSTEM

The theory of multi degree of freedom system is applicable to all kinds of structural systems whether discrete or continuous systems represented as discrete. Dynamic response is in more generalised form and realistic, if the structures are idealised as multi degree of freedom system. The structure represented by the sdof model does not discribe it adequately, most of the time.

To transform the building structure into a discrete number of degree of freedom with lumped masses at the floor level, following assumptions are necessary. i) The entire mass of the building is concentrated at the floor levels. ii) The axial forces do not contribute significantly for the deformation of structures and hence the stiffness iii) The floors with slabs and beams are infinitely rigid as compared to the columns and remain horizontal without rotation.

Equations of Motion (Free Vibration)



(a)Three storey frame (b)Lumped Masses (c) Conceptual model

$$[M]\{\ddot{x}\} + [K][x] = \{0\}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solution

$$[M]\{\ddot{x}\} + [K][x] = \{0\}$$

With the solution in the form,

$$x(t) = \{\varphi\} \sin(\omega t - \alpha),$$

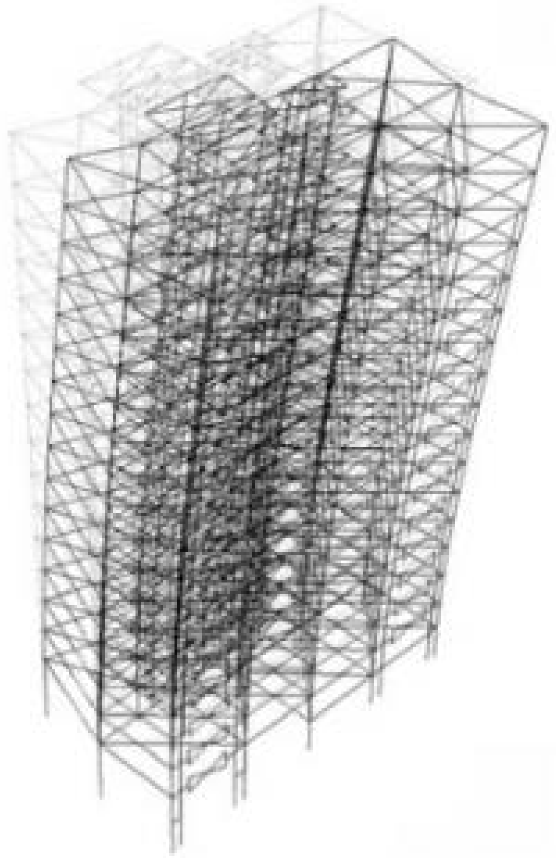
the equation of motion reduces to the following eigen value problem.

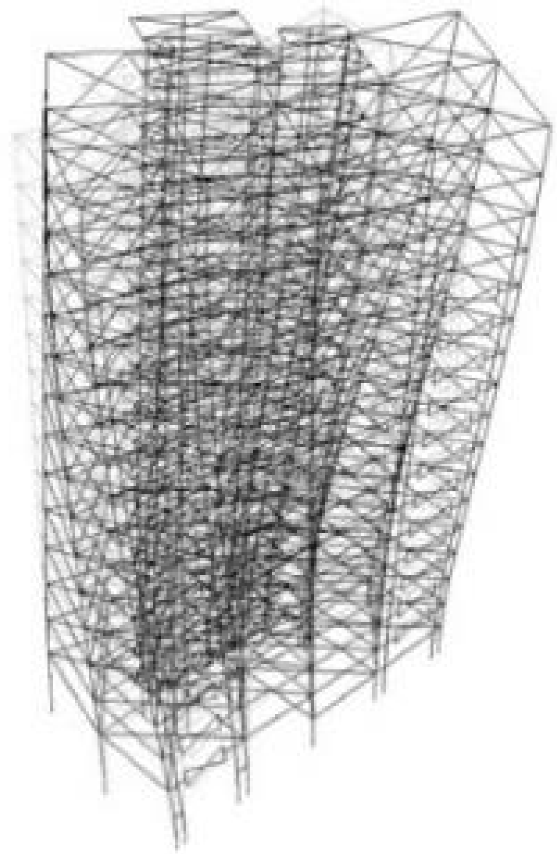
$$\{[K] - \omega^2 [M]\}\{\varphi\} = \{0\}.$$

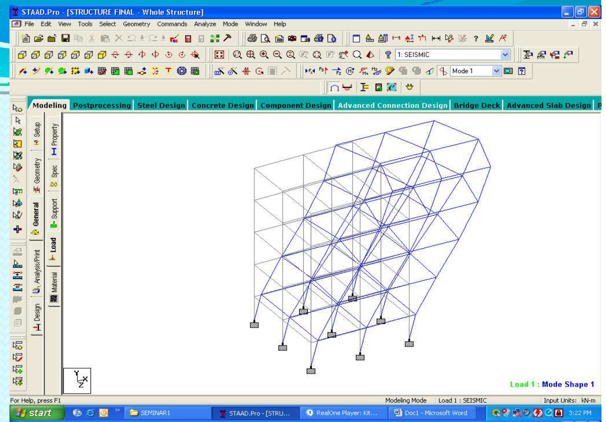
For the non trivial solution, determinant of

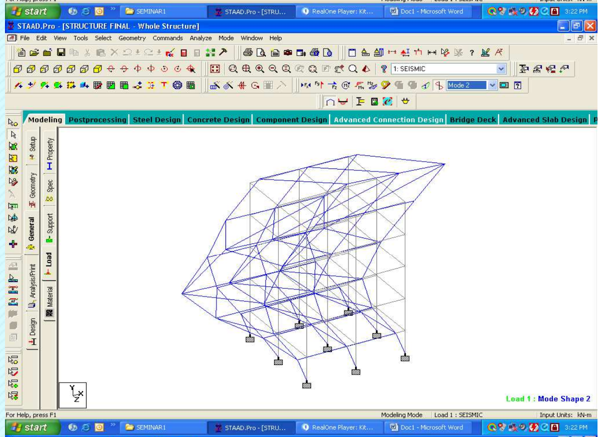
$$([K] - \omega^2 [M]) = 0,$$

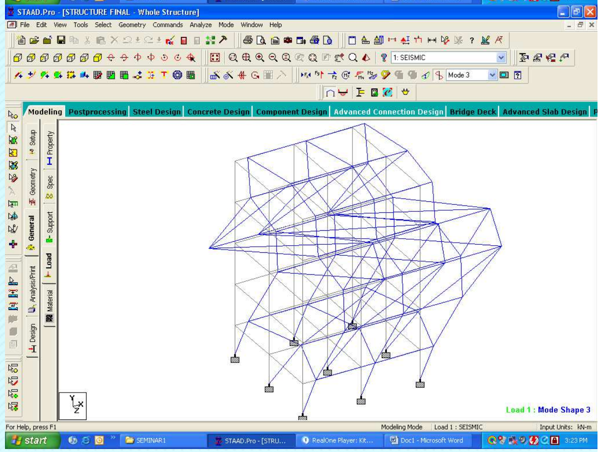
known to be a characteristic equation (an eigen value problem) in degree, n, with n eigen values ω^2 (ω, being the natural frequency). Associated with each eigen value, ω^2, there will be an eigen vector known to be the mode shape which is a characteristic deflected shape, $\{\varphi\}$.






STAAD.Pro - [STRUCTURE FINAL - Whole Structure]
Load 1 : Mode Shape 1


STAAD.Pro - [STRUCTURE FINAL - Whole Structure]
Load 1 : Mode Shape 2


STAAD.Pro - [STRUCTURE FINAL - Whole Structure]
Load 1 : Mode Shape 3

It is to be noted that the modes corresponding to the different natural frequencies satisfy the following orthogonality conditions.

$$\{\varphi\}_n^T\,[M]\{\varphi\}_r = 0 \;\; and$$

$$\{\varphi\}_n^T\,[K]\{\varphi\}_r \qquad\qquad for\ \{\varphi\}_n \neq \{\varphi\}_r$$

It is known from the elementary mechanics, that if the line of action of the force and the displacement are orthogonal to each other, the work done is zero since the component of displacement in the direction of force is zero. Therfore, the modal orthogonality property implies that the work done by the inertia forces pertaining to n^{th} mode in going through the displacements pertaining to r^{th} mode is zero.

Normal mode method :

Normal modes have the important property of making the system matrices (especially the stiffness matrix) diagonal, i.e. separating the degrees of freedom so that they can be treated as a series of single degrees of freedom. Such uncoupled system of equations is much easier to deal with than a coupled system. Calculation of normal modes, however, requires the use of eigenvalues, ω^2 and eigenvectors $\{\phi\}$.

Normalisation (Transformation to modal co-ordinates) of system matrices, $x(t) = [\phi]\{z(t)\}$ where, $x(t)$ is the vector of displacement of individual masses, $\{z(t)\}$, the vector of displacement at global co-ordinates and $[\phi]$, the transformation matrix.

$$[M]\{\ddot{x}\} + [c]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

$$x(t) = [\phi]\{z(t)\}$$

$$[M][\varphi]\{\ddot{z}\} + [c][\varphi]\{\dot{z}\} + [K][\varphi]\{z\} = \{F(t)\}$$

$$[\varphi]^T[M][\varphi]\{\ddot{z}\} + [\varphi]^T[c][\varphi]\{\dot{z}\} + [\varphi]^T[K][\varphi]\{z\} = [\varphi]^T\{F(t)\}$$

Normalisation of mass matrix, $[M]_{normalised} = [\phi]^T [M] [\phi]$
Normalisation of stiffness matrix, $[K]_{normalised} = [\phi]^T [K] [\phi]$
Normalisation of force matrix, $[F(t)]_{normalised} = [\phi]^T [F(t)]$

THANK YOU