MAT 137Y: Calculus with proofs
Assignment 3
Due on Thursday, Nov 16 by 11:59pm via GradeScope

Instructions

This problem set is based on Unit 3 and Unit 4 (4.1-4.4): Derivatives and Inverse Functions. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: ______________________________

Student number: ______________________________

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Student number: ______________________________

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ______________________________

2) ______________________________

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Assume that g is one-to-one function. Show that

$$\frac{d}{dx} f\left(g(x)\right) = g'(x)\, f'\left(g(x)\right)$$

without using any differentiation rules.

2. Let $a \in \mathbb{R}$. Let f be a function defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. If it is true, prove it directly. Hint: often times, the easiest way to prove something is false is by providing a counter example and proving that counter example satisfies the required conditions.

   (a) IF the limit $\lim_{h \to 0} \dfrac{f(a+h) - 2f(a) + f(a-h)}{h^2}$ exists,
   THEN f is twice differentiable at $x = a$.

◯ True ◯ False

(b) IF there exists a function $m(x)$ with domain $\mathbb{R}$ such that $f(x) - f(a) = m(x)(x - a)$, THEN $f(x)$ is differentiable at $x = a$.

◯ True ◯ False

3. Consider the function $f(x)$ given by the equation

$$f(x) = 2023 + \cfrac{2023}{2022 + \cfrac{2022}{2021 + \cfrac{2021}{2020 + \ddots + \cfrac{2}{1 + \frac{1}{x}}}}}$$

Find the equation of the line tangent to the graph of $f(x)$ at the point with x-coordinate 1.
Hint: Construct a sequence of functions $f_1, f_2, f_3, \cdots, f_{2023}$ such that $f_{2023} = f(x)$. Then use induction twice (to find $f'(1)$ and $f(1)$).

Extra Page

4. For this problem, let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that a non-empty set $U \subseteq \mathbb{R}$ is **ajar** if

$$\forall u \in U, \exists r > 0 \text{ such that } (u - r, u + r) \subseteq U.$$

(a) Show that $(0, 2)$ is ajar.

(b) Find a set that is not ajar. You don't need to prove it.
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In the following parts, we will use the follow two definitions.
For $A \subseteq \mathbb{R}$, we define $f(A) := \{f(a) : a \in A\}$.
For $B \subseteq \mathbb{R}$, we define $f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\}$.
Note that $f(A)$ and $f^{-1}(B)$ are both sets.

(c) Show that $\forall a \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0$ such that $f((a - \delta, a + \delta)) \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$ is equivalent to the definition of f is continuous everywhere.

(d) Assume that f is continuous everywhere. Show that for any non-empty subset $U \subseteq \mathbb{R}$, if U is ajar, then $f^{-1}(U)$ is ajar.

Hint: To prove this, you need to prove and use these facts 1) for all non-empty $A \subseteq \mathbb{R}$, $A \subseteq f^{-1}(f(A))$. This proof should be one line proof; 2) $f^{-1}(B_1) \subseteq f^{-1}(B_2)$ if non-empty sets B_1, B_2 satisfies $B_1 \subseteq B_2 \subseteq \mathbb{R}$. This proof should be very short; and 3) the results from part c).

(e) Assume that for any non-empty subset $U \subseteq \mathbb{R}$, if U is ajar, then $f^{-1}(U)$ is ajar. Prove that f is continuous everywhere.

Hint: Let $a \in \mathbb{R}$ and $\varepsilon > 0$ and consider the ajar set $(f(a) - \varepsilon,\ f(a) + \varepsilon)$ (you may assume this set is ajar without proof). You may also prove and use these facts 1) for all non-empty $B \subseteq \mathbb{R}$, $f\left(f^{-1}(B)\right) \subseteq B$; 2) $f(A_1) \subseteq f(A_2)$ if non-empty sets A_1, A_2 satisfies $A_1 \subseteq A_2 \subseteq \mathbb{R}$. Both proofs should be very short; and 3) the results of part c).