How to Safely Use Extensionality in Liquid Haskell (extended version)

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Refinement type checkers are a powerful way to reason about functional programs. For example, one can prove properties of a slow, specification implementation, porting the proofs to an optimized implementation that behaves the same. Without functional extensionality, proofs must relate functions that are fully applied. When data itself has a higher-order representation, fully applied proofs face serious impediments! When working with first-order data, fully applied proofs lead to noisome duplication when using higher-order functions.

While dependent type theories are typically consistent with functional extensionality axioms, refinement type systems with semantic subtyping treat naive phrasings of functional extensionality inconsistently, leading to *unsoundness*. We demonstrate this unsoundness and develop a new approach to equality in Liquid Haskell: we define a propositional equality in a library we call PEq. Using PEq avoids the unsoundness while still proving useful equalities at higher types; we demonstrate its use in several case studies. We validate PEq by building a small model and developing its metatheory. Additionally, we prove metaproperties of PEq inside Liquid Haskell itself using an unnamed folklore technique, which we dub 'classy induction'.

1 INTRODUCTION

Refinement types have been extensively used to reason about functional programs [Constable and Smith 1987; Rondon et al. 2008; Rushby et al. 1998; Swamy et al. 2016; Xi and Pfenning 1998]. Higher-order functions are a key ingredient of functional programming, so reasoning about function equality within refinement type systems is unavoidable. For example, Vazou et al. [2018a] prove function optimizations correct by specifying equalities between fully applied functions. Do these equalities hold in the context of higher order functions (e.g., maps and folds) or do the proofs need to be redone for each fully applied context? Without functional extensionality (a/k/a funext), one must duplicate proofs for each higher-order function. Worse still, all reasoning about higher-order representations of data requires first-order observations.

Most verification systems allow for function equality by way of functional extensionality, either built-in (e.g., Lean) or as an axiom (e.g., Agda, Coq). Liquid Haskell and F*, two major, SMT-based verification systems built on refinement types, are no exception: function equalities come up regularly. But, in both these systems, the first attempt to give an axiom for functional extensionality was wrong.¹ A naive funext axiom proves equalities between unequal functions.

Our first contribution is to expose why a naive encoding of unfext is inconsistent ($\S 2$). At first sight, function equality can be encoded as a refinement type stating that for functions f and g, if we can prove that f x equals g x for all x, then the functions f and g are equal:

funext ::
$$\forall$$
 a b. f:(a \rightarrow b) \rightarrow g:(a \rightarrow b) \rightarrow (x:a \rightarrow {f x = g x}) \rightarrow {f = g}

(The 'refinement proposition' {e} is equivalent to {_:() | e}.) On closer inspection, funext does not encode function equality, since it is not reasoning about equality on the domains of the functions. What if we instantiate the domain type parameter a's refinement to an intersection of the domains of the input functions or, worse, to an uninhabited type? Would such an instantiation of funext still prove equality of the two input functions? It turns out that this naive extensionality

 $^{^1}$ See https://github.com/FStarLang/FStar/issues/1542 for F*'s initial, wrong encoding and §7 for F*'s different solution. We explain the situation in Liquid Haskell in §2.

axiom is inconsistent with refinement types: in §2 we assume this naive funext and prove false—disaster! We work in Liquid Haskell, but the problem generalizes to any refinement type system that allows for semantic subtyping along with refinement polymorphism, i.e., refinements inferred from constraints [Rondon et al. 2008]. To be sound, proofs of function equality must carry information about the domain type on which the compared functions are equal.

Our second contribution is to define a type-indexed propositional equality as a Liquid Haskell library (§3), where the type indexing uses Haskell's GADTs and Liquid Haskell's refinement types. We call the propositional equality PEq and find that it adequately reasons about function equality: we can prove the theorems we want, and we can't prove the (non-)theorems we don't want. Further, we prove in Liquid Haskell <code>itself</code> that the implementation of PEq is an equivalence relation, i.e., it is reflexive, symmetric, and transitive. To conduct these proofs—which go by induction on the structure of the type index—we applied a heretofore-unnamed folklore proof methodology, which we dub <code>classy induction</code> (§3.3).

Our third contribution is to use PEq to prove equalities between functions (§4). As simple examples, we prove optimizations correct as equalities between functions (i.e., reverse), work carefully with functions that only agree on certain domains and dependent codomains, lift equalities to higher-order contexts (i.e., map), prove equivalences with multi-argument higher-order functions (i.e., fold), and showcase how higher-order, propositional equalities can co-exist with and speedup executable code. We also provide a more substantial case study, proving the monad laws for reader monads.

Our fourth and final contribution is to formalize λ^{RE} , a core calculus modeling PEq's two important features: type-indexed, functionally extensional propositional equality and refinement types with semantic subtyping (§5). We prove that λ^{RE} is sound and that propositional equality implies equality in a term model of equivalence (§6).

2 FUNCTIONAL EXTENSIONALITY IS INCONSISTENT IN REFINEMENT TYPES

Functional extensionality states that two functions are equal, if their values are equal at every argument: $\forall f,g:A\to B, \forall x\in A, f(x)=g(x)\Rightarrow f=g$. Most theorem provers consistently admit functional extensionality as an axiom, which we call funext throughout. Admitting funext is a convenient way to generate equalities on functions and reuse higher order proofs. For example, Agda defines functional extensionality as below in the standard library:

```
Extensionality : (a b : Level) \rightarrow Set _ -- Axiom.Extensionality.Propositional Extensionality a b = {A : Set a} {B : A \rightarrow Set b} {f g : (x : A) \rightarrow B x} \rightarrow (\forall x \rightarrow f x \equiv g x) \rightarrow f \equiv g
```

Having seen funext's success in other dependently typed languages, we naively admitted the funext axiom below in Liquid Haskell:

```
{-@ assume funext :: \forall a b. f:(a\rightarrowb) \rightarrow g:(a\rightarrowb) \rightarrow (x:a \rightarrow {f x = g x}) \rightarrow {f = g} @-} funext :: (a \rightarrow b) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow ()) \rightarrow () funext _f _g _pf = ()
```

The assume keyword introduces an axiom: Liquid Haskell will accept the refinement signature of funext wholesale and ignore its definition. Also, note that the = symbol in the refinements refers to SMT equality (see §3.4). Our encoding certainly *looks* like Agda's Extensionality axiom. But looks can be deceiving: in Liquid Haskell, we can use funext to prove false. Why?

Consider two functions on Integers: the incrInt function increases all integers by one; the incrPos function increases positive numbers by one, returning 0 otherwise:

```
incrInt, incrPos :: Integer \rightarrow Integer incrInt n = n + 1 incrPos n = if 0 < n then n + 1 else 0
```

Liquid Haskell easily proves that these two functions behave the same on positive numbers:

```
{-@ type Pos = {n:Integer | 0 < n } @-} 
{-@ incrSamePos :: n:Pos \rightarrow {incrPos n = incrInt n} @-} incrSamePos :: Integer \rightarrow () incrSamePos _n = ()
```

We can use funext to show that incrPos and incrInt are equal, using our proof incrSamePos on the domain of positive numbers.

```
{-@ incrExt :: {incrPos = incrInt} @-}
incrExt :: ()
incrExt = funext incrPos incrInt incrSamePos
```

Having incrExt to hand, it's easy to prove that every higher-order use of incrPos can be replaced with incrInt, which is much more efficient—it saves us a conditional branch! For example, incrMap shows that mapping over a list with incrPos is just the same as mapping over it with incrInt.

```
{-@ incrMap :: xs:[Pos] \rightarrow {map incrPos xs = map incrInt xs} @-} incrMap :: [Integer] \rightarrow () incrMap xs = incrExt
```

We could prove incrMap without function equality, i.e., if we only knew incrSamePos. To do so, we would write an inductive proof—and we'd have to redo the proof for every context in which we would rewrite incrPos to incrInt. So funext is in part about *modularity* and *reuse* in theorem proving. We don't give a full example here, but funext is particularly critical when trying to equate structures that are themselves higher order, like difference lists or streams.

Unfortunately, incrExt makes it *too* easy to prove equivalences... our system is inconsistent! Here's a proof that 0 is equal to -4:

```
{-@ inconsistencyI :: {incrPos (-5) = incrInt (-5)} @-} -- 0 = -4 inconsistencyI :: () inconsistencyI = incrExt
```

What happened here? How can we have that equality... that $\emptyset = -4$? Liquid Haskell looked at incrExt and saw the two functions were equal... without any regard to the domain on which incrExt proved incrPos and incrInt equal! We *forgot* the domain, and so incrExt generates a proof in SMT that those two functions are equal on *any* domain.

So funext is inconsistent in Liquid Haskell! The problem is that Liquid Haskell forgets the domain on which the two functions are proved equal, remembering only the equality itself.

We can exploit funext to find equalities between *any* two functions that share the same Haskell type on the *empty* domain, and Liquid Haskell will treat these functions as *universally* equal. Ouch!

For example, plus2 below defines a function that increases its input by 2 and is obviously not equal to incrInt on any nontrivial domain.

```
plus2 :: Integer \rightarrow Integer plus2 x = x + 2
```

Even so, we can use funext to prove that plus2 behaves the same as incrInt on the empty domain, i.e., for all inputs n that satisfy false.

```
{-@ type Empty = {v:Integer | false } @-} 
{-@ incrSameEmpty :: n:Empty \rightarrow {incrInt n = plus2 n} @-} 
incrSameEmpty :: Integer \rightarrow () 
incrSameEmpty _n = ()
```

Now incrSameEmpty provides enough evidence for funext to show that incrInt equals plus2, which we use to prove another egregious inconsistency.

```
{-@ incrPlus2Ext :: {incrInt = plus2} @-}
incrPlus2Ext :: ()
incrPlus2Ext = funext incrInt plus2 incrSameEmpty

{-@ inconsistencyII :: {incrInt 0 = plus2 0} @-} -- 1 = 2
inconsistencyII :: ()
inconsistencyII = incrPlus2Ext
```

Liquid Haskell isn't like most other dependent type theories: we can't just admit funext as phrased. But we still want to prove equalities between higher-order values! What can we do?

2.1 Refined, Type-Indexed, Extensional, Propositional Equality

If we're going to reason using functional extensionality in Liquid Haskell, we'll need to be careful to remember the type at which we show the functions produce equal results. What domains are involved when we use functional extensionality?

To prove two functions f and g extensionally equal, we must reason about four domains. Let \mathcal{D}_f and \mathcal{D}_g be the domain on which the functions f and g are respectively defined. Let \mathcal{D}_p be the domain on which the two functions are proved equal and \mathcal{D}_e the domain on which the resulting equality between the two functions is found. In our incrExt example above, the function domains are Integer ($\mathcal{D}_f = \mathcal{D}_g = \text{Integer}$), as specified by the function definitions, the domain of the proof is positive numbers ($\mathcal{D}_p = \text{Pos}$), as specified by incrSamePos, and, disastrously, the domain of the equality itself is unspecified in funext. Liquid Haskell will implicitly set the domain on which the functions are equal to the most general one where both functions can be called ($\mathcal{D}_e = \text{Integer}$).

Our funext encoding naively imposes no real constraints between these domains. In fact, funext only requires that \mathcal{D}_f , \mathcal{D}_g , and \mathcal{D}_p are supertypes of the empty domain (§5), which trivially holds for all types, leaving \mathcal{D}_e underconstrained.

To be consistent, we need a functional extensionality axiom that (1) captures the domain of function equality \mathcal{D}_e explicitly, (2) requires that the domain of the equality, \mathcal{D}_e , is a subtype of the domain of the proof, \mathcal{D}_p , which should be a subtype of the functions domains, \mathcal{D}_f and \mathcal{D}_g , and (3) ensures that the resulting equality between functions is only used on subdomains of \mathcal{D}_e .

Our solution is to define a refined, type-indexed, extensional propositional equality. We do so in the Liquid Haskell library PEq, which defines a propositional equality also called PEq. We write PEq a $\{e_l\}$ to mean that the expressions e_l and e_r are propositionally equal and of type a. We carefully crafted PEq's definition as a refined GADT (§3) to meet our three criteria.

1. PEq is Type-Indexed. The type index a in PEq a $\{e_I\}$ makes it easy to track types explicitly. PEq's constructor axiomatizing functional extensionality keeps careful track of types:

```
XEq :: f:(a \rightarrow b) \rightarrow g:(a \rightarrow b) \rightarrow (x:a \rightarrow PEq b \{f x\} \{g x\}) \rightarrow PEq (a \rightarrow b) \{f\} \{g\}
```

The result type of XEq explicitly captures the equality domain as the domain of the return type (i.e., a). The standard variance and type checking rules of Liquid Haskell ensure that the domains \mathcal{D}_f , \mathcal{D}_q , and \mathcal{D}_p are supertypes of \mathcal{D}_e . (See §5 for more detail on type checking.)

2. Generating Function Equalities. The XEq case of PEq generates equalities at function types using functional extensionality. Liquid Haskell will check the domains appropriately: it won't prove equality between functions at an inappropriate domain.

Returning to our concrete example of incrPos and incrInt, we can use XEq to find these functions equal on the domain Pos:

```
{-@ incrExtGood :: PEq (Pos \rightarrow Integer) {incrPos} {incrInt} @-} incrExtGood :: PEq (Integer \rightarrow Integer) incrExtGood = XEq incrPos incrInt incrEq
```

XEq checks that the domains of the functions incrPos and incrInt are supertypes of Pos, i.e., Pos <: Integer. Further it checks that the domain of the proof incrEq is supertype of Pos.

What might we define for incrEq? Here are three alternatives. Each alternative is either accepted or rejected by XEq as appropriate for the Pos \rightarrow Integer type index; each alternative is also possible or impossible to prove. (See §3 for more on how incrEq can be defined.)

```
incrEq :: n:Pos \rightarrow PEq Integer {incrPos n} {incrInt n} -- ACCEPTED and POSSIBLE incrEq :: n:Integer \rightarrow PEq Integer {incrPos n} {incrInt n} -- ACCEPTED and IMPOSSIBLE incrEq :: n:Empty \rightarrow PEq Integer {incrPos n} {incrInt n} -- REJECTED and POSSIBLE
```

The first two alternatives, n:Pos and n:Integer, will be accepted by XEq, since both Pos and Integer are supertypes of Pos... though it is impossible to actually construct a proof for the second alternative, i.e., a proof that incrPos n equals incrInt n for all integers n. On the other hand, the last proof on n:Empty is trivial, but XEq rejects it, because Empty is not a supertype of Pos. Liquid Haskell's checks on XEq's type indices prevents inconsistencies like inconsistencyII.

3. Using Function Equalities. Just as PEq's XEq constructor ensures that the right domains are checked and tracked for functional extensionality, we have a constructor for ensuring these equalities are used appropriately. The constructor CEq characterizes equality as valid in all contexts, i.e., if x and y are equal, they can be substituted in any context ctx and the results ctx x and ctx y will be equal:

```
CEq :: x:a \rightarrow y:a \rightarrow PEq \ a \ \{x\} \ \{y\} \rightarrow ctx: (a \rightarrow b) \rightarrow PEq \ b \ \{ctx \ x\} \ \{ctx \ y\}
```

It is easy to use CEq to apply functional equalities in higher order contexts. For example, we can prove that map <code>incrPos</code> equals map <code>incrInt</code>:

```
{-@ incrMapProp :: PEq ([Pos] → [Integer]) {map incrPos} {map incrInt} @-} incrMapProp :: PEq ([Integer] → [Integer]) incrMapProp = CEq incrPos incrInt incrExtGood (map)
```

We can more generally show that propositionally equal functions produce equal results on equal inputs. The trick is to *flip* the context, defining a function app that takes as input two functions f and g, a proof these functions are equal, and an argument x, returning a proof that f x = g x:

```
{-@ app :: f:(a → b) → g:(a → b) → PEq (a → b) {f} {g}

→ x:a → PEq b {f x} {g x} @-}

app :: (a → b) → (a → b) → PEq (a → b) → a → PEq b

app f g eq x = CEq f g eq (flip x)
```

The app lemma makes it easy to use function equalities while still checking the domain on which the function is applied. These checks prevent inconsistencies like inconsistency I. For instance, we can try to apply the functional equality incrextGood to a bad and a good input.

```
{-@ badFO ::PEq Integer {incrPos 0} {incrInt 0} @-} badFO = app incrPos incrInt incrExtGood 0 -- REJECTED 

{-@ goodFO :: x:{Integer | 42 < x } \rightarrow PEq Integer {incrPos x} {incrNat x} @-} goodFO x = app incrPos incrInt incrExtGood x -- ACCEPTED
```

Liquid Haskell rejects the bad input in badF0: the number 0 isn't in the Pos domain on which incrExtGood was proved. Liquid Haskell accepts the good input in goodF0, since any x greater than 42 is certainly positive. The goodF0 proof yields a first-order equality on any such x, here on Integer. Such first order equalities correspond neatly with the notion of equality used in the SMT solvers that buttress all of Liquid Haskell's reasoning. (For more information on how SMT equality relates to notions of equality in Liquid Haskell, see §3. For an example of how these first-order equalities can lead to runtime optimizations, see §4.5.)

2.2 Why Isn't funext Inconsistent in Agda?

At the beginning of $\S 2$, we present Agda's Extensionality axiom, whose return type is $f \equiv g$. Agda's equality appears to lack a type index. Why doesn't Agda also suffer from inconsistency?

Agda's equality only seems to be unindexed. In fact, Agda's built-in equality is the standard, type-indexed Leibniz equality used in most dependent type theories (omitting Level polymorphism):

```
data \equiv {A : Set} (x : A) : A \rightarrow Set a where refl : x \equiv x
```

The curly braces around the type index A marks it as *implicit*, i.e., to be inferred. If we were to explicitly give implicit arguments by wrapping them in curly braces, Agda's extensionality axiom returns (\equiv) {a \rightarrow b} f g.

Our XEq axiom recovers the type indexing in Agda's equivalence that's missing in our original funext encoding. Of course, (Liquid) Haskell's lack of implicit type indices makes reasoning about function equalities verbose. On the other hand, Liquid Haskell's subtyping can reinterpret functions at many domains (see §4.2). In Agda, however, it is much more complex to reinterpret functions and to generate heterogeneous equality relating incrInt and incrPos only on positive inputs.

3 PEQ: A LIBRARY AND GADT FOR EXTENSIONAL EQUALITY

We define the PEq library in Liquid Haskell, implementing the type-indexed propositional equality, also called PEq. First, we axiomatize equality for base types in the AEq typeclass (§3.1). Next, we define propositional equality for base and function types with the PEq GADT [Cheney and Hinze 2003; Xi et al. 2003] (§3.2). Refinements on the GADT enforce the typing rules of our formal model (§6), but we prove some of the metatheory in Liquid Haskell itself (§3.3). Finally, we discuss how AEq and PEq interact with Haskell's and SMT's equalities (§3.4).

3.1 The AEq typeclass, for axiomatized equality

We begin with by axiomatizing equality that can be ported to SMT: such an equality should be an equivalence relation that implies SMT equality. We use refinements on typeclasses [Liu et al. 2020] to define a typeclass AEq, which contains the (operational) equality method \equiv , three methods that encode the equality laws, and one method that encodes correspondence with SMT equality.

```
{-@ class AEq a where

(≡) :: x:a → y:a → Bool

reflP :: x:a → {x \equiv x}

symmP :: x:a → y:a → {x \equiv y \Rightarrow y \equiv x}
```

```
-- (1) Plain GADT
data PBEq :: * \rightarrow * where
        BEq :: AEq a \Rightarrow a \rightarrow a \rightarrow () \rightarrow PBEq a
        XEq :: (a \rightarrow b) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow PEq b) \rightarrow PBEq (a \rightarrow b)
        CEq :: a \rightarrow a \rightarrow PBEq \ a \rightarrow (a \rightarrow b) \rightarrow PBEq \ b
-- (2) Uninterpreted equality between terms e1 and e2
\{-\text{@ type PEq a e1 e2} = \{v: PBEq a \mid e1 \leq e2\} \text{ @-}\}
\{-\text{@ measure } (\cong) :: a \rightarrow a \rightarrow \text{Bool } @-\}
-- (3) Type refinement of the GADT
\{-\emptyset \text{ data PBEq } :: * \rightarrow * \text{ where }
        BEq :: AEq a \Rightarrow x:a \rightarrow y:a \rightarrow {v:() | x \equiv y}
               \rightarrow PEq a \{x\} \{y\}
        XEq :: f:(a \rightarrow b) \rightarrow g:(a \rightarrow b) \rightarrow (x:a \rightarrow PEq b \{f x\} \{g x\})
               \rightarrow PEq (a \rightarrow b) {f} {g}
        CEq :: x:a \rightarrow y:a \rightarrow PEq \ a \ \{y\} \rightarrow ctx:(a \rightarrow b)
               \rightarrow PEq b {ctx x} {ctx y} @-}
```

Fig. 1. Implementation of the propositional equality PEq as a refinement of Haskell's GADT PBEq.

```
transP :: x:a \rightarrow y:a \rightarrow z:a \rightarrow { (x \equiv y \&\& y \equiv z) \Rightarrow x \equiv z } smtP :: x:a \rightarrow y:a \rightarrow { x \equiv y } \rightarrow { x = y } @-}
```

To define an instance of AEq one has to define the method (\equiv) and provide explicit proofs that it is reflexive, symmetric, and transitive (reflP, symmP, and transP resp.); thus \equiv is, by construction, an equality. Finally, we require the proof smtP that captures that (\equiv) implies equality provable by SMT (e.g., structural equality).²

3.2 The PBEq GADT and its PEq Refinement

We use AEq to define our type-indexed propositional equality PEq a {e1} {e2} in three steps (Figure 1): (1) structure as a GADT, (2) definition of the refined type PEq, and (3) axiomatization of equality by refining of the GADT.

First, we define the structure of our proofs of equality as PBEq, an unrefined, i.e., Haskell, GADT (Figure 1, (1)). The plain GADT defines the structure of derivations in our propositional equality (i.e., which proofs are well formed), but none of the constraints on derivations (i.e., which proofs are valid). There are three ways to prove our propositional equality, each corresponding to a constructor of PBEq: using an AEq instance (constructor BEq); using funext (constructor XEq); and by congruence closure (constructor CEq).

Next, we define the refinement type PEq to be our propositional equality (Figure 1, (2)). Two terms e1 and e2 of type a are propositionally equal when (a) there is a well formed and valid PBEq proof and (b) we have e1 \simeq e2, where (\simeq) is an *uninterpreted* SMT function. Liquid Haskell uses curly braces for expression arguments in type applications, e.g., in PEq a {x} {y}, x and y are expressions, but a is a type.

²The three axioms of equality alone are not enough to ensure SMT's structural equality, e.g., one can define an instance x = y = True which satisfies the equality laws, but does not correspond to SMT equality.

Finally, we refine the type constructors of PBEq to axiomatize the uninterpreted (\mathfrak{D}) and generate proofs of PEq (Figure 1, (3)). Each constructor of PBEq is refined to return something of type PEq, where PEq a {e1} {e2} means that terms e1 and e2 are considered equal at type a. BEq constructs proofs that two terms, x and y of type a, are equal when $x \equiv y$ according to the AEq instance for a. XEq is the (type-indexed) funext axiom. Given functions f and g of type $a \to b$, a proof of equality via extensionality also needs a PEq-proof that f x and g x are equal for all x of type a. Such a proof has refined type x: $a \to PEq$ b {f x} {g x}. Critically, we don't lose any type information about f or g! CEq implements congruence closure: for x and y of type a that are equal—i.e., PEq a {x} {y}—and an arbitrary context with an a-shaped hole (ctx :: $a \to b$), filling the context with x and y yields equal results, i.e., PEq b {ctx x} {ctx y}.

Design Alternatives. The first design choice we made was to define PEq as a GADT and not an axiomatized opaque type. While there's no reason to pattern match on PEq terms, there's also no harm in it. A GADT provides a clean interface on how PEq can be generated: it collects all the axioms as data contructors and prevents the user from arbitrarily adding new constructors. The second choice we made was to define the type PEq using a fresh uninterpreted equality symbol (Figure 1, (2)) instead of SMT equality. Again, we made this decision to ensure that all PEq terms are constructed via the constructors and not implicit SMT automation. The final choice we made was to define the base case using the AEq constraints. We considered two alternatives:

```
BEq :: x:a \rightarrow y:a \rightarrow \{v:() \mid x = y \} \rightarrow PEq \ a \{x\} \{y\} -- alternative I
BEq :: Eq a \Rightarrow x:a \rightarrow y:a \rightarrow \{v:() \mid x = y \} \rightarrow PEq \ a \{x\} \{y\} -- alternative II
```

We rejected the first to ensure that the base case does not include functions (which don't generally have Eq instances) and to support our metatheory (§3.3). We rejected the second to exclude user-defined Eq instances that do not correspond to SMT equality (since in §3.4 we define a machanism to turn PEq to SMT equalities).

Example: Having seen AEq and the BEq case of PEq, we can define the incrEq function from §2:

```
{-@ incrEq :: x:Pos → PEq Integer {incrPos x} {incrInt x} @-} incrEq x = BEq (incrPos x) (incrInt x) (reflP (incrPos x))
```

We start from reflP (incrPos x) :: {incrPos $x \equiv incrPos x$ }, since x is positive, the SMT derives incrPos x = incrInt x, generating the BEq proof term {incrPos $x \equiv incrInt x$ }.

3.3 Equivalence Properties and Classy Induction

We can prove metaproperties of the actual implementation of PEq—reflexivity, symmetry, and transitivity—within Liquid Haskell itself.

Our proofs in Liquid Haskell go by induction on types. But "induction" in Liquid Haskell means writing a recursive function, which necessarily has a single, fixed type. To express that PEq is reflexive, we want a Liquid Haskell theorem refl :: $x:a \rightarrow PEq a \{x\} \{x\}$, but its proof goes by induction on the type a, which is not possible in ordinary Haskell functions.³

The essence of our proofs is a folklore method we call *classy induction* (see §7 for the history). To prove a theorem using classy induction on the PEq GADT, one must: (1) define a typeclass with a method whose refined type corresponds to the theorem; (2) prove the base case for types with AEq instances; and (3) prove the inductive case for function types, where typeclass constraints on smaller types generate inductive hypotheses. All three of our proofs follow this pattern. Here's the proof for reflexivity.

³A variety of GHC extensions allow case analysis on types (e.g., type families and generics), but, unfortunately, Liquid Haskell doesn't support such fancy type-level programming.

```
-- (1) Refined typeclass
{-@ class Reflexivity a where
  refl :: x:a → PEq a {x} {x} @-}

-- (2) Base case (AEq types)
instance AEq a ⇒ Reflexivity a where
  refl a = BEq a a (reflP a)
-- (3) Inductive case (function types)
instance Reflexivity b ⇒ Reflexivity (a → b) where
  refl f = XEq f f (\a → refl (f a))
```

For (1), the typeclass Reflexivity simply states the desired theorem type, refl :: $x:a \rightarrow PEq$ a $\{x\}$ $\{x\}$. For (2), given an AEq a instance, BEq and the reflP method are combined to define the refl method. To define such a general instance, we enabled the GHC extensions FlexibleInstances and UndecidableInstances. For (3), XEq can show that f is equal to itself by using the refl instance from the codomain constraint: the Reflexivity b constraint generates a method refl :: $x:b \rightarrow PEq$ b $\{x\}$ $\{x\}$. The codomain constraint Reflexivity b corresponds exactly to the inductive hypothesis on the codomain: we are doing induction!

At compile time, any use of refl $\, x \,$ when $\, x \,$ has type a asks the compiler to find a Reflexivity instance for a. If a has an AEq instance, the proof of refl $\, x \,$ will simply be BEq $\, x \, x \,$ (reflP a). If a is a function of type b $\, \to \,$ c, then the compiler will try to find a Reflexivity instance for the codomain c—and if it finds one, generate a proof using XEq and c's proof. The compiler's constraint resolver does the constructive proof for us, assembling the 'inductive tower' to give us a refl for our chosen type. That is, even though Liquid Haskell can't mechanically check that our inductive proofs are in general complete (i.e., the base and inductive cases cover all types), our refl proofs will work for types where the codomain bottoms out with an AEq instance, i.e., any type consisting of functions and AEq-equable types.

Our proofs of symmetry and transitivity follow the same pattern, but both also make use congruence closure. The full proofs can be found in supplementary material [2021]. Here is the inductive case from symmetry:

```
instance Symmetry b \Rightarrow Symmetry (a \rightarrow b) where
-- sym :: l:(a\rightarrowb) \rightarrow r:(a\rightarrowb) \rightarrow PEq (a\rightarrowb) {l} {r} \rightarrow PEq (a\rightarrowb) {r} {l}
sym l r pf = XEq r l $ \a \rightarrow sym (l a) (r a) (CEq l r pf ($ a) ? ($ a 1) ? ($ a r)))
```

Here 1 and r are functions of type $a \to b$ and we know that $1 \subseteq r$; we must prove that $r \subseteq 1$. We do so using: (a) XEq for extensionality, letting a of type a be given; (b) sym (1 a) (r a) as the IH on the codomain b on (c) CEq for congruence closure on $1 \subseteq r$ in the context (\$ a). The last step is the most interesting: if 1 is equal to r, then plugging them into the same context yields equal results; as our context, we pick (\$ a), i.e., $f \to f$ a, showing that 1 a $\subseteq r$ a; the IH on the codomain b yields r a $\subseteq 1$ a, and extensionality shows that $r \subseteq 1$, as desired. The operator ?, defined as x ? p = x, asks Liquid Haskell to encode 'p' into the SMT solver to help prove 'x'. Our use of ? unfolds the definitions \$ a 1 and \$ a r to help CEq.

3.4 Interaction of the different equalities.

We have four equalities in our system (Figure 2): SMT equality (=), the (\equiv) method of the AEq typeclass(§3.1), the refined GADT PEq (§3.2), and the (==) method of Haskell's Eq typeclass.

SMT Equality. The single equal sign (=) represents SMT equality, which satisfies the three equality axioms and is syntactically defined for data types. The SMT-LIB standard [Barrett et al. 2010] permits

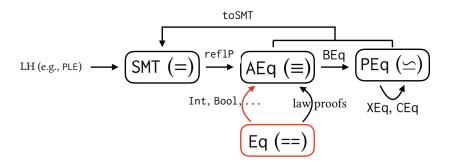


Fig. 2. The four different equalities and their interactions. Haskell equality is in red to highlight its potential for unsoundness.

the equality symbol on functions but does not specify its behavior. Implementations vary. CVC4 allows for functional extensionality and higher-order reasoning [Barbosa et al. 2019]. When Z3 compares functions for equality, it treats them as arrays, using the extensional array theory to incompletely perform the comparison. When asked if two functions are equal, Z3 typically answers unknown. To avoid this unpredictability, our system avoids SMT equality on functions.

Interactions of Equalities. SMT equalities are internally generated by Liquid Haskell using the reflection and PLE tactic of Vazou et al. [2018b] (see also §4.1). An $e_1 \equiv e_2$ equality can be generated one of three ways: (1) If SMT can prove an SMT equality $e_1 = e_2$, then the reflexivity ref1P method can generate that equality, i.e., ref1P e_1 proves $e_1 \equiv e_1$, which is enough to show $e_1 \equiv e_2$. (2) Our system provides AEq instances for the primitive Haskell types using the Haskell equality that we assume satisfies the four laws, e.g., the instance AEq Int is provided. (3) Using refinements in typeclasses [Liu et al. 2020] one can explicitly define instances of AEq, which may or may not coincide with Haskell Eq instances.

Constructors generate PEq proofs, bottoming out at AEq: BEq combined with an AEq term and XEq or CEq combined with other PEq terms.

Finally, we define a mechanism to convert PEq into an SMT equality. This conversion is useful when we want to derive an SMT equality f e = g e from a function equality PEq (a \rightarrow b) {f} {g} (see §4.5). The derivation requires that the domain b admits the axiomatized equality, AEq. To capture this requirement we define toSMT that converts PEq to SMT equality as a method of a class that requires an AEq constraint:

```
class AEq a \Rightarrow SMTEq a where
toSMT :: x:a \rightarrow y:a \rightarrow PEq a {x} {y} \rightarrow {x = y}
```

Non-interaction. Liquid Haskell maps Haskell's (==) to SMT equality by default. It is surely unsound to do so, as users can define their own Eq instances with (==) methods that do arbitrarily strange things. To avoid this built-in unsoundness, our implementation and case studies don't directly use Haskell's equality.

Equivalence Relation Axioms. Each of the four equalities has a different relationship to the equivalence relation axioms (reflexivity, symmetry, transitivity). AEq comes with explicit proof methods that capture the axioms. For PEq, we prove the equality axioms using classy induction (§3.3). For SMT equality, we simply trust implementation of the underlying solver. For Haskell's equality, there's no general way to enforce the equality axioms, though users can choose to prove them.

Two correct and one wrong implementations of reverse

```
slow, bad, fast :: [a] \rightarrow [a]
                                                   fast xs = fastGo [] xs
slow []
             = []
                                                   fastGo :: [a] \rightarrow [a] \rightarrow [a]
slow(x:xs) = slow xs ++ [x]
                                                   fastGo acc []
                                                                    = acc
bad xs = xs
                                                   fastGo\ acc\ (x:xs) = fastGo\ (x:acc)\ xs
First-Order Theorems relating fast and slow
reverseEq :: xs:[a] \rightarrow \{ fast xs = slow xs \}
           :: xs:[a] \rightarrow ys:[a] \rightarrow \{fastGo \ ys \ xs = slow \ xs ++ \ ys\}
           :: xs:[a] \rightarrow ys:[a] \rightarrow zs:[a] \rightarrow \{ (xs ++ ys) ++ zs = xs ++ (ys ++ zs) \}
assoc
rightId
           :: xs:[a] \rightarrow \{ xs ++ [] = xs \}
Proofs of the First-Order Theorems
reverseEq x
                 = lemma x [] ? rightId (slow x)
                                                                   rightId []
                                                                                     = ()
lemma []
             _ = ()
                                                                   rightId(_:x) = rightIdx
                                                                   assoc [] _ _ = ()
lemma (a:x) y = lemma x (a:y) ? assoc (slow x) [a] y
x ? _pf
                 = x
                                                                   assoc (\_:x) y z = assoc x y z
```

Fig. 3. Reasoning about list reversal.

Computability. Finally, the Eq and AEq classes define the computable equalities used in programs, (==) and (=) respectively. The PEq equality only contains proof terms, while the SMT equality lives entirely inside the refinements; neither can be meaningfully used in programs.

4 CASE STUDIES

We demonstrate our propositional equality in seven case studies. We start by moving from first-order equalities to equalities between functions (reverse, §4.1). Next, we show how PEq's type indices reason about refined domains and dependent codomains of functions (succ, §4.2). Proofs about higher-order functions demonstrate the contextual equivalence axiom (map, §4.3). Then, we see that PEq plays well with multi-argument functions (fold1, §4.4). Next, we present how a PEq proof can speedup code (spec, §4.5). Finally, we present two bigger case studies that prove the monoid laws for endofunctions (§4.6) and the monad laws for reader monads (§4.7). Complete code is available in the [supplementary material 2021].

4.1 Reverse: from First- to Higher-Order Equality

Consider three candidate definitions of the list-reverse function (Figure 3, top): a 'fast' one in accumulator-passing style, a 'slow' one in direct style, and a 'bad' one that is the identity.

First-Order Proofs. The reverseEq theorem neatly relates the two list reversals (Figure 3). The final theorem reverseEq is a corollary of a lemma and rightId, which shows that [] is a right identity for list append, (++). The lemma is the core induction, relating the accumulating fastGo and the direct slow. The lemma itself uses the inductive lemma assoc to show associativity of (++). All the equalities in the first order statements use the SMT equality, since they are automatically proved by Liquid Haskell's reflection and PLE tactic [Vazou et al. 2018b].

Higher-Order Proofs. Plain SMT equality isn't enough to prove that fast and slow are themselves equal. We need functional extensionality: the XEq constructor of the PEq GADT.

```
reverseH0 :: PEq ([a] \rightarrow [a]) {fast} {slow}
```

```
reverseHO = XEq fast slow reversePf
```

The job of the reversePf lemma is to prove fast xs propositionally equal to slow xs for all xs:

```
reversePf :: xs:[a] \rightarrow PEq [a] \{fast xs\} \{slow xs\}
```

There are several different ways to construct such a proof.

Style 1: Lifting First-Order Proofs. The first order equality proof reverseEq lifts directly into propositional equality, using the BEq constructor and the reflexivity property of AEq.

```
reversePf1 :: AEq [a] \Rightarrow xs:[a] \rightarrow PEq [a] {fast xs} {slow xs} reversePf1 xs = BEq (fast xs) (slow xs) (reverseEq xs ? reflP (fast xs))
```

Such proofs rely on SMT equality, which the ref1P call turns into axiomatized equality (AEq).

Style 2: Inductive Proofs. Alternatively, inductive proofs can be directly performed in the propositional setting, eliminating the AEq constraint. To give a sense of what these proofs are like, we translate lemma into lemmaP:

The proof goes by induction and uses the Reflexivity and Transitivity properties of PEq encoded as typeclasses (§3.3) along with assocP and rightIdP, the propositional versions of assoc and rightId (not shown). These typeclass constraints propagate to the reverseHO proof, via reversePf2.

Style 3: Combinations. One can combine the easy first order inductive proofs with the typeclass-encoded properties. Here refl sets up the propositional context; lemma and rightId complete the proof.

```
reversePf3 :: (Reflexivity [a]) \Rightarrow xs:[a] \rightarrow PEq [a] {fast xs} {slow xs} reversePf3 xs = refl (fast xs) ? lemma xs [] ? rightId (slow xs)
```

Bad Proofs. We could not use any of these styles to generate a bad (non-)proof: neither PEq ([a] \rightarrow [a]) {fast} {bad} nor PEq ([a] \rightarrow [a]) {slow} {bad} are provable.

4.2 Succ: Refined Domains and Dependent Codomains

Our propositional equality PEq naturally reasons about functions with refined domains and dependent codomains. For example, recall the functions incrInt and incrPos from §2:

```
incrInt, incrPos :: Integer \rightarrow Integer incrInt n = n + 1 incrPos n = if 0 < n then n + 1 else 0
```

In §2 we proved that the two functions are equal on the domain of positive numbers:

```
type Pos = \{x: Integer \mid 0 < x \}
posDom :: PEq (Pos \rightarrow Integer) \{incrInt\} \{incrPos\}
posDom = XEq incrInt incrPos \{x: A \rightarrow BEq (incrInt x) (incrPos x) (reflP (incrInt x))\}
```

We can also reason about how each function's domain affects its codomain. For example, we can prove that these functions are equal *and* they take Pos inputs to natural numbers.

```
posRng :: PEq (Pos \rightarrow {v:Integer | 0 <= v}) {incrInt} {incrPos} posRng = XEq incrInt incrPos x \rightarrow BEq (incrInt x \rightarrow BEq (in
```

Finally, we can prove properties of the function's codomain that depend on the inputs. Below we show that on positive arguments, the result is always increased by one.

```
type SPos x = {v:Pos | v = x + 1} 
depRng :: PEq (x:Pos \rightarrow SPos {x}) {incrInt} {incrPos} 
depRng = XEq incrInt incrPos \ \x \rightarrow \ BEq (incrInt x) (incrPos x) (reflP (incrInt x))
```

Equalities Rejected by Our System. Liquid Haskell correctly rejects various wrong, (non-)proofs of equality between the functions incrInt and incrPos. We highlight three:

```
badDom :: PEq ( Integer \rightarrow Integer) {incrInt} {incrPos} badCod :: PEq ( Pos \rightarrow {v:Integer | v < 0}) {incrInt} {incrPos} badDCod :: PEq (x:Pos \rightarrow {v:Integer | v = x+2}) {incrInt} {incrPos}
```

badDom expresses that incrInt and incrPos are equal for any Integer input, which is wrong, e.g., incrInt (-2) yields -1, but incrPos (-2) yields 0. Correctly constrained to positive domains, badCod specifies a negative codomain (wrong) while badDCod specifies that the result is increased by 2 (also wrong). Our system rejects all three with a refinement type error.

4.3 Map: Putting Equality in Context

Our propositional equality can be used in higher order settings: we prove that if f and g are propositionally equal, then map f and map g are also equal. Our proofs use the congruence closure equality constructor/axiom CEq.

Equivalence on the Last Argument. Direct application of CEq ports a proof of equality to the last argument of the context (a function). For example, mapEqP below states that if two functions f and g are equal, then so are the partially applied functions map f and map g.

```
\label{eq:period} \begin{split} \text{mapEqP} & :: \ f \colon (a \to b) \to g \colon (a \to b) \to PEq \ (a \to b) \ \{f\} \ \{g\} \\ & \to PEq \ ([a] \to [b]) \ \{\text{map } f\} \ \{\text{map } g\} \end{split} \label{eq:mapEqP} \\ \text{mapEqP} \ f \ g \ pf = CEq \ f \ g \ pf \ map \end{split}
```

Equivalence on an Arbitrary Argument. To show that map f xs and map g xs are equal for all xs, we use CEq with flipMap, i.e., a context that puts f and g in a 'flipped' context.

```
mapEq :: f:(a \rightarrow b) \rightarrow g:(a \rightarrow b) \rightarrow PEq (a \rightarrow b) {f} {g}

\rightarrow xs:[a] \rightarrow PEq [b] {map f xs} {map g xs}

mapEq f g pf xs = CEq f g pf (flipMap xs) ? fMapEq f xs ? fMapEq g xs

fMapEq :: f:\_ \rightarrow xs:[a] \rightarrow \{map f xs = flipMap xs f\}

fMapEq f xs = ()

flipMap xs f = map f xs
```

The mapEq proof relies on CEq with the flipped context and needs to know that map f xs = flipMap xs f. Liquid Haskell won't infer this fact on its own in the higher order setting of this proof; we explicitly provide this evidence with the fMapEq calls.

Proof Reuse in Context. Finally, we use the posDom proof (§4.2) to show how existing proofs can be reused with map.

```
client :: xs:[Pos] → PEq [Integer] {map incrInt xs} {map incrPos xs}
client = mapEq incrInt incrPos posDom

clientP :: PEq ([Pos] → [Integer]) {map incrInt} {map incrPos}
clientP = mapEqP incrInt incrPos posDom
```

client proves that map incrInt xs is equivalent to map incrPos xs for each list xs of positive numbers, while clientP proves that the partially applied functions map incrInt and map incrPos are equivalent on the domain of lists of positive numbers.

4.4 Fold: Equality of Multi-Argument Functions

As an example of equality proofs on multi-argument functions, we show that the directly tail-recursive foldl is equal to foldl', a foldr encoding of a left-fold via CPS. The first-order equivalence theorem is expressed as follows:

```
thm :: f:(b \rightarrow a \rightarrow b) \rightarrow b:b \rightarrow xs:[a] \rightarrow \{ foldl f b xs = foldl' f b xs \}
```

We lifted the first-order property into a multi-argument function equality by using XEq for all but the last arguments and BEq for the last, as below:

```
foldEq :: AEq b \Rightarrow PEq ((b \rightarrow a \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b) {foldl} {foldl'} foldEq = XEq foldl foldl' $\forall f \notin \text{VEq (foldl f) (foldl' f) $\forall f \notin \text{XS} \rightarrow \text{BEq (foldl f b xS) (foldl' f b xS)} (thm f b xs ? reflP (foldl f b xS))
```

One can avoid the first-order proof and the AEq constraint, by using the second, typeclass-oriented style of §4.1, (see supplementary material [2021] for details).

4.5 Spec: Function Equality for Program Efficiency

Function equality can be used to prove optimizations sound. For example, consider a critical function that, for safety, can only run on inputs that satisfy a specification spec, and fastSpec, a fast implementation to check spec.

```
spec, fastSpec :: a \rightarrow Bool critical :: x:{ a | spec x } \rightarrow a
```

A client function can soundly call critical for any input x by performing the runtime fastSpec x check, given a PEq proof that the functions fastSpec and spec are equal.

```
client :: PEq (a \rightarrow Bool) {fastSpec} {spec} \rightarrow a \rightarrow Maybe a client pf x = 
 if fastSpec x ? toSMT (fastSpec x) (spec x) (CEq fastSpec spec pf (\x f \rightarrow f x)) then Just (critical x) else Nothing
```

Monoid Instance for Endofunctions

```
type Endo a = a \rightarrow a
                                                  _{-} = = y = y
                                                  mappend :: Endo a \rightarrow Endo a \rightarrow Endo a
mempty :: Endo a
mempty a = a
                                                  mappend f g a = f (g a) -- a/k/a (<>)
Endofunction Monoid Laws
mLeftIdentity :: (Reflexivity a, Transitivity a)
                 \Rightarrow x:Endo a \rightarrow PEq (Endo a) {mappend mempty x} {x}
mRightIdentity :: (Reflexivity a, Transitivity a)
                 \Rightarrow x:Endo a \rightarrow PEq (Endo a) {x} {mappend x mempty}
mAssociativity :: (Reflexivity a, Transitivity a)
                 \Rightarrow x:(Endo a) \rightarrow y:(Endo a) \rightarrow z:(Endo a)
                 \rightarrow PEq (Endo a) {mappend (mappend x y) z} {mappend x (mappend y z)}
Proofs By Reflexivity and Transitivity
mLeftIdentity x = XEq (mappend mempty x) x \ \a \rightarrow
       refl (mappend mempty x a) ? (mappend mempty x a = \sim mempty (x a) = \sim x a *** QED)
mRightIdentity x = XEq x (mappend x mempty) \ \a \rightarrow
       refl (x a) ? (x a =~= x (mempty a) =~= mappend x mempty a *** QED)
mAssociativity x y z =
    XEq (mappend (mappend x y) z) (mappend x (mappend y z)) \ \a \rightarrow
       refl (mappend (mappend x y) z a) ?
            (mappend (mappend x y) z a =~= (mappend x y) (z a) =~= x (y (z a))
                                           =^= x (mappend y z a)
                                           =\sim= mappend x (mappend y z) a *** QED)
```

Fig. 4. Case study: Endofunction Monoid Proofs.

The toSMT call generates the SMT equality that fastSpec x = spec x, which, combined with the branch condition check fastSpec x, lets the path-sensitive refinement type checker decide that the call to critical x is safe in the then branch.

Our propositional equality (1) co-exists with practical features of refinement types, e.g., path sensitivity, and (2) can help optimize executable code.

4.6 Monoid Laws for Endofunctions

Endofunctions form a law-abiding monoid. A function f is an *endofunction* when its domain and codomain types are the same. A *monoid* is an algebraic structure comprising an identity element (mempty) and an associative operation (mappend). For the monoid of endofunctions, mempty is the identity function and mappend is function composition (Figure 4; top).

To be a monoid, mempty must really be an identity with respect to mappend (mLeftIdentity and mRightIdentity) and mappend must really be associative (mAssociativity) (Figure 4; middle).

Proving the monoid laws for endofunctions demands functional extensionality (Figure 4; bottom). For example, consider the proof that mempty is a left identity for mappend, i.e., mappend mempty x = x. To prove this equation between *functions*, we can't use SMT equality. With functional

Monad Instance for Readers

```
type Reader r a = r \rightarrow a
                                              pure :: a \rightarrow Reader r a
                                              pure a_r = a
  kleisli :: (a \rightarrow Reader r b)
           \rightarrow (b \rightarrow Reader r c)
                                              bind :: Reader r a \rightarrow (a \rightarrow Reader r b)
            \rightarrow a \rightarrow Reader r c
                                                    \rightarrow Reader r b
  kleisli f g x = bind (f x) g
                                              bind fra farb = \ \ \rightarrow \  farb (fra r) r
Reader Monad Laws
monadLeftIdentity :: Reflexivity b \Rightarrow a:a
                      \rightarrow f:(a \rightarrow Reader r b) \rightarrow PEq (Reader r b) {bind (pure a) f} {f a}
monadRightIdentity :: Reflexivity a
                      \Rightarrow m:(Reader r a) \rightarrow PEq (Reader r a) {bind m pure} {m}
monadAssociativity :: (Reflexivity c, Transitivity c)
                      \Rightarrow m:(Reader r a) \rightarrow f:(a \rightarrow Reader r b) \rightarrow g:(b \rightarrow Reader r c)
                      \rightarrow PEq (Reader r c) {bind (bind m f) g} {bind m (kleisli f g)}
Identity Proofs By Reflexivity
  monadLeftIdentity a f =
                                                   monadRightIdentity m =
    XEq (bind (pure a) f) (f a) \uparrow \r \rightarrow
                                                     XEq (bind m pure) m \ \r \rightarrow
     refl (bind (pure a) f r) ?
                                                        refl (bind m pure r) ?
       (bind (pure a) f r = f (pure a r) r
                                                      (bind m pure r =~= pure (m r) r
                 =~= f a r *** OED)
                                                                           =~= m r *** QED)
Associativity Proof By Transitivity and Reflexivity
  monadAssociativity m f g = XEq (bind (bind m f) g) (bind m (kleisli f g)) \ \r \rightarrow
  let { el = bind (bind m f) g r ; eml = g (bind m f r) r ; em = (bind (f (m r)) g) r
       ; emr = kleisli f g (m r) r ; er = bind m (kleisli f g) r }
  in trans el em er (trans el eml em (refl el) (refl eml))
                       (trans em emr er (refl em) (refl emr))
```

Fig. 5. Case study: Reader Monad Proofs.

extensionality, each proof reduces to three parts: XEq to take an input of type a; reflon the left-hand side of the equation, to generate an equality proof; and $(=\sim=)$ to give unfolding hints to the SMT solver. The $(=\sim=)$ operator is defined as $_=\sim==$ y = y, and it is unrefined, i.e., it is not checking equality of its arguments.

The Reflexivity constraints on the theorems make our proofs general in the underlying type a: endofunctions on the type a form a monoid whether a admits SMT equality or if it's a complex higher-order type (whose ultimate result admits equality). Haskell's typeclass resolution ensures that an appropriate refl method will be constructed whatever type a happens to be.

4.7 Monad Laws for Reader Monads

A *reader* is a function with a fixed domain r, i.e., the partially applied type Reader r (Figure 5, top left). Readers form a monad and their composition is a useful way of defining and composing functions that take some fixed information, like command-line arguments or configuration files. Our propositional equality can prove the monad laws for readers.

The monad instance for the reader type is defined using function composition (Figure 5, top). We also define Kleisli composition of monads as a convenience for specifying the monad. We prove that readers are in fact monads, i.e., their operations satisfy the monad laws (Figure 5, bottom). We also prove that they satisfy the functor and applicative laws in supplementary material [2021]. The reader monad laws are expressed as refinement type specifications using PEq. We prove the left and right identities following the pattern of §4.6, i.e., XEq, followed by reflexivity with (=~=) for function unfolding (Figure 5, middle). We use transitivity to conduct the more complicated proof of associativity (Figure 5, bottom).

Proof by Associativity and Error Locality. As noted earlier, the use of (=~=) in proofs by reflexivity is not checking intermediate equational steps. So, the proof either succeeds or fails without explanation. To address this problem, during proof construction, we employed transitivity. For instance, in the monadAssociativity proof, our goal is to construct the proof PEq _ {el} {er}. To do so, we pick an intermediate term em; we might attempt an equivalence proof as follows:

The refl el proof will produce a type error; replacing that proof with an appropriate trans to connect el and em via eml completes the monadAssociativity proof (Figure 5, bottom). Writing proofs in this trans/refl style works well: start with refl and where the SMT solver can't figure things out, a local refinement type error tells you to expand with trans (or look for a counterexample).

Our reader proofs use the Reflexivity and Transitivity typeclasses to ensure that readers are monads whatever the return type a may be (with the type of 'read' values fixed to r). Having generic monad laws is critical: readers are typically used to compose functions that take configuration information, but such functions usually have other arguments, too! For example, an interpreter might run readFile >>= parse >>= eval, where readFile :: Config \rightarrow String and parse :: String \rightarrow Config \rightarrow Expr and eval :: Expr \rightarrow Config \rightarrow Value. With our generic proof of associativity, we can rewrite the above to readFile >>= (kleisli parse eval) even though parse and eval are higher-order terms without Eq instances. Doing so could, in theory, trigger inlining/fusion rules that would combine the parser and the interpreter.

5 TYPE CHECKING XEQ: DID WE GET IT RIGHT?

We've seen that XEq is effective at proving equalities between functions (§4) and we've argued that we avoid the inconsistency with funext. Things seem to work in Liquid Haskell. But: Why do things go so wrong with funext? Does XEq really avoid funext's issues? We give a schematic example showing why Liquid Haskell works with XEq consistently but works with funext inconsistently. (We give a detailed, formal model of our propositional equality in §6.)

Suppose we have two functions h and k, defined on domains d_h and d_k and codomains r_h and r_k , respectively. Let's also assume we have some lemma that proves, for all x in some domain d_p , we have an equality $e_l \subseteq e_r$, where e_l and e_r are arbitrary expressions of type $\{v : \beta \mid r_p\}$.

```
\begin{array}{l} \mathsf{h} \; :: \; \mathsf{x} \colon \{\alpha \; | \; d_\mathsf{h}\} \; \to \; \{\mathsf{v} \colon \beta \; | \; r_\mathsf{h}\} \\ \mathsf{k} \; :: \; \mathsf{x} \colon \{\alpha \; | \; d_\mathsf{k}\} \; \to \; \{\mathsf{v} \colon \beta \; | \; r_\mathsf{k}\} \\ \mathsf{lemma} \; :: \; \mathsf{x} \colon \{\alpha \; | \; d_P\} \; \to \; \mathsf{PEq} \; \{\mathsf{v} \colon \beta \; | \; r_P\} \; \{e_l\} \; \{e_r\} \end{array}
```

We can pass these along to our XEq constructor (of §3) to form a proof that h equals k on some domain d_e :

Typing Environmennt

```
\mathsf{XEq} : \forall \alpha \beta. f : (\alpha \to \beta) \to g : (\alpha \to \beta) \to (x : \alpha \to \mathsf{PEq} \ \beta \{f \ x\} \{g \ x\}) \to \mathsf{PEq} \ (\alpha \to \beta) \{f\} \{g\}
                                                                                                                                  h: x: \{|d_h|\} \to \{|r_h|\}, k: x: \{|d_k|\} \to \{|r_k|\}, lemma: x: \{|d_p|\} \to PEq\{|r_p|\}\{e_l\}\{e_r\}
         Type Checking
         1.\Gamma \vdash XEq :: \forall \alpha \beta. f:(\alpha \rightarrow \beta) \rightarrow g:(\alpha \rightarrow \beta) \rightarrow (x:\alpha \rightarrow PEq \beta \{f x\}\{g x\}) \rightarrow PEq (\alpha \rightarrow \beta)\{f\}\{g\}
         2.\Gamma + \mathsf{XEq} \ @\{|\kappa_\alpha|\} :: \forall \beta.f : (\{|\kappa_\alpha|\} \to \beta) \to g : (\{|\kappa_\alpha|\} \to \beta) \to (x : \{|\kappa_\alpha|\} \to \mathsf{PEq} \ \beta \ \{f \ x\} \{g \ x\}) \to \mathsf{PEq} \ (\{|\kappa_\alpha|\} \to \beta) \{f\} \{g\} \to (x : \{|\kappa_\alpha|\} \to \beta) \{g \ x\})
         3.\Gamma \vdash \mathsf{XEQ} @\{ \lVert \kappa_\alpha \rVert \ @\{ \lVert \kappa_\beta \rVert : f : \{ \lVert \kappa_\alpha \rVert \to \{ \lVert \kappa_\beta \rVert \} \to g : \{ \lVert \kappa_\alpha \rVert \to \{ \lVert \kappa_\beta \rVert \} \to \{ \varkappa_\beta \rVert \} \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \{ \varkappa_\beta \rVert \} \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \{ \varkappa_\beta \rVert \} \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \{ \varkappa_\beta \rVert \} \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \{ \varkappa_\beta \rVert \} \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha \rVert \to \mathsf{PEQ} \ \{ \lVert \kappa_\alpha
           4.\Gamma \vdash \mathsf{XEq} \ @\{ |\kappa_{\alpha}|\} \ @\{ |\kappa_{\beta}|\} \ \mathsf{h} :: g : (\{|\kappa_{\alpha}|\} \ \to \{|\kappa_{\beta}|\}) \ \to (x : \{|\kappa_{\alpha}|\} \ \to \mathsf{PEq} \ \{|\kappa_{\beta}|\} \{\mathsf{h} \ x\} \{g \ x\}) \ \to \mathsf{PEq} \ (\{|\kappa_{\alpha}|\} \ \to \{|\kappa_{\beta}|\}) \{\mathsf{h}\} \{g\} \}
           5.\Gamma \vdash \mathsf{XEq} \ @\{\{\kappa_{\alpha}\}\} \ @\{\{\kappa_{\beta}\}\} \ \mathsf{h} \ \mathsf{k} :: (x: \{\{\kappa_{\alpha}\}\} \to \mathsf{PEq} \ \{\{\kappa_{\beta}\}\} \{\mathsf{h} \ x\} \{\mathsf{k} \ x\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{h}\} \{\mathsf{k} \}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{h}\} \{\mathsf{k} \}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{h}\} \{\mathsf{k} \}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{h}\} \{\mathsf{k} \}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{h}\} \{\mathsf{k}\}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\beta}\}\} \{\mathsf{k}\}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\}\} \to \{\{\kappa_{\alpha}\}\} \{\mathsf{k}\}) \to \mathsf{PEq} \ (\{\{\kappa_{\alpha}\} \{\mathsf{k}\} \{\mathsf{k
           6.\Gamma \vdash \mathsf{XEq} \ @\{ |\kappa_{\alpha}| \} \ @\{ |\kappa_{\beta}| \} \ \mathsf{h} \ \mathsf{k} \ \mathsf{lemma} :: \mathsf{PEq} \ (\{ |\kappa_{\alpha}| \} \ \to \ \{ |\kappa_{\beta}| \}) \{ \mathsf{h} \} \{ \mathsf{k} \}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           Sub-L
           7.\Gamma ⊢ XEq @{\kappa_{\alpha}} @{\kappa_{\beta}} h k lemma :: PEq ({\kappa_{\alpha}} → {\kappa_{\beta}}){h}{k}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             Sub-Sub
       8.\Gamma \vdash XEq @{\{\kappa_{\alpha}\}} @{\{\kappa_{\beta}\}} h k lemma :: PEq ({\{d_e\}} \rightarrow {\{r_e\}}){\{h\}}{\{k\}}
Subtyping Derivation Leaves
                                                                                                                                                                                                                                                                                \kappa_{\alpha} \Rightarrow r_{h} \Rightarrow \kappa_{\beta}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        \kappa_{\alpha} \Rightarrow r_{k} \Rightarrow \kappa_{\beta}
                                                              i. \kappa_{\alpha} \Rightarrow d_{\mathsf{h}}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 ii. \kappa_{\alpha} \Rightarrow d_{k}
                                                                                                                                                                                                                             \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ |r_{\mathsf{h}}| \} \leq \{ |\kappa_{\beta}| \}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          \Gamma \vdash \{|\kappa_{\alpha}|\} \leq \{|d_{\mathsf{k}}|\}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ |r_{k}| \} \leq \{ |\kappa_{\beta}| \}
                                    \Gamma \vdash \{|\kappa_{\alpha}|\} \leq \{|d_{\mathsf{h}}|\}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                         -Sub-H
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  Sub-K
                                                                                       \Gamma \vdash x : \{|d_{\mathsf{h}}|\} \longrightarrow \{|r_{\mathsf{h}}|\} \ \leq \ \{|\kappa_{\alpha}|\} \longrightarrow \{|\kappa_{\beta}|\}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             \Gamma \vdash x : \{|d_{\mathsf{k}}|\} \to \{|r_{\mathsf{k}}|\} \le \{|\kappa_{\alpha}|\} \to \{|\kappa_{\beta}|\}
                                                                                                                                                                                                                                                                                          \kappa_{\alpha} \Rightarrow r_{p} \Rightarrow \kappa_{\beta}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           \kappa_{\alpha} \Rightarrow \kappa_{\beta} \Rightarrow r_{p}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                \overline{\Gamma, x : \{ |\kappa_{\alpha}| \} + \{ |\kappa_{\beta}| \}} \leq \{ |r_p| \}
                                                                                                                                                                                                                                             \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ |r_{p}| \} \leq \{ |\kappa_{\beta}| \}
                                                          iii. \kappa_{\alpha} \Rightarrow d_{p}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           iv. \ \kappa_{\alpha} \Rightarrow e_l \backsimeq e_r \Rightarrow h \ x \backsimeq k \ x
                                   \Gamma \vdash \{|\kappa_\alpha|\} \ \leq \ \{|d_p|\}
                                                                                                                                                                                                                                                                                                                                                                                                     \Gamma, x: \{ |\kappa_\alpha| \} \vdash \mathsf{PEq} \ \{ |r_p| \} \{ e_l \} \{ e_r \} \ \leq \ \mathsf{PEq} \ \{ |\kappa_\beta| \} \{ \mathsf{h} \ x \} \{ \mathsf{k} \ x \}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    SUB-L
                                                                                                                                                                                                                                                 \Gamma \vdash x: \{ |d_p| \} \to \mathsf{PEq} \; \{ |r_p| \} \{ e_l \} \{ e_r \} \; \leq \; x: \{ |\kappa_\alpha| \} \to \mathsf{PEq} \; \{ |\kappa_\beta| \} \{ \mathsf{h} \; x \} \{ \mathsf{k} \; x \}
                                                                                                                                                                                                                                                            \kappa_{\alpha} \Rightarrow \kappa_{\beta} \Rightarrow r_e
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             \kappa_{\alpha} \Rightarrow r_e \Rightarrow \kappa_f
                                                                  vi.\ d_e \Rightarrow \kappa_\alpha
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      v.\;\kappa_\alpha \Longrightarrow d_e
                                              \Gamma \vdash \{|d_e|\} \leq \{|\kappa_{\alpha}|\}
                                                                                                                                                                                                                      \Gamma, x: \{|d_e|\} \vdash \{|\kappa_\beta|\} \ \leq \ \{|r_e|\}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ |r_e| \} \leq \{ |\kappa_{\beta}| \}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 \Gamma \vdash \{|\kappa_\alpha|\} \ \leq \ \overline{\{|d_e|\}}
                                                                                                     \Gamma \vdash \{ |\kappa_{\alpha}| \} \longrightarrow \{ |\kappa_{\beta}| \} \leq \{ |d_e| \} \longrightarrow \{ |r_e| \}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         \Gamma \vdash \{|d_e|\} \rightarrow \{|r_e|\} \leq \{|\kappa_\alpha|\} \rightarrow \{|\kappa_\beta|\}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          h \backsimeq k \Rightarrow h \backsimeq k
```

Fig. 6. Type checking XEq h k lemma. For space, we write $\{d\}$ to mean the refined type $\{v: t \mid d\}$.

 $\Gamma \vdash \mathsf{PEq}\; (\{\![\kappa_{\alpha}]\!] \to \{\![\kappa_{\beta}]\!]) \{\mathsf{h}\} \{\mathsf{k}\} \; \leq \; \mathsf{PEq}\; (\{\![d_e]\!] \to \{\![r_e]\!]) \{\mathsf{h}\} \{\mathsf{k}\}$

```
XEq h k lemma :: PEq (\{v:\alpha \mid d_e\} \rightarrow \{v:\beta \mid r_e\}) {h} {k}
```

When type checking this use of XEq, we need to check that the lemma equates the right expressions (i.e., forall x. $e_l \simeq e_r$ implies $h x \simeq k x$). Critically, type checking must also ensure that the final equality domain (d_e) is stronger than the domains for the functions (d_h, d_k) and for the lemma (d_p) .

Liquid Haskell goes through a complex series of steps to enforce both required checks (Figure 6). We haven't modified Liquid Haskell's typing rules or implementation *at all*; we merely defined PEq in such a way that the existing type checking rules in Liquid Haskell implement the right checks to soundly show extensional equality between functions.

It's easiest to understand how type checking works from top to bottom ("Type Checking", Figure 6). First, we look up XEq's type in the environment (1). Since the XEq is polymorphic, we instantiate the type arguments with the types, $\{v:\alpha\mid\kappa_\alpha\}$ (2) and $\{v:\beta\mid\kappa_\beta\}$ (3). (We write $\{|\kappa_\alpha|\}$ as a short for $\{v:\alpha\mid\kappa_\alpha\}$, since we focus on the refinements assuming the Haskell types match.) Here κ_α and κ_β are refinement type variables; type checking will generate constraints on them that liquid type inference will try to resolve [Rondon et al. 2008]. Next we apply each of the arguments: h (4), k (5), and lemma (6). Each application applies standard dependent function application, with consideration for subtyping. That is, each application (a) substitutes the applied argument in the codomain type and (b) checks that the type of the argument is a subtype of the function's domain type. Application leads to the subtyping constraints Sub-H, Sub-K, and Sub-L set off in boxes, resolved below. Now Liquid Haskell has *inferred* a type for the checked expression (7). To conclude the check, it introduces the final subtype constraint Sub-Sub: the inferred type should be a subtype of the type the user specified (8).

The four instances of subtyping during type checking generate 13 logical implications to resolve for the original expression to type check ("Subtyping Derivation Leaves", Figure 6). The six purple implications with Roman numerals place requirements on the domain; we'll ignore the others, which impose less interesting constraints on the functions' codomains. The Sub-H and Sub-K derivations require (via contravariance) that the refinement variable κ_{α} implies the refinements on the functions' domains, $d_{\rm h}$ and $d_{\rm k}$. Similarly, the derivation Sub-L requires that κ_{α} implies the proof domain $d_{\rm p}$. Since PEq is defined as refined type alias (§3), Sub-L also checks that the refinements given imply the top level refinements of PEq, i.e., that the result of the lemma is sufficient to show XEq's precondition. The Sub-Sub derivation checks subtyping of two PEq types, by treating the type arguments *invariantly*. (We mark covariant implications in red and contravariant implications in blue.) Liquid Haskell treats checks invariantly because PEq's definition uses its type parameter in both positive and negative positions. Sub-Sub will ultimately require that the refinement variable κ_{α} is *equivalent* to the equality domain $d_{\rm e}$.

To sum up, type checking imposes the following six implications as constraints:

Implications v and vi require the refinement variable κ_{α} to be equivalent to the equality domain d_e . Given that equality, implications i-iii state that the equality domain d_e should imply the domains of the functions (i and ii) and lemma (iii). Implication iv requires that the lemma's domain implies equality of the two functions for each argument x that satisfies the domain d_e . All together, these constraints exactly capture the requirements of functional extensionality.

Naive Functional Extensionality with funext. When, in §2, we use the non-type-indexed funext in Liquid Haskell, the typing derivation looks almost exactly the same, but one critical thing changes: the type-indexed PEq t $\{e_l\}$ $\{e_r\}$ is replaced by a refined unit $\{v:() \mid e_l = e_r\}$. This only affects the Sub-L and Sub-Sub derivations, which lose the red and blue parts and become:

$$\frac{iii'. \kappa_{\alpha} \Rightarrow d_{p}}{\Gamma \vdash \{ |\kappa_{\alpha}| \} \leq \{ |d_{p}| \}} \frac{iv'. \kappa_{\alpha} \Rightarrow e_{l} = e_{r} \Rightarrow h \ x = k \ x}{\Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ v : () \mid e_{l} = e_{r} \} \leq \{ v : () \mid h \ x = k \ x \}}$$

$$\Gamma \vdash x : \{ |d_{p}| \} \rightarrow \{ v : () \mid e_{l} = e_{r} \} \leq x : \{ |\kappa_{\alpha}| \} \rightarrow \{ v : () \mid h \ x = k \ x \}$$

$$\frac{h \ x = k \ x \Rightarrow h \ x = k \ x}{\Gamma \vdash \{ v : () \mid h \ x = k \ x \}} \xrightarrow{\text{Sub-Sub-Naive}}$$

Sub-L-Naive generates the implications iii' and iv' that are essentially the same as before. But, Sub-Sub-Naive won't generate any meaningful checks, because equality is just a unit type. We lost implications v and vi! We are now left with an implication system in which the refinement variable κ_{α} only appears in the assumptions. Since Liquid Haskell always tries to infer the most specific refinement possible, it will find a very specific refinement for κ_{α} : false! Having inferred false for κ_{α} , the entire use of funext trivially holds and can be used on other, nontrivial domains—with inconsistent results.

6 A REFINEMENT CALCULUS WITH BUILT-IN TYPE-INDEXED EQUALITY

Because funext is inconsistent in Liquid Haskell (§2), we developed PEq to reason consistently about extensional equality, using the GADT PBEq and the uninterpreted equality PEq (§3). We're able to prove some interesting equalities (§4) and Liquid Haskell's type checking seems to be doing the right thing (§5). But how do we know that our definitions suffice? Formalizing *all* of Liquid Haskell is a challenge, but we can build a model to check the features we use. We formalize a core calculus λ^{RE} with Refinement types, semantic subtyping, and type-indexed propositional Equality.

 $e \hookrightarrow e$

```
Constants
                                         c ::= true | false | unit | (==<sub>b</sub>) | (==<sub>(c,b)</sub>)
                                         Expressions
                                               ::= c \mid \lambda x : \tau. \ e \mid \mathsf{bEq}_b \ e \ e \ v \mid \mathsf{xEq}_{x : \tau \to \tau} \ e \ e \ v
                            Values
                    Refinements
                                               := e
                     Basic Types
                                         b ::= Bool \mid ()
                                             ::= \{x:b \mid r\} \mid x:\tau \to \tau \mid \mathsf{PEq}_{\tau} \{e\} \{e\}
                             Types
                                             ::= \emptyset \mid \Gamma, x : \tau
        Typing Environment
       Closing Substitutions
                                             ::= \emptyset \mid \theta, x \mapsto v
                                             = \emptyset \mid \delta, (v, v)/x
Equivalence Environments
                                              ::= \bullet \mid \mathcal{E} e \mid v \mathcal{E} \mid \mathsf{bEq}_b \ e \ e \ \mathcal{E} \mid \mathsf{xEq}_{x:\tau \to \tau} \ e \ e \ \mathcal{E}
         Evaluation Contexts &
```

Reduction

 $\mathcal{E}[e] \hookrightarrow \mathcal{E}[e'], \quad \text{if } e \hookrightarrow e'$ $(\lambda x : \tau. e) v \hookrightarrow e[v/x]$ $(==_b) c_1 \hookrightarrow (==_{(c_1,b)})$ $(==_{(c_1,b)}) c_2 \hookrightarrow c_1 = c_2, \quad \text{syntactic equality on constants}$

Fig. 7. Syntax and Dynamic Semantics of λ^{RE} .

 λ^{RE} contains just enough to check the core interactions between refinement types and a type-indexed propositional equality resembling our PBEq definition (§6.1). We omit plenty of important features from Liquid Haskell (e.g., algebraic data types): our purpose here is not to develop a complete formal model, but to check that our implementation holds together.

Using λ^{RE} 's static semantics (§6.2), we prove several metatheorems (§6.3). Most importantly, we define a logical relation that characterizes λ^{RE} equivalence and reflects λ^{RE} 's propositional equality. Propositional equivalence in λ^{RE} implies equivalence in the logical relation (Theorem 6.2); both are reflexive, symmetric, and transitive (Theorems 6.3 and 6.4).

6.1 Syntax and Semantics of λ^{RE}

We present λ^{RE} , a core calculus with Refinement types and type-indexed Equality (Figure 7).

Expressions. λ^{RE} expressions include constants (booleans, unit, and equality operations on base types), variables, lambda abstraction, and application. There are also two primitives to prove propositional equality: bEq $_b$ and xEq $_{x:\tau_x\to\tau}$ construct proofs of equality at base and function types, respectively. Equality proofs take three arguments: the two expressions equated and a proof of their equality; proofs at base type are trivial, of type (), but higher types use functional extensionality. These two primitives correspond to BEq and XEq constructors of §3; we did not encode congruence closure since it can be proved by induction on expressions, which is impossible in Haskell.

Values. The values of λ^{RE} are constants, functions, and equality proofs with converged proofs.

Types. λ^{RE} 's basic types are booleans and unit. Basic types are refined with boolean expressions r in refinement types $\{x:b \mid r\}$, which denote all expressions of base type b that satisfy the refinement r. In addition to refinements, λ^{RE} 's types also include dependent function types $x:\tau_x \to \tau$ with arguments of type τ_x and result type τ , where τ can refer back to the argument x. Finally, types include our propositional equality PEq_{τ} $\{e_1\}$ $\{e_2\}$, which denotes a proof of equality between the two expressions e_1 and e_2 of type τ . We write b to mean the trivial refinement type $\{x:b \mid \mathsf{true}\}$.

```
\begin{split} \|\{x:b\mid r\}\| &\ \doteq \ \{e\mid e\hookrightarrow^* v\wedge \vdash_B e::b\wedge r[e/x]\hookrightarrow^* \text{true}\}\\ \|x:\tau_x\to\tau\| &\ \doteq \ \{e\mid \forall e_x\in \|\tau_x\|.\ e\ e_x\in \|\tau[e_x/x]\|\}\\ \|\text{PEq}_b\ \{e_l\}\ \{e_r\}\| &\ \doteq \ \{e\mid \vdash_B e:: \text{PBEq}_b\wedge e\hookrightarrow^* \text{bEq}_b\ e_l\ e_r\ e_{pf}\wedge e_l==_b\ e_r\hookrightarrow^* \text{true}\}\\ \|\text{PEq}_{x:\tau_x\to\tau}\ \{e_l\}\ \{e_r\}\| &\ \doteq \ \{e\mid \vdash_B e:: \text{PBEq}_{\lfloor x:\tau_x\to\tau\rfloor}\wedge e\hookrightarrow^* \text{xEq}\_e_l\ e_r\ e_{pf}\\ &\ \wedge e_l, e_r\in \|x:\tau_x\to\tau\| \wedge \forall e_x\in \|\tau_x\|.\ e_{pf}\ e_x\in \|\text{PEq}_{\tau[e_x/x]}\ \{e_l\ e_x\}\ \{e_r\ e_x\}\|\} \end{split}
```

Fig. 8. Semantic typing: a unary syntactic logical relation interprets types.

We omit polymorphic types to avoid known and resolved metatheoretical problems [Sekiyama et al. 2017]. Yet, xEq equality primitive is defined as a family of operators, one for each refinement function type, capturing the essence of polymorphic function equality.

Environments. The typing environment Γ binds variables to types, the (semantic typing) closing substitution θ binds variables to values, and the (logical relation) pending substitution δ binds variables to pairs of equivalent values.

Runtime Semantics. The relation $\cdot \hookrightarrow \cdot$ evaluates λ^{RE} expressions using contextual, small step, call-by-value semantics (Figure 7, bottom). The semantics are standard with bEq_b and xEq_{x:\tau_x\rightarrow\tau} evaluating proofs but not the equated terms. Let $\cdot \hookrightarrow^* \cdot$ be the reflexive, transitive closure of $\cdot \hookrightarrow \cdot$.}

Type Interpretations. Semantic typing uses a unary logical relation to interpret types in a syntactic term model (closed terms, Figure 8; open terms, Figure 9).

The interpretation of the base type $\{x:b \mid r\}$ includes all expressions which yield b-value v that satisfy the refinement, i.e., r evaluates to true on v. To decide the unrefined type of an expression we use $\vdash_B e :: b$ (defined in §B.1). The interpretation of function types $x:\tau_x \to \tau$ is logical: it includes all expressions that yield τ -results when applied to τ_x arguments. The interpretation of base-type equalities $\mathsf{PEq}_b \{e_l\} \{e_r\}$ includes all expressions that satisfy the basic typing (PBEq_τ is the unrefined version of $\mathsf{PEq}_\tau \{e_l\} \{e_r\}$) and reduce to a basic equality proof whose first arguments reduce to equal b-constants. Finally, the interpretation of the function equality type $\mathsf{PEq}_{x:\tau_x\to\tau} \{e_l\} \{e_r\}$ includes all expressions that satisfy the basic typing (based on the $\lfloor \cdot \rfloor$ operator; §B.1). These expressions reduce to a proof whose first two arguments are functions of type $x:\tau_x\to\tau$ and the third, proof argument takes τ_x arguments to equality proofs of type $\mathsf{PEq}_{\tau[e_x/x]} \{e_l e_x\} \{e_r e_x\}$. We write these proofs as xEq , since the type index does not need to be syntactically equal to the index of the type.

Constants. For simplicity, λ^{RE} constants are only the two boolean values, unit, and equality operators for basic types. For each basic type b, we define the type indexed "computational" equality $==_b$. For two constants c_1 and c_2 of basic type b, $c_1==_b c_2$ evaluates in one step to $(==_{(c_1,b)}) c_2$, which then steps to true when c_1 and c_2 are the same and false otherwise.

Each constant c has the type TyCon(c), as defined below.

```
\begin{array}{lll} \operatorname{TyCon}(\operatorname{true}) & \doteq & \{x:\operatorname{Bool} \mid x ==_{\operatorname{Bool}} \operatorname{true} \} \\ \operatorname{TyCon}(\operatorname{false}) & \doteq & \{x:\operatorname{Bool} \mid x ==_{\operatorname{Bool}} \operatorname{false} \} \\ \operatorname{TyCon}(\operatorname{unit}) & \doteq & \{x:() \mid x ==_{()} \operatorname{unit} \} \\ \operatorname{TyCon}(==_b) & \doteq & x:b \to y:b \to \{z:\operatorname{Bool} \mid z ==_{\operatorname{Bool}} (x ==_b y) \} \end{array}
```

Our system could of course be extended with further constants, as long as they belong in the interpretation of their type. This requirement is formally defined by the Property 1 which, for the four constants of our system is proved in Theorem B.1

```
PROPERTY 1 (CONSTANTS). c \in [TyCon(c)]
```

Fig. 9. Typing of λ^{RE} .

6.2 Static Semantics of λ^{RE}

 λ^{RE} 's static semantics comes in two parts: as typing judgments (§6.2.1) and as a binary logical relation characterizing equivalence (§6.2.2).

6.2.1 Typing of λ^{RE} . Type checking in λ^{RE} uses three mutually recursive judgments (Figure 9): type checking, $\Gamma \vdash e :: \tau$, for when e has type τ in Γ ; well formedness, $\Gamma \vdash \tau$, for when when τ is well formed in Γ ; and subtyping, $\Gamma \vdash \tau_l \leq \tau_r$, for when when τ_l is a subtype of τ_r in Γ .

Type Checking. Beyond the conventional rules for refinement type systems [Knowles and Flanagan 2010; Ou et al. 2004; Rondon et al. 2008], the interesting rules are concerned with equality (TEQBASE, TEQFUN).

The rule TEQBASE assigns to the expression bEq_b e_l e_r e the type PEq_b $\{e_l\}$ $\{e_r\}$. To do so, we guess types τ_l and τ_r that fit e_l and e_r , respectively. Both these types should be subtypes of b that are strong enough to derive that if $l:\tau_l$ and $r:\tau_r$, then the proof argument e has type $\{:() \mid l==_b r\}$. Our formal model allows checking of strong, selfified types (rule TSELF), but does not define an algorithmic procedure to generate them. In Liquid Haskell, type inference [Rondon et al. 2008]

```
Value\ equivalence\ relation \\ c \sim c \ :: \{x:b \mid r\}; \delta \qquad \dot{=} \qquad \vdash_B c :: b \wedge \delta_1 \cdot r[c/x] \hookrightarrow^* \text{true} \wedge \delta_2 \cdot r[c/x] \hookrightarrow^* \text{true} \\ v_1 \sim v_2 :: x: \tau_x \to \tau; \delta \qquad \dot{=} \qquad \forall v_3 \sim v_4 :: \tau_x; \delta. \ v_1 \ v_3 \sim v_2 \ v_4 :: \tau; \delta, \ (v_3, v_4)/x \\ v_1 \sim v_2 :: \mathsf{PEq}_\tau \ \{e_l\} \ \{e_r\}; \delta \qquad \dot{=} \qquad \delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: \tau; \delta \\ \\ Expression\ equivalence\ relation \\ e_1 \sim e_2 :: \tau; \delta \qquad \dot{=} \qquad e_1 \hookrightarrow^* v_1, \quad e_2 \hookrightarrow^* v_2, \quad v_1 \sim v_2 :: \tau; \delta \\ \\ Open\ expression\ equivalence\ relation \\ \delta \in \Gamma \qquad \boxed{\Gamma \vdash e \sim e :: \tau} \\ \delta \in \Gamma \ \dot{=} \ \forall x : \tau \in \Gamma, \ \delta_1(x) \sim \delta_2(x) :: \tau; \delta \qquad \Gamma \vdash e_1 \sim e_2 :: \tau \ \dot{=} \ \forall \delta \in \Gamma, \ \delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \tau; \delta \\ \\ \Gamma \vdash e_1 \sim e_2 :: \tau \ \dot{=} \ \forall \delta \in \Gamma, \ \delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \tau; \delta \\ \\ C \rightarrow C :: \tau; \delta \rightarrow C :: \tau; \delta
```

Fig. 10. Definition of equivalence logical relation.

automatically and algorithmically derives such strong types. We don't encumber λ^{RE} with inference, since, formally speaking, we can always guess any type that inference can derive.

The rule TEQFUN gives the expression $\mathtt{xEq}_{x:\tau_x\to\tau}\ e_l\ e_r\ e$ type $\mathtt{PEq}_{x:\tau_x\to\tau}\ \{e_l\}\ \{e_r\}$. As in TEQBASE, we guess strong types τ_l and τ_r to stand for e_l and e_r such that with $l:\tau_l$ and $r:\tau_r$, the proof argument e should have type $x:\tau_x\to \mathtt{PEq}_\tau\ \{l\ x\}\ \{r\ x\}$, i.e., it should prove that l and r are extensionally equal. We require that the index $x:\tau_x\to \tau$ is well formed as technical bookkeeping.

Well Formedness. Refinements should be booleans (WFBASE); functions are treated in the usual way (WFFun); and the propositional equality $PEq_{\tau} \{e_l\} \{e_r\}$ is well formed when the expressions e_l and e_r are typed at the index τ , which is also well formed (WFEQ).

Subtyping. Basic types are related by set inclusion on the interpretation of those types (SBASE, and Figure 8). Concretely, for all closing substitutions (CEMP, CSUB) the interpretation of the left-hand side type should be a subset of the right-hand side type. The rule SFUN implements the usual (dependent) function subtyping. Finally, SEQ reduces subtyping of equality types to subtyping of the type indices, while the expressions to be equated remain unchanged. Even though covariant treatment of the type index would suffice for our metatheory, we treat the type index invariantly to be consistent with the implementation (§5) where the GADT encoding of PEq is invariant. Our subtyping rule allows equality proofs between functions with convertible types (§4.2).

6.2.2 Equivalence Logical Relation for λ^{RE} . We characterize equivalence with a term-model binary logical relation. We lift a relation on closed values to closed and then open expressions (Figure 10). Instead of directly substituting in type indices, all three relations use *pending substitutions* δ , which map variables to pairs of equivalent values.

Closed Values and Expressions. We read $v_1 \sim v_2 :: \tau; \delta$ as saying that values v_1 and v_2 are related under the type τ with pending substitutions δ . The relation is defined as a fixpoint on types, noting that the propositional equality on a type, $PEq_{\tau}\{e_1\}\{e_2\}$, is structurally larger than the type τ .

For refinement types $\{x:b \mid r\}$, related values must be the same constant c. Further, this constant should actually be a b-constant and it should actually satisfy the refinement r, i.e., substituting c for x in r should evaluate to true under either pending substitution (δ_1 or δ_2). Two values of function type are equivalent when applying them to equivalent arguments yield equivalent results. Since we have dependent types, we record the arguments in the pending substitution for later substitution in the codomain. Two proofs of equality are equivalent when the two equated expressions are equivalent in the logical relation at type-index τ —equality proofs 'reflect' the logical relation. Since

the equated expressions appear in the type itself, they may be open, referring to variables in the pending substitution δ . Thus we use δ to close these expressions, using the logical relation on $\delta_1 \cdot e_l$ and $\delta_2 \cdot e_r$. Following the proof irrelevance notion of refinement typing, the equivalence of equality proofs does not relate the proof terms—in fact, it doesn't even *inspect* the proofs v_1 and v_2 .

Two closed expressions e_1 and e_2 are equivalent on type τ with pending substitions δ , written $e_1 \sim e_2 :: \tau; \delta$, *iff* they respectively evaluate to equivalent values v_1 and v_2 .

Open Expressions. A pending substitution δ satisfies a typing environment Γ when its bindings are relates pairs of values at the type in Γ . Two open expressions, with variables from Γ are equivalent on type τ , written $\Gamma \vdash e_1 \sim e_2 :: \tau$, iff for each δ that satisfies Γ , we have $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \tau; \delta$. The expressions e_1 and e_2 and the type τ might refer to variables in the environment Γ . We use δ to close the expressions eagerly, while we close the type lazily: we apply δ in the refinement and equality cases of the closed value equivalence relation.

6.3 Metaproperties: PEq is an Equivalence Relation

Finally, we show various metaproperties of λ^{RE} . Theorem 6.1 proves soundness of syntactic typing with respect to semantic typing. Theorem 6.2 proves that propositional equality implies equivalence in the term model. Theorems 6.3 and 6.4 prove that both the equivalence relation and propositional equality define equivalences, i.e., satisfy the three equality axioms. All the proofs are in Appendix B. λ^{RE} is semantically sound: syntactically well typed programs are also semantically well typed.

Theorem 6.1 (Typing is Sound). If $\Gamma \vdash e :: \tau$, then $\Gamma \models e \in \tau$.

The proof goes by induction on the derivation tree. Our system could not be proved sound using purely syntactic techniques, like progress and preservation [Wright and Felleisen 1994], for two reasons. First, and most essentially, SBASE needs to quantify over all closing substitutions and purely syntactic approaches flirt with non-monotonicity (though others have attempted syntactic approaches in similar systems [Zalewski et al. 2020]). Second, and merely coincidentally, our system does not enjoy subject reduction. In particular, SEQ allows us to change the type index of propositional equality, but not the term index. Why? Consider the term:

$$(\lambda x: \{x: Bool \mid true\}. bEq_{Bool} x x ()) e$$

such that $e \hookrightarrow e'$ for some e'. The whole application has type $PEq_{Bool} \{e\}$ after we take a step, it will have type $PEq_{Bool} \{e'\} \{e'\}$. Subject reduction demands that the latter is a subtype of the former. We have

$$\mathsf{PEq}_{\mathsf{Bool}}\ \{e\}\ \{e\} \Longrightarrow \mathsf{PEq}_{\mathsf{Bool}}\ \{e'\}\ \{e'\}$$

so we could recover subject reduction by allowing a supertype's terms to parallel reduce (or otherwise convert) to a subtype's terms. Adding this condition would not be hard: the logical relations' metatheory already demands a variety of lemmas about parallel reduction, relegated to supplementary material(Appendix C) to avoid distraction and preserve space for our main contributions. We haven't made this change because subject reduction isn't necessary for our purposes.

```
Theorem 6.2 (PEq is Sound). If \Gamma \vdash e :: PEq_{\tau} \{e_1\} \{e_2\}, then \Gamma \vdash e_1 \sim e_2 :: \tau.
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The proof (see Theorem B.13) is a corollary of the fundamental property of the logical relation (Theorem B.22), i.e., if $\Gamma \vdash e :: \tau$ then $\Gamma \vdash e \sim e :: \tau$, which is proved in turn by induction on the typing derivation.

Theorem 6.3 (The logical relation is an Equivalence). $\Gamma \vdash e_1 \sim e_2 :: \tau$ is reflexive, symmetric, and transitive:

- Reflexivity: If $\Gamma \vdash e :: \tau$, then $\Gamma \vdash e \sim e :: \tau$.
- Symmetry: If $\Gamma \vdash e_1 \sim e_2 :: \tau$, then $\Gamma \vdash e_2 \sim e_1 :: \tau$.
- Transitivity: If $\Gamma \vdash e_2 :: \tau, \Gamma \vdash e_1 \sim e_2 :: \tau$, and $\Gamma \vdash e_2 \sim e_3 :: \tau$, then $\Gamma \vdash e_1 \sim e_3 :: \tau$.

Reflexivity is also called the *fundamental property* of the logical relation. The other proofs go by structural induction on τ (Theorem B.23). Transitivity requires reflexivity on e_2 , so we also assume that $\Gamma \vdash e_2 :: \tau$.

THEOREM 6.4 (PEq IS AN EQUIVALENCE). PEq $_{\tau}$ { e_1 } { e_2 } is reflexive, symmetric, and transitive on equable types. That is, for all τ that do not contain equalities themselves:

- Reflexivity: If $\Gamma \vdash e :: \tau$, then there exists v such that $\Gamma \vdash v :: \mathsf{PEq}_{\tau} \{e\} \{e\}$.
- Symmetry: If $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\}$, then there exists v_{21} such that $\Gamma \vdash v_{21} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_1\}$.
- Transitivity: If $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\}$ and $\Gamma \vdash v_{23} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_3\}$, then there exists v_{13} such that $\Gamma \vdash v_{13} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_3\}$.

The proofs go by induction on τ (Theorem B.24). Reflexivity requires us to generalize the inductive hypothesis to generate appropriate τ_l and τ_r for the PEq proofs.

7 RELATED WORK

Functional Extensionality and Subtyping with an SMT Solver. F*also uses a type-indexed funext axiom after having run into similar unsoundness issues [FStarLang 2018]. Their extensionality axiom makes a more roundabout connection with SMT: function equality uses ==, a proof-irrelevant, propositional Leibniz equality. They assume that their Leibniz equality coincides with SMT equality. Liquid Haskell can't just copy F*: there are no dependent, inductive type definitions, nor a dedicated notion of propositions. Our PEq GADT approximates F*'s approach, with different compromises.

Dafny's SMT encoding axiomatizes extensionality for data, but not for functions [Leino 2012]. Function equality is utterable but neither provable nor disprovable in their encoding into Z3.

Ou et al. [2004] introduce *selfification*, which assigns singleton types using equality (as in our TSelf rule). SAGE assigns selfified types to *all* variables, implying equality on functions [Knowles et al. 2006]. Dminor avoids the question: it lacks first-class functions [Bierman et al. 2012].

Extensionality in Dependent Type Theories. Functional extensionality (funext) has a rich history of study. Martin-Löf type theory comes in a decidable, intensional flavor (ITT) [Martin-Löf 1975] as well as an undecidable, extensional one (ETT) [Martin-Löf 1984]. NuPRL implements ETT [Constable et al. 1986], while Coq and Agda implement ITT [2008; 2020]. Lean's quotient-based reasoning can prove funext [de Moura et al. 2015]. Extensionality axioms are independent of the rules of ITT; funext is a common axiom, but is not consistent in every model of type theory [von Glehn 2014]. Hofmann [1996] shows that ETT is a conservative but less computational extension of ITT with funext and UIP. Pfenning [2001] and Altenkirch and McBride [2006] try to reconcile ITT and ETT.

Dependent type theories often care about equalities between equalities, with axioms like UIP (all identity proofs are the same), K (all identity proofs are ref1), and univalence (identity proofs are isomorphisms, and so not the same). If we allowed equalities between equalities, we could add UIP. Our propositional equality isn't exactly Leibniz equality, so axiom K would be harder to encode.

Zombie's type theory uses an adaptation of a congruence closure algorithm to automatically reason about equality [Sjöberg and Weirich 2015]. Zombie can do some reasoning about equalities on functions but cannot show equalities based on bound variables. Zombie is careful to omit a λ -congruence rule, which could be used to prove funext, "which is not compatible with [their] 'very heterogeneous' treatment of equality" [Ibid., §9].

Cubical type theory offers alternatives to our propositional equality [Sterling et al. 2019]. Such approaches may play better with F*'s approach using dependent, inductive types than the 'flatter'

approach we used for Liquid Haskell. Univalent systems like cubical type theory get funext 'for free'—that is, for the price of the univalence axiom or of cubical foundations.

Classy Induction: Inductive Proofs Using Typeclasses. We used 'classy induction' to prove metaproperties of PEq inside Liquid Haskell (§3.3), using ad-hoc polymorphism and general instances to generate proofs that 'cover' some class of types. We did not *invent* classy induction—it is a folklore technique that we named. We have seen five independent uses of "classy induction" in the literature [Boulier et al. 2017; Dagand et al. 2018; Guillemette and Monnier 2008; Tabareau et al. 2019; Weirich 2017].

Any typeclass system that accommodates ad-hoc polymorphism and a notion of proof can use classy induction. Sozeau [2008] generates proofs of nonzeroness using something akin to classy induction, though it goes by induction on the operations used to build up arithmetic expressions in the (dependent!) host language (§6.3.2); he calls this the 'programmation logique' aspect of typeclasses. Instance resolution is characterized as proof search over lemmas (§7.1.3). Sozeau and Oury [2008] introduce typeclasses to Coq; their system can do induction by typeclasses, but they do not demonstrate the idea in the paper. Earlier work on typeclasses focused on overloading [Nipkow and Prehofer 1993; Nipkow and Snelting 1991; Wadler and Blott 1989], with no notion of classy induction even in settings with proofs [Wenzel 1997].

8 CONCLUSION

In a refinement type system with subtyping a naive encoding of funext is inconsistent. We explained the inconsistency by examples (that proved false) and by standard type checking (where the equality domain is inferred as false). We implemented a type-indexed propositional equality that avoids this inconsistency and validated it with a model calculus. Several case studies demonstrate the range, effectiveness, and power of our work.

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How to Safely Use Extensionality in Liquid Haskell (extended version)

A COMPLETE TYPE CHECKING OF EXTENSIONALITY EXAMPLE

$$\frac{\Gamma(\operatorname{funext}) = \forall a \, b. \operatorname{Eq} \, b \Rightarrow f: (a \to b) \to g: (a \to b) \to (x: a \to \{f \, x = g \, x\}) \to \{f \, x \neq g\}}{\Gamma \vdash \operatorname{funext} :: \forall a \, b. \operatorname{Eq} \, b \Rightarrow f: (a \to b) \to g: (a \to b) \to (x: a \to \{f \, x = g \, x\}) \to \{f \, x = g \, x\}) \to \{f \, x = g \, x\}}$$

$$\frac{\Gamma \vdash \operatorname{funext} :: \forall a \, b. \operatorname{Eq} \, b \Rightarrow f: (\{k_{\alpha}\} \to b) \to g: (\{k_{\alpha}\} \to b) \to (x: \{k_{\alpha}\} \to \{f \, x = g \, x\}) \to \{f \, x = g \, x\}) \to \{f \, x = g \, x\}}{\Gamma \vdash \operatorname{funext} :: \{k_{\alpha}\} :: \exists g \, \{k_{\beta}\} :: \exists g$$

$$\frac{\kappa_{\alpha} \Rightarrow d_{h}}{\Gamma \vdash \{ |\kappa_{\alpha}| \} \leq \{v : \alpha \mid d_{h} \}} \qquad \frac{\kappa_{\alpha} \Rightarrow r_{h} \Rightarrow \kappa_{\beta}}{\Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{v : \beta \mid r_{h} \} \leq \{ |\kappa_{\beta}| \}} \qquad \text{Sub-H}$$

$$\Gamma \vdash x : \{ |d_{h}| \} \rightarrow \{ |r_{h}| \} \leq \{ |\kappa_{\alpha}| \} \rightarrow \{ |\kappa_{\beta}| \} \qquad \text{Sub-H}$$

$$\kappa_{\alpha} \Rightarrow d_{k} \qquad \kappa_{\alpha} \Rightarrow r_{k} \Rightarrow \kappa_{\beta}$$

$$\Gamma \vdash \{ |\kappa_{\alpha}| \} \leq \{v : \alpha \mid d_{k} \} \qquad \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{v : \beta \mid r_{k} \} \leq \{ |\kappa_{\beta}| \} \qquad \text{Sub-K}$$

$$\Gamma \vdash x : \{ |d_{k}| \} \rightarrow \{ |r_{k}| \} \leq \{ |\kappa_{\alpha}| \} \rightarrow \{ |\kappa_{\beta}| \} \qquad \text{Sub-K}$$

$$\kappa_{\alpha} \Rightarrow \text{true} \qquad \kappa_{\alpha} \Rightarrow p \Rightarrow h x == k x$$

$$\Gamma \vdash \{ |\kappa_{\alpha}| \} \leq \alpha \qquad \Gamma, x : \{ |\kappa_{\alpha}| \} \vdash \{ p \} \leq \{ |h x == k x \} \qquad \text{Sub-L}$$

$$\Gamma \vdash x : \{ |d_{p}| \} \rightarrow \{ p \} \leq x : \{ |\kappa_{\alpha}| \} \rightarrow \{ |h x == k x \} \qquad \text{Sub-L}$$

Fig. 11. Complete type checking of naive extensionality in theoremEq.

Expressions
$$e$$
 ::= as in λ^{RE}

Types t ::= Bool | () | PBEq $_t$ ee | $t \to t$

Typing Environment G ::= \emptyset | $G, x : t$

Basic Type checking

$$G \vdash_B e :: t$$

$$\frac{G \vdash_{B} e :: t_{x} \to t \quad G \vdash_{B} e_{x} :: t_{x}}{G \vdash_{B} e e_{x} :: t} \xrightarrow{\text{BTCon}} \frac{x : t \in G}{G \vdash_{B} x :: t} \xrightarrow{\text{BTVAR}}$$

$$\frac{G \vdash_{B} e :: t_{x} \to t \quad G \vdash_{B} e_{x} :: t_{x}}{G \vdash_{B} e e_{x} :: t} \xrightarrow{\text{BTAPP}} \frac{G, x : \lfloor \tau_{x} \rfloor \vdash_{B} e :: t}{G \vdash_{B} \lambda x : \tau_{x} . e :: \lfloor \tau_{x} \rfloor \to t} \xrightarrow{\text{BTLAM}}$$

$$\frac{G \vdash_{B} e :: ()}{G \vdash_{B} e_{1} :: b \quad G \vdash_{B} e_{2} :: b} \xrightarrow{\text{BTEQBASE}} \frac{G \vdash_{B} e_{1} :: \lfloor \tau_{x} \to \tau \rfloor \quad G \vdash_{B} e_{2} :: \lfloor \tau_{x} \to \tau \rfloor}{G \vdash_{B} \text{Eq}_{x : \tau_{x} \to \tau} e_{1} e_{2} e :: \text{PBEq}_{\lfloor \tau_{x} \to \tau \rfloor} e_{1} e_{2}} \xrightarrow{\text{BTEQFun}}$$

Fig. 12. Syntax and Typing of λ^E .

B PROOFS AND DEFINITIONS FOR METATHEORY

In this section we provide proofs and definitions ommitted from §6.

B.1 Base Type Checking

For completeness, we defined λ^E , the unrefined version of λ^{RE} , that ignores the refinements on basic types and the expression indices from the typed equality.

The function $\lfloor \cdot \rfloor$ is defined to turn λ^{RE} types to their unrefined counterparts.

$$\begin{array}{cccc} \lfloor \mathsf{Bool} \rfloor & \doteq & \mathsf{Bool} \\ & \lfloor () \rfloor & \doteq & () \\ \lfloor \mathsf{PEq}_\tau \; \{e_1\} \; \{e_2\} \rfloor & \doteq & \mathsf{PBEq}_{\lfloor \tau \rfloor} \\ & \lfloor \{v \mathpunct{:}b \mid r\} \rfloor & \doteq & b \\ & \lfloor x \mathpunct{:}\tau_x \to \tau \rfloor & \doteq & \lfloor \tau_x \rfloor \to \lfloor \tau \rfloor \\ \end{array}$$

Figure 12 defines the syntax and typing of λ^E that we use to define type denotations of λ^{RE} .

B.2 Constant Property

THEOREM B.1. For the constants c = true, false, unit, and $==_b$, Property 1 holds, i.e., $c \in [TyCon(c)]$.

PROOF. Below are the proofs for each of the four constants.

- $e \equiv \text{true}$ and $e \in [[\{x:Bool \mid x ==_{Bool} \text{ true}\}]]$. We need to prove the below three requirements of membership in the interpretation of basic types:
 - $-e \hookrightarrow^* v$, which holds because true is a value, thus v = true;
 - + B e :: Bool, which holds by the typing rule BTCoN; and
 - $-(x ==_{Bool} true)[e/x] \hookrightarrow^* true$, which holds because

$$(x ==_{\mathsf{Bool}} \mathsf{true})[e/x] = \mathsf{true} ==_{\mathsf{Bool}} \mathsf{true}$$
 $\hookrightarrow (==_{(\mathsf{true},\mathsf{Bool})}) \mathsf{true}$
 $\hookrightarrow \mathsf{true} = \mathsf{true}$
 $= \mathsf{true}$

- $e \equiv \text{false}$ and $e \in [\{x: \text{Bool} \mid x ==_{\text{Bool}} \text{false}\}]$. We need to prove the below three requirements of membership in the interpretation of basic types:
 - $-e \hookrightarrow^* v$, which holds because false is a value, thus $v = \mathsf{false}$;
 - $\vdash_B e :: Bool,$ which holds by the typing rule BTCoN; and
 - $-(x ==_{Bool} false)[e/x] \hookrightarrow^* true$, which holds because

```
(x ==_{Bool} false)[e/x] = false ==_{Bool} false
\hookrightarrow (==_{(false,Bool)}) false
\hookrightarrow false = false
= true
```

- $e \equiv \text{unit}$ and $e \in [\{x:() \mid x==() \text{ unit}\}]$. We need to prove the below three requirements of membership in the interpretation of basic types:
 - $-e \hookrightarrow^* v$, which holds because unit is a value, thus v = unit;
 - $\vdash_B e :: ()$, which holds by the typing rule BTCoN; and
 - $-(x ==_{()} \text{unit})[e/x] \hookrightarrow^* \text{true}$, which holds because

$$\begin{array}{rcl} (x ==_{()} \operatorname{unit})[e/x] & = & \operatorname{unit} ==_{()} \operatorname{unit} \\ & \hookrightarrow & (==_{(\operatorname{unit},())}) \operatorname{unit} \\ & \hookrightarrow & \operatorname{unit} = \operatorname{unit} \\ & = & \operatorname{true} \end{array}$$

• == $_b \in [x:b \to y:b \to \{z: Bool \mid z==_{Bool} (x==_b y)\}]$]. By the definition of interpretation of function types, we fix e_x , $e_y \in [b]$ and we need to prove that $e \equiv e_x ==_b e_y \in [(\{z: Bool \mid z==_{Bool} (x==_b y)\}]$. We prove the below three requirements of membership in the interpretation of basic types: $-e \hookrightarrow^* v$, which holds because

$$e$$
 = $e_x ==_b e_y$
 $\hookrightarrow^* v_x ==_b e_y$ because $e_x \in [\![b]\!]$
 $\hookrightarrow^* v_x ==_b v_y$ because $e_y \in [\![b]\!]$
 $\hookrightarrow (==_{(v_x,b)}) v_y$
 $\hookrightarrow v_x = v_y$
= v with $v = \text{true or } v = \text{false}$

− $\vdash_B e$:: Bool, which holds by the typing rule BTCon and because $e_x, e_y \in [\![b]\!]$ thus $\vdash_B e_x$:: b and $\vdash_B e_y$:: b; and

 $-(z ==_{\mathsf{Bool}} (x ==_b y))[e/z][e_x/x][e_y/y] \hookrightarrow^* \mathsf{true}.$ Since $e_x, e_y \in [\![b]\!]$ both expressions evaluate to values, say $e_x \hookrightarrow^* v_x$ and $e_y \hookrightarrow^* v_y$ which holds because

$$(z ==_{\mathsf{Bool}} (x ==_b y))[e/z][e_x/x][e_y/y] = e =_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$= (e_x ==_b e_y) ==_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$\Leftrightarrow (v_x ==_b e_y) ==_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$\Leftrightarrow (v_x ==_b v_y) ==_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$\Leftrightarrow ((e=_{(v_x,b)}) v_y) ==_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (e_x ==_b e_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (v_x ==_b e_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (v_x ==_b e_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (v_x ==_b v_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (v_x ==_b v_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} ((e=_{(v_x,b)}) v_y)$$

$$\Leftrightarrow (v_x = v_y) ==_{\mathsf{Bool}} (v_x = v_y)$$

B.3 Type Soundness

Theorem B.2 (Semantic soundness). If $\Gamma \vdash e :: \tau \text{ then } \Gamma \models e \in \tau$.

PROOF. By induction on the typing derivation.

TSUB By inversion of the rule we have

```
(1) \Gamma \vdash e :: \tau'
```

(2)
$$\Gamma \vdash \tau' \leq \tau$$

By IH on (1) we have

(3) $\Gamma \models e \in \tau'$

By Theorem B.6 and (2) we have

(4) $\Gamma \vdash \tau' \subseteq \tau$

By (3), (4), and the definition of subsets we directly get $\Gamma \models e \in \tau$.

TSelf Assume $\Gamma \vdash e :: \{z:b \mid z ==_b e\}$. By inversion we have

```
(1) \Gamma \vdash e :: \{z:b \mid r\}
```

By IH we have

(2) $\Gamma \models e \in \{z:b \mid r\}$

We fix $\theta \in ||\Gamma||$. By the definition of semantic typing we get

(3) $\theta \cdot e \in [\theta \cdot \{z:b \mid r\}]$

By the definition of denotations on basic types we have

(4)
$$\theta \cdot e \hookrightarrow^* v$$

(5)
$$\vdash_B \theta \cdot e :: b$$

(6)
$$\theta \cdot r[\theta \cdot e/z] \hookrightarrow^* \text{true}$$

Since θ contains values, by the definition of $==_b$ we have

(7)
$$\theta \cdot e ==_h \theta \cdot e \hookrightarrow^* \text{true}$$

Thus

(8)
$$\theta \cdot (z ==_b e)[\theta \cdot e/z] \hookrightarrow^* \text{true}$$

By (4), (5), and (8) we have

$$(9) \ \theta \cdot e \in [\theta \cdot \{z:b \mid z ==_b e\}]$$

Thus,
$$\Gamma \models e \in \{z:b \mid z ==_b e\}$$
.

TCon This case holds exactly because of Property B.1.

TVAR This case holds by the definition of closing substitutions.

TLAM Assume $\Gamma \vdash \lambda x : \tau_x . e :: x : \tau_x \to \tau$. By inversion of the rule we have $\Gamma, x : \tau_x \vdash e :: \tau$. By IH we get $\Gamma, x : \tau_x \models e \in \tau$.

We need to show that $\Gamma \models \lambda x : \tau_x. \ e \in x : \tau_x \to \tau$. Which, for some $\theta \in [\Gamma]$ is equivalent to $\lambda x : \theta \cdot \tau_x. \ \theta \cdot e \in [x : \theta \cdot \tau_x \to \theta \cdot \tau]$.

We pick a random $e_x \in [\theta \cdot \tau_x]$ thus we need to show that $\theta \cdot e[e_x/x] \in [\theta \cdot \tau[e_x/x]]$. By Lemma B.3, there exists v_x so that $e_x \hookrightarrow^* v_x$ and $v_x \in [\tau_x]$. By the inductive hypothesis, $\theta \cdot e[v_x/x] \in [\theta \cdot \tau[v_x/x]]$. By Lemma B.4, $\theta \cdot e[e_x/x] \in [\theta \cdot \tau[e_x/x]]$, which concludes our proof.

TAPP Assume $\Gamma \vdash e \ e_x :: \tau[e_x/x]$. By inversion we have

- (1) $\Gamma \vdash e :: x : \tau_x \to \tau$
- (2) $\Gamma \vdash e_x :: \tau_x$

By IH we get

- (3) $\Gamma \models e \in x : \tau_x \to \tau$
- (4) $\Gamma \models e_x \in \tau_x$

We fix $\theta \in \|\Gamma\|$. By the definition of semantic types

- (5) $\theta \cdot e \in [\theta \cdot x : \tau_x \to \tau]$
- $(6) \ \theta \cdot e_x \in [\![\theta \cdot \tau_x]\!]$

By (5), (6), and the definition of semantic typing on functions:

(7) $\theta \cdot e \ e_x \in [\theta \cdot \tau [e_x/x]]$

Which directly leads to the required $\Gamma \models e \ e_x \in \tau[e_x/x]$

TEQBASE Assume $\Gamma \vdash \mathsf{bEq}_b \ e_l \ e_r \ e :: \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}$. By inversion we get:

- (1) $\Gamma \vdash e_l :: \tau_l$
- (2) $\Gamma \vdash e_r :: \tau_r$
- (3) $\Gamma \vdash \tau_l \leq \{x:b \mid \mathsf{true}\}$
- (4) $\Gamma \vdash \tau_r \leq \{x:b \mid \mathsf{true}\}$
- (5) $\Gamma, r : \tau_r, l : \tau_l \vdash e :: \{x:() \mid l ==_b r\}$

By IH we get

- (4) $\Gamma \models e_l \in \tau_l$
- (5) $\Gamma \models e_r \in \tau_r$
- (6) $\Gamma, r : \tau_r, l : \tau_l \models e \in \{x:() \mid l ==_b r\}$

We fix $\theta \in [\Gamma]$. Then (4) and (5) become

- (7) $\theta \cdot e_l \in [\theta \cdot \tau_l]$
- (8) $\theta \cdot e_r \in [\theta \cdot \tau_r]$
- (9) $\Gamma \models e_r \in \tau_r$
- (10) $\Gamma, r : \tau_r, l : \tau_l \models e \in \{x:() \mid l ==_b r\}$

Assume

- (11) $\theta \cdot e_l \hookrightarrow^* v_l$
- $(12) \ \theta \cdot e_r \hookrightarrow^* v_r$

By (7), (8), (11), (12), and Lemma B.3 we get

- (13) $v_I \in [\theta \cdot \tau_I]$
- $(14) v_r \in [\![\theta \cdot \tau_r]\!]$

By (10), (11), and (12) we get

(15) $v_l ==_b v_r \hookrightarrow^* \mathsf{true}$

By (11), (12), (15), ane Lemma B.5 we have

(16) $\theta \cdot e_l ==_b \theta \cdot e_r \hookrightarrow^* \text{true}$

By (1-5) we get:

(17) $\vdash_B \theta \cdot \mathsf{bEq}_b \ e_l \ e_r \ e :: \mathsf{PBEq}_b$

Trivially, with zero evaluation steps we have:

(18) $\theta \cdot \mathsf{bEq}_b \ e_l \ e_r \ e \hookrightarrow^* \mathsf{bEq}_b \ (\theta \cdot e_l) \ (\theta \cdot e_l) \ (\theta \cdot e)$

By (16), (17), (18) and the definition of semantic types on basic equality types we have

(19) $\theta \cdot \mathsf{bEq}_b \ e_l \ e_r \ e \in [\theta \cdot \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}]$

Which leads to the required $\Gamma \models \mathsf{bEq}_b \ e_l \ e_r \ e \in \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}.$

TEQFUN Assume $\Gamma \vdash xEq_{x:\tau_r \to \tau} e_l e_r e :: PEq_{x:\tau_r \to \tau} \{e_l\} \{e_r\}$. By inversion we have

- (1) $\Gamma \vdash e_l :: \tau_l$
- (2) $\Gamma \vdash e_r :: \tau_r$
- (3) $\Gamma \vdash \tau_l \leq x : \tau_x \to \tau$
- (4) $\Gamma \vdash \tau_r \leq x : \tau_x \to \tau$
- (5) $\Gamma, r : \tau_r, l : \tau_l \vdash e :: (x : \tau_x \rightarrow \mathsf{PEq}_\tau \{l \ x\} \{r \ x\})$
- (6) $\Gamma \vdash x:\tau_x \to \tau$

By IH and Theorem B.6 we get

- (7) $\Gamma \models e_l \in \tau_l$
- (8) $\Gamma \models e_r \in \tau_r$
- $(9) \Gamma \vdash \tau_l \subseteq x : \tau_x \to \tau$
- $(10) \; \Gamma \vdash \tau_r \; \subseteq \; x : \tau_x \to \tau$
- (11) $\Gamma, r : \tau_r, l : \tau_l \models e \in (x : \tau_x \rightarrow \mathsf{PEq}_\tau \{l \ x\} \{r \ x\})$ By (1-5) we get
- (12) $\vdash_B \theta \cdot \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ e \ :: \mathsf{PBEq}_{\lfloor \theta \cdot (x:\tau_x \to \tau) \rfloor}$ Trivially, by zero evaluation steps, we get
- (13) $\theta \cdot \mathsf{xEq}_{x:\tau_x \to \tau} e_l e_r e \hookrightarrow^* \mathsf{xEq}_{x:\theta \cdot \tau_x \to \theta \cdot \tau} (\theta \cdot e_l) (\theta \cdot e_r) (\theta \cdot e)$ By (7-10) we get
- (14) $\theta \cdot e_l, \theta \cdot e_r \in [\theta \cdot x : \tau_x \to \tau]$

By (7), (8), (11), the definition of semantic types on functions, and Lemmata B.3 and B.4 (similar to the previous case) we have

$$- \forall e_x \in [\![\tau_x]\!] . e \ e_x \in [\![\mathsf{PEq}_{\tau[e_x/x]}] \ \{e_l \ e_x\} \ \{e_r \ e_x\}]\!]$$

By (12), (13), (14), and (15) we get

$$(19) \ \theta \cdot \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ e \in \left\| \theta \cdot \mathsf{PEq}_{x:\tau_x \to \tau} \ \{e_l\} \ \{e_r\} \right\|$$

Which leads to the required $\Gamma \models x \to q_{x:\tau_x \to \tau} e_l e_r e \in P \to q_{x:\tau_x \to \tau} \{e_l\} \{e_r\}.$

LEMMA B.3. If $e \in [\tau]$, then $e \hookrightarrow^* v$ and $v \in [\tau]$.

PROOF. By structural induction of the type τ .

LEMMA B.4. If $e_x \hookrightarrow^* v_x$ and $e[v_x/x] \in [|\tau[v_x/x]|]$, then $e[e_x/x] \in [|\tau[e_x/x]|]$.

PROOF. We can use parallel reductions (of §C) to prove that if $e_1
ightharpoonup e_2$, then (1) $||\tau[e_1/x]|| = ||\tau[e_2/x]||$ and (2) $e_1 \in ||\tau||$ iff $e_2 \in ||\tau||$. The proof directly follows by these two properties.

LEMMA B.5. If $e_x \hookrightarrow^* e_x'$ and $e[e_x'/x] \hookrightarrow^* c$, then $e[e_x/x] \hookrightarrow^* c$.

PROOF. As an instance of Corollary C.17.

We define semantic subtyping as follows: $\Gamma \vdash \tau \subseteq \tau'$ iff $\forall \theta \in [\Gamma] \cdot [\theta \cdot \tau] \subseteq [\theta \cdot \tau']$.

Theorem B.6 (Subtyping semantic soundness). If $\Gamma \vdash \tau \leq \tau'$ then $\Gamma \vdash \tau \subseteq \tau'$.

PROOF. By induction on the derivation tree:

SBASE Assume $\Gamma \vdash \{x:b \mid r\} \leq \{x':b \mid r'\}$. By inversion $\forall \theta \in [\![\Gamma]\!]$, $[\![\theta \cdot \{x:b \mid r\}]\!] \subseteq [\![\theta \cdot \{x':b \mid r'\}]\!]$, which exactly leads to the required.

SFUN Assume $\Gamma \vdash x:\tau_x \to \tau \leq x:\tau_x' \to \tau'$. By inversion

- $(1) \Gamma \vdash \tau_x' \leq \tau_x$
- (2) $\Gamma, x : \tau'_x \vdash \tau \leq \tau'$

By IH

- (3) $\Gamma \vdash \tau_x' \subseteq \tau_x$
- (4) $\Gamma, x : \tau'_x \vdash \tau \subseteq \tau'$

We fix $\theta \in \Gamma$. We pick e. We assume $e \in [\theta \cdot x : \tau_x \to \tau]$ and we will show that $e \in [\theta \cdot x : \tau_x' \to \tau']$. By assumption

(5) $\forall e_x \in [\theta \cdot \tau_x]$. $e \ e_x \in [\theta \cdot \tau[e_x/x]]$

We need to show $\forall e_x \in [\![\theta \cdot \tau_x']\!]$. $e \ e_x \in [\![\theta \cdot \tau'[e_x/x]]\!]$. We fix e_x . By (3), if $e_x \in [\![\theta \cdot \tau_x']\!]$, then $e_x \in [\![\theta \cdot \tau_x]\!]$ and (5) applies, so $e \ e_x \in [\![\theta \cdot \tau[e_x/x]]\!]$, which by (4) gives $e \ e_x \in [\![\theta \cdot \tau'[e_x/x]]\!]$. Thus, $e \in [\![\theta \cdot x:\tau_x' \to \tau']\!]$. This leads to $[\![\theta \cdot x:\tau_x \to \tau]\!] \subseteq [\![\theta \cdot x:\tau_x' \to \tau']\!]$, which by definition gives semantic subtyping: $\Gamma \vdash x:\tau_x \to \tau \subseteq x:\tau_x' \to \tau'$.

SEQ Assume $\Gamma \vdash \mathsf{PEq}_{\tau_i} \{e_l\} \{e_r\} \leq \mathsf{PEq}_{\tau_i'} \{e_l\} \{e_r\}$. We split cases on the structure of τ_i .

- If τ_i is a basic type, then τ_i is trivially refined to true. Thus, $\tau_i = \tau_i' = b$ and for each $\theta \in \Gamma$, $\|\theta \cdot \mathsf{PEq}_{\tau} \{e_l\} \{e_r\}\| = \|\theta \cdot \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}\|$, thus set inclusion reduces to equal sets.
- *If* τ_i is a function type, thus $\Gamma \vdash \mathsf{PEq}_{x:\tau_x \to \tau} \{e_l\} \{e_r\} \leq \mathsf{PEq}_{x:\tau_x' \to \tau'} \{e_l\} \{e_r\}$ By inversion
- $(1) \Gamma \vdash x:\tau_x \to \tau \leq x:\tau_x' \to \tau'$
- (2) $\Gamma \vdash x:\tau_x' \to \tau' \leq x:\tau_x \to \tau$

By inversion on (1) and (2) we get

- (3) $\Gamma \vdash \tau_x' \leq \tau_x$
- (4) $\Gamma, x : \tau'_x \vdash \tau \leq \tau'$
- (5) $\Gamma, x : \tau_x \vdash \tau' \leq \tau$

By IH on (1) and (3) we get

- (6) $\Gamma \vdash x:\tau_x \to \tau \subseteq x:\tau'_x \to \tau'$
- $(7) \Gamma \vdash \tau_x' \subseteq \tau_x$

We fix $\theta \in \Gamma$ and some e. If $e \in \|\theta \cdot \mathsf{PEq}_{x:\tau_x \to \tau} \{e_l\} \{e_r\}\|$ we need to show that $e \in \|\theta \cdot \mathsf{PEq}_{x:\tau_x \to \tau'} \{e_l\} \{e_r\}\|$. By the assumption we have

- (8) $\vdash_B e :: PBEq_{\lfloor \theta \cdot (x:\tau_x \to \tau) \rfloor}$
- (9) $e \hookrightarrow^* xEq (\theta \cdot e_l) (\theta \cdot e_r) e_{pf}$
- (10) $(\theta \cdot e_l), (\theta \cdot e_r) \in [\theta \cdot (x:\tau_x \to \tau)]$
- (11) $\forall e_x \in [\theta \cdot \tau_x]$ $e_{pf} e_x \in [\text{PEq}_{\theta \cdot (\tau[e_x/x])} \{(\theta \cdot e_l) e_x\} \{(\theta \cdot e_r) e_x\}]$ Since (8) only depends on the structure of the type index, we get
- (12) $\vdash_B e :: \mathsf{PBEq}_{\lfloor \theta \cdot (x:\tau'_x \to \tau') \rfloor}$

By (6) and (10) we get

(13) $(\theta \cdot e_l), (\theta \cdot e_r) \in [\theta \cdot (x:\tau_x' \to \tau')]$

By (4), (5), Lemma B.7, the rule SEQ and the IH, we get that $\|PEq_{\theta \cdot (\tau[e_x/x])} \{(\theta \cdot e_l) e_x\} \{(\theta \cdot e_r) e_x\}\| \subseteq \|PEq_{\theta \cdot (\tau[e_x/x])} \{(\theta \cdot e_l) e_x\} \{(\theta \cdot e_r) e_x\}\|$. By which, (11), (7), and reasoning similar to the SFun case, we get

Lemma B.7 (Strengthening). If $\Gamma_1 \vdash \tau_1 \leq \tau_2$, then:

- (1) If $\Gamma_1, x : \tau_2, \Gamma_2 \vdash e :: \tau \text{ then } \Gamma_1, x : \tau_1, \Gamma_2 \vdash e :: \tau$.
- (2) If $\Gamma_1, x : \tau_2, \Gamma_2 \vdash \tau \leq \tau'$ then $\Gamma_1, x : \tau_1, \Gamma_2 \vdash \tau \leq \tau'$.
- (3) If $\Gamma_1, x : \tau_2, \Gamma_2 \vdash \tau$ then $\Gamma_1, x : \tau_1, \Gamma_2 \vdash \tau$.
- (4) If $\vdash \Gamma_1, x : \tau_2, \Gamma_2$ then $\vdash \Gamma_1, x : \tau_1, \Gamma_2$.

Proof. The proofs go by induction. Only the TVAR case is insteresting; we use TSUB and our assumption. $\hfill\Box$

Lemma B.8 (Semantic typing is closed under parallel reduction in expressions). If $e_1 \rightrightarrows^* e_2$, then $e_1 \in [|\tau|]$ iff $e_2 \in [|\tau|]$.

PROOF. By induction on τ , using parallel reduction as a bisimulation (Lemma C.5 and Corollary C.15).

Lemma B.9 (Semantic typing is closed under parallel reduction in types). If $\tau_1 \rightrightarrows^* \tau_2$ then $||\tau_1|| = ||\tau_2||$.

PROOF. By induction on τ_1 (which necessarily has the same shape as τ_2). We use parallel reduction as a bisimulation (Lemma C.5 and Corollary C.15).

Lemma B.10 (Parallel reducing types are equal). If $\Gamma \vdash \tau_1$ and $\Gamma \vdash \tau_2$ and $\tau_1 \rightrightarrows^* \tau_2$ then $\Gamma \vdash \tau_1 \leq \tau_2$ and $\Gamma \vdash \tau_1 \leq \tau_2$.

PROOF. By induction on the parallel reduction sequence; for a single step, by induction on τ_1 (which must have the same structure as τ_2). We use parallel reduction as a bisimulation (Lemma C.5 and Corollary C.15).

Lemma B.11 (Regularity). (1) If $\Gamma \vdash e :: \tau \ then \vdash \Gamma \ and \ \Gamma \vdash \tau$.

- (2) If $\Gamma \vdash \tau$ then $\vdash \Gamma$.
- (3) If $\Gamma \vdash \tau_1 \leq \tau_2$ then $\vdash \Gamma$ and $\Gamma \vdash \tau_1$ and $\Gamma \vdash \tau_2$.

PROOF. By a big ol' induction.

Lemma B.12 (Canonical forms). *If* $\Gamma \vdash v :: \tau$, *then*:

- If $\tau = \{x:b \mid e\}$, then v = c such that $\mathsf{TyCon}(c) = b$ and $\Gamma \vdash \mathsf{TyCon}(c) \leq \{x:b \mid e\}$.
- If $\tau = x:\tau_x \to \tau'$, then $v = TLAMx\tau_x'e$ such that $\Gamma \vdash \tau_x \leq \tau_x'$ and $\Gamma, x:\tau_x' \vdash e :: \tau''$ such that $\tau'' \vdash \tau' \leq .$
- If $\tau = \text{PEq}_b \{e_l\} \{e_r\}$ then $v = \text{bEq}_b e_l e_r v_p$ such that $\Gamma \vdash e_l :: \tau_l \text{ and } \Gamma \vdash e_r :: \tau_r \text{ (for some } \tau_l \text{ and } \tau_r \text{ that are refinements of } b) \text{ and } \Gamma, r : \tau_r, l : \tau_l \vdash v_p :: \{x : () \mid l ==_b r\}.$
- If $\tau = \text{PEq}_{x:\tau_x \to \tau'} \{e_l\} \{e_r\}$ then $v = \text{xEq}_{x:\tau_x' \to \tau''} e_l e_r v_p$ such that $\Gamma \vdash \tau_x \leq \tau_x'$ and $\Gamma, x : \tau_x \vdash \tau'' \leq \tau'$ and $\Gamma \vdash e_l :: \tau_l$ and $\Gamma \vdash e_r :: \tau_r$ (for some τ_l and τ_r that are subtypes of $x:\tau_x' \to \tau''$) and $\Gamma, r : \tau_r, l : \tau_l \vdash v_p :: x:\tau_x' \to \text{PEq}_{\tau''} \{e_l x\} \{e_r x\}.$

B.4 The Binary Logical Relation

Theorem B.13 (EqRT soundness). If $\Gamma \vdash e :: PEq_{\tau} \{e_1\} \{e_2\}$, then $\Gamma \vdash e_1 \sim e_2 :: \tau$.

PROOF. By $\Gamma \vdash e :: \mathsf{PEq}_\tau \ \{e_1\} \ \{e_2\}$ and the Fundamental Property B.22 we have $\Gamma \vdash e \sim e :: \mathsf{PEq}_\tau \ \{e_1\} \ \{e_2\}$. Thus, for a fixed $\delta \in \Gamma$, $\delta_1 \cdot e \sim \delta_2 \cdot e :: \mathsf{PEq}_\tau \ \{e_1\} \ \{e_2\}$; δ . By the definition of the logical relation for EqRT, we have $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \tau$; δ . So, $\Gamma \vdash e_1 \sim e_2 :: \tau$.

Lemma B.14 (LR respects subtyping). If $\Gamma \vdash e_1 \sim e_2 :: \tau$ and $\Gamma \vdash \tau \leq \tau'$, then $\Gamma \vdash e_1 \sim e_2 :: \tau'$.

PROOF. By induction on the derivation of the subtyping tree.

SBASE By assumption we have

- (1) $\Gamma \vdash e_1 \sim e_2 :: \{x:b \mid r\}$
- (2) $\Gamma \vdash \{x:b \mid r\} \leq \{x':b \mid r'\}$

By inversion on (2) we get

(3) $\forall \theta \in \llbracket \Gamma \rrbracket$, $\llbracket \theta \cdot \{x:b \mid r\} \rrbracket \subseteq \llbracket \theta \cdot \{x':b \mid r'\} \rrbracket$

We fix $\delta \in \Gamma$. By (1) we get

(4) $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \{x:b \mid r\}; \delta$

By the definition of logical relations:

- (5) $\delta_1 \cdot e_1 \hookrightarrow^* v_1$
- (6) $\delta_2 \cdot e_2 \hookrightarrow^* v_2$
- (7) $v_1 \sim v_2 :: \{x:b \mid r\}; \delta$

By (7) and the definition of the logical relation on basic types we have

- (8) $v_1 = v_2 = c$
- $(9) \vdash_B c :: b$
- (10) $\delta_1 \cdot r[c/x] \hookrightarrow^* \text{true}$
- (11) $\delta_2 \cdot r[c/x] \hookrightarrow^* \text{true}$

By (3), (10) and (11) become

- (12) $\delta_1 \cdot r'[c/x'] \hookrightarrow^* \text{true}$
- (13) $\delta_2 \cdot r'[c/x'] \hookrightarrow^* \mathsf{true}$

By (8), (9), (12), and (13) we get

(14) $v_1 \sim v_2 :: \{x':b \mid r'\}; \delta$

By (5), (6), and (14) we have

(15) $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \{x':b \mid r'\}; \delta$ Thus, $\Gamma \vdash e_1 \sim e_2 :: \{x':b \mid r'\}.$

SFun By assumption:

- (1) $\Gamma \vdash e_1 \sim e_2 :: x:\tau_x \to \tau$
- (2) $\Gamma \vdash x:\tau_x \to \tau \leq x:\tau_x' \to \tau'$

By inversion of the rule (2)

- (3) $\Gamma \vdash \tau_x' \leq \tau_x$
- (4) $\Gamma, x : \tau'_x \vdash \tau \leq \tau'$

We fix $\delta \in \Gamma$. By (1) and the definition of logical relation

- (5) $\delta_1 \cdot e_1 \hookrightarrow^* v_1$
- (6) $\delta_2 \cdot e_2 \hookrightarrow^* v_2$
- (7) $v_1 \sim v_2 :: x:\tau_x \to \tau; \delta$

We fix v'_1 and v'_2 so that

(8) $v_1' \sim v_2' :: \tau_r'; \delta$

By (8) and the definition of logical relations, since the values are idempotent under substitution, we have

(9) $\Gamma \vdash v_1' \sim v_2' :: \tau_x'$

By (9) and inductive hypothesis on (3) we have

 $(10) \Gamma \vdash v_1' \sim v_2' :: \tau_X$

By (10), idempotence of values under substitution, and the definition of logical relations, we have

(11) $v_1' \sim v_2' :: \tau_x; \delta$

By (7), (11), and the definition of logical relations on function values:

(12) $v_1 v_1' \sim v_2 v_2' :: \tau; \delta, (v_1', v_2')/x$

By (9), (12), and the definition of logical relations we have

(12) $\Gamma, x : \tau'_x \vdash v_1 \ v'_1 \sim v_2 \ v'_2 :: \tau$

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By (12) and inductive hypothesis on (4) we have
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(13) $\Gamma, x : \tau'_x \vdash v_1 \ v'_1 \sim v_2 \ v'_2 :: \tau'$

By (8), (13), and the definition of logical relations, we have

(14) $v_1 v_1' \sim v_2 v_2' :: \tau'; \delta, (v_1', v_2')/x$

By (8), (14), and the definition of logical relations, we have

(15) $v_1 \sim v_2 :: x:\tau'_x \to \tau'; \delta$

By (5), (6), and (15), we get

(16) $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: x:\tau'_x \to \tau'; \delta$

So, $\Gamma \vdash e_1 \sim e_2 :: x:\tau'_x \to \tau'$.

SEQ By hypothesis:

- (1) $\Gamma \vdash e_1 \sim e_2 :: PEq_{\tau} \{e_l\} \{e_r\}$
- $(2) \Gamma \vdash \mathsf{PEq}_{\tau} \{e_l\} \{e_r\} \leq \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}$

We fix $\delta \in \Gamma$. By (1)

(3) $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \mathsf{PEq}_\tau \{e_l\} \{e_r\}; \delta$

By (3) and the definition of logical relations.

- (4) $\delta_1 \cdot e_1 \hookrightarrow^* v_1$
- (5) $\delta_2 \cdot e_2 \hookrightarrow^* v_2$
- (6) $v_1 \sim v_2 :: PEq_{\tau} \{e_l\} \{e_r\}; \delta$

By (6) and the definition of logical relations

(7) $\delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: \tau; \delta$

By (7) and the definition of logical relations.

(8) $\Gamma \vdash e_l \sim e_r :: \tau$

By inversion on (2)

- (9) $\Gamma \vdash \tau \leq \tau'$
- (10) $\Gamma \vdash \tau' \leq \tau$

By (8) and inductive hypothesis on (9)

(11) $\Gamma \vdash e_l \sim e_r :: \tau'$

Thus,

(12) $\delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: \tau'; \delta$

By (12), (4), (5), and determinism of operational semantics:

(12) $v_1 \sim v_2 :: PEq_{\tau'} \{e_l\} \{e_r\}; \delta$

By (4), (5), and (13)

(14) $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: PEq_{\tau'} \{e_l\} \{e_r\}; \delta$

So, by definition of logical relations, $\Gamma \vdash e_1 \sim e_2 :: PEq_{\tau'} \{e_l\} \{e_r\}$.

Lemma B.15 (Constant soundness). $\Gamma \vdash c \sim c :: \mathsf{TyCon}(c)$

PROOF. The proof follows the same steps as Theorem B.1.

Lemma B.16 (Selfification of constants). If $\Gamma \vdash e \sim e :: \{z:b \mid r\}$ then $\Gamma \vdash x \sim x :: \{z:b \mid z =_b x\}$.

PROOF. We fix $\delta \in \Gamma$. By hypothesis $(v_1, v_2)/x \in \delta$ with $v_1 \sim v_2 :: \{z:b \mid r\}; \delta$. We need to show that $\delta_1 \cdot x \sim \delta_2 \cdot x :: \{z:b \mid z ==_b x\}; \delta$. Which reduces to $v_1 \sim v_2 :: \{z:b \mid z ==_b x\}; \delta$. By the definition on the logical relation on basic values, we know $v_1 = v_2 = c$ and $\vdash_B c :: b$. Thus, we are left to prove that $\delta_1 \cdot ((z ==_b x)[c/z]) \hookrightarrow^*$ true and $\delta_2 \cdot ((z ==_b x)[c/z]) \hookrightarrow^*$ true which, both, trivially hold by the definition of $==_b$.

Lemma B.17 (Variable soundness). If $x : \tau \in \Gamma$, then $\Gamma \vdash x \sim x :: \tau$.

PROOF. By the definition of the logical relation it suffices to show that $\forall \delta \in \Gamma.\delta_1(x) \sim \delta_2(x) :: \tau; \delta$; which is trivially true by the definition of $\delta \in \Gamma$.

Lemma B.18 (Transitivity of Evaluation). If $e \hookrightarrow^* e'$, then $e \hookrightarrow^* v$ iff $e' \hookrightarrow^* v$.

PROOF. Assume $e \hookrightarrow^* v$. Since the \hookrightarrow is by definition deterministic, there exists a unique sequence $e \hookrightarrow e_1 \hookrightarrow \ldots \hookrightarrow e_i \hookrightarrow \ldots \hookrightarrow v$. By assumption, $e \hookrightarrow^* e'$, so there exists a j, so $e' \equiv e_j$, and $e' \hookrightarrow^* v$ following the same sequence.

Assume $e' \hookrightarrow^* v$. Then $e \hookrightarrow^* e' \hookrightarrow^* v$ uniquely evaluates e to v.

Lemma B.19 (LR closed under evaluation). If $e_1 \hookrightarrow^* e_1'$, $e_2 \hookrightarrow^* e_2'$, then $e_1' \sim e_2' :: \tau; \delta$ iff $e_1 \sim e_2 :: \tau; \delta$.

PROOF. Assume $e'_1 \sim e'_2 :: \tau; \delta$, by the definition of the logical relation on closed terms we have $e'_1 \hookrightarrow^* v_1, e'_2 \hookrightarrow^* v_2$, and $v_1 \sim v_2 :: \tau; \delta$. By Lemma B.18 and by assumption, $e_1 \hookrightarrow^* e'_1$ and $e_2 \hookrightarrow^* e'_2$, we have $e_1 \hookrightarrow^* v_1$ and $e_2 \hookrightarrow^* v_2$. By which and $v_1 \sim v_2 :: \tau; \delta$ we get that $e_1 \sim e_2 :: \tau; \delta$. The other direction is identical.

Lemma B.20 (LR closed under parallel reduction). If $e_1 \rightrightarrows^* e_1'$, $e_2 \rightrightarrows^* e_2'$, and $e_1' \sim e_2' :: \tau; \delta$, then $e_1 \sim e_2 :: \tau; \delta$.

Proof. By induction on τ , using parallel reduction as a backward simulation (Corollary C.15).

Lemma B.21 (LR Compositionality). If $\delta_1 \cdot e_x \hookrightarrow^* v_{x_1}$, $\delta_2 \cdot e_x \hookrightarrow^* v_{x_2}$, $e_1 \sim e_2 :: \tau; \delta, (v_{x_1}, v_{x_2})/x$, then $e_1 \sim e_2 :: \tau[e_x/x]; \delta$.

PROOF. By the assumption we have that

- (1) $\delta_1 \cdot e_x \hookrightarrow^* v_{x_1}$
- (2) $\delta_2 \cdot e_x \hookrightarrow^* v_{x_2}$
- $(3) e_1 \hookrightarrow^* v_1$
- $(4) e_2 \hookrightarrow^* v_2$
- (5) $v_1 \sim v_2 :: \tau; \delta, (v_{x1}, v_{x_2})/x$

and we need to prove that $v_1 \sim v_2 :: \tau[e_x/x]; \delta$. The proof goes by structural induction on the type τ .

- $\tau \doteq \{z:b \mid r\}$. For i=1,2 we need to show that if $\delta_i, [v_{x_i}/x] \cdot r[v_i/z] \hookrightarrow^*$ true then $\delta_i \cdot r[v_i/z][e_i/x] \hookrightarrow^*$ true. We have $\delta_i, [v_{x_i}/x] \cdot r[v_i/z] \rightrightarrows^* \delta_i \cdot r[v_i/z][e_i/x]$ because substituting parallel reducing terms parallel reduces (Corollary C.3) and parallel reduction subsumes reduction (Lemma C.4). By cotermination at constants (Corollary C.17), we have $\delta_i \cdot r[v_i/z][e_i/x] \hookrightarrow^*$ true.
- $\tau \doteq y:\tau_y' \to \tau'$. We need to show that if $v_1 \sim v_2 :: y:\tau_y' \to \tau'; \delta, (v_{x_1}, v_{x_2})/x$, then $v_1 \sim v_2 :: y:\tau_y' \to \tau'[e_x/x]; \delta$.

We fix v_{y_1} and v_{y_2} so that $v_{y_1} \sim v_{y_2} :: \tau'_{y_2}; \delta, (v_{x_1}, v_{x_2})/x$.

Then, we have that $v_1 \ v_{y_1} \sim v_2 \ v_{y_2} :: \tau'; \delta, (v_{x_1}, v_{x_2})/x, (v_{y_1}, v_{y_2})/y$.

By inductive hypothesis, we have that $v_1 \ v_{y_1} \sim v_2 \ v_{y_2} :: \tau'[e_x/x]; \delta, (v_{y_1}, v_{y_2})/y$.

By inductive hypothesis on the fixed arguments, we also get $v_{y_1} \sim v_{y_2} :: \tau'_y[e_x/x]; \delta$.

Combined, we get $v_1 \sim v_2 :: y : \tau'_y \to \tau'[e_x/x]; \delta$.

• $\tau \doteq \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}$. We need to show that if $v_1 \sim v_2 :: \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}; \delta, (v_{x_1}, v_{x_2})/x$, then $v_1 \sim v_2 :: \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\} [e_x/x]; \delta$.

This reduces to showing that if δ_1 , $[v_{x_1}/x] \cdot e_l \sim \delta_2$, $[v_{x_2}/x] \cdot e_r :: \tau'$; δ , then $\delta_1 \cdot e_l[e_x/x] \sim \delta_2 \cdot e_r[e_x/x] :: \tau'$; δ ; we find $\delta_1 \cdot e_l[e_x/x] \rightrightarrows^* \delta_1$, $[v_{x_1}/x] \cdot e_l$ and $\delta_2 \cdot e_r[e_x/x] \rightrightarrows^* \delta_2$, $[v_{x_2}/x] \cdot e_r$

because substituting multiple parallel reduction is parallel reduction (Corollary C.3). The logical relation is closed under parallel reduction (Lemma B.20), and so $\delta_1 \cdot e_l[e_x/x] \sim \delta_2 \cdot e_r[e_x/x] :: \tau'; \delta$.

Theorem B.22 (LR Fundamental Property). If $\Gamma \vdash e :: \tau$, then $\Gamma \vdash e \sim e :: \tau$.

PROOF. The proof goes by induction on the derivation tree:

TSUB By inversion of the rule we have

- (1) $\Gamma \vdash e :: \tau'$
- (2) $\Gamma \vdash \tau' \leq \tau$

By IH on (1) we have

(3) $\Gamma \vdash e \sim e :: \tau'$

By (3), (4), and Lemma B.14 we have $\Gamma \vdash e \sim e :: \tau$.

TCon By Lemma B.15.

TSELF By inversion of the rule, we have:

- (1) $\Gamma \vdash e :: \{z:b \mid r\}.$
- (2) By the IH on (1), we have: $\Gamma \vdash e \sim e :: \{z:b \mid r\}.$
- (3) We fix a δ such that:

$$\delta \in \Gamma$$
 and

$$\delta_1 \cdot e \sim \delta_2 \cdot e :: \{z:b \mid r\}; \delta$$

(4) There must exist v_1 and v_2 such that:

$$\delta_1 \cdot e \hookrightarrow^* v_1$$

$$\delta_2 \cdot e \hookrightarrow^* v_2$$

$$v_1 \sim v_2 :: \{z:b \mid r\}; \delta$$

(5) By definition, $v_1 = v_2 = c$ such that:

$$\vdash_B c :: b$$

$$\delta_1 \cdot r[c/x] \hookrightarrow^* \mathsf{true}$$

$$\delta_2 \cdot r[c/x] \hookrightarrow^* \mathsf{true}$$

(6) We find $v_1 \sim v_2 :: \{z:b \mid z ==_b e\}; \delta$, because:

$$\vdash_B c :: b \text{ by } (5)$$

$$\delta_1 \cdot (z ==_b e)[c/z] \hookrightarrow^* \text{true because } \delta_1 \cdot e \hookrightarrow^* v_1 = c \text{ by (4)}$$

$$\delta_2 \cdot (z = b \ e)[c/z] \hookrightarrow^* \text{true because } \delta_2 \cdot e \hookrightarrow^* v_2 = c \text{ by (4)}$$

TVAR By inversion of the rule and Lemma B.17.

TLAM By hypothesis:

(1)
$$\Gamma \vdash \lambda x : \tau_x . \ e :: x : \tau_x \to \tau$$

By inversion of the rule we have

- (2) Γ , $x : \tau_x \vdash e :: \tau$
- (3) $\Gamma \vdash \tau_x$

By inductive hypothesis on (2) we have

(4)
$$\Gamma, x : \tau_x \vdash e \sim e :: \tau$$

We fix a δ , v_{x_1} , and v_{x_2} so that

(5)
$$\delta \in \Gamma$$

(6)
$$v_{x_1} \sim v_{x_2} :: \tau_x; \delta$$

Let
$$\delta' \doteq \delta$$
, $(v_{x_1}, v_{x_2})/x$.

By the definition of the logical relation on open terms, (4), (5), and (6) we have

(7)
$$\delta'_1 \cdot e \sim \delta'_2 \cdot e :: \tau; \delta'$$

By the definition of substitution

(8)
$$\delta_1 \cdot e[v_{x_1}/x] \sim \delta_2 \cdot e[v_{x_2}/x] :: \tau; \delta'$$

By the definition of the logical relation on closed expressions

(9)
$$\delta_1 \cdot e[v_{x_1}/x] \hookrightarrow^* v_1$$
, $\delta_2 \cdot e[v_{x_2}/x] \hookrightarrow^* v_2$, and $v_1 \sim v_2 :: \tau; \delta'$

By the definition and determinism of operational semantics

(10)
$$\delta_1 \cdot (\lambda x : \tau_x. \ e) \ v_{x_1} \hookrightarrow^* v_1, \ \delta_2 \cdot (\lambda x : \tau_x. \ e) \ v_{x_2} \hookrightarrow^* v_2, \ \text{and} \ v_1 \sim v_2 :: \tau; \delta'$$

By (6) and the definition of logical relation on function values, $\frac{1}{2}$

(11)
$$\delta_1 \cdot \lambda x : \tau_x. \ e \sim \delta_2 \cdot \lambda x : \tau_x. \ e :: x : \tau_x \to \tau; \delta$$

Thus, by the definition of the logical relation, $\Gamma \vdash \lambda x : \tau_x. \ e \sim \lambda x : \tau_x. \ e :: x : \tau_x \rightarrow \tau$

TAPP By hypothesis:

(1)
$$\Gamma \vdash e \ e_x :: \tau[e_x/x]$$

By inversion we get

(2)
$$\Gamma \vdash e :: x : \tau_x \to \tau$$

(3)
$$\Gamma \vdash e_x :: \tau_x$$

By inductive hypothesis

(3)
$$\Gamma \vdash e \sim e :: x:\tau_x \to \tau$$

(4)
$$\Gamma \vdash e_x \sim e_x :: \tau_x$$

We fix a $\delta \in \Gamma$. Then, by the definition of the logical relation on open terms

(5)
$$\delta_1 \cdot e \sim \delta_2 \cdot e :: (x:\tau_x \to \tau); \delta$$

(6)
$$\delta_1 \cdot e_x \sim \delta_2 \cdot e_x :: \tau_x; \delta$$

By the definition of the logical relation on open terms:

(7)
$$\delta_1 \cdot e \hookrightarrow^* v_1$$

(8)
$$\delta_2 \cdot e \hookrightarrow^* v_2$$

(9)
$$v_1 \sim v_2 :: x:\tau_x \to \tau; \delta$$

$$(10) \ \delta_1 \cdot e_x \hookrightarrow^* v_{x_1}$$

(11)
$$\delta_2 \cdot e_x \hookrightarrow^* v_{x_2}$$

(12)
$$v_{x_1} \sim v_{x_2} :: \tau_x; \delta$$

By (7) and (10)

$$(13) \ \delta_1 \cdot e \ e_x \hookrightarrow^* v_1 \ v_{x_1}$$

By (8) and (11)

$$(14) \ \delta_2 \cdot e \ e_x \hookrightarrow^* v_2 \ v_{x_2}$$

By (9), (12), and the definition of logical relation on functions:

(15)
$$v_1 v_{x_1} \sim v_2 v_{x_2} :: \tau; \delta, (v_{x_1}, v_{x_2})/x$$

By (13), (14), (15), and Lemma B.19

(16)
$$\delta_1 \cdot e \ e_x \sim \delta_2 \cdot e \ e_x :: \tau; \delta, (v_{x_1}, v_{x_2})/x$$

By (10), (11), (16), and Lemma B.21

(17)
$$\delta_1 \cdot e \ e_x \sim \delta_2 \cdot e \ e_x :: \tau[e_x/x]; \delta$$

So from the definition of logical relations, $\Gamma \vdash e \ e_x \sim e \ e_x :: \tau[e_x/x]$.

TEOBASE By hypothesis:

(1)
$$\Gamma \vdash \mathsf{bEq}_b \ e_l \ e_r \ e :: \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}$$

By inversion of the rule:

- (2) $\Gamma \vdash e_1 :: \tau_r$
- (3) $\Gamma \vdash e_r :: \tau_1$
- (4) $\Gamma \vdash \tau_r \leq b$
- (5) $\Gamma \vdash \tau_l \leq b$
- (6) $\Gamma, r : \tau_r, l : \tau_l \vdash e :: \{x:() \mid l ==_b r\}$

By inductive hypothesis on (2), (3), and (6) we have

(7)
$$\Gamma \vdash e_l \sim e_l :: \tau_r$$

(2) $\Gamma \vdash e_l :: \tau_r$ (3) $\Gamma \vdash e_r :: \tau_l$

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(8) \Gamma \vdash e_r \sim e_r :: \tau_l
             (9) \Gamma, r : \tau_r, l : \tau_l \vdash e \sim e :: \{x:() \mid l ==_b r\}
                We fix \delta \in \Gamma. Then (7) and (8) become
           (10) \delta_1 \cdot e_l \sim \delta_2 \cdot e_l :: \tau_r; \delta
           (11) \delta_1 \cdot e_r \sim \delta_2 \cdot e_r :: \tau_l; \delta
                By the definition of the logical relation on closed terms:
           (12) \delta_1 \cdot e_l \hookrightarrow^* v_{l_1}
           (13) \delta_2 \cdot e_l \hookrightarrow^* v_{l_2}
           (14) v_{l_1} \sim v_{l_2} :: \tau_l; \delta
           (15) \ \delta_1 \cdot e_r \hookrightarrow^* v_{r_1}
           (16) \delta_2 \cdot e_r \hookrightarrow^* v_{r_2}
           (17) v_{r_1} \sim v_{r_2} :: \tau_r; \delta
                We define \delta' \doteq \delta, (v_{r_1}, v_{r_2})/r, (v_{l_1}, v_{l_2})/l.
                By (9), (14), and (17) we have
           (18) \delta'_1 \cdot e \sim \delta'_2 \cdot e :: \{x:() \mid l ==_b r\}; \delta'
                By the definition of the logical relation on closed terms:
           (19) \delta' \cdot e \hookrightarrow^* v_1
           (20) \delta' \cdot e \hookrightarrow^* v_2
           (21) v_1 \sim v_2 :: \{x:() \mid l ==_b r\}; \delta'
                By (21) and the definition of logical relation on basic values:
           (19) \delta'_1 \cdot (l ==_b r) \hookrightarrow^* \mathsf{true}
           (20) \delta'_2 \cdot (l ==_b r) \hookrightarrow^* \mathsf{true}
                By the definition of ==_b
           (21) v_{l_1} = v_{r_1}
           (22) \ v_{l_2} = v_{r_2}
                By (14) and (17) and since \tau_l and \tau_r are basic types
           (23) \ v_{l_1} = v_{l_2}
           (24) \ v_{r_1} = v_{r_2}
                By (21) and (24)
           (25) v_{l_1} = v_{r_2}
                By the definition of the logical relation on basic types
           (26) v_{l_1} \sim v_{r_2} :: b; \delta
                By which, (12), (16), and Lemma B.19
           (27) \delta_1 \cdot e_1 \sim \delta_2 \cdot e_r :: b; \delta
                By (12), (15), and (19)
           (28) \delta_1 \cdot \mathsf{bEq}_b \ e_l \ e_r \ e \hookrightarrow^* \mathsf{bEq}_b \ v_{l_1} \ v_{r_1} \ v_1
                By (13), (16), and (20)
           (29) \delta_2 \cdot \mathsf{bEq}_b \ e_l \ e_r \ e \hookrightarrow^* \mathsf{bEq}_b \ v_{l_2} \ v_{r_2} \ v_2
                By (27) and the definition of the logical relation on EqRT
           (30) \mathsf{bEq}_b \ v_{l_1} \ v_{r_1} \ v_1 \sim \mathsf{bEq}_b \ v_{l_2} \ v_{r_2} \ v_2 :: \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}; \delta.
                By (28), (29), and (30)
           (31) \delta_1 \cdot \mathsf{bEq}_b \ e_l \ e_r \ e \sim \delta_2 \cdot \mathsf{bEq}_b \ e_l \ e_r \ e :: \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}; \delta.
                So, by the definition on the logical relation, \Gamma \vdash \mathsf{bEq}_b \ e_l \ e_r \ e \sim \mathsf{bEq}_b \ e_l \ e_r \ e :: \mathsf{PEq}_b \ \{e_l\} \ \{e_r\}.
TEQFUN By hypothesis
             (1) \Gamma \vdash \mathsf{xEq}_{\tau_x:\tau \to} e_l \ e_r \ e :: \mathsf{PEq}_{x:\tau_x \to \tau} \ \{e_l\} \ \{e_r\}
                By inversion of the rule
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(4) \Gamma \vdash \tau_r \leq x : \tau_x \to \tau
 (5) \Gamma \vdash \tau_l \leq x : \tau_x \to \tau
 (6) \Gamma, r : \tau_r, l : \tau_l \vdash e :: (x : \tau_x \rightarrow \mathsf{PEq}_\tau \{l \ x\} \{r \ x\})
 (7) \Gamma \vdash x:\tau_x \to \tau
    By inductive hypothesis on (2), (3), and (6) we have
 (8) \Gamma \vdash e_l \sim e_l :: \tau_r
 (9) \Gamma \vdash e_r \sim e_r :: \tau_l
(10) \Gamma, r : \tau_r, l : \tau_l \vdash e \sim e :: (x:\tau_x \rightarrow \mathsf{PEq}_\tau \{l \ x\} \{r \ x\})
    By (8), (9), and Lemma B.14
(11) \Gamma \vdash e_l \sim e_l :: x:\tau_x \to \tau
(12) \Gamma \vdash e_r \sim e_r :: x:\tau_x \to \tau
    We fix \delta \in \Gamma. Then (11), and (12) become
(13) \delta_1 \cdot e_l \sim \delta_2 \cdot e_l :: x:\tau_x \to \tau; \delta
(14) \delta_1 \cdot e_r \sim \delta_2 \cdot e_r :: x:\tau_x \to \tau; \delta
    By the definition of the logical relation on closed terms:
(15) \delta_1 \cdot e_l \hookrightarrow^* v_l
(16) \delta_2 \cdot e_l \hookrightarrow^* v_{l_2}
(17) v_{l_1} \sim v_{l_2} :: x:\tau_x \to \tau; \delta
(18) v_{l_1} \sim v_{l_2} :: \tau_l; \delta
(19) \delta_1 \cdot e_r \hookrightarrow^* v_{r_1}
(20) \ \delta_2 \cdot e_r \hookrightarrow^* v_{r_2}
(21) v_{r_1} \sim v_{r_2} :: x:\tau_x \to \tau; \delta
(22) v_{r_1} \sim v_{r_2} :: \tau_r; \delta
    We fix v_{x_1} and v_{x_2} so that v_{x_1} \sim v_{x_2} :: \tau_x; \delta. Let \delta_x \doteq \delta, (v_{x_1}, v_{x_2})/x.
    By the definition on the logical relation on function values, (17) and (21) become
(23) v_{l_1} v_{x_1} \sim v_{l_2} v_{x_2} :: \tau; \delta_x
(24) v_{r_1} v_{x_1} \sim v_{r_2} v_{x_2} :: \tau; \delta_x
    Let \delta_{lr} \doteq \delta, (v_{r_1}, v_{r_2})/r, (v_{l_1}, v_{l_2})/l.
    By the definition of the logical relation on closed terms, (10) becomes:
(25) \delta_{lr} \cdot e \hookrightarrow^* v_1
(26) \delta_{lr} \cdot e \hookrightarrow^* v_2
(27) v_1 \sim v_2 :: x:\tau_x \rightarrow \mathsf{PEq}_\tau \{l \ x\} \{r \ x\}; \delta_{lr}
    By (27) and the definition of logical relation on function values:
(28) v_1 v_{x_1} \sim v_2 v_{x_2} :: PEq_{\tau} \{l x\} \{r x\}; \delta_{lr}, (v_{x_1}, v_{x_2})/x
    By the definition of the logical relation on EqRT
(29) \ v_{l_1} \ v_{x_1} \sim v_{r_2} \ v_{x_2} :: \tau; \delta_{lr}, (v_{x_1}, v_{x_2})/x
    By the definition of logical relations on function values
(30) v_{l_1} \sim v_{r_2} :: x:\tau_x \to \tau; \delta_{lr}
    By (7), l and r do not appear free in the relation, so
(31) v_{l_1} \sim v_{r_2} :: x:\tau_x \to \tau; \delta
    By which, (15), (20), and Lemma B.19
(32) \delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: x:\tau_x \to \tau; \delta
    By (15), (19), and (25)
(33) \delta_1 \cdot \mathsf{xEq}_{\tau_r : \tau \to} e_l \ e_r \ e \hookrightarrow^* \mathsf{xEq}_{\tau_r : \tau \to} v_{l_1} \ v_{r_1} \ v_1
    By (16), (20), and (26)
(34) \delta_2 \cdot \mathsf{xEq}_{\tau_x:\tau \to} e_l \ e_r \ e \hookrightarrow^* \mathsf{xEq}_{\tau_x:\tau \to} v_{l_2} \ v_{r_2} \ v_2
    By (32) and the definition of the logical relation on EqRT
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(35) $\mathsf{xEq}_{\tau_x:\tau\to} v_{l_1} v_{r_1} v_1 \sim \mathsf{xEq}_{\tau_x:\tau\to} v_{l_2} v_{r_2} v_2 :: \mathsf{PEq}_{x:\tau_x\to\tau} \{e_l\} \{e_r\}; \delta.$

(36) $\delta_1 \cdot \mathsf{xEq}_{\tau_x:\tau_{\to}} e_l \, e_r \, e \sim \delta_2 \cdot \mathsf{xEq}_{\tau_x:\tau_{\to}} \, e_l \, e_r \, e :: \mathsf{PEq}_{x:\tau_x\to\tau} \, \{e_l\} \, \{e_r\}; \delta.$ So, by the definition on the logical relation, $\Gamma \vdash \mathsf{xEq}_{\tau_x:\tau_{\to}} \, e_l \, e_r \, e \sim \mathsf{xEq}_{\tau_x:\tau_{\to}} \, e_l \, e_r \, e :: \mathsf{PEq}_{x:\tau_x\to\tau} \, \{e_l\} \, \{e_r\}.$

B.5 The Logical Relation and the EqRT Type are Equivalence Relations

Theorem B.23 (The logical relation is an equivalence relation). $\Gamma \vdash e_1 \sim e_2 :: \tau$ is reflexive, symmetric, and transivite.

- Reflexivity: If $\Gamma \vdash e :: \tau$, then $\Gamma \vdash e \sim e :: \tau$.
- Symmetry: If $\Gamma \vdash e_1 \sim e_2 :: \tau$, then $\Gamma \vdash e_2 \sim e_1 :: \tau$.
- Transitivity: If $\Gamma \vdash e_2 :: \tau$ and $\Gamma \vdash e_1 \sim e_2 :: \tau$ and $\Gamma \vdash e_2 \sim e_3 :: \tau$, then $\Gamma \vdash e_1 \sim e_3 :: \tau$.

PROOF. Reflexivity: This is exactly the Fundamental Property B.22.

Symmetry: Let $\bar{\delta}$ be defined such that $\bar{\delta}_1(x) = \delta_2(x)$ and $\bar{\delta}_2(x) = \delta_1(x)$. First, we prove that $v_1 \sim v_2 :: \tau; \delta$ implies $v_2 \sim v_1 :: \tau; \bar{\delta}$, by structural induction on τ .

- $\tau \doteq \{z:b \mid r\}$. This case is immediate: we have to show that $c \sim c :: \{z:b \mid r\}$; $\bar{\delta}$ given $c \sim c :: \{z:b \mid r\}$; δ . But the definition in this case is itself symmetric: the predicate goes to true under both substitutions.
- $\tau \doteq x : \tau'_x \to \tau'$. We fix v_{x_1} and v_{x_2} so that
- (1) $v_{x_1} \sim v_{x_2} :: \tau_x'; \delta$

By the definition of logical relations on open terms and inductive hypothesis

(2) $v_{x_2} \sim v_{x_1} :: \tau_x'; \bar{\delta}$

By the definition on logical relations on functions

(3) $v_1 v_{x_1} \sim v_2 v_{x_2} :: \tau'; \delta, (v_{x_1}, v_{x_2})/x$

By the definition of logical relations on open terms and since the expressions v_1 v_{x_1} and v_2 v_{x_2} are closed, By the inductive hypothesis on τ' :

(4) $v_2 v_{x_2} \sim v_1 v_{x_1} :: \tau'; \bar{\delta}, x : \tau'_x$

By (2) and the definition of logical relations on open terms

(5) $v_2 v_{x_2} \sim v_1 v_{x_1} :: \tau'; \bar{\delta}, (v_{x_2}, v_{x_1})/x$

By the definition of the logical relation on functions, we conclude that $v_2 \sim v_1 :: x:\tau'_x \to \tau'; \bar{\delta}$

- $\tau \doteq \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}$. By assumption,
- (1) $v_1 \sim v_2 :: PEq_{\tau'} \{e_l\} \{e_r\}; \delta$

By the definition of the logical relation on EqRT types

(2) $\delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: \tau'; \delta$

i.e., $\delta_1 \cdot (e_l) \hookrightarrow^* v_l$ and similarly for v_r such that $v_l \sim v_r :: \tau'; \delta$.

By the IH on τ' , we have:

(3) $v_r \sim v_l :: \tau'; \bar{\delta}$

And so, by the definition of the LR on equality proofs:

(4) $v_2 \sim v_1 :: \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}; \bar{\delta}$

Next, we show that $\delta \in \Gamma$ implies $\bar{\delta} \in \Gamma$. We go by structural induction on Γ .

- $\Gamma = \cdot$. This case is trivial.
- $\Gamma = \Gamma', x : \tau$. For $x : \tau$, we know that $\delta_1(x) \sim \delta_2(x) :: \tau; \delta$. By the IH on τ , we find $\delta_2(x) \sim \delta_1(x) :: \tau; \bar{\delta}$, which is just the same as $\bar{\delta}_1(x) \sim \bar{\delta}_2(x) :: \tau; \bar{\delta}$. By the IH on Γ' , we can use similar reasoning to find $\bar{\delta}_1(y) \sim \bar{\delta}_2(y) :: \tau'; \bar{\delta}$ for all $y : \tau' \in \Gamma'$.

Now, suppose $\Gamma \vdash e_1 \sim e_2 :: \tau$; we must show $\Gamma \vdash e_2 \sim e_1 :: \tau$. We fix $\delta \in \Gamma$; we must show $\delta_1 \cdot e_2 \sim \delta_2 \cdot e_1 :: \tau; \delta$, i.e., there must exist v_1 and v_2 such that $\delta_1 \cdot e_2 \hookrightarrow^* v_2$ and $\delta_2 \cdot e_1 \hookrightarrow^* v_1$ and

 $v_2 \sim v_1 :: \tau; \delta$. We have $\delta \in \Gamma$, and so $\bar{\delta} \in \Gamma$ by our second lemma. But then, by assumption, we have v_1 and v_2 such that $\bar{\delta}_1 \cdot e_1 \hookrightarrow^* v_1$ and $\bar{\delta}_2 \cdot e_2 \hookrightarrow^* v_2$ and $v_1 \sim v_2 :: \tau; \bar{\delta}$. Our first lemma then yields $v_2 \sim v_1 :: \tau; \delta$ as desired.

Transitivity: First, we prove an inner property: if $\delta \in \Gamma$ and $v_1 \sim v_2 :: \tau; \delta$ and $v_2 \sim v_3 :: \tau; \delta$, then $v_1 \sim v_3 :: \tau; \delta$. We go by structural induction on the type index τ .

- $\tau \doteq \{z:b \mid r\}$. Here all of the values must be the fixed constant c. Furthermore, we must have $\delta_1 \cdot r[c/x] \hookrightarrow^*$ true and $\delta_2 \cdot r[c/x] \hookrightarrow^*$ true, so we can immediately find $v_1 \sim v_3 :: \tau; \delta$.
- $\tau \doteq x: \tau_x' \to \tau'$. Let $v_l \sim v_r :: \tau_x'; \delta$ be given. We must show that $v_1 \sim v_3 :: \tau; \delta, (v_l, v_r)/x$. We know by assumption that: $v_1 \ v_l \sim v_2 \ v_r :: \tau'; \delta, (v_l, v_r)/x$ and $v_2 \ v_l \sim v_3 \ v_r :: \tau'; \delta, (v_l, v_r)/x$. By the IH on τ' , we find $v_1 \ v_l \sim v_3 \ v_r :: \tau'; \delta, (v_l, v_r)/x$; which gives $v_1 \sim v_3 :: \tau; \delta, (v_l, v_r)/x$.
- $\tau \doteq \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}$. To find $v_1 \sim v_3 :: \mathsf{PEq}_{\tau} \{e_l\} \{e_r\}$; δ , we merely need to find that $\delta_1 \cdot e_l \sim \delta_2 \cdot e_r :: \tau$; δ , which we have by inversion on $v_1 \sim v_2 :: \mathsf{PEq}_{\tau} \{e_l\} \{e_r\}$; δ .

With our proof that the value relation is transitive in hand, we turn our attention to the open relation. Suppose $\Gamma \vdash e_1 \sim e_2 :: \tau$ and $\Gamma \vdash e_2 \sim e_3 :: \tau$; we want to see $\Gamma \vdash e_1 \sim e_3 :: \tau$. Let $\delta \in \Gamma$ be given. We have $\delta_1 \cdot e_1 \sim \delta_2 \cdot e_2 :: \tau; \delta$ and $\delta_1 \cdot e_2 \sim \delta_2 \cdot e_3 :: \tau; \delta$. By the definition of the logical relations, we have $\delta_1 \cdot e_1 \hookrightarrow^* v_1$, $\delta_2 \cdot e_2 \hookrightarrow^* v_2$, $\delta_1 \cdot e_2 \hookrightarrow^* v_2'$, $\delta_2 \cdot e_3 \hookrightarrow^* v_3$, $v_1 \sim v_2 :: \tau; \delta$, and $v_2' \sim v_3 :: \tau; \delta$.

Moreover, we know that e_2 is well typed, so by the fundamental theorem (Theorem B.22), we know that $\Gamma \vdash e_2 \sim e_2 :: \tau$, and so $v_2 \sim v_2' :: \tau; \delta$.

By our transitivity lemma on the value relation, we can find that v_1 is equivalent to v_2 is equivalent to v_3 , and so $v_1 \sim v_3 :: \tau; \delta$.

$$\begin{array}{rcl} & \text{pf} & : & e \rightarrow e \rightarrow \tau \\ & \text{pf}(l,r,b) & = & \{x{:}() \mid l ==_b r\} \\ & \text{pf}(l,r,x{:}\tau_x \rightarrow \tau) & = & x{:}\tau_x \rightarrow \mathsf{PEq}_\tau \; \{l \; x\} \; \{r \; x\} \end{array}$$

Our propositional equality $PEq_{\tau} \{e_l\} \{e_r\}$ is a reflection of the logical relation, so it is unsurprising that it is also an equivalence relation. We can prove that our propositional equality is treated as an equivalence relation by the syntactic type system. There are some tiny wrinkles in the syntactic system: symmetry and transitivity produce normalized proofs, but reflexivity produces unnormalized ones in order to generate the correct invariant types τ_l and τ_r in the base case.

THEOREM B.24 (EqRT is an equivalence relation). PEq $_{\tau}$ { e_1 } { e_2 } is reflexive, symmetric, and transitive on equable types. That is, for all τ that contain only refinements and functions:

- Reflexivity: If $\Gamma \vdash e :: \tau$, then there exists e_p such that $\Gamma \vdash e_p :: PEq_{\tau} \{e\} \{e\}$.
- Symmetry: $\forall \Gamma, \tau, e_1, e_2, v_{12}$. if $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\}$, then there exists v_{21} such that $\Gamma \vdash v_{21} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_1\}$.
- Transitivity: $\forall \Gamma, \tau, e_1, e_2, e_3, v_{12}, v_{23}$. if $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\}$ and $\Gamma \vdash v_{23} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_3\}$, then there exists v_{13} such that $\Gamma \vdash v_{13} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_3\}$.

PROOF. **Reflexivity**: We strengthen the IH, simultaneously proving that there exist e_p , e_{pf} and $\Gamma \vdash \tau_l \leq \tau$ and $\Gamma \vdash \tau_r \leq \tau$ such that $\Gamma, l : \tau_l, r : \tau_r \vdash e_{pf} :: \mathsf{pf}(e, e, \tau)$ and $\Gamma \vdash e_p :: \mathsf{PEq}_\tau \{e\} \{e\}$ by induction on τ , leaving e general.

- $\tau \doteq \{x:b \mid e'\}$.
- (1) Let $e_{pf} = ()$.
- (2) Let $e_p = bEq_b e e e_{pf}$.

- (3) Let $\tau_l = \tau_r = \{x:b \mid x ==_b e\}$.
- (4) We have $\Gamma \vdash x ==_b e \leq \tau$ by SBASE and semantic typing.
- (5) We find $\Gamma \vdash e_p :: PEq_b \{e\} \{e\}$ by TEQBASE, with $e_l = e_r = e$. We must show:
 - (a) $\Gamma \vdash e_l :: \tau_l \text{ and } \Gamma \vdash e_r :: \tau_r, \text{ i.e., } \Gamma \vdash e :: \{x:b \mid x ==_b e\};$
 - (b) $\Gamma \vdash \tau_r \leq \{x:b \mid \text{true}\}\ \text{and}\ \Gamma \vdash \tau_l \leq \{x:b \mid \text{true}\};\ \text{and}\$
 - (c) $\Gamma, r : \tau_r, l : \tau_l + e_{pf} :: \{x:() \mid l ==_b r\}.$
- (6) We find (5a) by TSELF.
- (7) We find (5b) immediately by SBASE.
- (8) We find (5c) by TVAR, using TSUB to see that if $l, r : \{x:b \mid x ==_b e\}$ then unit will be typeable at the refinement where both l and r are equal to e.
- $\tau \doteq x : \tau_x \to \tau'$.
- (1) $\Gamma, x : \tau_x \vdash e \ x :: \tau[x/x]$ by TAPP and TVAR, noting that $\tau[x/x] = \tau$.
- (2) By the IH on $\Gamma, x : \tau_x \vdash e \ x :: \tau'[x/x] = \tau'$, there exist e_p', e_{nf}', τ_l' , and τ_r' such that:
 - (a) $x : \tau_x \vdash \tau'_l \leq \tau$ and $x : \tau_x \vdash \tau'_r \leq \tau$;
 - (b) $\Gamma, x : \tau_x, \dot{l} : \tau'_l, r : \tau'_r \vdash e'_{pf} :: pf(e \ x, e \ x, \tau');$ and
 - (c) $\Gamma, x : \tau_x \vdash e_p' :: \mathsf{PEq}_{\tau'} \{ e \ x \} \{ e \ x \}.$
- (3) If $\tau' = \{x:() \mid \tau'\}e \ xe \ x$, then pf $(e \ x, ex, b) = \{x:() \mid ex ==_b ex\}$; otherwise, pf $(l, r, x:\tau_x \rightarrow \tau) = x:\tau_x \rightarrow \mathsf{PEq}_\tau \{e \ x\} \{e \ x\}$.

In the former case, let $e_{\rm pf}^{\prime\prime}={\sf bEq}_b\ (e\ x)(e\ x)e_{\rm pf}^\prime.$ In the latter case, let $e_{\rm pf}^{\prime\prime}=e_{\rm pf}^\prime.$ Either way, we have $\Gamma,x:\tau_x,l:\tau_l^\prime,r:\tau_r^\prime\vdash e_{\rm pf}^{\prime\prime}::{\sf PEq}_{\tau^\prime}\ \{e\ x\}\ \{e\ x\}$ by Teqbase or Teqfun, respectively.

- (4) Let $e_{pf} = x : \tau_x \to e''_{pf}$.
- (5) Let $e_p = xEq_{x:\tau_x \to \tau} e e e_{pf}$.
- (6) Let $e_l = e_r = e$ and $\tau_l = x : \tau_x \to \tau_l'$ and $\tau_r = x : \tau_x \to \tau_r'$.
- (7) We find subtyping by SFun and (2a).
- (8) By TEQFUN. We must show:
 - (a) $\Gamma \vdash e_l :: \tau_l \text{ and } \Gamma \vdash e_r :: \tau_r$;
 - (b) $\Gamma \vdash \tau_l \leq x : \tau_x \rightarrow \tau$ and $\Gamma \vdash \tau_r \leq x : \tau_x \rightarrow \tau$;
 - (c) $\Gamma, r : \tau_r, l : \tau_l \vdash e_{\mathsf{pf}} :: (x : \tau_x \to \mathsf{PEq}_\tau \{l \ x\} \{r \ x\})$
 - (d) $\Gamma \vdash x:\tau_x \to \tau$
- (9) We find (8a) by assumption, TSUB, and (7).
- (10) We find (8b) by (7).
- (11) We find (8c) by TLAM and (2b).
- $\tau \doteq \mathsf{PEq}_{\tau'}\{e_1\}\{e_2\}$. These types are not equable, so we ignore them.

Symmetry: By induction on τ .

- $\tau \doteq \{x:b \mid e\}$.
- (1) We have $\Gamma \vdash v_{12} :: PEq_b \{e_1\} \{e_2\}.$
- (2) By canonical forms, $v_{12} = \mathsf{bEq}_b \ e_l \ e_r \ v_p$ such that $\Gamma \vdash e_l :: \tau_l \ \text{and} \ \Gamma \vdash e_r :: \tau_r \ (\text{for some} \ \tau_l \ \text{and} \ \tau_r \ \text{that are refinements of} \ b) \ \text{and} \ \Gamma, r : \tau_r, l : \tau_l \vdash v_p :: \{x : () \mid l ==_b r\} \ (\text{Lemma B.12}).$
- (3) Let $v_{21} = bEq_b e_r e_l v_p$.
- (4) By TEQBASE, swapping τ_l and τ_r from (2). We already have appropriate typing and subtyping derivations; we only need to see Γ , $l : \tau_l, r : \tau_r \vdash v_p :: \{x:() \mid r ==_b l\}$.
- (5) We have $\Gamma, l : \tau_l, r : \tau_r \vdash \{x:() \mid r ==_b l\} \leq \{x:() \mid l ==_b r\}$ by SBASE and symmetry of $(==_b)$.
- $\tau \doteq x : \tau_x \rightarrow \tau'$.
- (1) We have $\Gamma \vdash v_{12} :: PEq_{x:\tau_x \to \tau'} \{e_1\} \{e_2\}.$

- (2) By canonical forms, $v_{12} = \mathsf{xEq}_{x:\tau_x' \to \tau''} \ e_l \ e_r \ v_p$ such that $\tau_x \vdash \tau_x' \leq and \ \tau'' \vdash \tau' \leq and \ \Gamma \vdash e_l :: \tau_l \text{ and } \Gamma \vdash e_r :: \tau_r \text{ (for some } \tau_l \text{ and } \tau_r \text{ that are subtypes of } x:\tau_x' \to \tau'') \text{ and } \Gamma, r : \tau_r, l : \tau_l \vdash v_p :: x:\tau_x' \to \mathsf{PEq}_{\tau''} \{l \ x\} \{r \ x\}.$
- (3) By canonical forms, this time on v_p from (2), $v_p = \text{TLAM} x \tau_x' e_p$ such that $\Gamma \vdash \tau_x \leq \tau_x'$ and $\Gamma, r : \tau_r, l : \tau_l, x : \tau_x' \vdash e :: \tau'''$ such that $\Gamma, r : \tau_r, l : \tau_l, x : \tau_x' \vdash \tau''' \leq \text{PEq}_{\tau''} \{l \ x\} \{r \ x\}.$
- (4) By TSUB, (3), and the IH on $\mathsf{PEq}_{\tau''}$ { $l \ x$ } { $r \ x$ }, we know there exists some e'_p such that $\Gamma, l : \tau_l, r : \tau_r, x : \tau'_x \vdash e'_p :: \mathsf{PEq}_{\tau''}$ { $r \ x$ } { $l \ x$ }.
- (5) Let $v_p' = x : \tau_x' \to e_p'$.
- (6) By (4) and TLAM, and TSUB (using subtyping from (3) and (2)), $\Gamma, l : \tau_l, r : \tau_r \vdash v_p' :: PEq_{x:\tau_r \to \tau'} \{e_r \ x\} \{e_l \ x\}.$
- (7) Let $v_{21} = \mathsf{xEq}_{x:\tau_x \to \tau'} e_r e_l v_p'$.
- (8) By TEQBASE, with (6) for the proof and (3) and (2) for the rest.
- $\tau \doteq \mathsf{PEq}_{\tau'} \{e_1\} \{e_2\}$. These types are not equable, so we ignore them.

Transitivity: By induction on τ .

- $\tau \doteq \{x:b \mid e\}$.
- (1) We have $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\} \text{ and } \Gamma \vdash v_{23} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_3\}.$
- (2) By canonical forms, $v_{12} = \mathsf{bEq}_b \ e_1 \ e_2 \ v_{12}'$ such that $\Gamma \vdash e_1 :: \tau_1 \text{ and } \Gamma \vdash e_2 :: \tau_2$ (for some τ_1 and τ_2 that are refinements of b) and $\Gamma, r : \tau_2, l : \tau_1 \vdash v_{12}' :: \{x:() \mid l ==_b r\}$. and, similarly, $v_{23} = \mathsf{bEq}_b \ e_1 \ e_2 \ v_{23}'$ such that $\Gamma \vdash e_2 :: \tau_2' \text{ and } \Gamma \vdash e_3 :: \tau_3$ (for some τ_2' and τ_3 that are refinements of b) and $\Gamma, r : \tau_3, l : \tau_2' \vdash v_{23}' :: \{x:() \mid l ==_b r\}$.
- (3) By canonical forms again, we know that $v'_{12} = v'_{23} = \text{unit}$ and we have:

$$\Gamma, r : \tau_2, l : \tau_1 \vdash \{x:() \mid x ==_{()} \text{ unit}\} \le \{x:b \mid \{x:() \mid l ==_b r\}\}, \text{ and } \Gamma, r : \tau_3, l : \tau_2' \vdash \{x:() \mid x ==_{()} \text{ unit}\} \le \{x:b \mid \{x:() \mid l ==_b r\}\}.$$

(4) Elaborating on (3), we know that $\forall \theta \in [\Gamma, r : \tau_2, l : \tau_1]$, we have:

$$\left\|\theta\cdot\{x{:}()\mid x==_{()}\mathsf{unit}\}\right\|\subseteq \left\|\theta\cdot\{x{:}()\mid l==_br\}\right\|$$

and $\forall \theta \in [\Gamma, r : \tau_3, l : \tau_2']$, we have:

$$\|\theta \cdot \{x:() \mid x ==_{()} \text{unit}\}\| \subseteq \|\theta \cdot \{x:() \mid l ==_b r\}\|$$
.

- (5) Since {x:() | x ==() unit} contains all computations that terminate with unit in all models (Theorem B.1), the right-hand sides of the equations must also hold all unit computations. That is, all choices for l and r₂ (resp. l and r) that are semantically well typed are necessarily equal.
- (6) By (5), we can infer that in any given model, τ₁, τ₂, τ'₂, and τ₃ identify just one *b*-constant. Why must τ₂ and τ'₂ agree? In particular, e₂ has *both* of those types, but by semantic soundness (Theorem B.2), we know that it will go to a value in the appropriate type interpretation. By determinism of evaluation, we know it must be the *same* value. We can therefore conclude that ∀θ ∈ [Γ, r : τ₃, l : τ₁], ||θ · {x:() | x ==₍₎ unit}|| ⊆ [|θ · {x:() | l ==_b r}||.
- (7) By TEQBASE, using τ_1 and τ_3 and unit as the proof. We need to show $\Gamma, r : \tau_3, l : \tau_1 \vdash \text{unit} :: \{x:() \mid l ==_b r\}$; all other premises follow from (2).
- (8) By TSUB and SBASE, using (6) for the subtyping.
- $\bullet \ \tau \doteq x : \tau_x \to \tau'.$
- (1) We have $\Gamma \vdash v_{12} :: \mathsf{PEq}_{\tau} \{e_1\} \{e_2\} \text{ and } \Gamma \vdash v_{23} :: \mathsf{PEq}_{\tau} \{e_2\} \{e_3\}.$
- (2) By canonical forms, we have

$$v_{12} = x Eq_{x:\tau_x \to \tau'} e_1 e_2 v'_{12}$$

 $v_{23} = x Eq_{x:\tau_x \to \tau'} e_2 e_3 v'_{23}$

where there exist types τ_1 , τ_2 , τ_2' , and τ_3 subtypes of $x:\tau_x \to \tau'$ such that

$$\Gamma \vdash e_1 :: \tau_1 \qquad \Gamma \vdash e_2 :: \tau_2$$

$$\Gamma \vdash e_2 :: \tau_2' \qquad \Gamma \vdash e_3 :: \tau_3$$

and there exist types $\tau_{x_{12}}$, $\tau_{x_{23}}$, τ'_{12} , and τ'_{23} such that

$$\begin{split} & \Gamma, r: \tau_2, l: \tau_1 \vdash v_{p_{12}} :: x: \tau_{x_{12}} \rightarrow \mathsf{PEq}_{\tau'_{12}} \; \{l \; x\} \; \{r \; x\}, \\ & \Gamma, r: \tau_2, l: \tau_1 \vdash \tau_x \; \leq \; \tau_{x_{12}}, \\ & \Gamma, r: \tau_2, l: \tau_1, x: \tau_x \vdash \tau'_{12} \; \leq \; \tau', \\ & \Gamma, r: \tau_3, l: \tau'_2 \vdash v_{p_{23}} :: x: \tau'_x \rightarrow \mathsf{PEq}_{\tau'_{23}} \; \{l \; x\} \; \{r \; x\}, \\ & \Gamma, r: \tau_3, l: \tau'_2 \vdash \tau_x \; \leq \; \tau_{x_{23}}, \; \mathsf{and} \\ & \Gamma, r: \tau_3, l: \tau'_2, x: \tau_x \vdash \tau'_{23} \; \leq \; \tau'. \end{split}$$

(3) By canonical forms on $v_{p_{12}}$ and $v_{p_{23}}$ from (2), we know that:

$$v_{p_{12}} = \lambda x : \tau_{x_{12}}. e'_{12} \qquad v_{p_{23}} = \lambda x : \tau_{x_{23}}. e'_{23}$$

such that:

$$\begin{array}{l} \Gamma, r: \tau_2, l: \tau_1, x: \tau_{x_{12}} \vdash e'_{12} :: \tau''_{12}, \\ \Gamma, r: \tau_2, l: \tau_1, x: \tau_{x_{12}} \vdash \tau''_{12} \leq \tau'_{12}, \end{array}$$

$$\begin{array}{l} \Gamma, r: \tau_3, l: \tau_2', x: \tau_{x_{23}} \vdash e_{23}':: \tau_{23}'', \\ \Gamma, r: \tau_3, l: \tau_2', x: \tau_{x_{23}} \vdash \tau_{23}'' \ \le \ \tau_{23}', \ \text{and} \end{array}$$

(4) By strengthening (Lemma B.7) using (2), we can replace x's type with τ_x in both proofs, to find:

$$\Gamma, r : \tau_2, l : \tau_1, x : \tau_x \vdash e'_{12} :: \tau'_{12}$$
, and $\Gamma, r : \tau_3, l : \tau'_2, x : \tau_x \vdash e'_{23} :: \tau'_{23}$.

Then, by TSUB, we can relax the type of the proof bodies:

$$\Gamma, r : \tau_2, l : \tau_1, x : \tau_x \vdash e'_{12} :: \tau', \text{ and } \Gamma, r : \tau_3, l : \tau'_2, x : \tau_x \vdash e'_{23} :: \tau'.$$

- (5) By (4, (3), and the IH on $PEq_{\tau'}$ {l x} {r x}, we know there exists some proof body e'_{13} such that $\Gamma, r : \tau_3, l : \tau_1 \vdash e'_{13} :: PEq_{\tau'}$ {l x} {r x}.
- (6) Let $v_p = x : \tau_x \to e'_{13}$.
- (7) By (5), and TLAM.
- (8) Let $v_{13} = x Eq_{x:\tau_x \to \tau'} e_1 e_3 v_p$.
- (9) By TEQBASE, with (7) for the proof and (2) for the rest.
- $\tau \doteq \mathsf{PEq}_{\tau'} \{e_1\} \{e_2\}$. These types are not equable, so we ignore them.

C PARALLEL REDUCTION AND COTERMINATION

The conventional application rule for dependent types substitutes a term into a type, finding $e_1 e_2 : \tau[e_2/x]$ when $e_1 : x:\tau_x \to \tau$. We define two logical relations: a unary interpretation of types (Figure 8) and a binary logical relation characterizing equivalence (Figure 10). Both of these logical relations are defined as fixpoints on types. The type index poses a problem: the function case of these logical relations quantify over values in the relation, but we sometimes need to reason about expressions, not values. If $e \hookrightarrow^* v$, are $\tau[e/x]$ and $\tau[v/x]$ treated the same by our logical relations? We encounter this problem in particular in proof of logical relation compositionality, which is precisely about exchanging expressions in types with the values the expressions reduce to in closing substitutions: for the unary logical relation and binary logical relation (Lemma B.21).

The key technical device to prove these compositionality lemmas is *parallel reduction* (Figure 13). Parallel reduction generalizes our call-by-value relation to allow multiple steps at once, throughout a

term—even under a lambda. Parallel reduction is a bisimulation (Lemma C.5 for forward simulation; Corollary C.15 for backward simulation). That is, expressions that parallel reduce to each other go to identical constants or expressions that themselves parallel reduce, and the logical relations put terms that parallel reduce in the same equivalence class.

To prove the compositionality lemmas, we first show that (a) the logical relations are closed under parallel reduction (for the unary relation and Lemma B.20 for the binary relation) and (b) use the backward simulation to change values in the closing substitution to a substituted expression in the type.

Our proof comes in three steps. First, we establish some basic properties of parallel reduction (§C.1). Next, proving the forward simulation is straightforward (§C.2): if $e_1 \Rightarrow e_2$ and $e_1 \hookrightarrow e_1'$, then either parallel reduction contracted the redex for us and $e_1' \Rightarrow e_2$ immediately, or the redex is preserved and $e_2 \hookrightarrow e_2'$ such that $e_1' \Rightarrow e_2'$. Proving the backward simulation is more challenging (§C.3). If $e_1 \Rightarrow e_2$ and $e_2 \hookrightarrow e_2'$, the redex contracted in e_2 may not yet be exposed. The trick is to show a tighter bisimulation, where the outermost constructors are always the same, with the subparts parallel reducing. We call this relation *congruence* (Figure 14); it's a straightforward restriction of parallel reduction, eliminating β , eq1, and eq2 as outermost constructors (but allowing them deeper inside). The key lemma shows that if $e_1 \Rightarrow e_2$, then there exists $e_1' e_1 \hookrightarrow^* e_1'$ such that $e_1' \approx e_2$ (Lemma C.11). Once we know that parallel reduction implies reduction to congruent terms, proving that congruence is a backward simulation allows us to reason "up to congruence". In particular, congruence is a sub-relation of parallel reduction, so we find that parallel reduction is a backward simulation. Finally, we can show that $e_1 \Rightarrow e_2$ implies observational equivalence (§C.4); for our purposes, it suffices to find cotermination at constants (Corollary C.17).

One might think, in light of Takahashi's explanation of parallel reduction [Takahashi 1989], that the simulation techniques we use are too powerful for our needs: why not simply rely on the Church-Rosser property and confluence, which she proves quite simply? Her approach works well when relating parallel reduction to full β -reduction (and/or η -reduction): the transitive closure of her parallel reduction relation is equal to the transitive closure of plain β -reduction (resp. η -and $\beta\eta$ -reduction). But we're interested in programming languages, so our underlying reduction relation isn't full β : we use call-by-value, and we will never reduce under lambdas. But even if we were call-by-name, we would have the same issue. Parallel reduction implies reduction, but not to the *same* value, as in her setting. Parallel reduction yields values that are equivalent, *up to parallel reduction and congruence* (see, e.g., Corollary C.13).

C.1 Basic Properties

LEMMA C.1 (PARALLEL REDUCTION IS REFLEXIVE). For all e and τ , $e \Rightarrow e$ and $\tau \Rightarrow \tau$.

PROOF. By mutual induction on e and τ .

Expressions.

- $e \doteq x$. By var.
- $e \doteq c$. By const.
- $e = \lambda x : \tau$. By the IHs on τ and e' and lam.
- $e \doteq e_1 \ e_2$. By the IH on e_1 and e_2 and app.
- $e \doteq bEq_b \ e_l \ e_r \ e'$. By the IHs on e_l , e_r , and e' and beq.
- $e \doteq xEq_{x:\tau_x \to \tau} e_l e_r e'$. By the IHs on τ_x , τ , e_l , e_r , and e' and xeq.

Types.

- $\tau = \{x:b \mid r\}$. By the IH on r (an expression) and ref.
- $\tau \doteq x : \tau_x \to \tau'$. By the IHs on τ_x and τ' and fun.

$$\frac{r \Rightarrow \tau' \quad e \Rightarrow e'}{\lambda x : \tau. \ e \Rightarrow \lambda x : \tau'. \ e'} \lim \frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_2}{e_1 \ e_2 \Rightarrow e'_1 \ e'_2} \operatorname{app}$$

$$\frac{e \Rightarrow e' \quad v \Rightarrow v'}{(\lambda x : \tau. \ e) \ v \Rightarrow e'[v'/x]} \beta \qquad (==_b) \ c_1 \Rightarrow (==_{(c_1,b)}) \qquad (==_{(c_1,b)}) \ c_2 \Rightarrow c_1 = c_2$$

$$\frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \ e'_2}{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_2} \operatorname{beq}$$

$$\frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \ e'_2 \Rightarrow e'_2}{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2} \operatorname{beq}$$

$$\frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \ e'_2 \Rightarrow e'_2}{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \quad e'_2 \Rightarrow e'_2 \Rightarrow e'_2}$$

$$\frac{e_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \ e'_2 \Rightarrow e'_1 \Rightarrow e'_1 \quad e_2 \Rightarrow e'_1 \Rightarrow e'_2 \Rightarrow e'_1 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_1 \Rightarrow e'_1 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_1 \Rightarrow e'_1 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_2 \Rightarrow e'_1 \Rightarrow e'_2 \Rightarrow$$

Fig. 13. Parallel reduction in terms and types.

•
$$\tau \doteq \mathsf{PEq}_{\tau'} \{e_l\} \{e_r\}$$
. By the IHs on τ' , e_l , and e_r and eq.

Lemma C.2 (Parallel reduction is substitutive). If e
ightharpoonup e', then:

- (1) If $e_1 \Rightarrow e_2$, then $e_1[e/x] \Rightarrow e_2[e'/x]$.
- (2) If $\tau_1 \rightrightarrows \tau_2$, then $\tau_1[e/x] \rightrightarrows \tau_2[e'/x]$.

PROOF. By mutual induction on e_1 and τ_1 .

Expressions.

var $y \Rightarrow y$. If $y \neq x$, then the substitution has no effect and the case is trivial. If y = x, then x[e/x] = e and we have $e \Rightarrow e'$ by assumption. We have $e \Rightarrow e$ by reflexivity (Lemma C.1). const $c \Rightarrow c$. This case is trivial: the substitution has no effect.

lam $\lambda y : \tau$. $e' \Rightarrow \lambda y : \tau$. If $y \neq x$, then by the IH on e' and lam. If y = x, then the substitution has no effect and the case is trivial.

- app e_{11} $e_{12} \Rightarrow e_{21}$ e_{22} , where $e_{1i} \Rightarrow e_{2i}$ for i = 1, 2. By the IHs on e_{1i} and app.
- beta $(\lambda y : \tau. \ e') \ v \Rightarrow e'[v'/y]$, where $e' \Rightarrow e''$ and $v \Rightarrow v'$. If $y \neq x$, then $(\lambda y : \tau. \ e'[e/x]) \ v[e/x] \Rightarrow e''[e/x][v'[e/x]/y]$ by β . Since $y \neq x$, e''[e/x][v'[e/x]/y] = e''[v'/y][e/x] as desired. If y = x, then the substitution in the lambda has no effect, and we find $(\lambda x : \tau. \ e') \ v[e/x] \Rightarrow e''[v'[e/x]/x]$ by β . We have e''[v'[e/x]/x] = e''[v'/x][e/x] as desired.
- eq1 (==_b) $c_1 \Rightarrow (==_{(c_1,b)})$. This case is trivial by eq1, as the substitution has no effect.
- eq2 (== (c_1,b)) $c_2 \Rightarrow c_1 = c_2$. This case is trivial by eq2, as the substitution has no effect.
- beq bEq_b e_l e_r $e_p \Rightarrow$ bEq_b e'_l e'_r e'_p , where $e_l \Rightarrow e'_l$ and $e_r \Rightarrow e'_r$ and $e_p \Rightarrow e'_p$. By the IHs on e_l , e_r , and e_p and beq.
- $\operatorname{\mathsf{xeq}} \ \operatorname{\mathsf{xEq}}_{x:\tau_x\to\tau} e_l \ e_r \ e_p \rightrightarrows \operatorname{\mathsf{xEq}}_{x:\tau_x\to\tau} e_l' \ e_r' \ e_p', \text{ where } e_l \rightrightarrows e_l' \text{ and } e_r \rightrightarrows e_r' \text{ and } e_p \rightrightarrows e_p'.$ By the IHs on e_l, e_r , and e_p and $\operatorname{\mathsf{xeq}}$.

Types.

ref $\{y:b\mid r\} \rightrightarrows \{y:b\mid r'\}$ where $r\rightrightarrows r'$. If $y\neq x$, then $r[e/x]\rightrightarrows r'[e'/x]$ by the IH on r; we are done by ref.

If y = x, then the substitution has no effect, and the case is immediate by reflexivity (Lemma C.1).

fun $y:\tau_y\to\tau\rightrightarrows y:\tau_y'\to\tau'$ where $\tau_y\rightrightarrows\tau_y'$ and $\tau\rightrightarrows\tau'$. If $y\neq x$, then by the IH on τ_y and τ and fun

If y = x, then the substitution only has effect in the domain. The IH on τ_y finds $\tau_y[e/x] \Rightarrow \tau_y'[e/x]$ in the domain; reflexivity covers the codomain (Lemma C.1), and we are done by fun

eq
$$PEq_{\tau} \{e_l\} \{e_r\} \Longrightarrow PEq_{\tau'} \{e'_l\} \{e'_r\}$$
. By the IHs and eq.

Corollary C.3 (Substituting multiple parallel reduction is parallel reduction). If $e_1 \rightrightarrows^* e_2$, then $e[e_1/x] \rightrightarrows^* e[e_2/x]$.

PROOF. First, notice that $e \Rightarrow e$ by reflexivity (Lemma C.1). By induction on $e_1 \Rightarrow^* e_2$, using reflexivity in the base case (Lemma C.1); the inductive step uses substituting parallel reduction (Lemma C.2) and the IH.

Lemma C.4 (Parallel reduction subsumes reduction). If $e_1 \hookrightarrow e_2$ then $e_1 \rightrightarrows e_2$.

PROOF. By induction on the evaluation derivation, using reflexivity of parallel reduction to cover expressions and types that didn't step (Lemma C.1).

```
ctx \mathcal{E}[e] \hookrightarrow \mathcal{E}[e'], where e \hookrightarrow e'. By the IH, e \rightrightarrows e'. By structural induction on \mathcal{E}.
```

- \mathcal{E} \doteq **•**. By the outer IH.
- \mathcal{E} $\stackrel{.}{=}$ \mathcal{E}_1 e_2 . By the inner IH on \mathcal{E}_1 , reflexivity on e_2 , and app.
- \mathcal{E} \doteq v_1 \mathcal{E}_2 . By reflexivity on v_1 , the inner IH on \mathcal{E}_2 , and app.
- $-\mathcal{E} \doteq \mathsf{bEq}_b \ e_l \ e_r \ \mathcal{E}'$. By reflexivity on e_l and e_r , the inner IH on and \mathcal{E}' , and beq.
- $-\mathcal{E} \doteq \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ \mathcal{E}'$. By reflexivity on τ_x , τ , e_l and e_r , the inner IH on and \mathcal{E}' , and xeq . β ($\lambda x:\tau$. e) $v \hookrightarrow e[v/x]$. By reflexivity (Lemma C.1, $e \rightrightarrows e$ and $v \rightrightarrows v$. By beta, ($\lambda x:\tau$. e) $v \rightrightarrows e[v/x]$.

eq1 By eq1.

eq2 By eq2. □

C.2 Forward Simulation

Lemma C.5 (Parallel reduction is a forward simulation). If $e_1 \rightrightarrows e_2$ and $e_1 \hookrightarrow e_1'$, then there exists e_2' such that $e_2 \hookrightarrow^* e_2'$ and $e_1' \rightrightarrows e_2'$.

PROOF. By induction on the derivation of $e_1 \hookrightarrow e'_1$, leaving e_2 general.

- ctx By structural induction on \mathcal{E} , using reflexivity (Lemma C.1) on parts where the IH doesn't apply.
 - \mathcal{E} \doteq •. By the outer IH on the actual step.
 - \mathcal{E} $\stackrel{.}{=}$ \mathcal{E}_1 e_2 . By the IH on \mathcal{E}_1 , reflexivity on e_2 , and app.
 - \mathcal{E} $\stackrel{.}{=}$ v_1 \mathcal{E}_2 . By reflexivity on v_1 , the IH on \mathcal{E}_2 , and app.
 - $-\mathcal{E} \doteq \mathsf{bEq}_b \ e_l \ e_r \ \mathcal{E}'$. By reflexivity on e_l and e_r , the IH on \mathcal{E}' , and beq.
 - $-\mathcal{E} \doteq \mathsf{xEq}_{x:\tau_x \to \tau} e_l e_r \mathcal{E}'$. By reflexivity on τ_x , τ , e_l and e_r , the IH on \mathcal{E}' , and xeq.
 - $\beta \ (\lambda x : \tau. \ e) \ v \hookrightarrow e[v/x].$ One of two rules could have applied to find $e_1 \rightrightarrows e_2$: app or β .

In the app case, we have $e_2 = (\lambda x: \tau'. e') \ v'$ where $\tau \Rightarrow \tau'$ and $e \Rightarrow e'$ and $v \Rightarrow v'$. Let $e'_2 = e'[v'/x]$. We find $e_2 \hookrightarrow^* e'_2$ in one step by β . We find $e[v/x] \Rightarrow e'[v'/x]$ by substitutivity of parallel reduction (Lemma C.2).

In the β case, we have $e_2 = e'[v'/x]$ such that $e \rightrightarrows e'$ and $v \rightrightarrows v'$. Let $e'_2 = e_2$. We find $e_2 \hookrightarrow^* e'_2$ in no steps at all; we find $e'_1 \rightrightarrows e'_2$ by substitutivity of parallel reduction (Lemma C.2).

eq1 (==_b) $c_1 \hookrightarrow (==_{(c_1,b)})$. One of two rules could have applied to find (==_b) $c_1 \rightrightarrows e_2$: app or eq1.

In the app case, we must have $e_2 = e_1 = (==_b) c_1$, because there are no reductions available in these constants. Let $e_2' = (==_{(c_1,b)})$. We find $e_2 \hookrightarrow^* e_2'$ in a single step by our assumption (or eq1). We find parallel reduction by reflexivity (Lemma C.1).

In the eq2 case, we have $e_2 = e_1' = (==_{(c_1,b)})$. Let $e_2' = e_2$. We find $e_2 \hookrightarrow^* e_2'$ in no steps at all. We find parallel reduction by reflexivity (Lemma C.1).

eq2 (== (c_1,b)) $c_2 \hookrightarrow c_1 = c_2$. One of two rules could have applied to find (== (c_1,b)) $c_2 \rightrightarrows e_2$: app or eq2.

In the app case, we have $e_2 = e_1 = (==_{(c_1,b)}) c_2$, because there are no reductions available in these constants. Let $e_2' \doteq c_1 = c_2$, i.e. true when $c_1 = c_2$ and false otherwise. We find $e_2 \hookrightarrow^* e_2'$ in a single step by our assumption (or eq2). We find parallel reduction by reflexivity (Lemma C.1).

In the eq2 case, we have $e_2 = e_1' \doteq c_1 = c_2$, i.e. true when $c_1 = c_2$ and false otherwise. Let $e_2' = e_2$. We find $e_2 \hookrightarrow^* e_2'$ in no steps at all. We find parallel reduction by reflexivity (Lemma C.1).

C.3 Backward Simulation

Lemma C.6 (Reduction is substitutive). If $e_1 \hookrightarrow e_2$, then $e_1[e/x] \hookrightarrow e_2[e/x]$.

PROOF. By induction on the derivation of $e_1 \hookrightarrow e_2$.

ctx By structural induction on \mathcal{E} .

- \mathcal{E} \doteq **•**. By the outer IH.
- \mathcal{E} $\stackrel{.}{=}$ \mathcal{E}_1 e_2 . By the IH on \mathcal{E}_1 and ctx.
- \mathcal{E} \doteq v_1 \mathcal{E}_2 . By the IH on \mathcal{E}_2 and ctx.
- $-\mathcal{E} \doteq \mathsf{bEq}_b \; e_l \; e_r \; \mathcal{E}'$. By the IH on \mathcal{E}' and ctx.
- $-\mathcal{E} \doteq \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ \mathcal{E}'$. By the IH on \mathcal{E}' and ctx.
- β ($\lambda y : \tau$. e') $v \hookrightarrow e'[v/y]$. We must show ($\lambda y : \tau$. e')[e/x] $v[e/x] \hookrightarrow e'[v/y][e/x]$.

The exact result depends on whether y = x. If $y \neq x$, the substitution goes through, and we have $(\lambda y:\tau.\ e')[e/x] = \lambda y:\tau[e/x].\ e'[e/x]$. By β , $(\lambda y:\tau[e/x].\ e'[e/x])\ v[e/x] \hookrightarrow e'[e/x][v[e/x]/y]$. But e'[e/x][v[e/x]/y] = e'[v/y][e/x], and we are done.

If, on the other hand, y = x, then the substitution has no effect in the body of the lambda, and $(\lambda y:\tau.e')[e/x] = \lambda y:\tau[e/x].e'$. By β again, we find $(\lambda y:\tau[e/x].e')v[e/x] \hookrightarrow e'[v[e/x]/y]$. Since y = x, we really have e'[v[e/x]/x] which is the same as e'[v/x][e/x] = e'[v/y][e/x], as desired.

eq1 The substitution has no effect; immediate, by eq1.

eq2 The substitution has no effect; immediate, by eq2.

Corollary C.7 (Multi-step reduction is substitutive). If $e_1 \hookrightarrow^* e_2$, then $e_1[e/x] \hookrightarrow^* e_2[e/x]$.

PROOF. By induction on the derivation of $e_1 \hookrightarrow^* e_2$. The base case is immediate ($e_1 = e_2$, and we take no steps). The inductive case follows by the IH and single-step substitutivity (Lemma C.6). \square

$$\frac{\tau \rightrightarrows \tau' \quad e \rightrightarrows e'}{\lambda x : \tau. \ e} \lim \frac{e_1 \rightrightarrows e'_1 \quad e_2 \rightrightarrows e'_2}{e_1 \ e_2 \lessapprox e'_1 \ e'_2} = pp$$

$$\frac{e_l \rightrightarrows e'_l \quad e_r \rightrightarrows e'_r \quad e \rightrightarrows e'}{\text{bEq}_b \ e_l \ e_r \ e} \lim \frac{\tau \rightrightarrows \tau' \quad e \rightrightarrows e'}{\lambda x : \tau. \ e} \lim \frac{e_1 \rightrightarrows e'_1 \quad e_2 \rightrightarrows e'_2}{e_1 \ e_2 \lessapprox e'_1 \ e'_2} = pp$$

$$\frac{\tau_x \rightrightarrows \tau'_x \quad \tau \rightrightarrows \tau' \quad e_l \rightrightarrows e'_l \quad e_r \rightrightarrows e'_r \quad e \rightrightarrows e'}{\text{xEq}_{x : \tau_x \to \tau} \ e_l \ e_r \ e} \exp \frac{\tau_x \rightrightarrows \tau'_x \quad \tau \rightrightarrows \tau' \quad e_l \rightrightarrows e'_l \quad e_r \rightrightarrows e'_r \quad e \rightrightarrows e'_r \qquad e \rightrightarrows e$$

Fig. 14. Term congruence.

We say terms are *congruent* when they (a) have the same outermost constructor and (b) their subparts parallel reduce to each other.⁴ That is, ≈ ⊆ ⇒, where the outermost rule must be one of var, const, lam, app, beq, or xeq and cannot be a *real* reduction like β , eq1, or eq2.

Congruence is a key tool in proving that parallel reduction is a backward simulation. Parallel reductions under a lambda prevent us from having an "on-the-nose" relation, but reduction can keep up enough with parallel reduction to maintain congruence.

LEMMA C.8 (CONGRUENCE IMPLIES PARALLEL REDUCTION). If $e_1 \stackrel{\text{$\sim$}}{\sim} e_2$ then $e_1 \stackrel{\text{$\sim$}}{\rightarrow} e_2$.

PROOF. By induction on the derivation of $e_1 \approx e_2$.

var $x \approx x$. By var.

const $c \approx c$. By const.

lam $\lambda x:\tau$. $e \cong \lambda x:\tau'$. e', with $\tau \Rightarrow \tau'$ and $e \Rightarrow e'$. By lam.

app $e_1 e_2 \approx e'_1 e'_2$, with $e_1 \Rightarrow e'_1$ and $e_2 \Rightarrow e'_2$. By app.

beq bEq_b e_l e_r $e \stackrel{?}{\approx}$ bEq_b e_l' e_r' e, with $e_l \Rightarrow e_l'$ and $e_r \Rightarrow e_r'$ and $e \Rightarrow e'$. By beq. xeq By xeq. xEq_{x: $\tau_x \to \tau$} e_l' e_r' $e \approx x$ xEq_{x: $\tau_x \to \tau$} e_l' e_r' e, with $\tau_x \Rightarrow \tau_x'$ and $\tau \Rightarrow \tau'$ and $e_l \Rightarrow e_l'$ and $e_r \rightrightarrows e'_r$ and $e \rightrightarrows e'$. By xeq.

We need to strengthen substitutivity (Lemma C.2) to show that it preserves congruence.

COROLLARY C.9 (CONGRUENCE IS SUBSTITUTIVE). If $e_1 \approx e_1'$ and $e_2 \approx e_2'$, then $e_1[e_2/x] \approx$ $e_2[e_2'/x].$

PROOF. By cases on e_1 .

- $e_1 = y$. It must be that $e_2 = y$ as well, since only var could have applied. If $y \neq x$, then the substitution has no effect and we have $y \approx y$ by assumption (or var). If x = y, then $e_1[e_2/x] = e_2$ and we have $e_2 \approx e_2'$ by assumption.
- $e_1 = c$. It must be that $e_2 = c$ as well. The substitution has no effect; immediate by var.
- $e_1 = \lambda y : \tau$. e. It must be that $e_2 = \lambda y : \tau'$. e' such that $\tau \Rightarrow \tau'$ and $e \Rightarrow e'$. If $y \neq x$, then we must show $\lambda y:\tau[e_2/x]$. $e[e_2/x] \stackrel{\sim}{\sim} \lambda y:\tau'[e_2'/x]$. $e'[e_2'/x]$, which we have immediately by lam and Lemma C.2 on our two subparts. If y = x, then we must show $\lambda y : \tau[e_2/x]$. $e \approx \lambda y : \tau'[e_2/x]$. e', which we have immediately by lam, Lemma C.2 on our $\tau \rightrightarrows \tau'$, and the fact that $e \rightrightarrows e'$.
- $e_1 = e_{11} e_{12}$. It must be that $e_2 = e_{21} e_{22}$, such that $e_{11} \rightrightarrows e_{21}$ and $e_{12} \rightrightarrows e_{22}$. By app and Lemma C.2 on the subparts.
- $e_1 = bEq_b \ e_l \ e_r \ e$. It must be the case that $e_2 = bEq_b \ e'_l \ e'_r \ e'$ where $e_l \implies e'_l$ and $e_r \implies e'_r$. By beq and Lemma C.2 on the subparts.
- $e_1 = \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ e$. It must be the case that $e_2 = \mathsf{xEq}_{x:\tau_x' \to \tau'} \ e_l' \ e_r' \ e'$ where $e_l \implies e_l'$ (and similarly for τ_x , τ , e_r , and e). By xeq and Lemma C.2 on the subparts.

 $^{^4}$ Congruent terms are related to Takahashi's \tilde{M} operator: in that they characterize parallel reductions that preserve structure. They are not the same, though: Takahashi's \tilde{M} will do $\beta\eta$ -reductions on outermost redexes.

Lemma C.10 (Parallel reduction of values implies congruence). If $v_1
ightharpoonup v_2$ then $v_1
ightharpoonup v_2$.

PROOF. By induction on the derivation of $v_1 \rightrightarrows v_2$.

var Contradictory: variables aren't values.

const Immediate, by const.

lam Immediate, by lam.

app Contradictory: applications aren't values.

beq Immediate, by beq.

xeq Immediate, by xeq.

 β Contradictory: applications aren't values.

eq1 Contradictory: applications aren't values.

eq2 Contradictory: applications aren't values.

LEMMA C.11 (PARALLEL REDUCTION IMPLIES REDUCTION TO CONGRUENT FORMS). If $e_1
ightharpoonup e_2$, then there exists $e_1' e_1
ightharpoonup e_1'$ such that $e_1'
ightharpoonup e_2$.

PROOF. By induction on $e_1 \rightrightarrows e_2$.

Structural rules.

```
var x \Rightarrow x. We have e_1 = e_2 = x by var.
```

const $c \Rightarrow c$. We have $e_1 = e_2 = c$ by const.

lam $\lambda x:\tau$. $e \Rightarrow \lambda x:\tau'$. e', where $\tau \Rightarrow \tau'$ and $e \Rightarrow e'$. Immediate, by lam.

app e_{11} $e_{12} \rightrightarrows e_{21}$ e_{22} , where $e_{11} \rightrightarrows e_{21}$ and $e_{12} \rightrightarrows e_{22}$. Immediate, by app.

beq bEq_b e_l e_r $e \Rightarrow$ bEq_b e'_l e'_r e' where $e_l \Rightarrow e'_l$ and $e_r \Rightarrow e'_r$ and $e \Rightarrow e'$. Immediate, by beq.

 $\operatorname{xeq} \operatorname{xEq}_{x:\tau_x \to \tau} e_l e_r e \rightrightarrows \operatorname{xEq}_{x:\tau_x' \to \tau'} e_l' e_r' e'$ where $\tau_x \rightrightarrows \tau_x'$ and $\tau \rightrightarrows \tau'$ and $e_l \rightrightarrows e_l'$ and $e_r \rightrightarrows e_r'$ and $e \rightrightarrows e'$. Immediate, by xeq.

Reduction rules. These are the more interesting cases, where the parallel reduction does a reduction step—ordinary reduction has to do more work to catch up.

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\beta (\lambda x : \tau. e) v \Rightarrow e'[v'/x], where e \Rightarrow e'' and v \Rightarrow v''.
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We have $(\lambda x:\tau.\ e)\ v\hookrightarrow e[v/x]$ by β . By the IH on $e\rightrightarrows e''$, there exists e' such that $e\hookrightarrow^* e'$ such that $e' \approx e''$. We *ignore* the IH on $v \rightrightarrows v''$, noticing instead that parallel reducing values are congruent (Lemma C.10) and so $v \approx v''$. Since reduction is substitutive (Corollary C.7), we can find that $e[v/x] \hookrightarrow^* e'[v/x]$. Since congruence is substitutive (Lemma C.9), we have $e'[v/x] \approx e''[v''/x]$, as desired.

eq1 (==_b) $c_1 \rightrightarrows (==_{(c_1,b)})$. We have (==_b) $c_1 \hookrightarrow (==_{(c_1,b)})$ in a single step; we find congruence by const.

eq2 $(==_{(c_1,b)})$ $c_2 \Rightarrow c_1 = c_2$. We have $(==_{(c_1,b)})$ $c_2 \hookrightarrow c_1 = c_2$ in a single step; we find congruence by const.

Lemma C.12 (Congruence to a value implies reduction to a value). If $e \approx v'$ then $e \hookrightarrow^* v$ such that $v \approx v'$.

PROOF. By induction on v'.

- $v' \doteq c$. It must be the case that e = c. Let v = c. By const.
- $v' \doteq \lambda x : \tau'$. It must be the case that $e = \lambda x : \tau$. e' such that $\tau \rightrightarrows \tau'$ and $e \rightrightarrows e''$. By lam.
- $v \doteq b Eq_b \ e'_l \ e'_r \ v'_p$. It must be the case that $e = b Eq_b \ e_l \ e_r \ e_p$ where $e_l \rightrightarrows e'_l$ and $e_r \rightrightarrows e'_r$ and $e_p \rightrightarrows v'_p$. Since parallel reduction implies reduction to congruent forms (Lemma C.11), we have $e_p \hookrightarrow^* e'_p$ and $e'_p \lesssim v'_p$. By the IH on v'_p , we know that $e'_p \hookrightarrow^* v_p$ such that $v_p \lesssim v'_p$.

By repeated use of ctx, we find $\mathsf{bEq}_b\ e_l\ e_r\ e_p \hookrightarrow^* \mathsf{bEq}_b\ e_l\ e_r\ v_p$. Since its proof part is a value, this term is a value. We find $\mathsf{bEq}_b\ e_l\ e_r\ v_p \stackrel{\text{\sim}}{\approx} \mathsf{bEq}_b\ e_l'\ e_r'\ v_p'$ by ebeq.

• $v \doteq \mathsf{xEq}_{x:\tau_x' \to \tau} \ e_l' \ e_r' \ v_p'$. It must be the case that $e = \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ e_p$ where $\tau_x \rightrightarrows \tau_x'$ and $\tau \rightrightarrows \tau'$ and $e_l \rightrightarrows e_l'$ and $e_r \rightrightarrows e_r'$ and $e_p \rightrightarrows v_p'$. Since parallel reduction implies reduction to congruent forms (Lemma C.11), we have $e_p \hookrightarrow^* e_p'$ and $e_p' \approx v_p'$. By the IH on v_p' , we know that $e_p' \hookrightarrow^* v_p$ such that $v_p \approx v_p'$. By repeated application of ctx, we find $\mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ e_p \hookrightarrow^* \mathsf{xEq}_{x:\tau_x \to \tau} \ e_l \ e_r \ v_p$. Since its proof part is a value, this term is a value. We find $\mathsf{xEq}_{\tau_x:\tau_x \to \tau} \ e_l \ e_r \ v_p \approx \mathsf{xEq}_{x:\tau_x \to \tau'} \ e_l' \ e_r' \ v_p'$ by exeq.

Corollary C.13 (Parallel reduction to a value implies reduction to a related value). If $e \Rightarrow v'$ then there exists v such that $e \hookrightarrow^* v$ and $v \approx v'$.

PROOF. Since parallel reduction implies reduction to congruent forms (Lemma C.11), we have $e \hookrightarrow^* e'$ such that $e' \approx v'$. But congruence to a value implies reduction to a value (Lemma C.12), so $e' \hookrightarrow^* v$ such that $v \approx v'$. By transitivity of reduction, $e \hookrightarrow^* v$.

LEMMA C.14 (CONGRUENCE IS A BACKWARD SIMULATION). If $e_1 \approx e_2$ and $e_2 \hookrightarrow e_2'$ then there exists e_1' where $e_1 \hookrightarrow^* e_1'$ such that $e_1' \approx e_2'$.

PROOF. By induction on the derivation of $e_2 \hookrightarrow e_2'$.

ctx $\mathcal{E}[e] \hookrightarrow \mathcal{E}[e']$, where $e \hookrightarrow e'$.

- \mathcal{E} \doteq **•**. By the outer IH.
- − $\mathcal{E} \doteq \mathcal{E}_1$ e_2 . It must be that $e_1 = e_{11}$ e_{12} , where $e_{11} \rightrightarrows \mathcal{E}_1[e]$ and $e_{12} \rightrightarrows e_2$. By the IH on \mathcal{E}_1 , finding evaluation with ctx and congruence with app.
- $-\mathcal{E} \doteq v_1' \mathcal{E}_2$. It must be that $e_1 = e_{11} e_{12}$, where $e_{11} \rightrightarrows v_1'$ and $e_{12} \rightrightarrows \mathcal{E}_2[e_2]$. We find that $e_{11} \hookrightarrow^* v_1$ such that $v_1 \stackrel{\sim}{\approx} v_1'$ by Corollary C.13. By the IH on \mathcal{E}_2 and evaluation with ctx and congruence with app.
- $-\mathcal{E} \doteq \mathsf{bEq}_b \ e'_l \ e'_r \ \mathcal{E}'$. It must be the case that $e_1 = \mathsf{bEq}_b \ e_l \ e_r \ e_p$ where $e_l \Rightarrow e'_l$ and $e_r \Rightarrow e'_r$. By the IH on \mathcal{E}' ; we find the evaluation with ctx and congruence with beq.
- $-\mathcal{E} \doteq \mathsf{xEq}_{x:\tau_x' \to \tau'} e_l' e_r' \mathcal{E}'$. It must be the case that $e_1 = \mathsf{xEq}_{x:\tau_x \to \tau} e_l e_r e_p$ such that $\tau_x \rightrightarrows \tau_x'$ and $\tau \rightrightarrows \tau'$ and $e_l \rightrightarrows e_l'$ and $e_r \rightrightarrows e_r'$. By the IH on \mathcal{E}' ; we find the evaluation with ctx and congruence with xeq.
- β ($\lambda x:\tau'$. e') $v' \hookrightarrow e'[v'/x]$. Congruence implies that $e_1 = e_{11}$ e_{12} such that $e_{11} \rightrightarrows \lambda x:\tau'$. e' and $e_{12} \rightrightarrows v'$. Parallel reduction to a value implies reduction to a congruent value (Corollary C.13), $e_{11} \hookrightarrow^* v_{11}$ such that $v'_{11} \stackrel{\sim}{\leadsto} \lambda x:\tau'$. e', i.e., $v_{11} = \lambda x:\tau$. e such that $\tau \rightrightarrows \tau'$ and $e \rightrightarrows e'$. Similarly, $e_{12} \hookrightarrow^* v$ such that $v \stackrel{\sim}{\leadsto} v'$.
 - By β , we find $(\lambda x:\tau.\ e)\ v \hookrightarrow^* e'[v/x]$; by transitivity of reduction, we have $e_1 = e_{11}\ e_{12} \hookrightarrow^* e'[v/x]$. Since congruence is substitutive (Corollary C.9), we have $e[v/x] \stackrel{\approx}{\approx} e'[v'/x]$.
- eq1 (==_b) $c_1 \hookrightarrow (==_{(c_1,b)})$. Congruence implies that $e_1 = e_{11}$ e_{12} such that $e_{11} \rightrightarrows (==_b)$ and $e_{12} \rightrightarrows c_1$. Parallel reduction to a value implies reduction to a related value (Corollary C.13), $e_{11} \hookrightarrow^* v_{11}$ such that $v_{11} \stackrel{\sim}{\lesssim} (==_b)$ (and similarly for e_{12} and c_1). But the each constant is congruent only to itself, so $v_{11} = (==_b)$ and $v_{12} = c_1$. We have $(==_b) c_1 \hookrightarrow (==_{(c_1,b)})$ by assumption. So $e_1 = e_{11} e_{12} \hookrightarrow^* (==_{(c_1,b)})$ by transitivity, and we have congruence by const.
- eq2 $(==_{(c_1,b)})$ $c_2 \hookrightarrow c_1 = c_2$. Congruence implies that $e_1 = e_{11}$ e_{12} such that $e_{11} \rightrightarrows (==_{(c_1,b)})$ c_2 and $e_{12} \rightrightarrows c_2$. Parallel reduction to a value implies reduction to a related value (Corollary C.13), $e_{11} \hookrightarrow^* v_{11}$ such that $v_{11} \rightrightarrows (==_{(c_1,b)})$ c_2 (and similarly for e_{12} and c_2). But the each constant is congruent only to itself, so $v_{11} = (==_{(c_1,b)})$ c_2 and $v_{12} = c_2$. We have $(==_{(c_1,b)})$ $c_2 \hookrightarrow c_1 = c_2$ already, by assumption. So $e_1 = e_{11}$ $e_{12} \hookrightarrow^* c_1 = c_2$ by transitivity, and we have congruence by const.

COROLLARY C.15 (PARALLEL REDUCTION IS A BACKWARD SIMULATION). If $e_1
ightharpoonup e_2$ and $e_2 \hookrightarrow e_2'$, then there exists e_1' such that $e_1 \hookrightarrow^* e_1'$ and $e_1'
ightharpoonup e_2'$.

PROOF. Parallel reduction implies reduction to congruent forms, so $e_1 \hookrightarrow^* e_1'$ such that $e_1' \stackrel{\sim}{\approx} e_2$. But congruence is a backward simulation (Lemma C.14), so $e_1' \hookrightarrow^* e_1''$ such that $e_1'' \stackrel{\sim}{\approx} e_2'$. By transitivity of evaluation, $e_1 \hookrightarrow^* e_1''$. Finally, congruence implies parallel reduction (Lemma C.8), so $e_1'' \stackrel{\sim}{\Rightarrow} e_2'$, as desired.

C.4 Cotermination

Theorem C.16 (Cotermination at constants). If $e_1
ightharpoonup e_2$ then $e_1 \hookrightarrow^* c$ iff $e_2 \hookrightarrow^* c$.

PROOF. By induction on the evaluation steps taken, using direct reduction in the base case (Corollary C.13) and using parallel reduction as a forward and backward simulation (Lemmas C.5 and Corollary C.15) in the inductive case.

Corollary C.17 (Cotermination at constants (multiple parallel steps)). If $e_1 \rightrightarrows^* e_2$ then $e_1 \hookrightarrow^* c$ iff $e_2 \hookrightarrow^* c$.

PROOF. By induction on the parallel reduction derivation. The base case is immediate ($e_1 = e_2$); the inductive case follows from cotermination at constants (Theorem C.16) and the IH.