Homework #1 is due: Tuesday next week (May 30, 2017) at 10:05 am at the latest.
What is due?

- All problems from Chapter 1 and parts of Chapter 2 that we will cover next week.
- Distance Learners must turn in homeworks by the same deadline highlighted above.
- After the homework has been turned in, solutions will be released.

Quiz #1 will be on Thursday 06/01/2017!

Last lecture, we covered sampling with or without replacement, permutation, etc. We explored case of uniform probability (that is, probability involving random experiments).

Exercise 1:
A tank contains 7 red, 5 blue and 3 yellow fish.

a) Three are drawn without replacement. What is the probability of getting 2 yellow and 1 red fish?
b) Solve the same question as a), but with replacement.

Solution:
The 1st thing we need to notice is that, for question a), it is asking for a uniform probability. We are drawing each fish not caring what the color is (we aren't aiming to draw a particular color).

a) 2 yellow and 1 red fish can be: YYR, YRY, or RYY. The thought process is:
   - {2 yellow and 1 red} corresponds to the sample space {YYR, YRY, RYY}
   - P({2 yellow and 1 red})= P({YYR, YRY, RYY})
   - These possibilities are mutually exclusive, so P({YYR, YRY, RYY}) = P({YYR}U{YRY}U{RYY})
   - P({YYR}U{YRY}U{RYY}) = P({YYR}) + P({YRY}) + P({RYY})

   In the case without replacement:
   - P({YYR}) = (3/15) * (2/14) * (7/13)
   - P({YRY}) = (3/15) * (7/14) * (2/13)
   - P({RYY}) = (7/15) * (3/14) * (2/13)
   - Multiply these three components individually then add them up to get the answer.

   Answer: P({2 yellow and 1 red}) = 3/65

   There is another way to solve this problem:
   P({2 yellow and 1 red}) = the number of ways of picking {2 yellow and 1 red} divided by the number of ways there are to pick any three fish.

   - The total number of combination (ways) to pick any three fish:
     = C(15,3) or $\binom{15}{3}$
   - The total number of ways to pick one red fish:

$= C(7,1)$ or $\binom{7}{1}$

- The total number of ways to pick 2 yellow fish:

$= C(3,2)$ or $\binom{3}{2}$

- Thus:

Probability = $\dfrac{\binom{7}{1}\binom{3}{2}\binom{5}{0}}{\binom{15}{3}} = \dfrac{3}{65}$

b) In the case WITH replacement:

- P({YYR}) = 3/15 * 3/15 * 7/15
- P({YRY) = 3/15 * 7/15 * 3/15
- P({RYY}) = 7/15 * 3/15 * 3/15
- P({2 yellow and 1 red}) = $\dfrac{3*3*3*7}{15^3}=\dfrac{7}{5^3}=\dfrac{7}{125}$

Alternative solution

How many ways can I pick 3 fish where 2 are yellow and 1 is red?

- Total ways to pick any 3 fish = 15^3
- (number of ways to pick 2Y and 1R)*(number of ways of picking a Y)*(number of ways of picking Y)*(number of ways of picking an R) = 3*3*3*7

P = $\dfrac{(Number\ of\ favorable\ cases)}{(Total\ number\ of\ all\ possible\ cases \in existence)} = \dfrac{3*3*3*7}{15^3} = \dfrac{7}{125}$

For any k = 1, 2, …, n, $\binom{n}{k}$ ← “n choose k” is called a binomial coefficient.

Important note:is $\binom{n}{k}$ the same as $\binom{n}{n-k}$:

Proof:

$= \binom{n}{k} \quad \left(\dfrac{n!}{k!(n-k)!}\right)$ and $\binom{n}{n-k} = \left(\dfrac{n!}{(n-k)!(n-(n-k))!}\right) = \left(\dfrac{n!}{(n-k)!k!}\right)$

Recall: $\forall \quad a,b \in \mathbb{R}$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-1}$ =

$$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Exercise:

Some things are so complicated to compute, we have to find other ways to compute it. A room contains n people. What should be the value of n, to be sure w/ a probability of at least ½ (or 0.5) that at least 2 people in the room have the same birthday?

P(Coincident B-day) must = ½

Assumption #1: First, let's clarify what birthday means. "Birthday" means the month and day of birth. Years do not matter to us in this case.

Assumption #2: We assume every birthday is equally likely.

Assumption #3: No leap year (365 possible birthdays this year).

What is meant by "at least 2 people have the same birthday"? It means we can have greater than or equal to 2 people who have the same birthday. We have to consider the scenarios where 3 people have the same birthday or if 25 people have the same birthday. When we hear the words "at least", we have to consider the complimentary event.

P(At least 2 people have the same birthday) = 1 – P(None have the same birthday)

(In other words: The probability that at least 2 people have the same birthday equals one minus the probability that none of the people have the same birthday).

We will denote P(At least 2 people have the same birthday) as P_n . We call it P_n because the probability (P) depends on n.
P_n = 1 - $(P_n)^c$, where $(P_n)^c$ is the complement of P_n

$(P_n)^c$ represents the probability that no two people among the n people have the same birthday.

For each person, each has 365 possible days that could be his or her birthday.

P = 1 - $\frac{(365)(364)(363)\ldots(365-n+1)}{(365)^n}$ $\leftarrow \textit{decrements by 1 for each person}$ $\leftarrow \textit{total number of birthdys for the n number of people}$

Once a birthday coincides w/ another person's birthday, it is no longer a choice (when we are looking at $(P_n)^c$).

What should n be in order to ensure that $P_n \approx \frac{1}{2}$?

Let's consider that 366 people ensures that no one has their birthday on the same day.

Note that $P_{366} = 1 - \frac{(365)(364)(363)...(1)}{365^{366}}$ = 1

If we guess:

- n=10, P = 1-0.88 = 0.17
- n=30, P=0.706
- n=50, P=0.970

Between n=50 and n=366, the probability only grows 3%, from 0.970 to 1.00.
The function of the probability is nonlinear.

In class, the professor surveyed every student and listed everyone's birthday on the white board as a fun experiment. There were about 21 to 23 students in class. No one had matching birthdays.

Conditional Probability and Independence
The two notions are very much linked, so we cover them at the same time.
Example:
A family has 2 children, one of them is a girl. What is the probability that the other is a girl?
Assumptions: A family has either a boy or a girl, and each gender has an equally likely chance of occurring.
Ω = {(Boy, Girl), (Girl, Girl), (Boy, Boy), (Girl, Boy)}

"Intuitively" (meaning, especially likely):

P({Girl, Girl}) = P({(Girl, Boy)}) = P({(Boy, Girl)}) = P({(Boy, Boy)}) = ¼

We are told one child is already a girl, so (Boy, Boy) does not count in Ω. So we have a restricted sample space here, denoted by $\tilde{\Omega}$. $\tilde{\Omega}$ = {(Girl, Girl), (Girl, Boy), (Boy, Girl)}.
In $\tilde{\Omega}$, P({(Girl, Girl)}) = 1/3

Conditional probability and independence are very counter-intuitive, so we MUST go through these problems systematically.

In the question, when it says one of them is a girl, it does not mean that the first one is necessarily a girl!

Suppose the question was: One of the children is a girl born on a *Monday.* What is the probability that the other one is a girl?

The answer is NOT 1/3, surprisingly!
(There's not a really good explanation in existence as to why this is, however.)

Definition: Let A and B be two events such that probability of B > 0. The conditional probability of A given B, written P(A|B), is defined by:

P(A|B) = $\frac{P(A\cap B)}{P(B)}$ ← *Here we restrict the sample space so it only includes outcomes* $\in B$

In the previous example:

A = {(Girl, Girl)}

B = {(Girl, Girl), (Girl, Boy), (Boy, Girl)}

P(B)=3/4 and $A\cap B$ = A. We caee that the only element in B that intersects A is (G, G). $A\cap B$ is A. So P(A) = ¼.

P(A|B) = $\frac{P(A\cap B)}{P(B)}$ = $\frac{1}{4}\Big/\frac{3}{4}$ = $\frac{1}{3}$

Definition:

Two events in B are said to be independent (written $A\perp B$) if $P(A\cap B)=P(A)P(B)$.

(Note: Orthogonality and independence have a relationship, that's why the symbol used is the same.)

Otherwise, A and B are called **dependent**.

Probability of independent events is just a product of the 2 events. We can also express the union of the two independent events.

If A and B are independent, then $P(A\cap B)=P(A)P(B)$

Intuitively, we expect that if A and B are independent, the P(A|B) = P(A) and P(B|A)=P(B). Because if the two events are truly independent, the probability of one event will not be affected given that the other event has occurred.

To show this:

P(A|B) = $\frac{P(A\cap B)}{P(B)}$ ⇔ P(A|B)*P(B) = $P(A\cap B)$

But if they are independent, then $P(A\cap B)$ = P(A)*P(B).

So this:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Becomes this:

P(A|B)*P(B) = P(A)*P(B)
P(A|B) = P(A), given P(B) > 0.

Similarly, P(B|A) = P(B), provided that P(A) > 0.

If A and B are independent, then so are:
- A and B'
- A' and B
- A' and B'

The converse is also true. If A and B' are independent, then so are:
- A and B
- A' and B
- A' and B'

Proof:
We need to show that if $P(A \cap B)$ = P(A)*P(B), then $P(A \cap B')$ = P(A)*P(B').

A = $(A \cap B) \cup (A \cap B')$

$(A \cap B)$ and $(A \cap B')$ are both pairwise disjoint in relation to each other.

P(A) = $P(A \cap B)$ + $P(A \cap B')$
If A and B are independent events, then: P(A) = P(A)*P(B)+ $P(A \cap B')$

i.e.: P(A) – P(A)*P(B) = $P(A \cap B')$ $\Rightarrow$ P(A)*(1-P(B)) = P(A)*P(B')

A, B and C are independent if:
- $P(A \cap B)$ = P(A)*P(B)
- $P(A \cap C)$ = P(A)*P(C)
- $P(B \cap C)$ = P(B)*P(C)

These pairs are all exhibiting pairwise independence (also known as "total independence" or "complete independence").

$$P(A \cap B \cap C) = P(A) * P(B) * P(C)$$

Pairwise independence is independence exhibited within a pair of events. (Each element in the pair is independent to the other).

If three events are independent, then they are pairwise independent. But the reverse is not true.

Example:
Flip a coin twice. Let A = 1st flip is tails. B = 2nd flip is heads, C = both flips are the same.
A = {(T, T), (T, H)}
B = {(T, H), (H, H)}
C = {(T, T), (H, H)}

Thus:

- $A \cap B$ = {(T, H)}
- $A \cap C$ = {(T, H)}
- $B \cap C$ = {(H, H)}

$P(A \cap B \cap C) = \varnothing$

$P(A \cap B)$ = ¼ = P(A)*P(B)
$P(A \cap C)$ = ¼ = P(A)*P(C)
$P(B \cap C)$ = ¼ = P(B)*(C)

$P(A \cap B \cap C) = 0 \neq \frac{1}{8}$

Exercise to introduce Baye's Theorem:
A drug test is 98% accurate (i.e. if a person is a drug user, then the person will test positive 98% of the time. If the person is not a drug user, then the test will give a negative 98% of the time). A population contains 2% of drug users. A person picked at random in the population tests positive. What is the probability that the person is a drug user?

A = The person is a drug user = 0.01
B = The person tests positive = $(tests\ positive, is\ drug\ user) \cup (tests\ positive\ but\ not\ a\ drug\ user)$

We are interested in P(A|B) = $\frac{P(A \cap B)}{P(B)}$

We can rewrite B like so: B = $A \cap B$ + $A' \cap B$

P(B) = $P(A \cap B)$ + $P(A' \cap B)$

($A \cap B$ and $A' \cap B$ are pairwise disjoint.)

So P(A|B) = $\frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B)}{P(A\cap B)+P(A'\cap B)}$

P(B|A) = 0.98 → but $P(A\cap B)$ = P(B|A)*P(A)

P(A|B) = $\frac{P(A\cap B)}{P(A\cap B)+P(A'\cap B)} = \frac{\boldsymbol{P(A\cap B)}}{\boldsymbol{P(A\cap B)+P(A'\cap B)}}$

$$P(B|A)=\frac{(0.98)*(0.02)}{(0.98)*(0.02)+(0.02)(0.98)}=\frac{(0.98)*(0.02)}{2(0.98)*(0.02)}=\frac{1}{2}$$

(Professor handed out printed notes for us to bring to class next time).