Polynomials and geometry

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Example constructions

Line (by gradient and intercept): $y = mx + c$

Line (by two points):

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

Circle: $(y - y_c)^2 + (x - x_c)^2 = r^2$

Polynomial algebra

We work with a ring $\mathbb{C}[x_1, \ldots, x_n]$, the ring of (multivariate) polynomials in variables $x_1, \ldots, x_n$. If there are a small number of variables, we'll call them x, y, z rather than x_1, x_2, x_3.

A polynomial is a sum of *terms*, each of which is a *coefficient* multiplied by a *monomial*. So for example, if I have

$$37x_1^2x_5 - x_2 + 10$$

then the term $37x_1^2x_5$ has the coefficient 37 and the monomial $x_1^2x_5$.

Instead of working with equations, we simply remember which things are "supposed to be zero". So instead of $y = mx + c$ we work with $y - mx - c = 0$.

An *ideal* $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is a set of polynomials with the following properties:

1. If $f, g \in I$, then $f + g \in I$;

2. If $f \in I$, and $g \in \mathbb{C}[x_1, \ldots, x_n]$, then $fg \in I$.

For a simple example of an ideal in $\mathbb{C}[x, y]$, (if someone tells you that $x = 0$), the set of polynomials which are multiples of x is an ideal. $x^3 + xy \in I$, but $x^3 + xy - 7y \notin I$.

The general case looks like this: given elements

$$f_1, \ldots, f_d \in \mathbb{C}[x_1, \ldots, x_n]$$

I can form an ideal

$$\langle f_1, \ldots, f_d \rangle = \{f_1 g_1 + \cdots + f_d g_d \mid g_1, \ldots, g_d \in \mathbb{C}[x_1, \ldots, x_n]\}$$

My simple example a moment ago was $\langle x \rangle$.

The ideal $\langle f_1, \ldots, f_d \rangle$ is the smallest ideal containing $f_1, \ldots, f_d$.

In other words, if someone comes along and tells us that $f_1, \ldots, f_d = 0$, then $\langle f_1, \ldots, f_d \rangle$ is the set of polynomials we deduce are also zero by polynomial algebra.

So the basic problem is, now, given polynomials $f_1, \ldots, f_d, g$, can we tell if $g \in \langle f_1, \ldots, f_d \rangle$? It turns out that this is a bit fiddly.

Consider $I = \langle x^{99}y^{100} + x, x^{100}y^{99} \rangle$. This ideal contains

$$x(x^{99}y^{100} + x) - y(x^{100}y^{99}) = x^2.$$

As a result it also contains x, and in fact $I = \langle x \rangle$.

Is there a means of calculating that makes it obvious that x is in this ideal? Yes, there is!

It's a two part plan:

1. start by transforming the ideal so that it has "nice generators",

2. persuade ourselves that, if it has "nice generators" then we can tell what's in it easily.

The notion of "nice generators" is called being a *Gröbner basis*.

To do...

Work out how to reproduce this in SAGE: work out to define a polynomial ring $\mathbb{C}[x, y]$, and define the ideal $\left\langle x^{99}y^{100} + x, x^{100}y^{99} \right\rangle$, and get SAGE to check that a Gröbner basis for it is $\langle x \rangle$.

Mess around a bit with other examples to get a feel.

Take a very small geometry problem, and see if you can completely describe it as a question of polynomial algebra. Keep count of the number of variables and equations. Can you get SAGE to do it with a Grobner basis?

Proposed example: Thales's theorem (this that says that the angle in a semicircle is a right angle).