Math 114 - Fall 2016 - Assignment 6

Due: Friday, December 2nd at 4:30PM in dropbox 7 outside of MC 4066

Eigenvectors and Eigenvalues

1. Let $A = \begin{bmatrix} -4 & 0 & 0 \\ 2 & -8 & 4 \\ -4 & 5 & 0 \end{bmatrix}$. Find the algebraic multiplicity, geometric multiplicity, and eigenspace of each of the eigenvalues of A.

2. The characteristic polynomial of a square matrix B is $C(\lambda) = 2\lambda^3 + \lambda^2 - 8\lambda + 5$. Given that $\lambda = 1$ is an eigenvalue of B, find all eigenvalues of B and state their algebraic multiplicities.

Diagonalization

3. Let $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 0 \\ 3 & 6 & 1 \end{bmatrix}$. Determine the eigenvalues and eigenvectors of A, and provide a matrix P and a diagonal matrix D such that $P^{-1}AP = D$. You may skip calculating P^{-1}.

4. Let A be a 3×3 matrix with characteristic polynomial $C(\lambda) = \lambda^3 - 14\lambda^2 + 49\lambda$. Further suppose $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 21 \end{bmatrix}$ and that $A \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \\ 21 \end{bmatrix}$.
   (a) By discussing the algebraic and geometric multiplicities of the eigenvalues of A, prove that A must be diagonalizable.
   (b) Use the geometric multiplicities of the eigenvalues of A and the invertible matrix theorem to prove that A has no inverse.

5. Suppose matrix B has characteristic polynomial $C(\lambda) = 7331\lambda^7 - e^{\pi}\lambda^3 + 1337\lambda$. By considering geometric multiplicity, argue that $B\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$. Is B invertible? Justify your answer.

6. For the following problem, the 3×3 matrix A is held constant.
   (a) Suppose you're told P diagonalizes A, $P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$, and that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. List the eigenvalues of A and their associated eigenspaces.
   (b) Let $Q = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -2 \\ 0 & 4 & -3 \end{bmatrix}$. Prove Q diagonalizes A and provide the diagonalized matrix product $D_2 = Q^{-1}AQ$.

7. The following matrix will not have real eigenvalues or eigenvectors, but the problem is still possible if we allow ourselves to use complex numbers. Use complex numbers to diagonalize the following matrix: $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$. You may skip calculating P^{-1}.

8. In this problem we'll introduce a new proof technique, "Proof by Induction", to describe a property of diagonal matrices.

   (a) Show that the determinant of a 2×2 diagonal matrix $\text{diag}(d_1, d_2)$ is the product $d_1 * d_2$.

   (b) By first invoking the definition of the determinant of a 3×3 matrix (using a cofactor expansion of your choice), show that the determinant of $\text{diag}(d_1, d_2, d_3) = d_3 * \det(\text{diag}(d_1, d_2))$.

   (c) Suppose your exhausted friend approaches you and claims to have confirmed that the determinant of a 99×99 diagonal matrix $\text{diag}(d_1, d_2, ..., d_{99})$ is the product $(d_1 * d_2 * .... * d_{99})$. By invoking the definition of the $n \times n$ determinant, show that $\det(\text{diag}(d_1, ..., d_{100})) = d_{100} * \det(\text{diag}(d_1, ..., d_{99}))$, and that $\det(\text{diag}(d_1, ..., d_{100}))$ must therefore be $(d_1 * d_2 * .... * d_{100})$.

   (d) Assuming someone else has shown that the determinant of a $k \times k$ diagonal matrix $\text{diag}(d_1, ..., d_k)$ is $(d_1 * d_2 * ...d_k)$, invoke the definition of the $n \times n$ determinant to show that the determinant of a $(k+1) \times (k+1)$ diagonal matrix $\text{diag}(d_1, ..., d_{k+1})$ must be $(d_1 * d_2 * ... * d_{k+1})$.

       Steps b) and c) were actually unnecessary. The general idea behind proof by induction is to solve a simple base case (or small number of such cases) and then show that the proof to a more complicated example can be expressed as an algorithm in terms of simpler sub-problems that cascade all the way back to the base case(s). Since the $n \times n$ determinant of a diagonal matrix can always be stated in terms of an $(n-1) \times (n-1)$ sub-problem (by invoking the definition of the determinant), we've actually proved that the determinant of an $n \times n$ diagonal matrix will be the product of the diagonal entries *no matter what value n takes*.

   (e) Prove that if an $n \times n$ matrix A is similar to an uninvertible diagonal matrix D (i.e. D^{-1} doesn't exist), then A must have 0 as an eigenvalue.

   (f) Prove that if an $n \times n$ matrix A is similar to a diagonal matrix D, then the determinant of A is the product of its eigenvalues (repeated according to their multiplicity).