

Math 114 - Fall 2016 - Assignment 4

Due: Friday, November 4 at 4:30PM in dropbox 7 outside of MC 4066

Matrices

1. Evaluate the following matrix-vector products:

$$\text{a) } \begin{bmatrix} 2 & -4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

2. Evaluate the following matrix-matrix products:

$$\text{a) } \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & 4 & 2 \\ 1 & -4 & -6 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$$

3. For this question, let A be a 1×3 matrix and B be a 3×1 matrix given by

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note, we would normally refer to these as row and column vectors. The point is that a vector is really just a special case of a matrix.

- (a) For the matrices above, compute both AB . What familiar operation does matrix multiplication reproduce in this case?
- (b) The operation from part (a) is also known as the **inner product** when working with vectors. Now use the same rules of matrix multiplication to compute BA . (This operation between vectors is called the **outer product** and turns out to be useful for describing the moment of inertia of rotating bodies and for various statistical analyses.)

Geometrical Transformations

4. Your friend tells you they found the vector $(-1, \sqrt{2}, 0)$ after applying a rotation of 45° about the x -axis to some input vector, $\vec{v} = (x, y, z)$. In other words, $(-1, \sqrt{2}, 0) = R_{x, \pi/4} \vec{v}$. Determine the input vector \vec{v} (assuming your friend applied the rotation correctly).
5. Find a matrix, A , to describe each of the transformations in \mathbb{R}^2 given below. In each case, it will be helpful to work out what the output vector, $\vec{v}' = (x'_1, x'_2)$ will should look like in terms of the components of an arbitrary input $\vec{v} = (x_1, x_2)$ using a sketch.
- (a) Scale a vector down to half its initial length (while preserving the orientation).
 - (b) Reverse the direction of a vector.
 - (c) Reflect (flip) a vector across the line $x_1 = x_2$.
 - (d) Project a vector onto the x -axis.
6. In the early evening, you notice a couple of interesting objects in the night sky - the Linear Nebula and the Algebras Constellation. Using the rotational axis of the Earth as the z -axis of a coordinate system (and two orthogonal directions for x and y) you can label the positions of these objects with the unit direction vectors

$$\vec{v}_L = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{v}_A = \begin{bmatrix} 1/\sqrt{8} \\ \sqrt{3}/2 \\ 1/\sqrt{8} \end{bmatrix}.$$

As time passes, the Earth rotates about what you have defined to be the z -axis but since you move with the Earth you see the night sky rotating through the **negative** of the angle the Earth actually rotates. With this in mind, find the directions of the Linear Nebula and Algebras Constellation 6 hours after your initial measurement. Does the angle *between* your direction vectors change over this time?

7. Consider the rotation matrix in \mathbb{R}^2 given by $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.
- Show that the columns of R_θ are orthogonal. (Note, this means the columns form a basis for \mathbb{R}^2 - a useful property).
 - Show that the magnitude of each column is equal to 1. (Note, this means the columns vectors are unit vectors - also a useful property for a basis).
 - We can actually test parts (a) and (b) simultaneously using matrix multiplication but first we must define the transpose of a matrix. So, given an $m \times n$ matrix A , the transpose is denoted A^T and defined by $(A^T)_{ij} = (A)_{ji}$. (Basically, just flip the matrix across the main (top-left to bottom-right) diagonal.) Now, show that $R_\theta^T R_\theta = I$. (Notice, 0's appearing off the main diagonal tell us what we found in part (a) - that the columns are orthogonal - and 1's appearing on the main diagonal tell us what we found in part (b) - that the columns are unit vectors.

We call a matrix satisfying $A^T A = I$ an **orthogonal matrix**. These come in very handy for finding bases in real-world problems.

Columnspace and Nullspace

8. Let $Q = \begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{bmatrix}$.
- Find a vector equation for the solution set of Q (i.e., the solution to the homogeneous equation $Q\vec{x} = \vec{0}$). What geometric shape does this vector equation represent?
 - Let $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ be three vectors from \mathbb{R}^3 that aren't in the solution set. Compute the matrix-vector products $Q\vec{x}$, $Q\vec{y}$, $Q\vec{z}$. Identify a simple geometric shape that contains these three output vectors.
 - Confirm that the vector equation for the shape you found in (b) is equivalent to $\text{Col}(Q)$.
9. Let $M = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 3 & -5 & -1 \end{bmatrix}$.
- Put M in RREF. What is the rank of M ? What is the solution set if M represents a set of homogeneous equations?
 - Let's denote the columns of M by \vec{m}_1 , \vec{m}_2 , and \vec{m}_3 . Show that the columns are linearly dependent by finding the general solution to $a_1\vec{m}_1 + a_2\vec{m}_2 + a_3\vec{m}_3 = \vec{0}$.
 - What is the nullspace of M ?
 - What is the columnspace of M ?
 - What is the sum of the dimensions of (c) and (d)? What geometric shapes do (c) and (d) represent?
 - Provide a basis for the columnspace of M .