MATH 114 - Fall 2016 - Assignment 4

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Problem 1. Matrix-Vector Products

a) 2x2 $\cdot$ 2x1 = 2x1

$$\begin{bmatrix} 2 & -4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} (2 \cdot 4) + (-4 \cdot -2) \\ (5 \cdot 4) + (-3 \cdot -2) \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \end{bmatrix}$$

b) 2x3 $\cdot$ 3x1 = 2x1

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

Problem 2. Matrix-Matrix Products

a) 2x2 $\cdot$ 2x3 = 2x3

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & 4 & 2 \\ 1 & -4 & -6 \end{bmatrix} = \begin{bmatrix} -6+2 & 8+-8 & 4+-12 \\ -9+3 & 12-12 & 6-18 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -8 \\ -6 & 0 & -12 \end{bmatrix}$$

b) 3x2 $\cdot$ 2x2 = 3x2

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3+4 & -1+5 \\ 6+8 & -2+10 \\ 9+12 & -3+15 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 14 & 8 \\ 21 & 12 \end{bmatrix}$$

Problem 3. Matrix Multiplications

a)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+4+9 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Problem 4. Geometric Transformations

By rotating the result by the inverse of R_θ we will reverse the rotation made to get $(-1, \sqrt{2}, 0)$. 3x3 $\cdot$ 3x1 $=$ 3x1

$$\begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{2}-2}{2} \\ \frac{-\sqrt{2}+2}{2} \end{bmatrix}$$

Problem 5. Geometric Transformations

For all questions below, I assumed that the matrix A was to be multiplied with the input vector $\vec{v}$. Thus A is the matrix that will change $\vec{v}$ to get the desired transformation.

a)

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

b)

This is a simplification of having π as θ in the rotation vector in R_θ

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 6. Geometric Transformations

By rotating the vectors $\vec{v}_L$ and $\vec{v}_A$ by $\frac{-\pi}{2}$ about the z axis we will acheive the rotation relative to us 6 hours later.

$\vec{v}_L$

$$\begin{bmatrix} \cos\frac{-\pi}{2} & -\sin\frac{-\pi}{2} & 0 \\ \sin\frac{-\pi}{2} & \cos\frac{-\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$\vec{v}_A$

$$\begin{bmatrix} \cos\frac{-\pi}{2} & -\sin\frac{-\pi}{2} & 0 \\ \sin\frac{-\pi}{2} & \cos\frac{-\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{8}} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{8}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}$$

The angle between the two vectors initially was $\frac{\pi}{3}$. After the 6 hours had passed, the angle between was $\frac{\pi}{3}$. Thus no change had been made.

Problem 7. Geometric Transformations

a)

The set of the columns is $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$ and by substituting any value for θ that is within the range of 0 to 2π will produce the scale that we can then reduce down to RREF. To simplify it, I'll use 2π so I don't have to make it into RREF.

$$S = \left\{ \begin{bmatrix} \cos 2\pi \\ \sin 2\pi \end{bmatrix}, \begin{bmatrix} -\sin 2\pi \\ \cos 2\pi \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

The set S is a basis for $\mathbb{R}^2$ because it is linearly independant, thus they are orthagonol.

b)

Continuing from the previous question, and using the same result, the magnitude of each column is 1.

$$S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\|S_1\| = \sqrt{1^2 + 0^2} = 1$$

$$S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\|S_1\| = \sqrt{0^2 + 1^2} = 1$$

c)

$$R_\theta^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_\theta^T \cdot R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta^2 + \sin\theta^2 & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin\theta^2 + \cos\theta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 8. Geometric Transformations

$$Q = \begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{bmatrix} R_1 - \tfrac{1}{3}R_3$$

a)

$$\sim \begin{bmatrix} 3 & -6 & 3 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{bmatrix} R_3 - R_1$$

$$\sim \begin{bmatrix} 3 & -6 & 3 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ \\ \frac{1}{2}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This represents a plane. There are two free variables. This is because all three equations are the same plane, thus they intersect in infinitely many points.

$$x_1 - 2x_2 + x_3 = 0$$

b)

$$Q\vec{x} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$Q\vec{y} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

$$Q\vec{x} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix}$$

A simple geometric shape that contains these three vectors is a line.

c)

All three vectors found are scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ which is equivelant to the columnspace of Q because the columnspace is also scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 9. Geometric Transformations

$$M = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 3 & -5 & -1 \end{bmatrix}$$

a)

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 3 & -5 & -1 \end{bmatrix} R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & -8 & -16 \end{bmatrix} R_3 + 8R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

M is of rank 2. With 1 free variable. This means that M's solution set represents a line.

b)

$$a_1\vec{m_1} + a_2\vec{m_2} + a_3\vec{m_3} = 0$$

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$-3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$a_1 = -3, a_2 = -2, a_3 = 1$$

Because the linear combination of the columns have a solution where a_1, a_2, a_3 are not all 0, the columnspace of M is linearly dependant.

c)

The nullspace of M is $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$

d)

The columnspace of M is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

e)

The sum of dim(null(M)) and dim(col(M)) is 3.
dim(null(M)) = 1, and dim(col(M)) = 2

f)

The basis for Col(M) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$