Math 114 - Fall 2016 - Assignment 5

Due: Friday, November 18 at 4:30PM in dropbox 7 outside of MC 4066

Inverses

1. Find the inverse of the following matrices or show that the inverse does not exist:

a) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 1 \\ -2 & 5 & -5 \end{bmatrix}$

2. Consider the following system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= b_1 \\ x_2 + 2x_3 &= b_2 \\ -x_1 + 3x_2 + 12x_3 &= b_3. \end{aligned}$$

By finding the inverse of the coefficient matrix associated with the system of equations, find an expression for the solution $\vec{x}$ in terms of arbitrary b_1, b_2, and b_3.

3. Suppose A is an $n \times n$ invertible matrix and your friend claims to have found two different matrices B and C that work as inverses. Prove that your friend is mistaken - that is, prove that there is only one inverse of A. (Hint: Assume your friend's claim is true but then show that this implies $B = C$.)

4. Suppose A and B are $n \times n$ matrices such that $AB = I$. Must it be true that $\texttt{rank}(A) = \texttt{rank}(B)$? Justify your answer.

5. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ and A is an $n \times n$ invertible matrix. Prove that the set $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ is also linearly independent. (Hint: Assume that the latter set is linearly dependent and show that this gives rise to a contradiction.)

Determinants

6. Determine if the following matrices are invertible by finding the determinant (you do not need to find the inverse):

a) $\begin{bmatrix} 1 & 5 & -3 \\ 2 & 13 & -7 \\ 3 & -3 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

7. For what values of s is the matrix $\begin{bmatrix} 1 & s & 1 \\ s & -3 & -2s \\ 1 & 2 & -1 \end{bmatrix}$ invertible?

8. It turns out all of our results regarding inverses apply equally well to complex-valued matrices. This includes using the determinant to check if a matrix is invertible. With this in mind, determine if the following matrix is invertible:

$$\begin{bmatrix} 1+i & 1-i \\ 1+3i & 3-i \end{bmatrix}.$$

9. In the previous assignment we defined the transpose of a matrix A - denoted A^T - by $(A^T)ij = (A)_{ji}$. Our goal now is to argue that, for an arbitrary $n \times n$ matrix we have $\texttt{det}(A^T) = \texttt{det}(A)$. Consider the cofactor expansion of an arbitrary 3×3 matrix A along its first column and compare it to the cofactor expansion of A^T along its first row. Are the determinants the same? Will this argument generalize to an $n \times n$ matrix? Justify your answers.

10. In tutorial, we established that determinants obey the property that for any two $n \times n$ matrices A and B, we have $\texttt{det}(AB) = \texttt{det}(A)\texttt{det}(B)$. Use this result to solve the following questions:

    (a) Consider three matrices A, B, and C. Given that $\texttt{det}(AB) = 6$, $\texttt{det}(BC) = 12$, and $\texttt{det}(ABC) = 24$, find the $\texttt{det}(B)$.

    (b) Suppose a matrix A satisfies $A^3 = A$. What are the possible values for the determinant of A?

    (c) In assignment 4 we defined an orthogonal matrix as one that satisfies $A^T A = I$. What values can the determinant of an orthogonal matrix have? (Hint: Use the result of question 9.)

    (d) When we study diagonalization, we will introduce the notion of **similar** matrices. In particular, if A and B are similar then there an invertible matrix P such that $B = P^{-1}AP$. In this case, show that $\texttt{det}(A) = \texttt{det}(B)$.

Eigenvalues and Eigenvectors

11. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ performs an unequal scaling on an input vector. In particular, it doubles the x_1 component and triples the x_2 component.

    (a) Consider the following three vectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $A\vec{v}_1$, $A\vec{v}_2$, and $A\vec{v}_3$.

    (b) On three separate graphs, sketch each input along with its output (e.g., on one graph, sketch $\vec{v}_1$ and $A\vec{v}_1$).

    (c) Which of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are eigenvectors? What are their associated eigenvalues?

12. Find the eigenvalues and associated eigenvectors for the following matrices:

    a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$