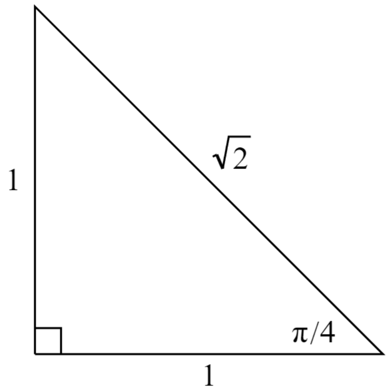
Markscheme

ID: 3014

Mathematics: analysis and approaches

Higher level

1. (a) Draw an isosceles right-angled triangle M1



We have

$$\sin(\pi/4) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$ A1A1

(b)

(i) Use the double angle identity M1

$$2\sin(x/2)\cos(x/2) = \sin(2 \times x/2) = \sin x$$ A1

(ii) Use the double angle identity M1

$$2\cos^2(x/2) - 1 = \cos(2 \times x/2) = \cos x$$ A1

(c) We have

$$2\cos^2(\pi/8) - 1 = \frac{\sqrt{2}}{2}$$ M1

So

$$\cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$$ A1

(d) We have

$$2\cos^2(\pi/16) - 1 = \frac{\sqrt{2+\sqrt{2}}}{2}$$ M1

So

$$\cos^2(\pi/16) = \frac{2+\sqrt{2+\sqrt{2}}}{4}$$ A1

Giving

$$\cos(\pi/16) = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$ A1

(e) In (b) part (i) we have shown it is true for $n = 1$. A1

Assume it is true for $n = k$. So

$$\sin x = 2^k \sin\left(\frac{x}{2^k}\right) \cdot \cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2^2}\right) \cdot \cdots \cdot \cos\left(\frac{x}{2^k}\right) \quad \text{A1}$$

When $n = k + 1$ we have

$$2^{k+1} \sin\left(\frac{x}{2^{k+1}}\right) \cdot \cos\left(\frac{x}{2}\right) \cdot \cdots \cdot \cos\left(\frac{x}{2^{k+1}}\right) = 2^k \sin\left(2 \times \frac{x}{2^{k+1}}\right) \cdot \frac{\sin x}{2^k \sin\left(\frac{x}{2^k}\right)} = \sin x \quad \text{M1A1A1}$$

So it is true for $n = k + 1$. A1

By the principle of mathematical induction it must be true for all $n \in \mathbb{Z}^+$. R1

(f) We have

$$\lim_{n \to \infty} \frac{\sin\left(\frac{c}{2^n}\right)}{2^{-n}} = \frac{-c(\ln 2)2^{-n} \cos\left(\frac{c}{2^n}\right)}{-(\ln 2)2^{-n}} = c \quad \text{M1A1A1}$$

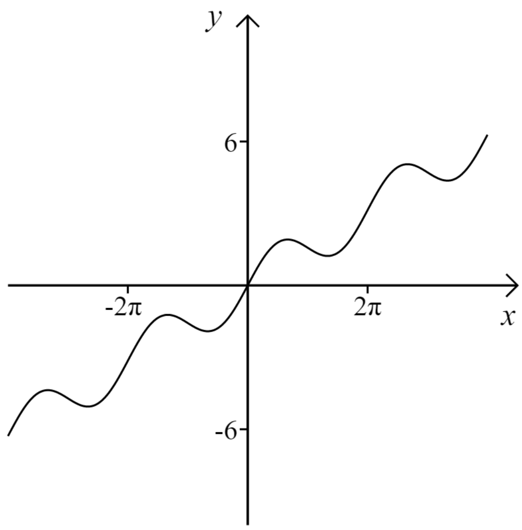
(g) We have

$$\sin(\pi/2) = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \cdots \quad \text{M1}$$

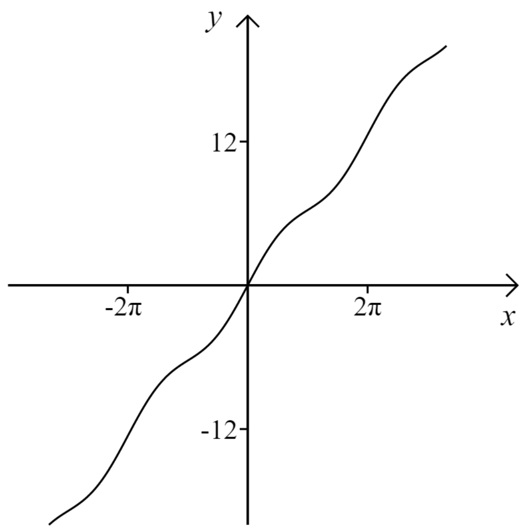
So

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \cdots \quad \text{A1}$$

2. (a) When $a = 1/2$ the shape is approximately correct. A1A1



When $a = 2$ the shape is approximately correct. A1A1



(b) For an inverse function to exist the function must be one-to-one. R1

So when $a = 2$ the function has an inverse. A1

(c) The function must be either increasing, or decreasing. So $f(x)$ cannot change sign. R1

We have

$$f'(x) = a + \cos x$$ A1

Since $|\cos x| \leq 1$ we must have $|a| \geq 1$. R1

So $a \leq -1$ or $a \geq 1$. A1

(d) We have

$$x = y + \sin y \qquad \text{A1}$$

Use implicit differentiation M1

$$1 = \frac{dy}{dx} + \frac{dy}{dx}\cos y = \frac{dy}{dx}(1+\cos y) \qquad \text{A1A1}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{1+\cos y} \qquad \text{A1}$$

(e) We have

$$\frac{d^2y}{dx^2} = \left(-\frac{dy}{dx}\sin y\right)\left(-\frac{1}{(1+\cos y)^2}\right) = \frac{\sin y}{(1+\cos y)^3} \qquad \text{A1A1A1}$$

(f) We have

$$\frac{d^3y}{dx^3} = \frac{\frac{dy}{dx}\cos y\,(1+\cos y)^3 + 3\sin^2 y\,\frac{dy}{dx}\,(1+\cos y)^2}{(1+\cos y)^6} \qquad \text{A1A1A1}$$

Factorise and simplify

$$\frac{d^3y}{dx^3} = \frac{(1+\cos y)\left(\cos y\,(1+\cos y) + 3\sin^2 y\right)}{(1+\cos y)^6} = \frac{\cos y + \cos^2 y + 3\sin^2 y}{(1+\cos y)^5} \qquad \text{M1A1}$$

Use the Pythagorean identity to rewrite

$$\frac{d^3y}{dx^3} = \frac{-2\cos^2 y + \cos y + 3}{(1+\cos y)^5} \qquad \text{A1}$$

Factorise and simplify

$$\frac{d^3y}{dx^3} = \frac{(3-2\cos y)(1+\cos y)}{(1+\cos y)^5} = \frac{3-2\cos y}{(1+\cos y)^4} \qquad \text{M1A1}$$

(g) When $x = 0$ then $y = 0$. A1

The first non-zero term is

$$\frac{1}{1+\cos 0}\cdot x = \frac{x}{2} \qquad \text{M1A1}$$

The second non-zero term is

$$\frac{3-2\cos 0}{3!(1+\cos 0)^4}\cdot x^3 = \frac{x^3}{96} \qquad \text{M1A1}$$