Mathematics: analysis and approaches
Higher level
Paper 3

2021 Mock

ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Friday, August 27th (morning)

1 hours

Candidate number

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Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

$\frac{39}{55} = 71\%$

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$
- the even values $n = 2$ and $n = 4$

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$ [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$ [2]

Consider the graph of $y = x^n(a-x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a-x)^n = k$ has four solutions for x. [5]

2. [Maximum mark: 24]

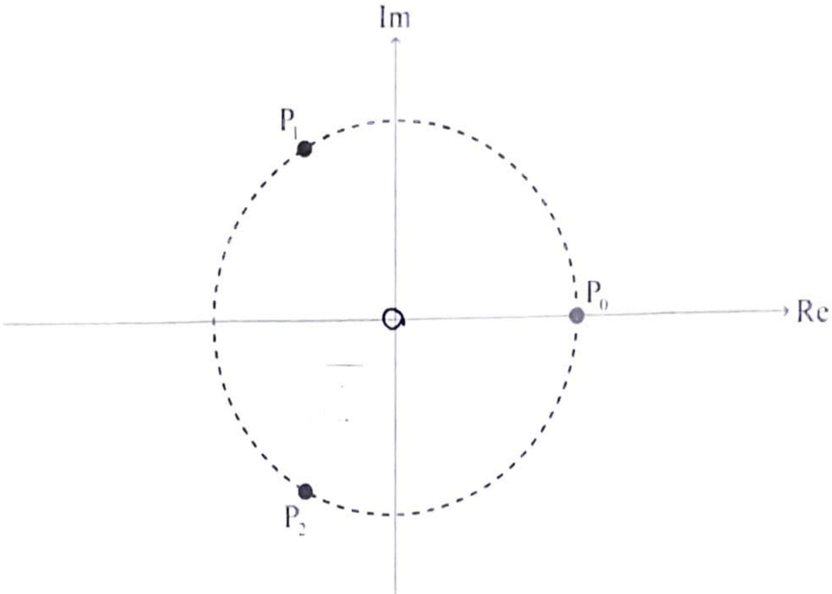
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \ldots \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point P_0, P_1, P_2, …, P_{n-1}, respectively on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

The roots of the equation $\boxed{z^3 = 1}$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre O $(0, 0)$.



(a) (i) Show the $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

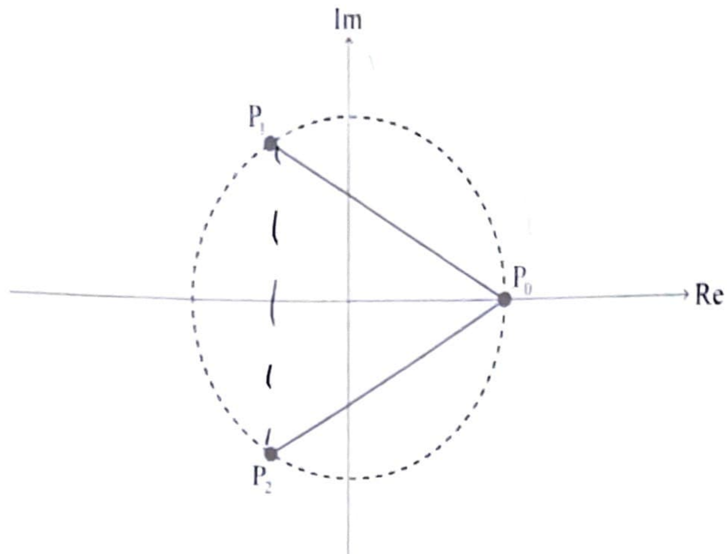
(This question continues on the following page)

$z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$

$z^3 - 1 = 0$

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

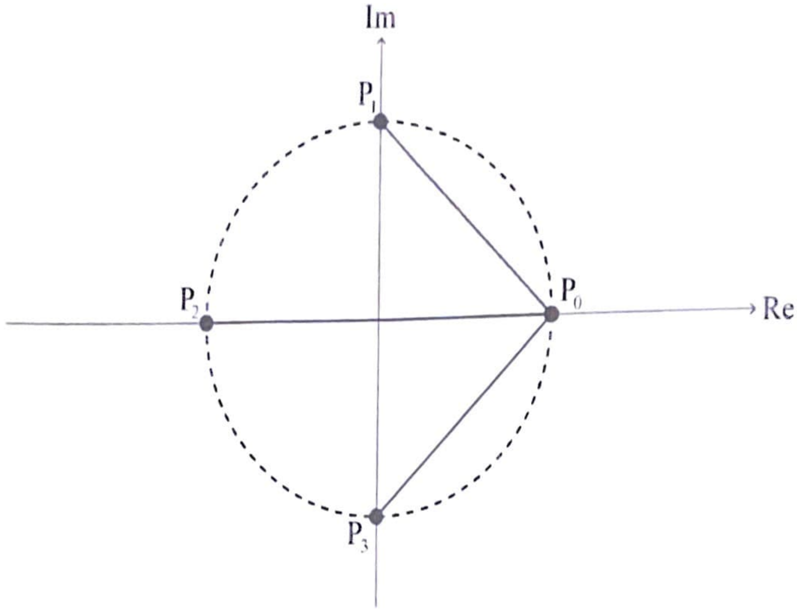
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorising $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0 , P_1 , P_2 and P_3 lie on a circle of radius 1 unit with centre O $(0,0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1 , ω , ω^2 , ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 , ..., ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0 , P_1 , P_2 , ..., P_{n-1} respectively where $[P_0P_1]$, $[P_0P_2]$, ..., $[P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre O $(0,0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times \ldots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω . [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω . [1]

Consider $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \cdots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

End of paper 3

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$\frac{39}{55} = 71\%$

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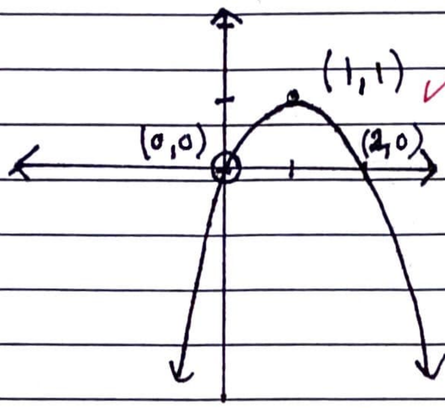
1 2 3 4 5 6 7 8 9 10

1

a) $f_1(x) = x(2-x)$



3

b) $f_n(x) = x^n(2-x)^n$

	Max	Min	Inflexion
$n=3, n=5$	1	0	2
$n=2, n=4$	2	2 1	0

6

c) $f_n(x) = x^n(a-x)^n$

$\therefore f_n'(x) = x^n[(a-x)^n(-1)] + (a-x^n)(x^{n-1})$

$=$

$f_n'(x) = (a-x)^n(nx^{n-1}) + x^n \times n(a-x)^{n-1}(-1)$ 3

$= nx^{n-1}(a-x)^n \cancel{+} x \times x^{n-1} \times -n(a-x)^{n-1}$

$= nx^{n-1}[(a-x)^n \cancel{\ } - \cancel{\ }\, x(a-x)^{n-1} \cancel{\ }]$

$= nx^{n-1}[$

12/

d) $nx^{n-1}(a-2x)(a-x)^{n-1} = 0$

$nx^{n-1} = 0 \qquad a-2x=0 \qquad (a-x)^{n-1}=0$

$\therefore x^n/x = 0 \qquad x = a/2 \qquad x = a$

$\therefore \quad x = 0$

2

$\therefore \quad x=0, \quad x = a/2, \quad x = a$

e) $\cancel{f'_n(a/2) = n\left(\frac{a}{2}\right)^{n-1}\left(a-\frac{2a}{2}\right)\left(a-a/2\right)^{n-1}}$

$f_n(a/2) = x^n(a-a/2)^n$
$= x^n\left(\frac{2a-a}{2}\right)^n$
$= x^n(a/2)^n$
$= \left(\frac{a}{2}\right)^n\left(\frac{a}{2}\right)^n$
$= \left(\frac{a}{2}\right)^{2n} > 0$

3

$\therefore \left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the x-axis

f) $\cancel{f'_n\left(\frac{a}{4}\right) =}$ ~~a/4 is~~

$f'_n(a/4) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{2a}{4}\right)\left(a-a/4\right)^{n-1}$
$= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{4a-2a}{4}\right)\left(\frac{4a-a}{4}\right)^{n-1}$
$= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3}{4}a\right)^{n-1}$

$\left(\frac{a}{4}\right)^{n-1} > 0$
$n > 0$

1

$\rightarrow$ as $a \in \mathbb{R}^+$, $f'_n(a/4) > 0$
$n \geq 1$

$\frac{a}{2} > 0$
$\left(\frac{3a}{4}\right)^{n-1} > 0$

6/

g) $\frac{a}{4} = -1 \rightarrow a = -4$

$f_n'(-1) = \nearrow$ negative $\leftarrow$ derive $\cancel{n(a-2)(a-1)^{n-1}}$

(2)

i) ~~$f_n'(2n)$~~

For values of $n \in$ even :

$f_n'(0) = \cancel{n x^{n-1}}$ $2n \times 0 \times \cdots = 0$

$f_n'(x)$ $\leftarrow$ from (f)

$\therefore f_n'(-1) < 0$

$\nwarrow$ need to derive

– + | 0

ii) $f_{2n-1}(0) = (2n-1)(0^{2n-1-1})(a)(a)^{2n-1-1}$

$= 0$

$f_n'(-1) = n(-1)^{n-1}(a+2)(a+1)^{n-1}$

derive $(-1)^{n-1} > 0$ {$n-1$ is even}

~~As $f_n'(0) < 0$~~

As $f_n'(-1) > 0$ for n is odd :

$\leftarrow$ from (f)

(1)

+ + | 0

h) As $(0,0)$ is a local min for even graphs, $\frac{a}{2} > k > 0$

(2)

As $n \rightarrow$ odd has at most two solutions, $n \in$ even

$\therefore k > 0, \quad n = 2, 4, 6, \ldots$

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2

a) i) $(\omega-1)(\omega^2+\omega+1)$

$= \omega^3+\omega^2+\omega$

$-\omega^2-\omega-1$

$= \omega^3-1$

2

ii) ~~$\omega^2+\omega+1 = \dfrac{\omega^3-1}{\omega-1}$~~

~~ω^2~~

~~$\omega-1 \,\big)\, \omega^3+0\omega^2+0\omega-1$~~

$\dfrac{\omega^3-1}{\omega-1} =$

2/

~~$P_0 = (0, +)$~~

$\omega^2 + \omega + 1 = 0$

$\therefore \omega = -\frac{1}{2} - \sqrt{3}/2 \, i$

$\omega = -\frac{1}{2} + \sqrt{3}/2 \, i$

P_2

P_1

$\therefore$ ~~$P_0P_1 =$~~

$\rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\rightarrow \quad P_1 = \begin{pmatrix} -\frac{1}{2} \\ \sqrt{3}/2 \end{pmatrix}$

$\rightarrow \quad P_2 = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$

$\therefore P_0P_1 = \begin{pmatrix} -\frac{1}{2} - 0 \\ \frac{\sqrt{3}}{2} - 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ \frac{\sqrt{3}-2}{2} \end{pmatrix}$

$\therefore P_0P_2 = \begin{pmatrix} -\frac{1}{2} - 0 \\ -\sqrt{3}/2 - 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ \frac{-\sqrt{3}-2}{2} \end{pmatrix}$

$\therefore P_0P_1 \times P_0P_2 = \begin{pmatrix} -1/2 \\ \frac{\sqrt{3}-2}{2} \end{pmatrix} \times \begin{pmatrix} -1/2 \\ \frac{-\sqrt{3}-2}{2} \end{pmatrix}$

$[P_0] =$

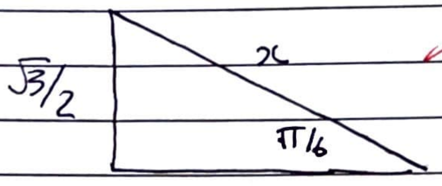
b) $\omega^2 + \omega + 1 = 0$

[G.D.C]

$\rightarrow \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (P_1)

$\rightarrow \omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (P_2)

3



$\sin \pi/6 = \frac{\sqrt{3}}{2} \div x$

$x \therefore = \frac{\sqrt{3}}{2} \div \sin \pi/6$

$= 1.73205....$

using GDC: $(1.7320S...)^2$

$= 3$

Both same length

c) $z^4 = 1 + 0i$ 2

$= cis(0 + 2k\pi)$

$\therefore z = cis(2k\pi/4) = cis(k\pi/2)$

$\therefore z = cis(0), cis(\pi/2), cis(\pi), cis(3\pi/2)$

$= 1, i, -1, -i$

$\therefore \omega^3 + \omega^2 + \omega + 1 = -i - 1 + i + 1 = 0$

other:

$(z^4 - 1) = (z^2 + 1)(z^2 - 1)$

$= (z+1)(z-1)(z^2-1)$

$\therefore \omega^4 - 1 = (\omega+1)(\omega^3 + \omega^2 + \omega + 1)$

5/

d) $P_0P_2 = 2$ units

$P_0P_1 = \sqrt{1^2+1^2}$ units $= \sqrt{2}$ units

$P_0P_3 = \sqrt{2}$ units

4

$\therefore P_0P_2 \times P_0P_1 \times P_0P_3 = 2 \times \sqrt{2} \times \sqrt{2}$

$= 4$ units^3

e) $P_0P_1 \times P_0P_2 \times \ldots \times P_0P_{n-1} = n$

1

~~f)~~

f) i) $P_0P_1 = |1-\omega|$

$P_0P_2 = |1-\omega^2|$

2

$P_0P_3 = |1-\omega^3|$

ii) $P_0P_{n-1} = |1-\omega^{n-1}|$

1

g) i) $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$

$(z^{n-1} + z^{n-2} + \cdots + z + 1) = \dfrac{z^n - 1}{z-1}$

roots: $(z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

ii) $|1-\omega||1-\omega^2||1-\omega^3|\cdots|1-\omega^{n-1}|$

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