Mathematics
Higher level
Paper 1

Tuesday 10 May 2016 (afternoon)

2 hours

Candidate session number

16	M	TZ	2	P	1	M	A	H	L

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

$\frac{98}{120} = 81.7\%$

13/10/22

2216–7205

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following system of equations represents three planes in space.

$$x + 3y + z = -1$$
$$x + 2y - 2z = 15$$
$$2x + y - z = 6$$

Find the coordinates of the point of intersection of the three planes.

(1) $x + 3y + z = -1$
$(x + 2y - 2z = 15) -$
$y + 3z = -16$

(2) $2x + 6y + 2z = -2$
$(2x + y - z = 6) -$
$5y + 3z = -8$

(3) $y + 3z = -16$
$(5y + 3z = -8) -$
$-4y = -8$
$\therefore \boxed{y = 2}$

(4) $3z = -16 - 2$
$\therefore 3z = -18$
$\therefore \boxed{z = -6}$

(5) $x + 3(2) - 6 = -1$
$\therefore \boxed{x = -1}$

Hence, point of intersection is $\boxed{(-1, 2, -6)}$

6

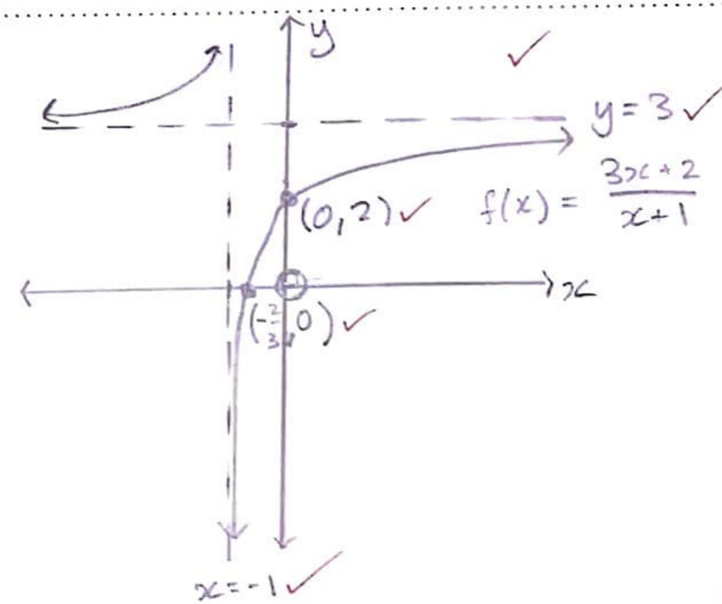
2. [Maximum mark: 5]

The function f is defined as $f(x) = \frac{3x + 2}{x + 1}$, $x \in \mathbb{R}, x \neq -1$.

Sketch the graph of $y = f(x)$, clearly indicating and stating the equations of any asymptotes and the coordinates of any axes intercepts.

$f(x) = \frac{3x+2}{x+1}$

	x	y
asymp.	$x = -1$	$y = 3$
intercept	$(-\frac{2}{3}, 0)$	$(0, 2)$



5

3. [Maximum mark: 5]

(a) Show that $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

(b) Hence find $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx$, $0 < \alpha < \frac{\pi}{2}$. [4]

(a) $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$

$= \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)}$ ✓ $\quad \left\{ \begin{array}{l} \sin\alpha = \cos(\pi/2 - \alpha) \\ \cos\alpha = \sin(\pi/2 - \alpha) \end{array} \right.$

$= \tan(\pi/2 - \alpha)$

1

(b) $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\,dx = \left[\arctan x\right]_{\tan\alpha}^{\cot\alpha}$ ✓

✓

$= \arctan\left(\tan\left(\frac{\pi}{2}-\alpha\right)\right) - \arctan(\tan\alpha)$

$= \frac{\pi}{2} - \alpha - \alpha$ ✓ 4

$= \pi/2 - 2\alpha$ ✓

5

4. [Maximum mark: 6]

The function f is defined as $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

Hayley conjectures that $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f'(x_2) + f'(x_1)}{2}$, $x_1 \neq x_2$.

Show that Hayley's conjecture is correct.

$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{ax_2^2 + bx_2 + c - ax_1^2 - bx_1 - c}{x_2 - x_1}$ ✓

~~$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1}$~~

~~$= a + b$~~

$\frac{2ax_2 + b + 2ax_1 + b}{2}$

$= ax_2 + ax_1 + b$

(1)

$\Rightarrow f'(x_2) = 2ax_2 + b$ ✓

$f'(x_1) = 2ax_1 + b$

~~$= ax_1(x_1 - x$~~ ✗

$= \frac{ax_2^2 + bx_2 + c - 2ax_2 - b - ax_1^2 - bx_1 - c - 2ax_1 - b + f'(x_2) + f'(x_1)}{x_2 - x_1}$

$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1) - 2ax_2 - 2ax_1 + f'(x_2) + f'(x_1)}{x_2 - x_1}$

$= a(x_2^2 - x_1^2 - 2x_2 - 2ax_1) + b(x_2 - x_1) + f'(x_2) + f'(x_1)$

$= \frac{a(x_2 - x_1)^2 + b(x_2 - x_1) + f'(x_2) + f'(x_1)}{x_2 - x_1}$

$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1} = \frac{a(x_2 + x_1)(x_2 - x_1) + b(x_2 - x_1)}{x_2 - x_1}$

$= \frac{ax_2 + ax_1 + b}{\cancel{2x_2 x_1}} = \text{eq.}(1)$

2

Turn over

5. [Maximum mark: 8]

A biased coin is tossed five times. The probability of obtaining a head in any one throw is p.

Let X be the number of heads obtained.

(a) Find, in terms of p, an expression for $\mathrm{P}(X = 4)$. [2]

(b) (i) Determine the value of p for which $\mathrm{P}(X = 4)$ is a maximum.

(ii) For this value of p, determine the expected number of heads. [6]

(a) $P(\text{Head}) = p$ $\quad X \sim B(5, p)$ ✓

$\therefore P(X=4) = \binom{5}{4}(p)^4(1-p)^1$

$= \frac{5!}{4!\,1!}p^4(1-p)^1$ ✓ 2

$= 5p^4 - 5p^5$

~~$= 5p(1-$~~

(b)(i) $\Rightarrow$ let $f(p) = 5p^4 - 5p^5$

$\therefore f'(p) = 20p^3 - 25p^4 = 0$ ✓✓ (Max/Min)

$\therefore 4 - 5p = 0$

$\therefore p = 4/5$ ✓✓

~~$f''(p) = 60p^2 - 100p^3 = 0$~~

~~$\therefore 100p = -60$~~ 4

checking concavity $\Rightarrow f''(4/5) = 60\left(\frac{4}{5}\right)^2 - 100\left(\frac{4}{5}\right)^3$

$= \left(\frac{4}{5}\right)^2(60 - 80) < 0$

$\therefore$ concavity is < 0 → maximum

(ii) $E(x) = np$ ✓ 2

$= 5(4/5)$

$\therefore E(X) = 4$ ✓

8

6. [Maximum mark: 8]

Consider the expansion of $(1 + x)^n$ in ascending powers of x, where $n \geq 3$.

(a) Write down the first four terms of the expansion. [2]

The coefficients of the second, third and fourth terms of the expansion are consecutive terms of an arithmetic sequence.

(b) (i) Show that $n^3 - 9n^2 + 14n = 0$.

(ii) Hence find the value of n. [6]

(a) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3$

$= 1 + nx + {}^nC_2 x^2 + {}^nC_3 x^3$ ✓✓ 2

(b)(i) $u_1 = n \quad u_2 = {}^nC_2 = \frac{n!}{2!(n-2)!} \quad u_3 = {}^nC_3 = \frac{n!}{3!(n-3)!}$

Hence, $\frac{n!}{3!(n-3)!} - \frac{n!}{2!(n-2)!} = \frac{n!}{2!(n-2)!} - n$ ✓

$\therefore \frac{2!\,n!\,(n-2)!}{3\cdot 2!(n-3)!(n-2)!} - n! = n! - 2n(n-2)!$

$\therefore$ ~~$n(n-1)(n-2)(n-3)$~~

$\therefore \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} = \frac{n(n-1)}{2} - n$ ✓

$\therefore n(n-1)(n-2) - 3n(n-1) = 3n(n-1) - 6n$

$\therefore n(n^2-2n-n+2) - 3n^2 + 3n = 3n^2 - 3n - 6n$

~~(ii) $n^3 - 9n^2 + 14n = 0$~~ AG

$\therefore n^3 - 2n^2 - n^2 + 2n - 3n^2 + 3n - 3n^2 + 3n + 6n = 0$

$\therefore n^3 - 9n^2 + 14n = 0$ ✓

(ii) $(n-3)(n^2 + an + b) = 0$

$\therefore n^3 + an^2 + bn - 3n^2 - 3an - 3b = 0$

$\therefore n^3 + (a-3)n^2 + (b-3a)n - 3b = 0$

→ $a - 3 = -9$

$\therefore a = -6$

$-3b = 0 \rightarrow b = 0.$

~~4~~ 3

$\therefore n^2 - 6n = 0$ ✗

$\therefore n(n-6) = 0$

$\therefore n = 0,\ n = 6 \rightarrow \boxed{n = 6},\ n \geq 3.$

5

~~6~~

Turn over

7. [Maximum mark: 6]

A and B are independent events such that $P(A) = P(B) = p, p \neq 0$.

(a) Show that $P(A \cup B) = 2p - p^2$. [2]

(b) Find $P(A | A \cup B)$ in simplest form. [4]

(a) $P(A) = P(B) = p$

$\therefore P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$= p + p - p^2$

$= 2p - p^2$ ✓ 2

(b) $P(A | A \cup B) = \dfrac{P(A \cap (A \cup B))}{P(A \cup B)}$ ✓

$= \dfrac{P(A)}{P(A \cup B)}$ ✓

$= \dfrac{p}{2p - p^2}$ ✓ 4

$\therefore P(A | A \cup B) = \dfrac{1}{2-p}$ ✓

6

8. [Maximum mark: 8]

Use mathematical induction to prove that $n(n^2 + 5)$ is divisible by 6 for $n \in \mathbb{Z}^+$.

$n(n^2+5) = 6A$ for $n \in \mathbb{Z}^+$

$\Rightarrow$ Step 1: prove for $n = 1$: $1(1^2+5) = 6A$

$\therefore 6 = 6A$

$\therefore A = 1$ ✓

$\therefore$ true for ~~A~~ $n = 1$

$\Rightarrow$ Step 2: assume true for $n = k$: $k(k^2+5) = 6A$, $A \in \mathbb{Z}$ ✓

$\Rightarrow$ Step 3: consider when $n = k+1$:

$(k+1)((k+1)^2+5) = 6Q$ ✓ $Q \in \mathbb{Z}$

$\therefore LHS = (k+1)(k^2+2k+6)$ ✓

$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$= k^3 + 3k^2 + 8k + 6$

$= \cancel{k^3 + k^2 + 5k + 2k^2 + 3k + 6}$

$= \cancel{k(k^2+5) + 2k^2} + 3k + 6$

$= 6A +$

$= k^3 + 5k + 3k^2 + 3k + 6$

$= \cancel{k(k^2+5) + 3k(k^2+k)}$

$= 6A + 3k(k+1) + 6$

$\Rightarrow$ $6A + 6$ is divisible by 6

$\Rightarrow$ $3k(k+1)$ is divisible by 6 as $k(k+1)$ is even.

$\therefore$ true by induction

4

9. [Maximum mark: 8]

Consider the equation $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$, $0<x<\frac{\pi}{2}$. Given that $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$

(a) verify that $x=\frac{\pi}{12}$ is a solution to the equation; [3]

(b) hence find the other solution to the equation for $0<x<\frac{\pi}{2}$. [5]

(a) $\frac{4(\sqrt{3}-1)}{\sqrt{6}-\sqrt{2}}+\frac{4(\sqrt{3}+1)}{\sqrt{6}+\sqrt{2}}=\frac{4(\sqrt{3}-1)(\sqrt{6}+\sqrt{2})}{6-2}+\frac{4(\sqrt{3}+1)(\sqrt{6}-\sqrt{2})}{6-2}$

$=(\sqrt{3}-1)(\sqrt{6}+2)+\frac{1}{2}(\sqrt{3}+1)(\sqrt{6}-2)$

$=\sqrt{18}+\sqrt{2}\sqrt{3}-\sqrt{6}-\sqrt{2}+\frac{1}{2}(\sqrt{18}-\sqrt{2}\sqrt{3}+\sqrt{6}-2)$

$\cancel{=3\sqrt{2}+2\sqrt{3}-\sqrt{6}-2+\sqrt{2}-\sqrt{3}+\frac{1}{2}\sqrt{6}-1}$

$=\sqrt{18}+\sqrt{2}\sqrt{3}-\sqrt{6}-\sqrt{2}+\sqrt{18}-\sqrt{2}\sqrt{3}+\sqrt{6}-\sqrt{2}$

$=2\sqrt{18}-2\sqrt{2}$

$=6\sqrt{2}-2\sqrt{2}$

$=4\sqrt{2}$ 3

$\therefore x=\pi/12$ is a solution

(b) $\frac{(\sqrt{3}-1)\cos x+(\sqrt{3}+1)(\sin x)}{\sin x\cos x}=4\sqrt{2}$

$\therefore \sqrt{3}\cos x-\cos x+\sqrt{3}\sin x+\sin x=4\sin x\cos x\sqrt{2}$

$\cancel{\therefore \sqrt{3}(\cos x+\sin x)-(\cos x-\sin x)=4\sin x\cos x\sqrt{2}}$

$\therefore \sqrt{3}-1+\sqrt{3}\tan x+\tan x=4\sin x\sqrt{2}$

3

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

A line L has equation $\frac{x-2}{p} = \frac{y-q}{2} = z-1$ where $p, q \in \mathbb{R}$.

A plane Π has equation $x + y + 3z = 9$.

(a) Show that L is not perpendicular to Π. [3]

(b) Given that L lies in the plane Π, find the value of p and the value of q. [4]

Consider the different case where the acute angle between L and Π is θ where $\theta = \arcsin\left(\frac{1}{\sqrt{11}}\right)$.

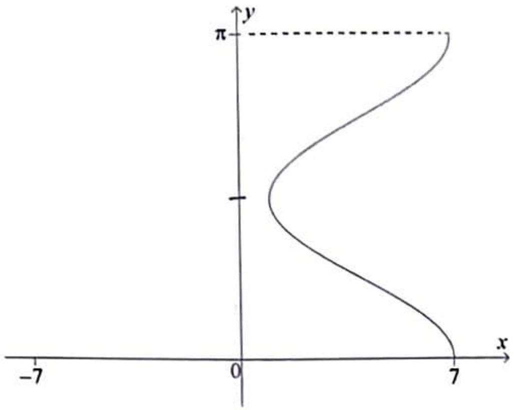
(c) (i) Show that $p = -2$.

(ii) If L intersects Π at $z = -1$, find the value of q. [11]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

The following graph shows the relation $x = 3\cos 2y + 4,\ 0 \le y \le \pi$.



The curve is rotated 360° about the y-axis to form a volume of revolution.

(a) Calculate the value of the volume generated. [8]

A container with this shape is made with a solid base of diameter 14 cm. The container is filled with water at a rate of $2\text{ cm}^3\text{ min}^{-1}$. At time t minutes, the water depth is $h\text{ cm}$, $0 \le h \le \pi$ and the volume of water in the container is $V\text{ cm}^3$.

(b) (i) Given that $\frac{dV}{dh} = \pi(3\cos 2h + 4)^2$, find an expression for $\frac{dh}{dt}$.

(ii) Find the value of $\frac{dh}{dt}$ when $h = \frac{\pi}{4}$. [4]

(c) (i) Find $\frac{d^2h}{dt^2}$.

(ii) Find the values of h for which $\frac{d^2h}{dt^2} = 0$.

(iii) By making specific reference to the shape of the container, interpret $\frac{dh}{dt}$ at the values of h found in part (c)(ii). [7]

Do **not** write solutions on this page.

12. [Maximum mark: 23]

Let $w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$.

(a) Verify that w is a root of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$. [3]

(b) (i) Expand $(w-1)\left(1 + w + w^2 + w^3 + w^4 + w^5 + w^6\right)$.

(ii) Hence deduce that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$. [3]

(c) Write down the roots of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram. [3]

Consider the quadratic equation $z^2 + bz + c = 0$ where $b, c \in \mathbb{R},\ z \in \mathbb{C}$. The roots of this equation are α and α^* where α^* is the complex conjugate of α.

(d) (i) Given that $\alpha = w + w^2 + w^4$, show that $\alpha^* = w^6 + w^5 + w^3$.

(ii) Find the value of b and the value of c. [10]

(e) Using the values for b and c obtained in part (d)(ii), find the imaginary part of α, giving your answer in surd form. [4]

Please **do not** write on this page.

Answers written on this page will not be marked.

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At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla.

Example / Ejemplo 27 [2][7]   Example / Ejemplo 3 [ ][3]

[1][0]

(a) $l: \frac{x-2}{p} = \frac{y-9}{2} = z-1$

$\Rightarrow \vec{b} = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} \Rightarrow r = \begin{pmatrix} 2 \\ 9 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix}$

$\Pi : x+y+3z = 9$

$\Rightarrow \vec{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

~~assume that $\vec{n}\cdot\vec{b} = 0$ is true~~

~~$\therefore (1)(p) + (1)(2) + (3)(1) = 0$~~

~~$\therefore p + 2 + 3 = 0$~~

~~$\therefore p = -5$~~

If $\vec{b} = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix}$ were $\perp$ to Π, then $\vec{b} \parallel \vec{n}$

However, $\vec{b}$ and $\vec{n}$ are not scalar multiples.

$\vec{b} \neq k\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, k \in \mathbb{R}$.

$\therefore$ not perpendicular to Π

(b) $r = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\q\\1\end{pmatrix} + \lambda\begin{pmatrix}p\\2\\1\end{pmatrix}$

Hence, $x=2$, $y=q$, $z=1$ must:

$2+q+3(1)=9$

$\therefore q+5 = 9$

$\therefore \boxed{q = 4}$

~~As $x=2+\lambda p$, $y=q+2\lambda$, $z=1+\lambda$~~

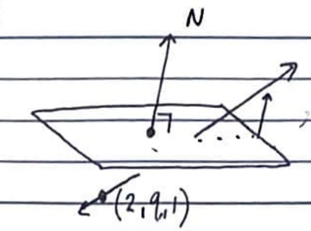
$\begin{pmatrix}p\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix} = 0$

$\therefore p+2+3 = 0$

$\boxed{\therefore p = -5}$

4

(c) (i)



$\sin\theta = \dfrac{|\vec{n}\cdot\vec{b}|}{|\vec{n}||\vec{b}|}$ $\left\{\vec{b} = \begin{pmatrix}p\\2\\1\end{pmatrix} \quad \vec{n} = \begin{pmatrix}1\\1\\3\end{pmatrix}\right\}$

$\therefore \dfrac{1}{\sqrt{11}} = \dfrac{p+2+3}{\sqrt{p^2+4+1}\sqrt{1+1+9}}$

$= \dfrac{5+p}{\sqrt{p^2+5}\sqrt{11}}$

Hence, $\dfrac{5+p}{\sqrt{p^2+5}} = 1$

$\therefore 5+p = \sqrt{p^2+5}$

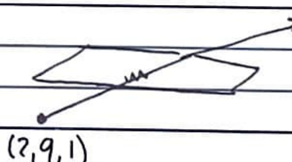
$\therefore 25+10p+p^2 = p^2+5$

$\therefore \cancel{25+10}\ 20+10p = 0$

$\therefore \boxed{p = -2}$

(ii) When $z=-1$, $x+y-3=9 \Rightarrow x+y=12$ ...(1)

when $z=-1$, $\dfrac{x-2}{p} = -2$ and $\dfrac{y-q}{2} = -2$ ...(2)



15 8

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Example / Ejemplo 27 [2 7] Example / Ejemplo 3 [3]

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(a)

$$V = \pi\int_0^\pi x^2\, dy$$

$$= \pi\int_0^\pi (3\cos 2y + 4)^2\, dy$$

$$= \pi\int_0^\pi \left(9\cos^2 2y + 16 + 24\cos 2y\right) dy.$$

$$= \pi\int_0^\pi \left(9\left(\frac{1}{2}\cos 4y + \frac{1}{2}\right) + 16 + 24\cos 2y\right) dy$$

$$= \pi\int_0^\pi \left(\frac{9}{2}\cos 4y + 24\cos 2y + \frac{41}{2}\right) dy$$

$$= \pi\left[\frac{9}{2}\sin 4y\left(\frac{1}{4}\right) + 24\sin 2y\left(\frac{1}{2}\right) + \frac{41}{2}y\right]_0^\pi$$

$$= \pi\left[\frac{9}{8}\sin 4y + 12\sin 2y + \frac{41}{2}y\right]_0^\pi$$

$$= \pi\left(\frac{9}{8}\sin 4\pi + 12\sin 2\pi + \frac{41}{2}\pi\right) - \left(\frac{9}{8}\sin 0 + 12\sin 0 + 0\right)$$

$$= \frac{41\pi^2}{2} \text{ units}^3$$

8

(b)(i) $\frac{dV}{dh} = \pi(3\cos 2h+4)^2$

$\frac{dV}{dt} = 2$

$dh/dt = \frac{dh}{dV} \times \frac{dV}{dt}$ $\cancel{\frac{dV}{dh} = \frac{dV}{dt} \times \frac{dt}{dh}}$

$= \frac{dV}{dt} \div \frac{dV}{dh}$

$= \frac{2}{\pi(3\cos 2h+4)^2}$ 2

(ii) When $h = \pi/4$ cm,

$\frac{dh}{dt} = \frac{2}{\pi(3\cos(\pi/2)+4)^2}$

$= \frac{2}{\pi(4)^2}$

$= \frac{2}{16\pi}$ 2

$= 1/(8\pi)$ cm s^{-1}

(C)(i) $\frac{d^2h}{dt^2} = \frac{d}{dh}\left((3\cos 2h+4)^{-2}\right) \times \frac{2}{\pi}$ M1

$= (-2)(3\cos 2h+4)^{-3}(-3\sin 2h)(2)$

$= \frac{(-2)(-3)(2)(\sin 2h)}{(3\cos 2h+4)^3}$

$= \frac{2}{\pi}\frac{12\sin 2h}{(3\cos 2h+4)^3}$

1

(ii) $d^2h/dt^2 = 0$

$\therefore 12\sin 2h = 0$

$\therefore 2h = 0, \pi, \ldots$

$\therefore h = 0, \pi/2$ $\{0 \le y \le \pi\}$

1

(iii) at $h=0$, $dh/dt = \frac{2}{\pi(3\cos 0+4)^2}$

$= \frac{2}{\pi(7)^2}$

$= \frac{2}{49\pi}$ cm s^{-1}

at $h = \pi/2$, $dh/dt = \frac{2}{\pi(3\cos\pi+4)^2}$ 2

$= \frac{2}{\pi(-3+4)^2}$

$= 2/\pi$ cm s^{-1}

As $2/\pi > 2/49\pi$, the ~~wate~~ height increases faster at $h = \pi/2$. This is because the container is much thinner at $h=\pi/2$, meaning at a constant volume increase rate of 2 $cm^3 min^{-1}$ the height must increase drastically faster here.

16

12

(a)

$w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$= \text{cis}^{2\pi/7}$

$z^7 = 1$

$= 1 \cdot \text{cis}(0 + 2k\pi)$

$\therefore z = \text{cis}(^{2k\pi}/_7)$ {De M's}, $k \in \{0, 1, 2, 3, 4, 5, 6\}$

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Hence, w is a root of z when $k = 1$

(b) (i) $w + \cancel{w^2} + \cancel{w^3} + \cancel{w^4} + \cancel{w^5} + \cancel{w^6} + w^7 - 1 - \cancel{w} - \cancel{w^2} + \cancel{w^3} - \cancel{w^4}$
$- \cancel{w^5} - \cancel{w^6}$

$= w^7 - 1$

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(ii) $(w-1)(1 + w + w^2 + w^3 + w^4 + w^5 + w^6) = w^7 - 1$

Hence, $(w-1)(1 + w + w^2 + w^3 + w^4 + w^5 + w^6) = 0$

$\therefore \quad 1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

{dividing by $(w-1)$ as $w \neq 1$}

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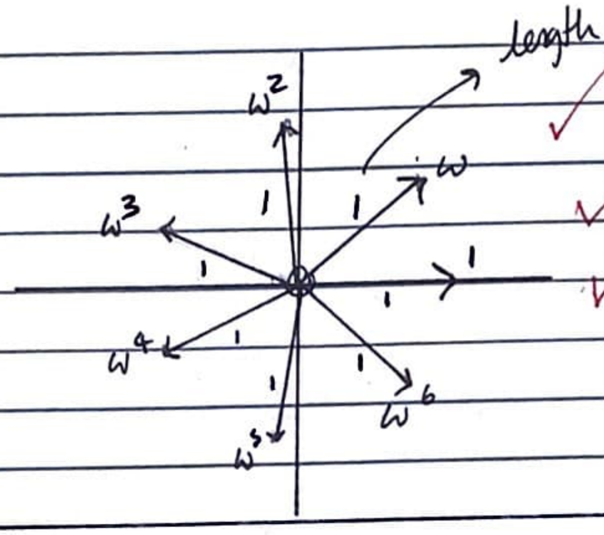
Example Ejemplo 27 | 27 | Example Ejemplo 3 | 3

(c) $z_1 = 1$
$z_2 = \omega$
$z_3 = \omega^2$
$z_4 = \omega^3$
$z_5 = \omega^4$
$z_6 = \omega^5$
$z_7 = \omega^6$



3

~~$z_2 = \omega = \cos 2\pi/7 + i\sin 2\pi/7$~~
~~$z_3 = \omega^2 = \cos(4\pi/7) + i\sin 4\pi/7$~~
~~$z_4 = \omega^3 = \cos(6\pi/7) + i\sin(6\pi/7) \rightarrow 6\pi/7 \quad \pi - 6\pi/7 = (7\pi - 6\pi)/7 = \pi/7$~~

04AX01

(d)(i) $z^2 + bz + c = 0$

$$\alpha = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)$$

$$\alpha^* = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} - i\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)$$

~~As $\cos 2\pi/7 = -\cos 5\pi/7$~~

Geometrically using part (c)

roots w^6, w^5, w^3 are simply the same as w, w^2, w^4 respectively but reflected over the y-axis hence, the "sin" or imaginary component must have a flipped sign.

$$\therefore \alpha^* = \left(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}\right) + \left(\cos\frac{8\pi}{7} - i\sin\frac{8\pi}{7}\right)$$

$$\therefore \alpha^* = w^6 + w^5 + w^3$$

~~3~~ 2

(ii)

$$(z - \alpha)(z - \alpha^*) = z^2 + bz + c$$

~~$-b = \text{SUM} = \alpha + \alpha^*$~~
~~$= Re(\alpha) + Re(\alpha^*)$ {imag. cancel}~~
~~$= 2Re(\alpha)$~~
~~$\therefore b = -2Re(\alpha)$~~
~~$= -2(\cos 2\pi/7 + \cos 4\pi/7 + \cos 8\pi/7)$~~
~~$= 0$ {x-components cancel out}~~

$c = \text{PRODUCT} = \alpha\alpha^*$

~~$= (w + w^2 + w^4)(w^6 + $~~

~~$= (Re(\alpha) + i\,Im(\alpha))(Re(\alpha) - i\,Im(\alpha))$~~

~~$= Re(\alpha)^2 + Im(\alpha)^2$~~

~~$= 0^2 +$~~

$= (w + w^2 + w^4)(w^6 + w^5 + w^3)$

$= w^7 + w^6 + w^4 + w^8 + w^7 + w^5 + w^{10} + w^9 + w^7$

$= 3w^7 + w^6 + w^4 + w^8 + w^5 + w^9 + w^{10}$

$= w^4 + w^5 + w^6 + 3w^7 + w^8 + w^9 + w^{10}$

$= 3 + w^1 + w^2 + w^3 + w^4 + w^5 + w^6$

$= 2 + 1 + w^1 + w^2 + w^3 + w^4 + w^5 + w^6$

$\therefore \boxed{c = 2}$

$-b = \text{SUM} = \alpha + \alpha^*$

$= w + w^2 + w^3 + w^4 + w^5 + w^6$

$= 0 - 1$

$= -1$

$\boxed{\therefore b = 1}$

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(e) $b=1 \quad c=2$.

$\therefore z^2+z+2=0$

$\therefore$

$\therefore \quad z = \dfrac{-1 \pm \sqrt{1-8}}{2}$

$= \dfrac{-1 \pm \sqrt{7}\,i}{2}$

$= -\frac{1}{2} + \frac{\sqrt{7}}{2}i \;,\; -\frac{1}{2} - \frac{\sqrt{7}}{2}$

$\mathrm{Im}(\alpha) = \sin\frac{12\pi}{7} + \sin\frac{10\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$

3 must always reason why removing one of the roots.

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