Mathematics: analysis and approaches

Higher level

Additional Practice

Counterexample & Contradiction (Non-Calculator)

ID: 4002

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[125 marks]**.

1. [Maximum points: 8]

Let $f(x) = x^2$ and $g(x)$ be any function. Prove or disprove the following.

(a) All solutions of $g(x) = 2$ are also solutions of $(f \circ g)(x) = 4$. [3]

(b) All solutions of $(f \circ g)(x) = 4$ are also solutions of $g(x) = 2$. [5]

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2. [Maximum points: 4]

Consider events A and B where $P(A) > 0$ and $P(B) > 0$. Prove that the events cannot be both mutually exclusive and independent at the same time.

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3. [Maximum points: 6]

Prove that $\log_4 10$ is irrational.

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4. [Maximum points: 6]

Prove by contradiction that $2^{1/3}$ is irrational.

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5. [Maximum points: 6]

(a) Prove by contradiction that $\frac{x^2+1}{x} \geq 2$ for $x > 0$. [4]

(b) Hence determine the largest value of a such that $\frac{(x^2+1)(y^2+1)}{xy} \geq a$ for $x, y > 0$. [2]

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6. [Maximum points: 8]

Consider the series $S_n = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$ where $n \in \mathbb{Z}^+$.

(a) Write down an expression for the value of S_n in terms of x and n. [1]

(b) Hence find $f(x)$ if $x^n - 1 = f(x)S_n$. [1]

(c) Show that [3]

$$2^{ab} - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{a(b-2)} + 2^{a(b-1)})$$

where $a, b \in \mathbb{Z}^+$.

(d) Prove by contradiction that if $2^n - 1$ is prime then n must be prime. [3]

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7. [Maximum points: 8]

Let $N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$.

Assume that the value of N is an integer.

(a) Show that $-\frac{1}{2} = 4 + 2 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{4}{9} - 4N$ [2]

(b) When the right side of the equation in part (a) is written as a single fraction in the lowest terms determine whether the denominator is even or odd. [2]

(c) Hence prove by contradiction that N is not an integer. [2]

Let $M = \sum_{n=1}^{20} \frac{1}{n}$.

(d) Explain how we can modify the step of the proof in part (a) to show that M is not an integer. All other parts of the proof are identical. [2]

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8. [Maximum points: 9]

Let $f(x)$ be an odd function and $g(x)$ be an even function.

Prove or disprove the following

(a) $(f(x))^2$ is always even [3]

(b) $f(x) \cdot g(x)$ is always odd [2]

(c) $f(x) + g(x)$ is always odd [4]

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9. [Maximum points: 12]

The table below shows the values of x^n for various values of x and n.

n	3^n	5^n	7^n	9^n
1	3	5	7	9
2	9	25	c	81
3	27	125	343	d
4	81	625	2401	6561
5	a	b	16807	59049

(a) Write down the values of a, b, c and d. [2]

(b) Find the final digit of 7^{165}. [3]

(c) Prove by contradiction that $\sqrt{5}$ is irrational. [7]

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10. [Maximum points: 12]

(a) Prove that if a prime number p divides x^2, where $x \in \mathbb{Z}$, then p must also divide x. [4]

(b) Prove by contradiction that $\sqrt{11}$ is irrational. [8]

10. [Maximum points: 12]

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11. [Maximum points: 12]

(a) Prove that if a prime number p divides x^2, where $x \in \mathbb{Z}$, then p must also divide x. [4]

(b) Prove by contradiction that $\sqrt{7}$ is irrational. [8]

11. [Maximum points: 12]

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12. [Maximum points: 12]

Any positive integer greater than 1 is either prime, or can be written as the product of prime numbers. For example $18 = 2 \times 3 \times 3$ and $60 = 2 \times 2 \times 3 \times 5$.

(a) Write 120 as the product of prime numbers. [1]

Assume there are a finite number of prime numbers. Let these numbers be represented by $p_1, p_2, \ldots, p_n$ where $p_1 < p_2 < \ldots < p_n$.

(b) Write down the values of p_1, p_2 and p_3. [1]

(c) Determine whether the following are prime or composite [3]

(i) $p_1 + 1$

(ii) $p_1 p_2 + 1$

(iii) $p_1 p_2 p_3 + 1$

Consider the integer $P = p_1 p_2 \cdots p_n + 1$.

(d) Explain why none of the prime numbers $p_1, p_2, \ldots, p_n$ divide P. [3]

(e) Hence prove that there are an infinite amount of prime numbers. [2]

(f) Prove or disprove the following statement: [2]

If p and q are distinct primes then $pq + 1$ is also prime.

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13. [Maximum points: 22]

Let $f(x) = \sin x \tan x$.

(a) Find $f'(x)$. [3]

(b) Show that $f''(x) = \dfrac{\cos^4 x - \cos^2 x + 2}{\cos^3 x}$. [5]

Consider the graph of $y = f(x)$ on the interval $[-3\pi/2, 3\pi/2]$.

(c) Find the equations of the vertical asymptotes. [3]

(d) Find the coordinates of any turning points, justifying whether they are maximum or minimum points. [5]

(e) Show that the graph has no points of inflection. [3]

(f) Sketch the graph of $y = f(x)$. [3]

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