Mathematics: analysis and approaches
Higher level
Paper 2
TZ1

Friday 7 May 2021 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

$\frac{102}{110} = 92.7\%$

20/10/22.

2221–7107

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

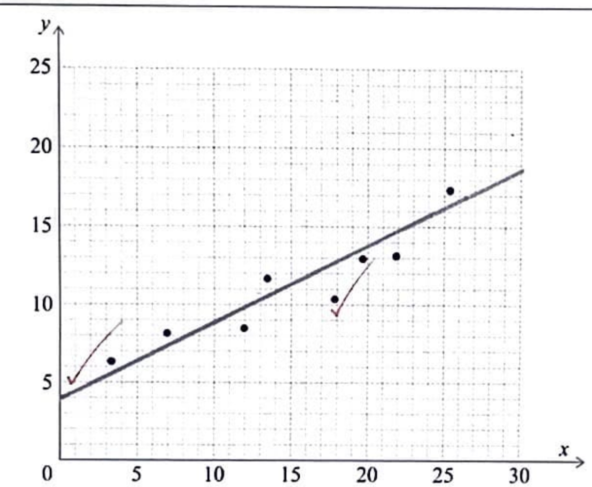
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

The following table shows the data collected from an experiment.

x	3.3	6.9	11.9	13.4	17.8	19.6	21.8	25.3
y	6.3	8.1	8.4	11.6	10.3	12.9	13.1	17.3

The data is also represented on the following scatter diagram.



(This question continues on the following page)

(Question 1 continued)

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$, where a, $b \in \mathbb{R}$.

(a) Write down the value of a and the value of b. [2]

(b) Use this model to predict the value of y when $x = 18$. [2]

(c) Write down the value of $\bar{x}$ and the value of $\bar{y}$. [1]

(d) Draw the line of best fit on the scatter diagram. [2]

(a) $y = 0.433157x + 4.50265$
$\therefore a = 0.433$ $b = 4.50$ 2

(b) $y = 0.433157(18) + 4.50265$
$= 12.2995$
$\therefore y \approx 12.3$ 2

(c) $\bar{x} = 15$ $\bar{y} = 11$ 1

(d) See diagram. 2

7

2. [Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of 100 bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. [3]

(a) $X \sim N(1000, 3.5^2)$ ✓

$P(X \le 995) = \text{normcdf}(-9E999, 995, 1000, 3.5)$

$= 0.076564$

$\therefore \underline{P(X \le 995) \approx 0.0766}$ ✓ 2

(b) $Y \sim B(100, 0.076564)$

$\therefore E(Y) = (100)(0.076564)$

$= 7.6564$

$\therefore \underline{E(Y) \approx 8 \text{ bags.}}$ ✓ 1

(c) P(not rejected) $= 1 - 0.076564 = 0.923436$ ✓

$P(X \ge 1005) = \cancel{0.0766}\ 0.076564$

$$\therefore P(X \ge 1005 \mid \text{not rejected}) = \frac{P(X \ge 1005)}{P(\text{not rejected})} ✓$$

$$= \frac{0.076564}{0.923436}$$

$$= 0.082912$$

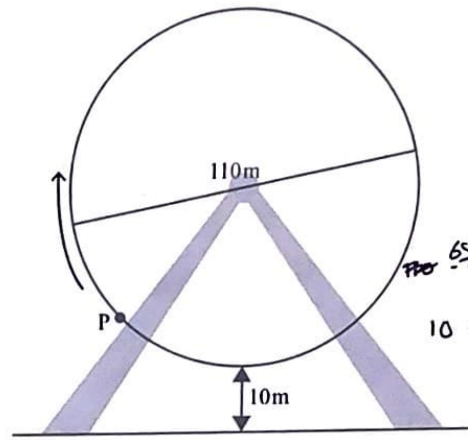
$$\approx \underline{0.0829}$$ ✓ 3

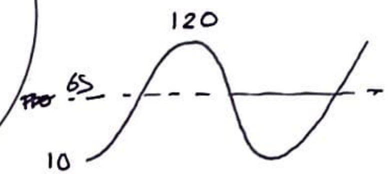
6

3. [Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale





The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where $a, b, c \in \mathbb{R}$.

Find the values of a, b and c.

~~$b = 2\pi/T = 2\pi/20 = \pi/10 \rightarrow \underline{b = \pi/10}$~~

~~a = amplitude = 100 → $\underline{a = -100}$ due to start pos~~

~~c = vertical offset = 110 → $\underline{c = 110}$~~

~~$\therefore h(t) = 100\cos(\frac{\pi}{10}x) + 110$~~

$b = 2\pi/T$ ✓ M1

$\therefore \underline{b = 2\pi/20 = \pi/10}$ ✓ A1

$\underline{a = -55}$ {amplitude} ✓ A1 M1

$\underline{c = 65}$ ✓ A1

$h(t) = -55\cos(\frac{\pi}{10}x) + 65$

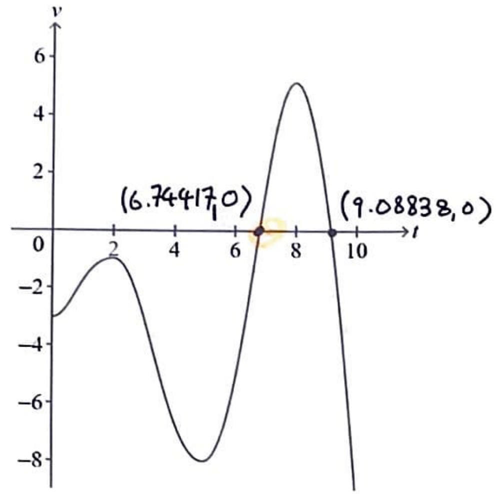
5

Turn over

4. [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\text{ ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.



(a) Find the smallest value of t for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

(a) zero's as indicated : $t = 6.74417$ s ✓M1

$\therefore t \approx 6.74$ s ✓A1 2

(b) $s = \int_0^{10} |v(t)|\,dt$ ✓A1

$= 37.0969$ m

$\therefore s \approx 37.1$ m ✓A1 2

(c) $a(t) = t\cos t + \sin t$ ✓M1

$\therefore a(7) = 7\cos 7 + \sin 7$

$= 5.93430$ 2

$\therefore a(7) \approx 5.93\ \text{ms}^{-2}$ ✓A1

6

5. [Maximum mark: 5]

Consider the expansion of $(3 + x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is 20412, find the value of n.

$(3+x^2)(3+x^2)^n$

$T_{r+1} = \binom{n}{r}(3^{n-r})(x^2)^r$

$= \binom{n}{r}(3^{n-r})x^{2r}$ M1 A1

$x^4 \to r=2$ $\quad x^2 \to r=1$ ✓✓

$\therefore$ coefficients of x^4 is:

$(3)\binom{n}{2}(3^{n-2})x^{2(2)} + (x^2)\binom{n}{1}(3^{n-1})x^2$

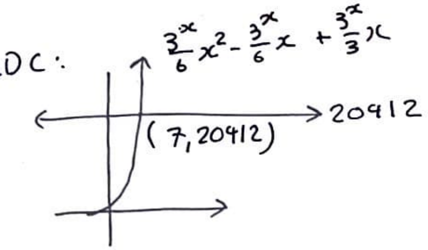
$= x^4\left(\frac{3\cdot 3^n \cdot n!}{3^2\cdot 2!\cdot(n-2)!} + \frac{3^n n!}{3\cdot 1!(n-1)!}\right)$

$= x^4\left(\frac{3^n(n)(n-1)(n-2)!}{3\cdot 2(n-2)!} + \frac{3^n}{3}n\right)$

$\therefore 20412 = \frac{3^n}{6}n(n-1) + \frac{3^n}{3}n$ ✓M1

$\therefore 20412 = \frac{3^n}{6}n^2 - \frac{3^n}{6}n + \frac{3^n}{3}n$

GDC: $\frac{3^x}{6}x^2 - \frac{3^x}{6}x + \frac{3^x}{3}x$



$\therefore n = 7$ ✓A1

5

6. [Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1 : 3x + 2y + z = 6$

$\Pi_2 : x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

(a) $n_1 = \begin{pmatrix}3\\2\\1\end{pmatrix}$ $\quad n_2 = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

$\therefore n_3 = \begin{pmatrix}3\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}2-(-2)\\1-3\\-6-2\end{pmatrix}$ ✓M1

$= \begin{pmatrix}4\\-2\\-8\end{pmatrix}$ ✓A1

$\therefore 4x - 2y - 8z = d$

$\therefore 2x - y - 4z = 0$ ✓A1 $\{d = 0$ due to origin$\}$

3

(b) GDC:

$\text{linSolve}\left(\begin{cases}3x+2y+z=6\\ x-2y+z=4\\ 2x-y-4z=0\end{cases}, \{x,y,z\}\right)$ ✓M1

$\left\{\frac{41}{21}, \frac{-10}{21}, \frac{23}{21}\right\}$

$\therefore$ Intersection coordinate $P\left(\frac{41}{21}, -\frac{10}{21}, \frac{23}{21}\right)$ ✓A1

2

5

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{(x^2+k)^3}} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}^+$.

(a) Show that $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of k. [2]

(a) $\int_0^4 \frac{x}{(\sqrt{(x^2+k)^3})} dx = \frac{1}{2}\int_0^4 \frac{du}{u^{3/2}}$ ✓M1 $\quad$ let $u = x^2+k$ $\therefore du = 2x\,dx$ A1

$= \frac{1}{2}\int_0^4 u^{-3/2} du$ → need to change limits or ✗ do definite integral later.

$= \frac{1}{2}\left[u^{-1/2}\right]_0^4 (-2)$

$= -\left[u^{-1/2}\right]_0^4$

$= -\left[(x^2+k)^{-1/2}\right]_0^4$ ✓A1

$\therefore \left[\left(\frac{1}{x^2+k}\right)^{1/2}\right]_0^4 = -1$

$\therefore \frac{1}{\sqrt{16+k}} - \frac{1}{\sqrt{k}} = -1$

$\therefore \sqrt{k} - \sqrt{16+k} = -\sqrt{k}\sqrt{16+k}$ ✓A1

$\therefore \sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$

5

(b) $\cancel{(\sqrt{16+k} - \sqrt{k})^2 = k(16+k)}$

$\cancel{(16+k) - 2\sqrt{k}\sqrt{16+k} + k^2 = 16k + k^2}$

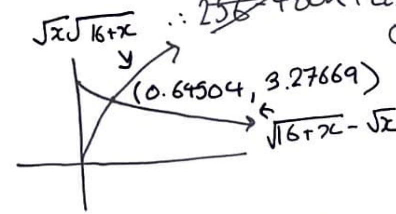
$\cancel{\therefore k - 16k + 16 = 2\sqrt{k}\sqrt{16+k}}$

$\cancel{\therefore (16 - 15k)^2 = 4k(16+k)}$

$\cancel{\therefore 256 - 480k + 225k = 64k + 4k^2}$

$\cancel{0 = 4k^2 + 319k - 256}$

GDC:



✓M1

$\therefore k \approx 0.64504$

$\therefore k \approx 0.645$ ✓A1 2

7

8. [Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of zw. [1]

(b) Find the argument of zw in terms of k. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of k.

(ii) For the value of k found in part (i), find the value of zw. [4]

(a) $|zw| = |z||w|$
$= (2)(8)$
$\therefore |zw| = 16$ ✓ A1 1

(b) $\arg(zw) = \arg(z) + \arg(w)$
$= \pi/5 - 2\pi k/5$ ✓M1 negative
$= \frac{\pi(1-2k)}{5}$ ✓A1 2

(c)(i) $zw = 16\,\mathrm{cis}\left(\frac{\pi-2k\pi}{5}\right)$
$\therefore \sin\left(\frac{\pi-2k\pi}{5}\right) = 0$
$\therefore \frac{\pi-2k\pi}{5} = 0+\pi n$ ✓M1
$\therefore \pi-2k\pi = 5\pi n \longrightarrow 2k\pi = \pi-5\pi n$ ✓M1
$\therefore \cancel{2k\pi = \pi-10\pi n}$ $\therefore 2k = 1-5n$
$\therefore \cancel{k = \frac{\pi-10\pi n}{2\pi}}$, ~~$n \in \{0,1,2,\dots\}$~~ $\rightarrow \therefore k = +3$ when $n=1$ ✓A1
$\therefore \cancel{k = 1/2, -9\pi/2\pi, -19\pi/2\pi},$ 3

(c)(ii) $zw = \cancel{(16)\left(\cos\left(\frac{\pi-\pi}{5}\right)+i\sin\left(\frac{\pi-\pi}{5}\right)\right)}$
$= 16\left(\cos(-5\pi/5) + i\sin(-5\pi/5)\right)$ 1
$\therefore zw = 16(-1+0i) = -16$ ✓A1

7

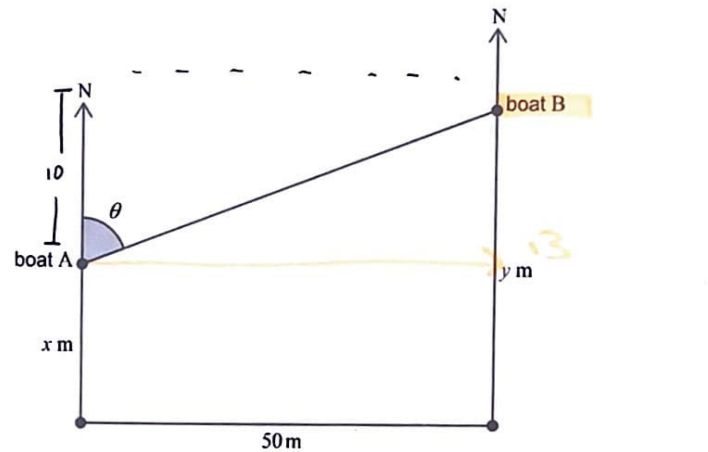
9. [Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

(a) $\cot\theta = (y-x)/50$ ✓A1
$\therefore 50\cot\theta = y-x$
$\therefore y = x+50\cot\theta$ 1

(b) $y-x = 10$ m
$d\theta/dt = -0.1$ rad s^{-1}
$dy/dt = 2dx/dt$

$\frac{dy}{dt} = \frac{dx}{dt} - 50\,\mathrm{cosec}^2\theta\, d\theta/dt$ M1 A1
$2dx/dt = dx/dt - 50\,\mathrm{cosec}^2\theta\, d\theta/dt$ A1
$dx/dt = -50(-0.1)\,\mathrm{cosec}^2\theta$ ✓A1
$\therefore dx/dt = 5\,\mathrm{cosec}^2\theta$
$\Rightarrow$ as $\cot\theta = 10/50$, $\theta = \cot^{-1}(1/5)$ ✓A1
$\Rightarrow dx/dt = 5\,\mathrm{cosec}^2(\cot^{-1}(1/5)) = 5.2$ ms^{-1} A1 6

7

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90\mathrm{e}^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

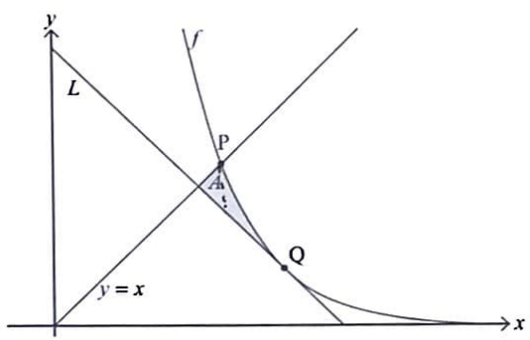
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

(ii) Hence, find the area of A. [5]

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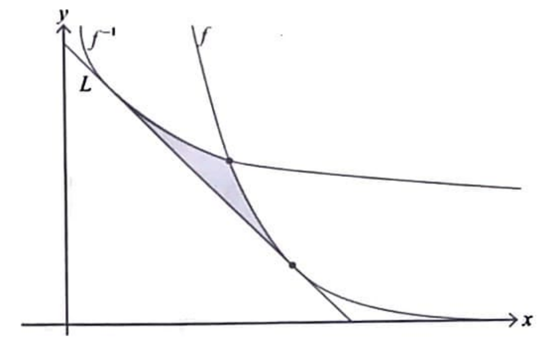
(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

11. [Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Do **not** write solutions on this page.

12. [Maximum mark: 20]

The function f has a derivative given by $f'(x) = \dfrac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\dfrac{a}{x} + \dfrac{b}{k-x}$, where a, $b \in \mathbb{R}$. Find a and b in terms of k. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P = \dfrac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k, giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

$$\frac{1200ke^{\frac{1}{5}t}}{(k-1200)+1200e^{\frac{1}{5}t}}$$

References:

Please **do not** write on this page.

Answers written on this page will not be marked.

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1

$\frac{102}{110} = 92.7\%$

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4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno:

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Candidate name: / Nom du candidat: / Nombre del alumno:

At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla.

Example / Ejemplo 27 | 2 | 7 |

Example / Ejemplo 3 | | 3 |

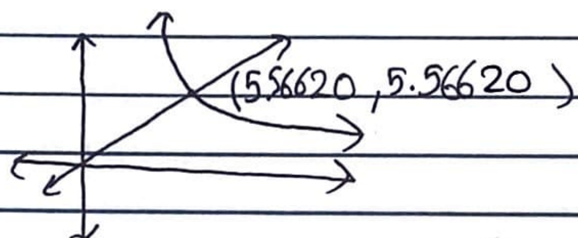
10

(a) $\cancel{f(x) = 90e^{-0.5x} = x}$

$\cancel{\therefore \ln 90 + \ln e^{-0.5x} = \ln x}$

$\cancel{\therefore -0.5x = \ln x - \ln 90}$

$\cancel{\therefore \ln x = -\frac{1}{2}x + \ln 90}$



$\therefore x = 5.56620$ ✓M1 2

$\therefore x \approx 5.57$ ✓A1 {x-coord of P}.

(b) $f'(x) = (-0.5)(90)e^{-0.5x}$

$= -45e^{-0.5x}$ ✓A1

$\therefore -45e^{-0.5x} = -1$ ✓M1

$\therefore e^{-0.5x} = 1/45$

$\therefore -0.5x = \ln(1) - \ln(45)$

$\therefore x = 2\ln 45$

$f(2\ln 45) = 90e^{-\frac{1}{2}2\ln 45}$

$= 90e^{-\ln 45}$

$= 90/45 = 2$

Q(2ln45, 2) ✓A1 ✓A1

4

(c) $y - 2 = (-1)(x - 2\ln 45)$ ✓ M1

$\therefore \; y = -x + 2\ln 45 + 2$ ✓ A1

2

(d)(i) $x = -x + 2\ln 45 + 2$

$\therefore 2x = 2\ln 45 + 2$

$\therefore \; x = \ln 45 + 1$

$\therefore \; x = \ln 45 + \ln e$

$\therefore \; x = \ln 45e$

$\therefore \; x \approx 4.80666$

$\therefore x \approx 4.81$ ✓ A1 1

(d)(ii) ✓ M1 $A_1 = \int_{\ln 45e}^{5.56620} \left(x - (-x + 2\ln 45 + 2)\right) dx$

$= 2\int_{\ln 45e}^{5.56620} (x \mp \ln 45 \mp 1)\, dx$ ✗ A0

$= 15.1803$ units^2

$A_2 = \int_{5.56620}^{2\ln 45} \left(90e^{-\frac{1}{2}x} - (-x + 2\ln 45 + 2)\right) dx$

$= \int_{5.56620}^{2\ln 45} \left(90e^{-\frac{1}{2}x} + x - 2\ln 45 - 2\right) dx$ ✓ A1

$= 0.942757$

3

$\therefore A = 15.1803 + 0.942757$

$= 16.123$ units^2

$\therefore A \approx 16.1$ units^2

$A = 1.52$ units^2

(e)

By symmetry, $A = 2 \times 1.52$ units^2

$= 3.04$ units^2

12

(a) $f(x) = \dfrac{3x + 2}{4x^2 - 1}$

✓ M1

denominator $= (2x+1)(2x-1)$

$\therefore x \neq -1/2$ and $x \neq 1/2$

$\therefore p = -1/2 \cdot \quad q = 1/2$ ✓ A1

2

(b)

$$f'(x) = \frac{(4x^2-1)(3) - (3x+2)(8x)}{(4x^2-1)^2}$$ ✓M1

$$= \frac{3(2x+1)(2x-1) - 24x^2 - 16x}{(2x+1)^2(2x-1)^2}$$

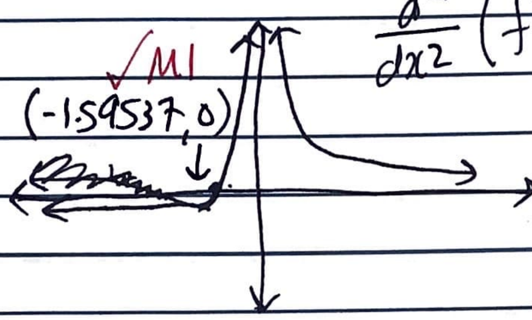
$$= \frac{12x^2 - 3 - 24x^2 - 16x}{(4x^2-1)^2}$$ ✓A1

$$= \frac{-12x^2 - 16x - 3}{(4x^2-1)^2}$$ ✓A1

3

(c) ✓M1

$\frac{d^2}{dx^2}(f(x))$

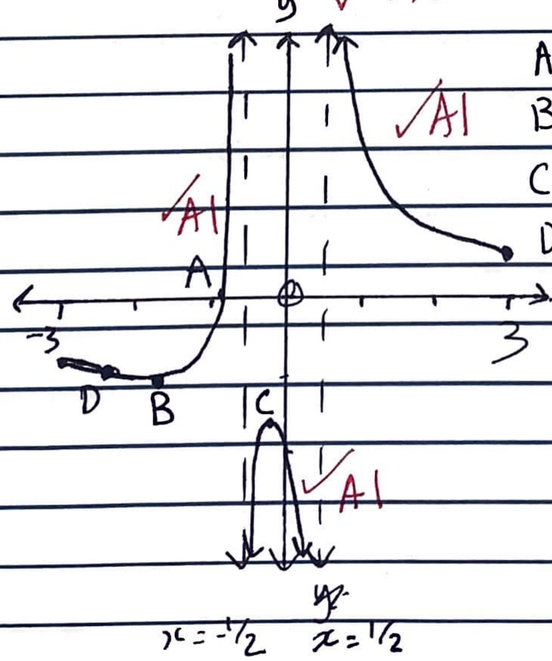


$\therefore x = -1.59537$

$\therefore x \approx -1.60$ ✓A1

2

(d) ✓A1



A: zero @ $(-\frac{2}{3}, 0)$

B: min @ $(-1.10763, -0.33856)$

C: max @ $(-0.22571, -1.66144)$

D: poi @ $(-1.59537, -0.30347)$

✓A1 ✓A1 ✓A1

$y = 0 \times A0$

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ANSWER BOOKLET ②
LIVRET DE RÉPONSES
CUADERNILLO DE RESPUESTAS

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4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno: 21 M TZ 1 P2 - M A H L

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At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla.

Example Ejemplo 27 | 2 7 | Example Ejemplo 3 | 3 |

| 1 1 |

(e)

$g(x) = 1/f(x)$

vertical asymptote: $x = -2/3$ ✓A1

oblique asymptote: ✓A1

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ (3x+2)\overline{)\,4x^2 + 0x - 1} \\ (4x^2 + \frac{8}{3}x) - \\ -\frac{8}{3}x - 1 \\ (-\frac{8}{3}x - \frac{16}{9}) - \\ \hline \frac{25}{9}. \end{array}$$ ✓M1

$\therefore\ y = \frac{4}{3}x - \frac{8}{9}$ ✓ A1

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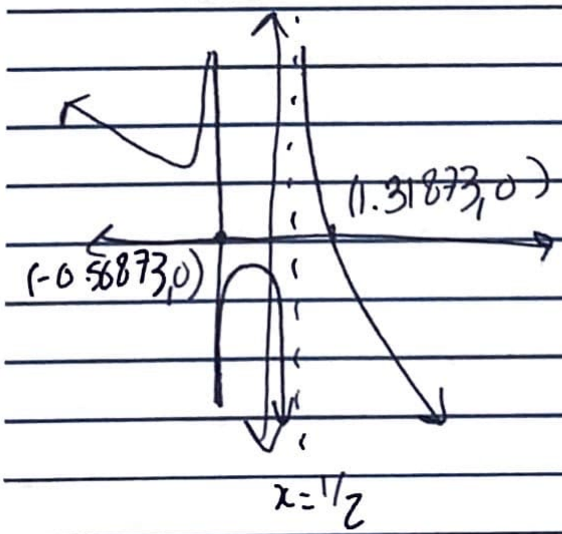
(f) $f(x) < g(x)$ when $g(x) - f(x) > 0$



$$\boxed{\begin{array}{l} \therefore x < -0.56873 \\ \frac{1}{2} < x < 1.31873. \end{array}}$$ ✗

0

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04AX01

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(a)

$$f'(x) = \frac{1}{x(k-x)}$$

$$\Rightarrow \frac{1}{x(k-x)} = \frac{a}{x} + \frac{b}{k-x} \quad \checkmark M1$$

$$\therefore \quad 1 = a(k-x) + bx \quad \checkmark A1$$

when $x = 0$, $1 = ka$

$\therefore a = 1/k$

when $x = k$, $bk = 1$

$\therefore b = 1/k$ $\checkmark A1$

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$$\therefore f'(x) = \frac{1}{kx} + \frac{1}{k^2 - kx}$$

(b) $f(x) = \int \frac{1}{kx} dx + \frac{1}{k} \int \frac{1}{k-x} dx$ $\checkmark M1$

$$= \cancel{\frac{1}{k}\ln|x| + \frac{1}{k}\ln|k-x|(-1) + c}$$

$$= \cancel{\frac{1}{k}\ln|xk - x^2| + c}$$

$$= \frac{1}{k} \ln \left| \frac{x}{k-x} \right| + c \quad \checkmark\checkmark A1 A1$$

~~(c)~~

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(c) $\frac{dP}{dt} = \frac{P(k-P)}{5k}$

$$\therefore \left(\frac{1}{P(k-P)}\right)\frac{dP}{dt} = \frac{1}{5k} \quad \checkmark M1$$

$$\therefore \frac{1}{k}\left(\frac{1}{P} + \frac{1}{k-P}\right)\frac{dP}{dt} = \frac{1}{5k}$$

$$\therefore \frac{1}{k}\int\left(\frac{1}{P} + \frac{1}{k-P}\right)dP = \int \frac{1}{5k} dt$$

$$\cancel{\therefore \ln(Pk - P^2) = \frac{1}{5}t + c}$$

$$\cancel{\therefore Pk - P^2 = Ae^{\frac{1}{5}t}, \quad A = \pm e^c}$$

$$\therefore \ln\left(\frac{P}{k-P}\right) = \frac{1}{5}t + c \quad \checkmark A1$$

$$\therefore \frac{P}{k-P} = Ae^{\frac{1}{5}t} \quad \checkmark A1 \quad A = \pm e^c$$

when $P = 1200$, $t = 0$. $\checkmark M1$

$$\therefore \frac{1200}{k-1200} = Ae^0$$

$$\therefore A = 1200/k-1200 \quad \checkmark A1$$

$$\therefore P = kAe^{\frac{1}{5}t} - PAe^{1/5 t} \quad \checkmark M1$$

$$P(1 + Ae^{1/5 t}) = kAe^{1/5 t} \quad \checkmark A1$$

$$\therefore P = \frac{kAe^{1/5 t}}{1 + Ae^{1/5 t}} \quad \checkmark A1$$

$$= \frac{k\left(\frac{1200}{k-1200}\right)e^{\frac{1}{5}t}}{1 + \left(\frac{1200}{k-1200}\right)e^{1/5 t}}$$

$$= \frac{1200ke^{1/5 t}}{k - 1200 + 1200e^{1/5 t}} \quad \checkmark A1$$

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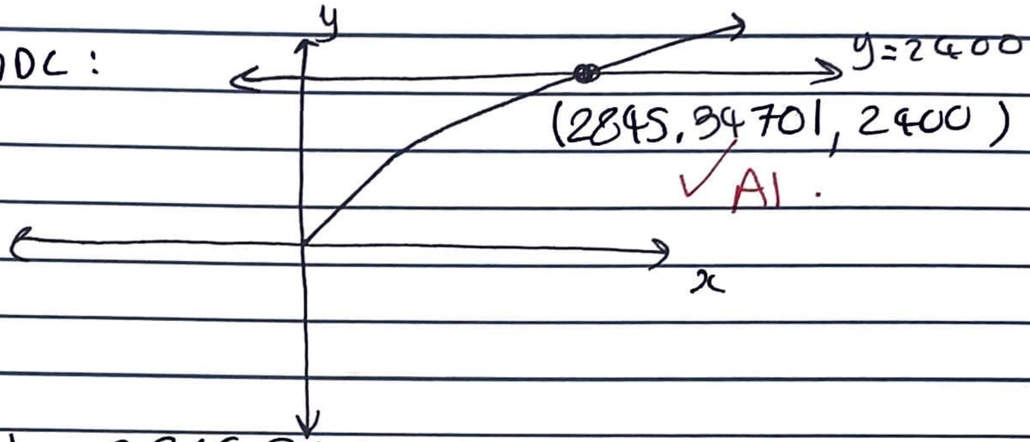
$\therefore P = \dfrac{1200k}{(k-1200)e^{-1/3 t} + 1200}$ AG

(d) $2400 = \dfrac{1200k}{(k-1200)e^{-2} + 1200}$ ✓M1 $\quad y = \dfrac{1200x}{(x-1200)e^{-10/5} + 1200}$

GDC:



✓A1

$\therefore k = 2845.34701$

$\therefore k \approx 2845$ ✓A1

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(e) $P = \dfrac{1200(2845)}{(2845-1200)e^{-t/5} + 1200} = 0$

$\therefore \dfrac{d^2P}{dt^2} = 0 \rightarrow t = 1.57709$ days ✓M1

$\therefore t \approx 1.58$ days ✓A2

{nsolve}.

3.

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