Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Friday 6 May 2022 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

14 pages

2222–7111

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. [1]

(b) Given that the n^{th} term of this sequence is -33, find the value of n. [2]

(c) Find the common difference, d. [2]

2. [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n+1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

3. [Maximum mark: 8]

A function f is defined by $f(x)=\dfrac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

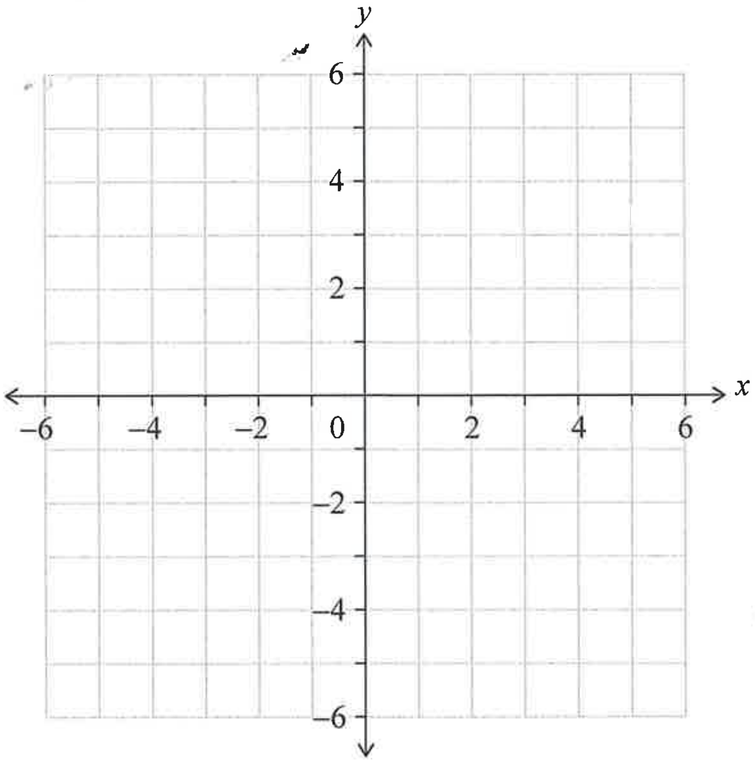
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\dfrac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\dfrac{2|x|-1}{|x|+1}<2$. [2]

(This question continues on the following page)

(Question 3 continued)

4. [Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

5. [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

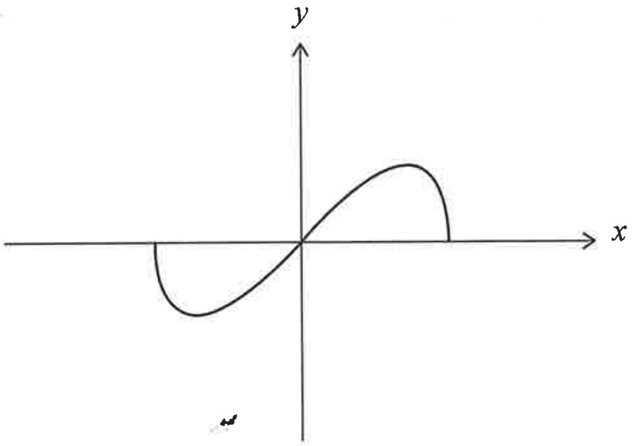
The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

6. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

7. [Maximum mark: 6]

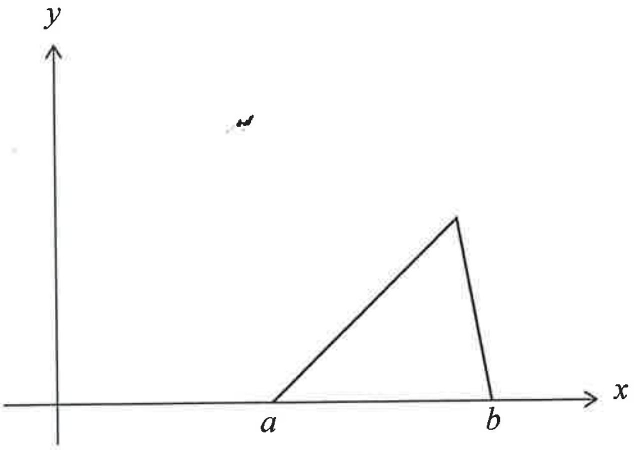
By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, dx$ in terms of n, where n is a non-zero real number.

8. [Maximum mark: 6]

A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\ \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\\ 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.



Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

9. [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die with faces labelled $1, 2, 3$ and 4 is rolled and the result recorded. Let X be the result obtained when the die is rolled. The probability distribution for X is given in the following table where p and q are constants.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	0.3	q	0.1

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair. Both yellow dice have faces labelled $1, 2, 3$ and 4. Let S represent the sum obtained by rolling the two yellow dice. The probability distribution for S is shown below.

s	2	3	4	5	6	7	8
$\mathrm{P}(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The first red die has faces labelled $1, 2, 2$ and 3. The second red die has faces labelled $1, a, a$ and b, where $a<b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of b. [2]

(e) Find the value of a, providing evidence for your answer. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 20]

A function f is defined by $f(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x)=\dfrac{1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x)=1+\dfrac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x)=\arctan\dfrac{x}{2}$, where $x \in \mathbb{R}$.

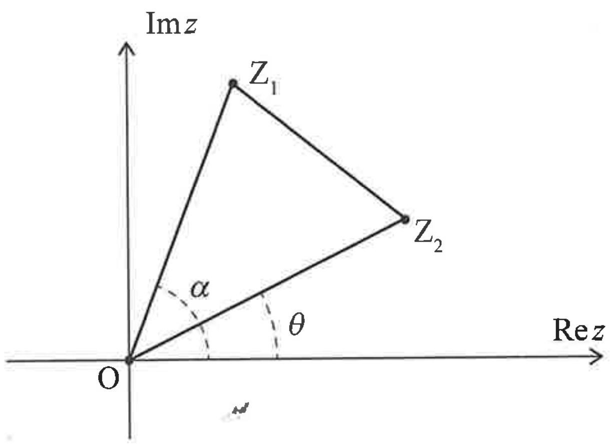
(c) Given that $(h \circ g)(a)=\dfrac{\pi}{4}$, find the value of a.

Give your answer in the form $p+\dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2{}^* = r_1 r_2 e^{i(\alpha - \theta)}$ where $z_2{}^*$ is the complex conjugate of z_2. [2]

(b) Given that $\text{Re}(z_1 z_2{}^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1{}^2 + z_2{}^2 = z_1 z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]