Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3008

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

1. [Maximum points: 22]

In this problem you will investigate the shortest distance between two lines in 3-dimensions.

A space shuttle is approaching a space station. The shuttle travels along line L_1 described by the equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

The position vector of the station at time t, where t is in seconds, is given by the equation

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$$

All units of coordinates are km.

The shuttle needs to get as close to the station as possible.

Let the coordinates of the space shuttle be $A\,(-1+3\lambda, 2+\lambda, 2-\lambda)$ and the coordinates of the station be $B\,(3+5t, 1+3t, -1-t)$.

(a) Determine $\overrightarrow{AB}$ in terms of λ and t. [2]

The line representing the shortest distance between the path of the space shuttle and the path of the space station is perpendicular to both paths.

(b) Show that for the shortest distance between the two lines we must have [6]

(i) $11\lambda - 19t = 14$

(ii) $19\lambda - 35t = 20$

(c) Solve the two equations from (b) to determine the values of λ and t to 5 significant figures. [3]

(d) Hence determine the shortest distance between the paths. [6]

(e) At $t = 0$ the position of the shuttle is $(-1,2,2)$. Find the speed the shuttle should travel in order to get as close to the station as possible. Write your answer in km/s. [5]

Copyright © 2022 ibtaskmaker.com. All rights reserved.

2. [Maximum points: 28]

In this problem you will investigate a coin flipping game with four players.

Let the random variable X represent the number of times a coin is flipped until a tail appears. For example if the outcomes are H, H, T then $X = 3$.

(a) Write down the value of $P(X = 1)$. [1]

(b) Find the exact value of [4]

(i) $P(X \leq 2)$

(ii) $P(X \leq 3)$

(c) Prove by induction that $P(X \leq n) = \dfrac{2^n - 1}{2^n}$. [8]

Four people play a game where they each take it in turns to flip a coin. Anyone who flips a tail is eliminated. They continue taking it in turns to flip until one person remains.

Player A flips first, then player B, then player C and then player D.

For example in the following game player B wins.

Player	A	B	C	D	A	B	A
Outcome	H	H	T	T	H	H	T

Consider the case when player D wins. Let a, b and c represent the number of flips made by players A, B and C respectively. Let the random variable Y represent the maximum value of a, b and c.

(d) Write down an expression for [2]

(i) $P(Y \leq n)$

(ii) $P(Y \leq n + 1)$

(e) Hence show that $P(Y = n + 1) = \left(\dfrac{2^{n+1} - 1}{2^{n+1}}\right)^3 - \left(\dfrac{2^n - 1}{2^n}\right)^3$. [2]

(f) Show that the probability that person D wins is equal to $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2^n - 1}{2^n}\right)^3 \times \left(\frac{1}{2}\right)^n$. [4]

(g) Find the exact value of the probability in part (d). [5]

(h) Find the probability that player A wins. [2]

Copyright © 2022 ibtaskmaker.com. All rights reserved.

1. (a) Subtract the coordinates of A from the coordinates of B. M1

$$\overrightarrow{AB} = \begin{pmatrix} 3+5t-(-1+3\lambda) \\ 1+3t-(2+\lambda) \\ -1-t-(2-\lambda) \end{pmatrix} = \begin{pmatrix} 4-3\lambda+5t \\ -1-\lambda+3t \\ -3+\lambda-t \end{pmatrix}$$ A1

(b)

(i) Calculate the dot product of $\overrightarrow{AB}$ and the direction of L_1 and set equal to zero. M1

$$\begin{pmatrix} 4-3\lambda+5t \\ -1-\lambda+3t \\ -3+\lambda-t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 0$$

So

$$12-9\lambda+15t-1-\lambda+3t+3-\lambda+t=0$$ A1

Giving

$$11\lambda-19t=14$$ A1

(ii) Calculate the dot product of $\overrightarrow{AB}$ and the direction of the space station and set equal to zero. M1

$$\begin{pmatrix} 4-3\lambda+5t \\ -1-\lambda+3t \\ -3+\lambda-t \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = 0$$

So

$$20-15\lambda+25t-3-3\lambda+9t+3-\lambda+t=0$$ A1

Giving

$$19\lambda-35t=20$$ A1

(c) Solve by substitution, eliminating variables, graphing etc. M1

$$\lambda = 4.5833$$ A1

$$t = 1.9167$$ A1

Copyright © 2022 ibtaskmaker.com. All rights reserved.

(d) Use the values from (c) to determine the coordinates of the point on each line that is closest to the other line.

On line L_1 this is

$$(-1+3\times 4.5833, 2+4.5833, 2-4.5833)$$ M1

Giving

$$(12.7499, 6.5833, -2.5833)$$ A1

On line $\boldsymbol{r}$ this is

$$(3+5\times 1.9167, 1+3\times 1.9167, -1-1.9167)$$ M1

Giving

$$(12.5835, 6.7501, -2.9167)$$ A1

Use the distance formula to determine the distance between these points. M1

$$\sqrt{(12.7499-12.5835)^2+(6.7501-6.5833)^2+(-2.9167-(-2.5833))^2}$$

Which is equal to 0.408 km. A1

(e) The shuttle has to travel from point $(-1,2,2)$ to the point $(12.7499, 6.5833, -2.5833)$ in 1.9167 seconds. R1

Use the distance formula to calculate this distance. M1

$$\sqrt{(12.7499-(-1))^2+(6.5833-2)^2+(-2.5833-2)^2}$$

This is equal to 15.201 km. A1

The speed is therefore

$$\frac{15.201}{1.9167}=7.93\,\text{km/s}$$ M1A1

Copyright © 2022 ibtaskmaker.com. All rights reserved.

2. (a) $\frac{1}{2}$ A1

(b)

(i) $\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{3}{4}$ M1A1

(ii) $\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{7}{8}$ M1A1

(c) When $n=1$ we have

$$P(X\le 1)=\frac{2^1-1}{2^1}=\frac{1}{2}$$ M1

So it is true for $n=1$. A1

Assume it is true for $n=k$. So $P(X\le k)=\frac{2^k-1}{2^k}$. A1

When $n=k+1$ we have

$$P(X\le k+1)=P(X\le k)+P(X=k+1)$$ A1

This is equal to

$$\frac{2^k-1}{2^k}+\left(\frac{1}{2}\right)^{k+1}=\frac{2^{k+1}-2+1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}$$ M1A1

So it is true for $n=k+1$. A1

By the principle of mathematical induction it must be true for all positive integers n. R1

(d)

(i) $P(Y\le n)=\left(\frac{2^n-1}{2^n}\right)^3$ A1

(ii) $P(Y\le n+1)=\left(\frac{2^{n+1}-1}{2^{n+1}}\right)^3$ A1

(e) $P(Y=n+1)=P(Y\le n+1)-P(Y\le n)=\left(\frac{2^{n+1}-1}{2^{n+1}}\right)^3-\left(\frac{2^n-1}{2^n}\right)^3$ M1A1

Copyright © 2022 ibtaskmaker.com. All rights reserved.

(f) For player D to win then player D needs to flip one less head than the value of Y. R1

The probability is therefore

$$\sum_{n=0}^{\infty}\left[\left(\frac{2^{n-1}-1}{2^{n+1}}\right)^3-\left(\frac{2^n-1}{2^n}\right)^3\right]\times\left(\frac{1}{2}\right)^n \qquad \text{M1}$$

This is equal to

$$\left(\frac{2^1-1}{2^1}\right)^3-0+\left(\frac{2^2-1}{2^2}\right)^3\times\frac{1}{2}-\left(\frac{2^1-1}{2^1}\right)^3\times\frac{1}{2}+\left(\frac{2^3-1}{2^3}\right)^3\times\frac{1}{4}-\left(\frac{2^2-1}{2^2}\right)^3\frac{1}{4}+\cdots \qquad \text{A1}$$

This simplifies to

$$\sum_{n=1}^{\infty}\left(\frac{2^n-1}{2^n}\right)^3\cdot\left(\frac{1}{2}\right)^n \qquad \text{A1}$$

(g) Expand and simplify

$$\sum_{n=1}^{\infty}\frac{2^{3n}-3\cdot 2^{2n}+3\cdot 2^n-1}{2^{4n}}=\sum_{n=1}^{\infty}2^{-n}-3\cdot 2^{-2n}+3\cdot 2^{-3n}-2^{-4n} \qquad \text{M1A1}$$

Use the infinite geometric series formula to evaluate M1

$$\frac{1/2}{1-1/2}-\frac{3/4}{1-1/4}+\frac{3/8}{1-1/8}-\frac{1/16}{1-1/16} \qquad \text{A1}$$

This is equal to

$$1-1+\frac{3}{7}-\frac{1}{15}=\frac{38}{105} \qquad \text{A1}$$

(h) Player A needs to flip a tail to stay in the game. The probability of this is 1/2. If this happens this is the same as playing the game in the order B, C, D, A. From part (g) the last person to flip has a probability of 38/105 of winning. R1

So the probability that player A wins is $\frac{1}{2}\times\frac{38}{105}=\frac{19}{105}$. A1

Copyright © 2022 ibtaskmaker.com. All rights reserved.