Mathematics: analysis and approaches

Higher level

Paper 3

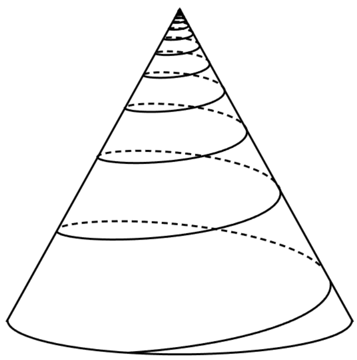
ID: 3002

Instructions to candidates

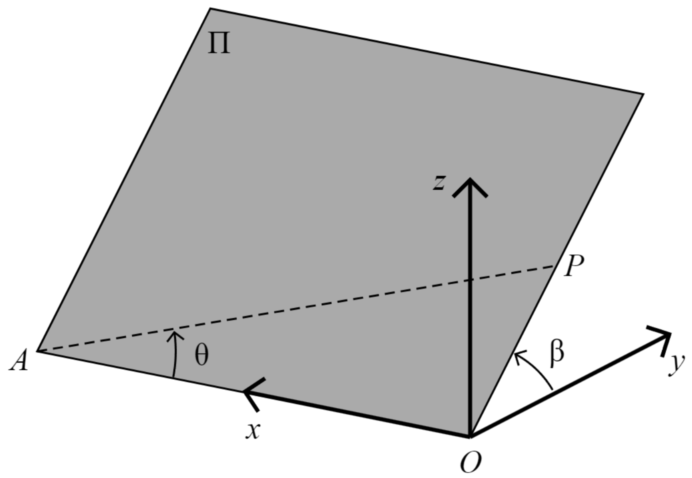
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[54 marks]**.

1. [Maximum points: 30]

In this problem you will investigate the path taken by a hiker climbing at a constant gradient up a mountain in the shape of a cone. The path of the hiker is shown in the diagram below.



Plane Π is inclined at an angle of β to the xy-plane. Points P, A and O lie on plane Π and $\angle PAO = \theta$. This is shown in the diagram below.



Let $|AP| = d$.

(a) Find $|OP|$ in terms of d and θ. [2]

Let the angle between line AP and the xy-plane be equal to ϕ.

(b) Show that $\sin\phi = \sin\theta \sin\beta$. [3]

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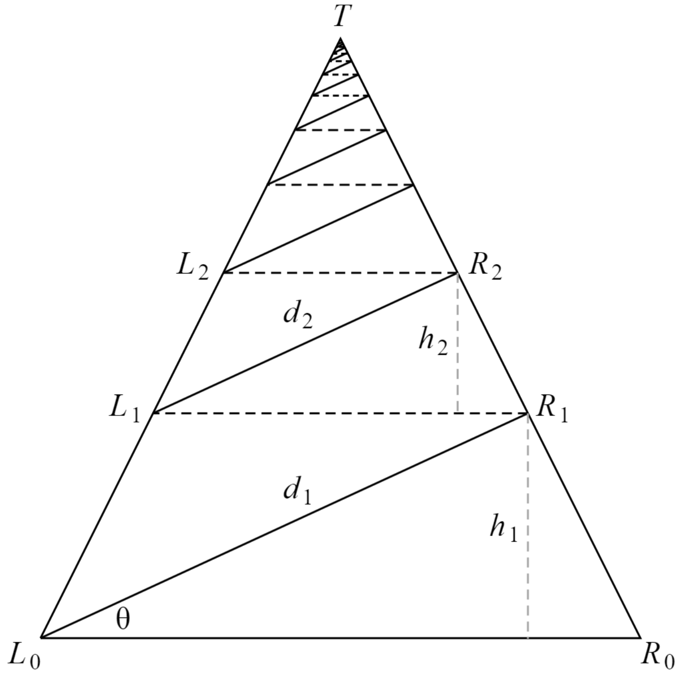
The diagram below shows isosceles triangle TL_0R_0 divided into infinitely many smaller triangles using dashed and solid lines. Each dashed line meets the isosceles triangle at L_n and R_n where $n \in \mathbb{Z}^+$.



For $n \in \mathbb{N}$ every triangle of the form TL_nR_n is similar, and all lines of the form L_nR_{n+1} are parallel.

Let $|TL_0| = |TR_0| = c$, $|L_0R_0| = b$ and $\angle R_1L_0R_0 = \theta$.

For $n \in \mathbb{N}$ let $|L_nR_{n+1}| = d_{n+1}$ and the height of $\Delta R_{n+1}L_nR_n$ be equal to h_{n+1}.

(c) Find the height of $\triangle TL_0R_0$ in terms of b and c. [2]

(d) Find the length of d_n in terms of θ and h_n. [2]

(e) Hence show that [2]

$$\sum_{n=1}^{\infty} d_n = \frac{\sqrt{4c^2 - b^2}}{2\sin\theta}$$

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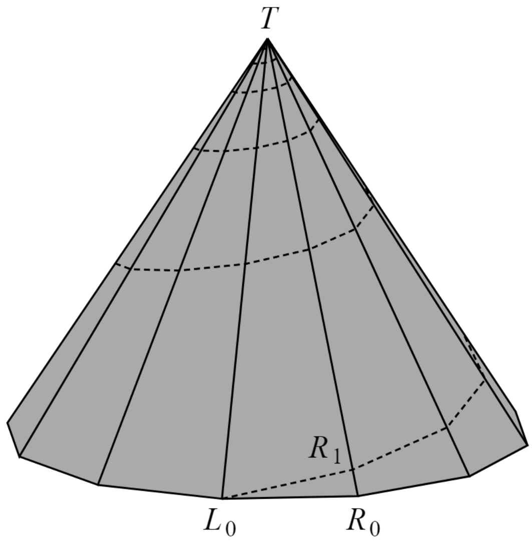
A *regular n-gon pyramid* is a pyramid with a base in the shape of a regular polygon with n sides. The top of the pyramid is directly above the centre of the base.

A mountain is in the shape of a cone with a height of 1000 m and a base of radius 500 m. A hiker climbs the mountain by following a path which remains at a fixed angle to the xy-plane.

The path of the hiker can be approximated by treating the mountain as a regular n-gon pyramid. This is shown in the diagram below where the dotted line represents the path of the hiker. The base of the pyramid is on the xy-plane.



Let the height of the pyramid be 1000 m and the base be the largest regular n-sided polygon which fits inside a circle of radius 500 m.

One of the faces of the pyramid is labelled $\triangle TL_0R_0$. Let $\angle R_1L_0R_0 = \theta$ and the angle between ΔTL_0R_0 and the xy-plane be equal to β.

(f) As $n \to \infty$ write down what type of shape the regular n-gon pyramid will become. [1]

(g) Hence show that $\lim\limits_{n \to \infty} \beta = \arcsin(2 \cdot 5^{-1/2})$. [3]

(h) Write down the value of $\lim\limits_{n \to \infty} \left| L_0R_0 \right|$. [1]

(i) Hence find an expression for the total distance the hiker must climb to reach the top of the **conical** mountain in terms of θ. [2]

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The speed s that the hiker can climb in m/s depends on the angle ϕ between the path of the hiker and the xy-plane and is given by $s = 0.5\cos\phi$.

(j) Show that the time t taken in seconds for the hiker to reach the top of the mountain is given by [5]

$$t = \frac{4000}{\sin 2\phi}$$

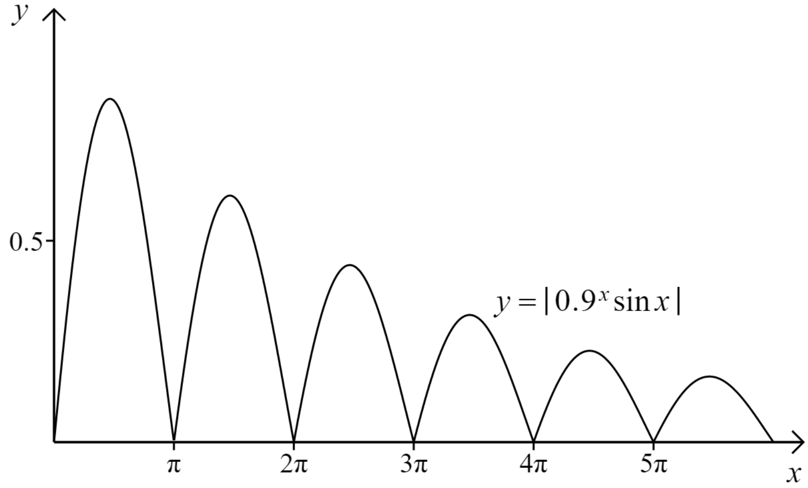
(k) Hence find the minimim amount of time it takes for the hiker to reach the top of the mountain and the corresponding value of ϕ. Write your answers as exact values. [5]

(l) By considering the path as the hiker gets closer to the summit explain why this model is not entirely realistic. [2]

2. [Maximum points: 24]

(a) Show that $\frac{d}{dx}(0.9^x) = \ln 0.9 \times 0.9^x$ [3]

The graph below shows the function $y = |0.9^x \sin x|$.



(b) Use repeated integration by parts to show that [8]

$$\int 0.9^x \sin x \, dx = \frac{0.9^x(\ln 0.9 \sin x - \cos x)}{1 + (\ln 0.9)^2}$$

(c) If n is a non-negative integer determine an expression for [7]

(i) $\int_{n\pi}^{(n+1)\pi} |0.9^x \sin x| \, dx$

(ii) $\int_{(n+1)\pi}^{(n+2)\pi} |0.9^x \sin x| \, dx$

(d) If the areas of each region bound by the function and the x-axis are calculated separately show that the areas form a geometric sequence. [3]

(e) Hence for $n \in \mathbb{Z}$ evaluate $\lim_{n \to \infty} \int_0^{n\pi} |0.9^x \sin x| \, dx$. [3]

1. (a) Use right-angled trigonometry M1

$$|OP| = d\sin\theta$$ A1

(b) Use right-angled trignometry to determine the z-coordinate of point P. This gives M1

$$d\sin\theta\sin\beta$$ A1

So we have

$$\sin\phi = \frac{d\sin\theta\sin\beta}{d} = \sin\theta\sin\beta$$ A1

(c) Use the Pythagorean theorem M1

$$h = \sqrt{c^2 - \frac{b^2}{4}} = \frac{\sqrt{4c^2 - b^2}}{2}$$ A1

(d) Use right-angled trigonometry M1

$$\sin\theta = \frac{h_n}{d_n}$$

So

$$d_n = \frac{h_n}{\sin\theta}$$ A1

(e) We have

$$\sum_{n=1}^{\infty} d_n = \frac{1}{\sin\theta}\sum_{n=1}^{\infty} h_n = \frac{\sqrt{4c^2 - b^2}}{2\sin\theta}$$ M1A1

(f) A cone A1

(g) The slope length of the cone is $\sqrt{1000^2 + 500^2} = \sqrt{1250000}$. A1

So

$$\sin\beta = \frac{1000}{\sqrt{1250000}} = \frac{2}{\sqrt{5}}$$ M1

Therefore

$$\beta = \arcsin\left(\frac{2}{\sqrt{5}}\right)$$ A1

(h) 0 A1

(i) Using the result from (f) we have $c = \sqrt{1000^2 + 500^2} = 500\sqrt{5}$. So

$$\lim_{b\to 0}\frac{\sqrt{4(1000^2 + 500^2) - b^2}}{2\sin\theta} = \frac{500\sqrt{5}}{\sin\theta}$$ M1A1

(j) Using time = distance ÷ speed we have

$$t = \frac{1000\sqrt{5}}{\sin\theta\cos\phi}$$ A1

From parts (b) and (g) we also have M1

$$\sin\theta = \frac{\sqrt{5}\sin\phi}{2}$$ A1

So

$$t = \frac{2000}{\sin\phi\cos\phi} = = \frac{4000}{2\sin\phi\cos\phi} = \frac{4000}{\sin 2\phi}$$ M1A1

(k) The smallest value of t will occur at the largest value of $\sin 2\phi$. So we need R1

$$\sin 2\phi = 1$$ M1

So

$$\phi = \frac{\pi}{4}$$ A1

The time taken is therefore

$$t = \frac{4000}{\sin(\pi/2)} = 4000 \text{ sec}$$ M1A1

(l) As the hiker approaches the summit the path with circle the summit infinitely many times with a smaller and smaller radius. A1A1

2. (a) Rewrite 0.9^x as $e^{x\ln 0.9}$. A1

Differentiate using the chain rule. M1

$$\frac{d}{dx}\left(e^{x\ln 0.9}\right) = \ln 0.9 \times e^{x\ln 0.9}$$

Which is equal to $\ln 0.9 \times 0.9^x$. A1

(b) Use integration by parts with

$$u = 0.9^x$$

and

$$v' = \sin x$$ M1

so

$$u' = \ln 0.9 \times 0.9^x$$

and

$$v = -\cos x$$ A1

giving

$$\int 0.9^x \sin x\, dx = -0.9^x \cos x + \ln 0.9 \int 0.9^x \cos x\, dx$$ A1

Use integration by parts again with

$$u = 0.9^x$$

and

$$v' = \cos x$$ M1

so

$$u' = \ln 0.9 \times 0.9^x$$

and

$$v = \sin x$$ A1

giving

$$\int 0.9^x \sin x\, dx = \ln 0.9 \times 0.9^x \sin x - 0.9^x \cos x - (\ln 0.9)^2 \int 0.9^x \sin x$$ A1

Take the integrals to the left side and factorise. M1

$$\left(1 + (\ln 0.9)^2\right) \int 0.9^x \sin x\, dx = 0.9^x (\ln 0.9 \sin x - \cos x)$$

So

$$\int 0.9^x \sin x\, dx = \frac{0.9^x (\ln 0.9 \sin x - \cos x)}{1 + (\ln 0.9)^2}$$ A1

(c)

(i) Use the expression from (b) to evaluate. M1

$$\left[\frac{0.9^{x}(\ln 0.9\sin x-\cos x)}{1+(\ln 0.9)^{2}}\right]_{n\pi}^{(n+1)\pi}$$

This is equal to

$$\frac{0.9^{(n+1)\pi}(-1)^{n+1}}{1+(\ln 0.9)^{2}}-\frac{0.9^{n\pi}(-1)^{n}}{1+(\ln 0.9)^{2}}$$ A1A1

which simplifies to

$$\frac{-(-1)^{n}0.9^{n\pi}(0.9^{\pi}+1)}{1+(\ln 0.9)^{2}}$$ A1

(ii) Follow the same step as part (ii) to get

$$\frac{0.9^{(n+2)\pi}(-1)^{n+2}}{1+(\ln 0.9)^{2}}-\frac{0.9^{(n+1)\pi}(-1)^{n+1}}{1+(\ln 0.9)^{2}}$$ A1A1

which simplifies to

$$\frac{-(-1)^{n+1}0.9^{(n+1)\pi}(0.9^{\pi}+1)}{1+(\ln 0.9)^{2}}$$ A1

(d) We have

$$\int_{n\pi}^{(n+1)\pi}|0.9^{x}\sin x|\,dx=\frac{0.9^{n\pi}(0.9^{\pi}+1)}{1+(\ln 0.9)^{2}}$$ A1

and

$$\int_{(n+1)\pi}^{(n+2)\pi}|0.9^{x}\sin x|\,dx=\frac{0.9^{(n+1)\pi}(0.9^{\pi}+1)}{1+(\ln 0.9)^{2}}$$ A1

Call these A_n and A_{n+1}.

Notice that

$$\frac{A_{n+1}}{A_n}=0.9^{\pi}$$

so the area of each region forms a geometric sequence. R1

(e) The area of the first region is

$$\frac{0.9^{\pi} + 1}{1 + (\ln 0.9)^2}$$ A1

Use the infinite geometric series formula to determine the sum to infinity. M1

$$S_{\infty} = \frac{0.9^{\pi} + 1}{\left(1 + (\ln 0.9)^2\right)(1 - 0.9^{\pi})}$$ A1