Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 9 November 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Attempt 1: $\frac{46}{40}$ = 100 %.

8821–7103

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z)=\dfrac{e^z+e^{-z}}{2}$ and $g(z)=\dfrac{e^z-e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t,\ u \in \mathbb{R}$.

(a) Verify that $u=f(t)$ satisfies the differential equation $\dfrac{d^2u}{dt^2}=u$. [2]

(b) Show that $(f(t))^2+(g(t))^2=f(2t)$. [3]

(c) Using $e^{iu}=\cos u+i\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(iu)$; [3]

(ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2+(g(iu))^2$. [2]

(e) Show that $(f(t))^2-(g(t))^2=(f(iu))^2-(g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta, \sin\theta)$ defines points on the unit circle with equation $x^2+y^2=1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2-y^2=1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2-y^2=1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2-y^2=1$ can be rotated to coincide with the curve defined by $xy=k$, $k\in\mathbb{R}$.

(g) Find the possible values of k. [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]

(ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

can have many variables ...

(b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t, show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]

(ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

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① $\frac{46}{46} = 100\%$.

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Q1 : $\frac{16}{16}$ Q2 : $\frac{30}{30}$

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1 2 3 4 5 6 7 8 9 10

1

a) $f(t) = \frac{1}{2}(e^{t} + e^{-t}) = u$

$f'(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$

$f''(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} = u$ ②

$\therefore \frac{d^2u}{dt^2} = u$

b) $(f(t))^2 + (g(t))^2$

$= \frac{1}{2}^2(e^{t} + e^{-t})^2 + \frac{1}{2}^2(e^{t} - e^{-t})^2$

$= \frac{1}{4}(e^{2t} + 2e^{t}e^{-t} + e^{-2t}) + \frac{1}{4}(e^{2t} - 2e^{t}e^{-t} + e^{-2t})$

$= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) + \frac{1}{4}(e^{2t} - 2 + e^{-2t})$

$= \frac{1}{4}(e^{2t} + 2 + e^{-2t} + e^{2t} - 2 + e^{-2t})$

$= \frac{1}{4}(2e^{2t} + 2e^{-2t})$

$= \frac{1}{2}(\cancel{2}e^{2t} + e^{-2t})$ ③

$= f(2t)$

c)(i) $f(iu) = \frac{e^{iu} + e^{-iu}}{2}$

$= \frac{\cos u + i\sin u + \cos u - i\sin u}{2}$ ③

~~$\therefore f(iu) = \cos u$~~

~~7~~ 8/

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c ii) $g(iu) = \frac{e^{iu} - e^{-iu}}{2}$

$= \frac{\cos u + i\sin u - (\cos u - i\sin u)}{2}$

$\therefore g(iu) = i\sin u$

2

d) $(f(iu))^2 + (g(iu))^2 = \cos^2 u + (i\sin u)^2$
$= \cos^2 u - \sin^2 u$
$= \cos(2u)$

2

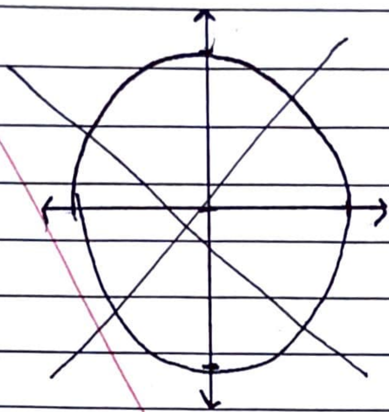
e) LHS $= (f(t))^2 - (g(t))^2$
$= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) - \frac{1}{4}(e^{2t} - 2 + e^{-2t})$
$= \frac{1}{4}(e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t})$
$= \frac{1}{4}(4)$
$= 1$

RHS $= (f(iu))^2 - (g(iu))^2$
$= \cos^2 u - (i\sin u)^2$
$= \cos^2 u + \sin^2 u$
$= 1$
$=$ LHS.

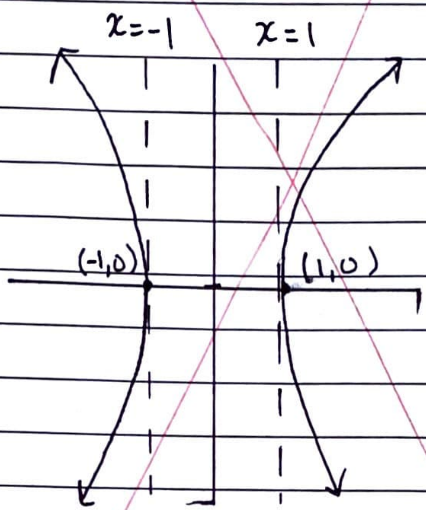
4

8/

f)



→ not assessed.



$x^2 - y^2 = 1$

$\therefore \quad y^2 = x^2 - 1$

x-int: $0 = x^2 - 1$

$\therefore x = \pm 1$

y-int: $y^2 = 0 - 1$

$\therefore y = \sqrt{-1}$ DNE

~~asymptote~~ Domain: $|x| \geq 1$

g) Not assessed.

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ANSWERS ON FOLLOWING ~~PG~~ PAGE

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2

$\frac{dx}{dt} = x - y \qquad \frac{dy}{dt} = ax + y$

ai) $\frac{dy}{dt} = y$

$\therefore \frac{1}{y}\frac{dy}{dt} = 1$

$\therefore \int \frac{1}{y}dy = \int dt$

$\therefore \ln|y| = t + c$

$\therefore y = \pm e^{t+c}$

$\therefore y = Ae^{t} \quad \{A = \pm e^{c}\} \quad ..(1)$

3

aii) $\frac{dx}{dt} - x = x - y - x$

$= -y$

$= -Ae^{t}$ from (1)

1

aiii) $\frac{dx}{dt} - x = Ae^{t}$

$I(x) = e^{-\int dt}$

$= e^{-t}$

$\cancel{\therefore e^{-t}\frac{dx}{dt} - xe^{-t} = Ae^{t}e^{-t}}$

$\cancel{\therefore \frac{d}{dt}[e^{-t}x - 1] = A}$

$\cancel{-e^{-t} = \int A\,dt}$

$\cancel{\therefore -e^{-t} = At + c}$

$\cancel{\therefore e^{-t} = c - At}$

4

$\frac{d}{dt}(xe^{-t}) = Ae^{t}e^{-t}$

$\therefore xe^{-t} = A\int dt$

$\therefore xe^{-t} = At + d$

$\therefore x = e^{t}(At + d), \; A = \pm e^{c}$

8/

b i) $\frac{dy}{dt} = -x + y$

$\therefore \frac{d^2y}{dt^2} = \cancel{-x\frac{dx}{dt} + \frac{y}{y}}\ -x + dy/dt$

$2\frac{dy}{dt} = -2x + 2y$

$\cancel{\frac{dy}{dt} = -e^{t}At + de^{t}} + y$

$\cancel{= -A(e^{t} + te^{t}) + de^{t}}$

$=$

$\frac{d^2y}{dt^2} = \frac{d}{dt}(-x + y)$

$= dy/dt$

$\frac{dy}{dt} = -x + y$

$\therefore \frac{d^2y}{dt^2} = -dx/dt + dy/dt$

$= \cancel{(-(x-y) + y})$

$= \cancel{-x + 2y}$

$= -(x - y) + dy/dt$

$= dy/dt + dy/dt$

$= 2\,dy/dt$

3

3/

~~bi $2Y = d^2y/dt^2$~~

~~$= \int d^2y/dt^2 \, dt$~~

bii) $\frac{d^2y}{dt^2} = 2Y$

$\therefore \frac{dY}{dt} = 2y$

$\therefore \frac{1}{y}\frac{dy}{dt} = 2$

$\therefore \ln|y| = 2t + c$

$\therefore Y = Be^{2t}$, $B = \pm e^{c}$

3

biii) $Y = \frac{dy}{dt} = Be^{2t}$

$\therefore y = B\int e^{2t} dt$

$\therefore y = \frac{1}{2}Be^{2t} + c$

2

biv) $\frac{dy}{dt} = -x + y$ $\rightarrow$ ~~$y = \frac{dy}{dt} + x$~~

$\therefore x = y - dy/dt$

$= \frac{1}{2}Be^{2t} + \cancel{A} - Be^{2t} + C$

$= -\frac{1}{2}Be^{2t} + C$

3

8

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②

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2

c)(i) When $a=-4$, $\frac{dy}{dt} = -4x+y$

$$\therefore \frac{d^2y}{dt^2} = -4\frac{dx}{dt} + \frac{dy}{dt}$$
$$= -4(x-y) + dy/dt$$
$$= -4x+4y + dy/dt$$
$$= -4x+y+3y+dy/dt$$
$$= 2\,dy/dt + 3y$$

$$\therefore \frac{d^2y}{dt^2} - 2\,dy/dt - 3y = 0$$

3

3

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c ii) $y = Fe^{\lambda t}$

$\therefore dy/dt = \lambda Fe^{\lambda t}$

$\therefore d^2y/dt^2 = \lambda^2 Fe^{\lambda t}$

$\therefore$ when $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$, then

$\lambda^2 Fe^{\lambda t} - 2\lambda Fe^{\lambda t} - 3Fe^{\lambda t} = 0$

$\therefore \lambda - 2\lambda - 3 = 0$

$\therefore (\lambda - 3)(\lambda + 1) = 0$

$\therefore \lambda = 3, \quad \lambda = -1$

4

c iii) $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$

$= Fe^{3t} + Ge^{-t}$ .. (1)

$\therefore dy/dt = 3Fe^{3t} - Ge^{-t}$ .. (2)

$\therefore d^2y/dt^2 = 9Fe^{3t} + Ge^{-t}$ .. (3)

Substitute (1), (2) and (3) into :

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$

$\therefore 9Fe^{3t} + Ge^{-t} - 2(3Fe^{3t} - Ge^{-t}) - 3(Fe^{3t} + Ge^{-t}) = 0$

LHS $= 9Fe^{3t} - 6Fe^{3t} - 3Fe^{3t} + Ge^{-t} + 2Ge^{-t} - 3Ge^{-t}$

$= 0 + 0$

$=$ RHS.

4

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