Markscheme

ID: 3003

Mathematics: analysis and approaches

Higher level

1. (a) $\overrightarrow{OQ} = \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix}$ A1A1

(b) We have

$$\overrightarrow{QP} = \begin{pmatrix} 3\cos 4t \\ 3\sin 4t \end{pmatrix}$$ A1A1

Since $\overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OP}$ we have M1

$$\boldsymbol{r} = \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix} + \begin{pmatrix} 3\cos 4t \\ 3\sin 4t \end{pmatrix}$$ A1

(c) The gradient of line OP is

$$\frac{8\sin t + 3\sin 4t}{8\cos t + 3\cos 4t}$$ A1

If (x,y) represents the coordinates of point P then

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$ M1

So

$$\frac{dy}{dx} = \frac{-8\cos \mathrm{t} - 12\cos 4t}{8\sin \mathrm{t} + 12\sin 4t} = \frac{-2\cos t - 3\cos 4t}{2\sin t + 3\sin 4t}$$ A1A1

Therefore

$$\frac{-2\cos \mathrm{t} - 3\cos 4t}{2\sin \mathrm{t} + 3\sin 4t} = \frac{8\sin t + 3\sin 4t}{8\cos t + 3\cos 4t}$$ M1

Rearrange and replace t with T

$$\frac{2\cos T + 3\cos 4T}{2\sin T + 3\sin 4T} = -\frac{8\sin T + 3\sin 4T}{8\cos T + 3\cos 4T}$$ A1

(d) We have

$$(2\cos T + 3\cos 4T)(8\cos T + 3\cos 4T) = -(8\sin T + 3\sin 4T)(2\sin T + 3\sin 4T)$$ A1

Expand

$$16\cos^2 T + 9\cos^2 4T + 30\cos T\cos 4T = -(16\sin^2 T + 9\sin^2 4T + 30\sin T\sin 4T)$$ A1

Rearrange and use the Pythagorean identity to simplify M1

$$-25 = 30(\cos T\cos 4T + \sin T\sin 4T)$$ A1

Use the compound angle identity to rewrite M1

$$-\frac{5}{6} = \cos 3T$$ A1

So $T = \dfrac{\arccos(-5/6)}{3}$. A1

(e) Let D represent the length of OP. We have

$$D^2 = (8\cos t + 3\cos 4t)^2 + (8\sin t + 3\sin 4t)^2 \qquad \text{A1}$$

Expand and simplify using Pythagorean and compound angle identities M1

$$D^2 = 64\cos^2 t + 9\cos^2 4t + 48\cos t\cos 4t + 64\sin^2 t + 9\sin^2 4t + 48\sin t\sin 4t \qquad \text{A1}$$

So

$$D^2 = 73 + 48\cos 3t \qquad \text{A1}$$

(f) Use implicit differentiation M1

$$2D\frac{dD}{dt} = -144\sin 3t \qquad \text{A1}$$

Therefore

$$\frac{dD}{dt} = -\frac{72\sin(\arccos(-5/6))}{\sqrt{73 + 48\cos(\arccos(-5/6))}} \qquad \text{M1}$$

This is equal to -6.93 m/s or $-4\sqrt{3}$ m/s. A1

2. (a) Use the arithmetic series formula M1

$$\frac{n(n+1)}{2}$$ A1

(b) When $n = 1$ we have

$$\sum_{r=1}^{1} r^2 = 1$$ M1

And

$$\frac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = 1$$

So it is true for $n = 1$. A1

Assume it is true for $n = k$. So

$$\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$$ A1

For $n = k + 1$ we have

$$\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6}$$ M1A1

Expand

$$\frac{(k+1)(2k^2 + 7k + 3)}{6}$$ M1

Factorise

$$\frac{(k+1)(k+2)\big(2(k+1)+1\big)}{6}$$ A1

So it is true for $n = k + 1$. A1

By the principle of mathematical induction it must be true for all positive integers n. R1

(c) $4 - 8 + 5 + 9 - 12 + 5 + 16 - 16 + 5 + 25 - 20 + 5 = 18$ M1A1

(d) The width of each rectangle is $\frac{4}{n}$. A1

(e) The x-coordinate of the left side of each rectangle is $2 + \dfrac{4(k-1)}{n}$. A1

The total area is therefore

$$A = \frac{4}{n}\sum_{k=1}^{n}\left(2 + \frac{4(k-1)}{n}\right)^2 - 4\left(2 + \frac{4(k-1)}{n}\right) + 5 \qquad \text{M1}$$

This can be written as

$$A = \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16(k-1)^2}{n^2} + 1\right) \qquad \text{A1}$$

(f) We have

$$A = \frac{64}{n^3}\sum_{k=1}^{n}k^2 - \frac{128}{n^3}\sum_{k=1}^{n}k + \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16}{n^2} + 1\right) \qquad \text{M1}$$

Using parts (a) and (b) this gives

$$A = \frac{32(n+1)(2n+1)}{3n^2} - \frac{64(n+1)}{n^2} + \frac{4(16+n^2)}{n^2} \qquad \text{A1A1A1}$$

(g) We have

$$\lim_{n\to\infty} A = \lim_{n\to\infty} \frac{32\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{3} - \frac{64\left(\frac{1}{n}+\frac{1}{n^2}\right)}{1} + \frac{4\left(\frac{16}{n^2}+1\right)}{1} \qquad \text{M1}$$

This is equal to $\dfrac{76}{3}$. A1

(h) $\displaystyle\int_2^6 x^2 - 4x + 5\,dx = \left[\frac{x^3}{3} - 2x^2 + 5x\right]_2^6 = 72 - 72 + 30 - \frac{8}{3} + 8 - 10 = \frac{76}{3}$ M1A1A1