Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 3

TZ2

Thursday 12 May 2022 (morning)

1 hour

**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

$\frac{44}{55} = 80\%$.

5 pages

2222–7113

**ANSWER BOOKLET**
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$\frac{44}{55} = \frac{4}{5} = 80\%$.

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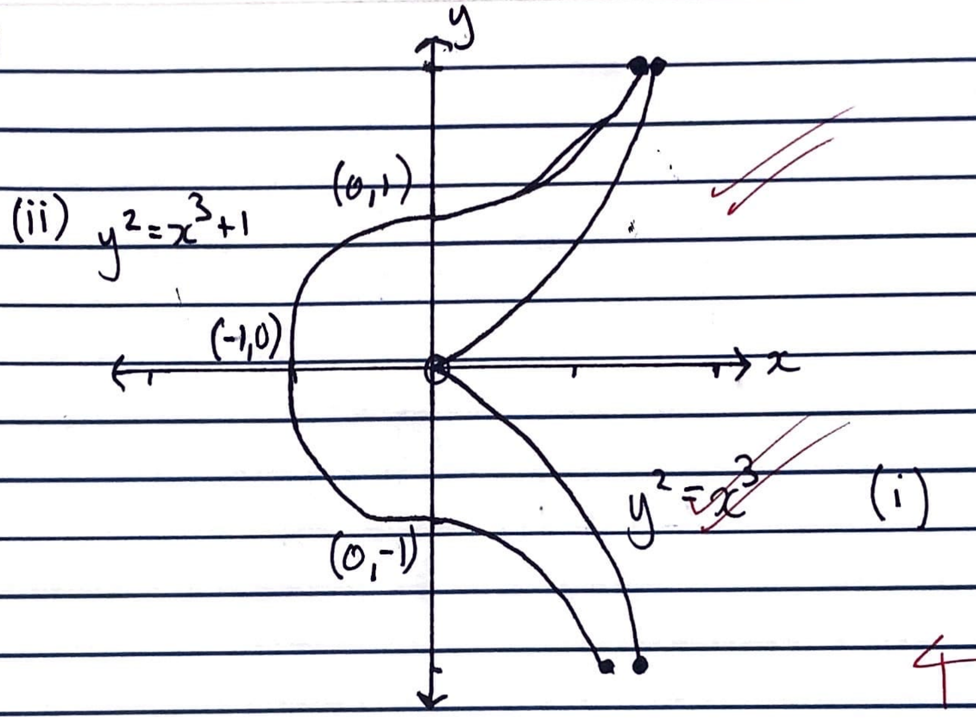
Example Ejemplo 27 | 2 7 — Example Ejemplo 3 | 3

1

(a)




(b)(i) Point of inflexion: $d^2y/dx^2 = 0$

$\Rightarrow$ $P_1(0,1)$

$\Rightarrow$ $P_2(0,-1)$ 1

(ii) • The y-intercept. (1 vs 0)
• The point of inflexion(s) (2 vs 0)

~~2~~ 1

(c) $y^2 = x^3 + b$ $x \geq -\sqrt[3]{b}$ $b \in \mathbb{Z}^+$

$\Rightarrow$ the shape, always with 2 points of inflexion

$\Rightarrow$ symmetry over the $x$ axis.

2

(d)(i) $y^2 = x^3 + x$

$\therefore 2y \frac{dy}{dx} = 3x^2 + 1$

~~$\therefore \frac{dy}{dx} = 3x^2 + 1$~~

$\therefore \frac{dy}{dx} = \frac{3x^2+1}{2y}$

$\therefore \frac{dy}{dx} = \pm \frac{3x^2+1}{2\sqrt{x^3+x}}$

3

(ii) Local min/max occurs at $dy/dx = 0$

$\therefore \frac{3x^2+1}{2\sqrt{x^3+x}} = 0$

$\therefore 3x^2+1 = 0$

$\therefore x = \sqrt{-1/3}$

$\therefore x$ does not have a real solution

1

~~(d)~~ (e) $\frac{dy}{dx} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$

$\therefore \frac{d^2y}{dx^2} = \frac{(2\sqrt{x^3+x})(6x) - (3x^2+1)(3x^2+1)(x^3+x)^{-1/2}}{4(x^3+x)} = 0$

$\therefore 12x\sqrt{x^3+x} - \frac{(3x^2+1)(3x^2+1)}{\sqrt{x^3+x}} = 0$

$\therefore 12x(x^3+x) - (3x^2+1)(3x^2+1) = 0$

$\therefore 12x^4 + 12x^2 - (9x^4 + 3x^2 + 3x^2 + 1) = 0$

$\therefore 12x^4 + 12x^2 - 9x^4 - 3x^2 - 3x^2 - 1 = 0$

$\therefore \cancel{-4x^4 - 3x^2 - 1} = 0$

$3x^4 + 6x^2 - 1 = 0$

$\therefore x^2 = \frac{-6 \pm \sqrt{36 + 12}}{6}$

$= \frac{-6 \pm 4\sqrt{3}}{6}$

$\therefore x = \sqrt{\frac{4\sqrt{3}-6}{6}}$ $\left(\begin{array}{l} x > 0 \\ 4\sqrt{3} - 6 > 0 \end{array}\right)$

simplify

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Example / Ejemplo 27 [2][7] Example / Ejemplo 3 [ ][3]

1

(f) $C: \quad y^2 = x^3 + 2 \qquad x \geq \sqrt[3]{2}$

$\leftarrow \quad P(-1,-1)$

$y^2 = x^3 + 2$

$\therefore 2y \, dy/dx = 3x^2 + 0$

$\therefore \dfrac{dy}{dx} = \dfrac{3x^2}{2y}$

$\therefore$ At $P(-1,-1)$, $\dfrac{dy}{dx} = \dfrac{3}{-2}$

$= -3/2$

$\therefore y-(-1) = -\frac{3}{2}(x - -1)$ 2

$\therefore y = -3/2 x - 3/2 - 1$

$\therefore y = -3/2 x - 5/2$

(f) (ii) $y = -3/2\, x - 5/2$

Substitute:

$$\left(-\frac{3}{2}x - \frac{5}{2}\right)^2 = x^3 + 2$$

$\therefore \quad x = 4.25 \approx 17/4$ (nsolve)

1

When $x = 4.25$, $y = \pm\sqrt{4.25^3 + 2}$
$= 8.875$
$\approx 71/8 \rightarrow -71/8$

$\therefore$ ~~Q is $\left(\frac{71}{8}, \frac{17}{4}\right)$~~ Q is $\left(\frac{17}{4}, \frac{71}{8}\right)$

(g) ~~$dy/dx = 3x^2/y$~~

Line is $[QS] \rightarrow \vec{QS} = \begin{pmatrix} -1 - 17/4 \\ 1 - 71/8 \end{pmatrix}$

M1M1

$= \begin{pmatrix} -21/4 \\ -63/8 \end{pmatrix}$

$\therefore \; x = -1 - 21/4\, t$
$y = 1 - 63/8\, t$

2

$\therefore \; (1 - 63/8\, t)^2 = (-1 - 21/4\, t)^3 + 2$

$\therefore \quad t = 0$

$\therefore$ coordinate is $\begin{pmatrix} -21/4 \\ -63/8 \end{pmatrix} \rightarrow \left(-\frac{21}{4}, -\frac{63}{8}\right)$

22

2

(a) ~~$x=0$~~ $(x-\alpha)(x-\beta)(x-\gamma)$

$= (x^2-\beta x-\alpha x+\alpha\beta)(x-\gamma)$

$= x^3-\beta x^2-\alpha x^2+\alpha\beta x-\gamma x^2+\gamma\beta x+\gamma\alpha x-\alpha\beta\gamma$

$= x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$

Equating coefficients:

$p = -(\alpha+\beta+\gamma)$ ✓

$q = \alpha\beta+\beta\gamma+\gamma\alpha$ ✓

$r = -\alpha\beta\gamma$ ✓

3

(b)(i) $p^2-2q = (-\alpha-\beta-\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ✓

$= (\alpha+\beta+\gamma)(\alpha+\beta+\gamma)-2\alpha\beta-2\beta\gamma-2\gamma\alpha$

$= \alpha^2+\cancel{\alpha\beta}+\cancel{\alpha\gamma}+\cancel{\alpha\beta}+\beta^2+\cancel{\beta\gamma}+\cancel{\alpha\gamma}+\cancel{\beta\gamma}+\gamma^2-\cancel{2\alpha\beta}-\cancel{2\beta\gamma}-\cancel{2\gamma\alpha}$

$= \alpha^2+\beta^2+\gamma^2$ ✓

3

(ii) $2p^2-6q = 2(p^2-3q)$

$= 2(p^2-2q-q)$

$= 2(p^2-2q)-2q$ ✓

$= 2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ✓

$= \alpha^2+\alpha^2+\beta^2+\beta^2+\gamma^2+\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha$

$= (\alpha^2-2\alpha\beta+\beta^2)+(\alpha^2-2\alpha\gamma+\gamma^2)+(\beta^2-2\beta\gamma+\gamma^2)$ ✓

$= (\alpha-\beta)^2+(\beta-\gamma)^2+\cancel{(\alpha-\gamma)^2}\ (\gamma-\alpha)^2$

3

(c) $p^2 < 3q$

$\therefore \cancel{(\alpha+\beta+\gamma)^2 < 3q}$

$p^2 < 3q$

$\therefore p^2 - 3q < 0$

$\therefore 2p^2 - 6q < 0$

$\therefore (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$

$\Rightarrow$ for this to be true, at least one of the terms must have $i$ in it because $i^2 = -1$

2

$\therefore$ ~~either~~

(d) $\cancel{x^3 - 7x^2 + 9x + 1 = 0}$

$\cancel{p^2 < 51}$

$\cancel{\therefore |p| < \sqrt{51}}$

$\cancel{\therefore -\sqrt{51} < p < \sqrt{51}}$

$\cancel{\therefore -\sqrt{51} < -(\alpha+\beta+\gamma) < \sqrt{51}}$

$\cancel{(\alpha+\beta+\gamma)^2 < 3(\alpha\beta+\beta\gamma+\gamma\alpha)}$

$\cancel{\therefore \alpha^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma + \beta^2 + \gamma^2 < 3\alpha\beta + 3\beta\gamma + 3\gamma\alpha}$

$\cancel{\therefore \alpha^2 + \beta^2 + \gamma^2 < \alpha\beta + \beta\gamma + \gamma\alpha}$

$p^2 = 49 \qquad 3q = 51$

$\therefore p^2 < 51$

$\Rightarrow$ has at least one complex root.

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Example Ejemplo 27 | 2 | 7 |

Example Ejemplo 3 | | 3 |

2 ~~(f)(ii)~~

(e)(i) $p^2 \geqslant 3q$

~~$q = 1$~~

$\cancel{p^2 \geqslant 3q}$

$\cancel{\therefore 49 \geqslant 3q}$

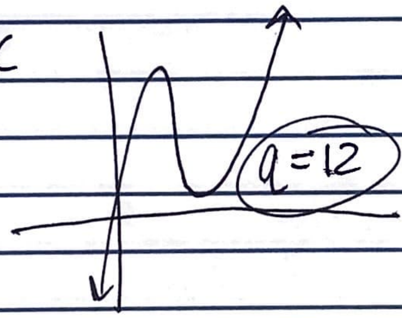
$\cancel{\therefore q \leq 49/3}$

~~$2p^2 \geqslant 3q$~~

~~$\therefore 6q \leq 2 \times 49$~~

~~$\therefore q \leq 16.333$~~

GDC




$\therefore q = 12$ is smallest value.

(ii) each complex root must have a conjugate pair.

(f) (i) $\alpha^2+\beta^2+\gamma^2+\delta^2 = p^2 - \cancel{3q}\ 2q$

~~If $p^2-2q<0$~~

$\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$= \alpha^2+\alpha\beta+\alpha\gamma+\alpha\delta+\alpha\beta+\beta^2+\beta\gamma+\beta\delta+\cdots-2(\alpha\beta+\alpha\gamma+\cdots)$

$= \alpha^2+\beta^2+\gamma^2+\delta^2$

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 = p^2-2q$ 3

(ii) $p^2-2q < 0$

$\therefore p^2 < 2q$ 1

(g) $p^2 < 2q \rightarrow p=-2 \quad q=3$

$\therefore$ LHS $= 4$

RHS $= 6 >$ LHS

$\therefore$ at least one ~~root~~ complex

BOD

(h) (i) $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$

$$\boxed{\begin{array}{ll} p = -9 & q = 24 \\ p^2 = 81 & 2q = \cancel{72}\ 48. \end{array}}$$

As $p^2 > 2q$, ~~there are no~~

$\Rightarrow$ nothing can be deduced. ~~complex roots.~~

(ii) $\therefore x = -1$ is the integer root.

(iii) $(x+1)(ax^3 + bx^2 + cx + d) = \cdots$

$\therefore ax^4 + bx^3 + cx^2 + dx$

$+ax^3 + bx^2 + cx + d$

$= ax^4 + (b+a)x^3 + (c+b)x^2 + (d+c)x + d$

$$\boxed{\begin{array}{l} b + a = -9 \\ \therefore b = -9 - 1 \\ d + c = 22 \\ \therefore c = 22 - (-12) \end{array}}$$

$\therefore d = -12$

$a = 1$

$b = -9 - 1 = -10$

$c = 22 + 12 = 34$

$\Rightarrow$ cubic factor is $x^3 - 10x^2 + 34x - 12$

$\Rightarrow$ from part (c): $(-10)^2 < 3(34)$

$\therefore 100 < 102$

$\therefore$ at least one complex root.

4

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