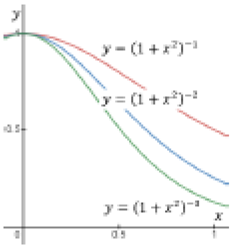
Maths AA HL Semester 2 Exam 2022.

Paper 3 Solutions

Question 1.

(a)



A1A1A1

(b) $\int_0^1 (1+x^2)^{-1}\,\mathrm{d}x = [\arctan x]_0^1 = \frac{\pi}{4}$ **M1A1A1**

(c) $x = \tan\theta \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ **A1**

$\int (1+x^2)^{-2}\,\mathrm{d}x = \int (1+\tan^2\theta)^{-2}\sec^2\theta\,\mathrm{d}\theta$ **A1**

$= \int \cos^2\theta\,\mathrm{d}\theta$ **A1**

$= \int \frac{\cos 2\theta + 1}{2}\mathrm{d}\theta$ **A1**

$= \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + c$ **A1**

$\int_0^1 (1+x^2)^{-2}\,\mathrm{d}x = \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{4}}$ **A1**

$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$ **M1**

$= \frac{\pi}{8} + \frac{1}{4}$ **A1**

(d) (i) $I_n = \int_0^1 (1+x^2)^{-n}\,(1)\mathrm{d}x$

$= [x(1+x^2)^{-n}]_0^1 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,\mathrm{d}x$ **M1A1A1**

$= 2^{-n} + 2n\int_0^1 (1+x^2-1)(1+x^2)^{-n-1}\,\mathrm{d}x$ **M1M1**

$= 2^{-n} + 2n(I_n - I_{n+1})$ **A1**

$2nI_{n+1} = (2n-1)I_n + 2^{-n}$ **A1**

$I_{n+1} = \left(1 - \frac{1}{2n}\right)I_n + \frac{2^{-n-1}}{n}$ **AG**

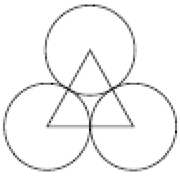
(ii) $I_3 = \left(1 - \frac{1}{4}\right)I_2 + \frac{2^{-3}}{2} = \frac{3}{4}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{16} = \frac{3\pi}{32} + \frac{1}{4}$ **M1A1A1**

(e) $\int_0^1 (x^2 - 2x + 2)^{-3}\,\mathrm{d}x = \int_0^1 (1 + (x-1)^2)^{-3}\,\mathrm{d}x = \int_{-1}^0 (1+x^2)^{-3}\,\mathrm{d}x$ **A1A1**

$= \int_0^1 (1+x^2)^{-3}\,\mathrm{d}x = \frac{3\pi}{32} + \frac{1}{4}$ **A1A1**

Question 2.

(a)



$$\text{Maximum proportion} = \frac{A_{\text{sectors in triangle}}}{A_{\text{triangle}}} \; (M1)$$

$$= \frac{\frac{1}{2}\pi r^2}{\frac{1}{2}(2r)^2 \sin\frac{\pi}{3}} = \frac{\pi}{2\sqrt{3}} \qquad \textbf{\textit{A1A1A1}}$$

(b)

(b) $1+2+3+4+ \ldots + k = \frac{k(k+1)}{2}$

Prove true for $k=1$

$LHS = 1 \qquad RHS = \frac{1(1+1)}{2}$

$= 1$

$\therefore$ true for $k=1$.

Assume true for $n=k$

Prove true for $n=k+1$

ie $1+2+3+4+\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

$LHS = \frac{k(k+1)}{2} + k+1$ from assumption

$= \frac{k(k+1)+2(k+1)}{2}$

$= \frac{k^2+k+2k+2}{2}$

$= \frac{(k+1)(k+2)}{2} = RHS$

$\therefore$ true for $n=k$.

As it is true for $n=k$ and $n=k+1$

it is true for $n=1+1=2$,

$n=2+1=3$

and so on for $n=\mathbb{Z}^+$

(c)

(i) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$ **M1A1**

(ii) $$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 \qquad \textbf{\textit{A1}}$$

$$\sum_{k=1}^{n}k^2 = \frac{1}{3}\left(\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}k^3 - 3\sum_{k=1}^{n}k - \sum_{k=1}^{n}1\right) = \frac{1}{3}\left((n+1)^3 - 1 - \frac{3n(n+1)}{2} - n\right) \qquad \textbf{\textit{M1A1}}$$

$$= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \qquad \textbf{\textit{A1AG}}$$

(d)

(i) $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ **M1A1**

(ii) $$\sum_{k=1}^{n}\frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k\right) \qquad \textbf{\textit{M1A1}}$$

$$= \frac{1}{2}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n(n+1)\right) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \qquad \textbf{\textit{M1A1AG}}$$

(e) Edge length $= n - 1$ $(A1)$

Area of base $= \frac{1}{2}(n-1)^2 \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}(n-1)^2$ **M1A1**

Height $= \sqrt{(n-1)^2 - \left(\frac{2}{\sqrt{3}}\left(\frac{n-1}{2}\right)\right)^2} = \sqrt{\frac{2}{3}}(n-1)$ **M1A1**

Volume $= \frac{1}{3}\left(\frac{\sqrt{3}}{4}\right)(n-1)^2\sqrt{\frac{2}{3}}(n-1) = \frac{\sqrt{2}}{12}(n-1)^3$ **M1A1AG**

(f) Volume of each ball $= \frac{4\pi}{3}\left(\frac{1}{2}\right)^3 = \frac{\pi}{6}$ **A1**

Proportion $= \lim_{n\to\infty} \frac{\frac{\pi}{6}\left(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right)}{\frac{\sqrt{2}}{12}(n-1)^3} = \frac{\pi}{3\sqrt{2}}$ **M1A1A1**