Practice Set A: Paper 3 Mark scheme

1 **a** $F_3 = 2$ A1
$F_4 = 3$ A1
$F_5 = 5$ A1
[3 marks]

b $F_{12} = 144$ A1
This is another Fibonacci number which is a perfect square R1
[2 marks]

c Check that the statement is true for $n = 1$: M1
$LHS = 1^2 = 1$ $RHS = 1 \times 1 = 1$ A1
Assume true for $n = k$
$\sum_{i=1}^{i=k} (F_i)^2 = F_k F_{k+1}$ A1
Then
$\sum_{i=1}^{i=k+1} (F_i)^2 = \sum_{i=1}^{i=k} (F_i)^2 + (F_{k+1})^2$ M1
$= F_k F_{k+1} + (F_{k+1})^2$
$= F_{k+1}(F_k + F_{k+1})$
$= F_{k+1} F_{k+2}$ A1
So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$ therefore it works for all $n \in \mathbb{Z}^+$. R1
[6 marks]

d Smallest such k is 5 A1
Check that the statement is true for $n = 5$ and $n = 6$: M1
$F_5 = 5, F_6 = 8$ A1
Assume true for $n = k$ and $n = k + 1$ M1
$F_k \geqslant k, F_{k+1} \geqslant k + 1$ A1
Then
$F_{k+2} = F_k + F_{k+1} \geqslant 2k + 1 > k + 2$ since $k > 1$ A1
So if the statement works for $n = k$ and $n = k + 1$ then it works for $n = k + 2$ and it works for $n = 5$ and $n = 6$ therefore it works for all integers $n \geqslant 5$ R1
[7 marks]

e $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$ M1
Dividing by α^n since $\alpha \neq 0$: $\alpha^2 = \alpha + 1$ or $\alpha^2 - \alpha - 1 = 0$ A1
Using the quadratic formula $\alpha = \dfrac{1 \pm \sqrt{5}}{2}$ A1A1
[4 marks]

f $F_n + F_{n+1} = A\alpha_1^n + B\alpha_2^n + A\alpha_1^{n+1} + B\alpha_2^{n+1}$ M1
$A(\alpha_1^n + \alpha_1^{n+1}) + B(\alpha_2^n + \alpha_2^{n+1})$
$A\alpha_1^{n+2} + B\alpha_2^{n+2} = F_{n+2}$ A1
[2 marks]

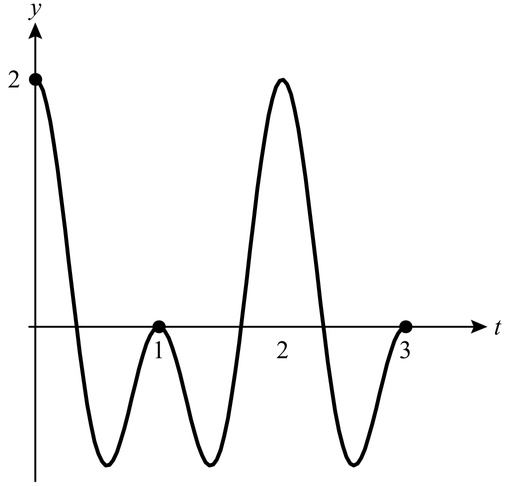
g $F_1 = A\alpha_1 + B\alpha_2 = 1$ A1
$F_2 = A\alpha_1^2 + B\alpha_2^2 = 1$ A1
Since $\alpha^2 = \alpha + 1$:
$A(\alpha_1 + 1) + B(\alpha_2 + 1) = 1$ M1
$A\alpha_1 + B\alpha_2 + A + B = 1$
$A + B = 0$
$A = -B$
Substituting into first equation: $A(\alpha_1 - \alpha_2) = 1$
$A = \dfrac{1}{\alpha_1 - \alpha_2} = \dfrac{1}{\sqrt{5}}$
$F_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$ A1
[4 marks]

h As n gets large, $\left(\dfrac{1-\sqrt{5}}{2}\right)^n \to 0$ M1
$\dfrac{F_{n+1}}{F_n} \approx \dfrac{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right)}{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)} = \dfrac{1+\sqrt{5}}{2}$ A1
[2 marks]
Total [30 marks]

2 **a** 2 A1

[1 mark]

b **i**



A1

ii 2 A1

[2 marks]

c **i** $A = 4$ A1

$B = 8$ A1

$C = 20$ A1

ii $T = 2n$ A1

[4 marks]

d $f(t + 2n) = \cos\left(\pi\left(t + 2n\right)\right) + \cos\left(\pi\left(1 + \frac{1}{n}\right)\left(t + 2n\right)\right)$ M1

$= \cos\left(\pi t + 2n\pi\right) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t + 2\pi\left(n + 1\right)\right)$ A1

$= \cos\left(\pi t\right) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right) = f\left(t\right)$

Since $\cos(x + 2\pi k) = \cos x$ if k is an integer R1

[3 marks]

e **i** $\cos(A + B) + \cos(A - B)$

$= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$

$= 2 \cos A \cos B$ A1

ii If $P = A + B$ and $Q = A - B$ then

$A = \frac{P + Q}{2}, B = \frac{P - Q}{2}$ M1

$\cos P + \cos Q = 2 \cos\left(\frac{P + Q}{2}\right) \cos\left(\frac{P - Q}{2}\right)$ A1

[3 marks]

f $f(t) = 2 \cos\left(\pi\left(1 + \frac{1}{2n}\right)t\right) \cos\left(\frac{\pi}{2n} t\right)$ A1

The graph of $\cos\left(\pi\left(1 + \frac{1}{2n}\right)t\right)$ provides the high frequency oscillations. R1

Their amplitude is determined/enveloped by the lower frequency curve $\cos\left(\frac{\pi}{2n} t\right)$ R1

[3 marks]

g $\frac{d^2x}{dt^2} = -\omega^2 \cos \omega t$ M1A1

The DE becomes:

$-\omega^2 \cos \omega t + 4 \cos \omega t = 0$ M1

This is solved when $\omega^2 = 4$ so $\omega = 2$ A1

[4 marks]

h $\frac{d^2x}{dt^2} = -4\cos 2t - k^2 g(k)\cos kt$ M1

The DE becomes:

$-4\cos 2t - k^2 g(k)\cos kt + 4\cos 2t + 4g(k)\cos kt = \cos kt$ M1

$(4g(k) - k^2 g(k))\cos kt = \cos kt$

This is true for all t when $g(k)(4 - k^2) = 1$

$g(k) = \frac{1}{4 - k^2}$ A1

[3 marks]

i When $k = 2$ A1

Since $\frac{1}{4 - k^2} \to \infty$ as $k \to 2$ R1

[2 marks]

Total [25 marks]