Mathematics: analysis and approaches
Higher level
Paper 3

1 hours

Candidate session number

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where *a* = 2.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+, n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$
- the even values $n = 2$ and $n = 4$

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+, n > 1$.

(c) Show that $f_n{}'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(f) Hence, or otherwise, show that $f_n{}'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n{}'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n , where $n > 1$ and $a \in \mathbb{R}^+$ [3]

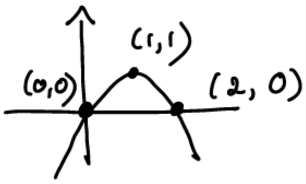
(ii) a point of inflexion with zero gradient for odd values of n , where $n > 1$ and $a \in \mathbb{R}^+$ [2]

Consider the graph of $y = x^n(a-x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a-x)^n = k$ has four solutions for x . [5]

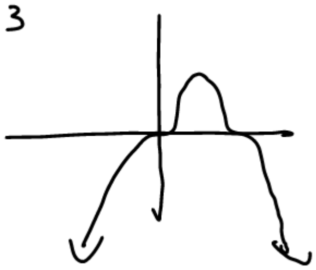
Question 1: $f_n(x) = x^n(2-x)^n$

a) $f_1(x) = x(2-x)$

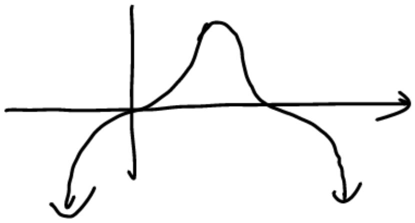


A1: shape
A1: local max
A1: x-intercepts

b) $f_3(x) = x^3(2-x)^3$



$f_5(x) = x^5(2-x)^5$



$f_4(x) = x^4(2-x)^4$

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$	1 - A1	0 - A1	2 - A1
$n = 2$ and $n = 4$	1 - A1	2 - A1	0 - A1

c) $f_n(x) = x^n(a-x)^n \quad a \in \mathbb{R}^+,\ n \in \mathbb{Z}^+$, $n > 1$

$u = x^n$ M1 $\qquad v = (a-x)^n$ M1

$u' = nx^{n-1} \qquad v' = -n(a-x)^{n-1}$

$f'(x) = u'v + v'u$

$= nx^{n-1}(a-x)^n - nx^n(a-x)^{n-1}$ M1

$= nx^{n-1}(a-x)^{n-1}(a-x-x)$ M1

$= nx^{n-1}(a-x)^{n-1}(a-2x)$ A1

d) $f'_n(x) = 0 = nx^{n-1}(a-x)^{n-1}(a-2x)$

$\boxed{x=0}$ $\quad a-x=0 \quad a-2x=0$ → M1

$\boxed{a=x}$ $\quad 2x=a$

$\boxed{x=\frac{a}{2}}$ → A1 : answer

e) $f_n\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^n\left(a-\frac{a}{2}\right)^n$

$= \left(\frac{a}{2}\right)^n\left(\frac{2a-a}{2}\right)^n = \left(\frac{a}{2}\right)^{2n}$ → M1

$a \in \mathbb{R}^+ \therefore a>0$ and $n>1$ → R1

$\therefore \left(\frac{a}{2}\right)^{2n} > 0$ → A1

as $f_n\left(\frac{a}{2}\right) > 0$, the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$

is always above x/ horizontal

f) $f'_n\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{a}{4}\right)^{n-1}\left(a-2\times\frac{a}{4}\right)$

$= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{3a}{4}\right)^{n-1}\left(\frac{a}{2}\right)$

$= n\left(\frac{a}{4}\right)^{n-1}\times 3^{n-1}\left(\frac{a}{4}\right)^{n-1}\times 2\left(\frac{a}{4}\right)$ M1

$= 2n\times 3^{n-1}\left(\frac{a}{4}\right)^{n-1+n-1+1}$

$= 2n\times 3^{n-1}\left(\frac{a}{4}\right)^{2n-1}$

since $a>0$, $n\in\mathbb{Z}^+ (n>0)$ ⇒ $\left(\frac{a}{4}\right)^{2n-1} > 0$ and $2^n > 0$, $3^{n-1} > 0$ R1

$\therefore f'_n\left(\frac{a}{4}\right) > 0$

g) i) a local min for even values of n. $(n>1,\ a\in\mathbb{R}^+)$

$f_n'(-1) = n(-1)^{n-1}(a+2)(a+1)^{n-1}$

If n is even: $n-1$ is odd $\Rightarrow (-1)^{n-1} < 0$] M1

but $(a+1)^{n-1} > 0$ as $a \in \mathbb{R}$

$\Rightarrow f_n'(-1) < 0$

$f_n'(0) = n\,0^{n-1}(a-2\times0)(a-0)^{n-1}$

$= 0$

From (f) $\Rightarrow f_n'\left(\frac{a}{4}\right) > 0$

R1

$f_n'(-1)$	$f_n'(0)$	$f'\left(\frac{a}{4}\right)$
$-$	0	$+$

$\therefore$

$\Rightarrow f'(x)$ changes from negative to positive, hence $(0,0)$ is a local min] A1

ii) If n is odd: $f_n'(-1) = n(-1)^{n-1}(a+2)(a+1)^{n-1}$

$\therefore n-1$ is even $\therefore (-1)^{n-1}$ is positive

$\therefore f_n'(-1)$ is positive.] R1

$f_n'(-1)$	$f_n'(0)$	$f_n'\left(\frac{a}{4}\right)$
$+$	0	$+$

A 1

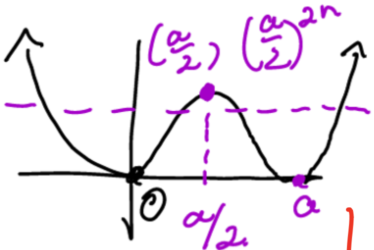
$f'(x)$ does not change signs at $x=0$

$\therefore (0,0)$ must be an inflection point with zero gradient

h) $y = x^n(a-x)^{n} - k \qquad n \in \mathbb{Z}^+, \; a \in \mathbb{R}^+, \; k \in \mathbb{R}$

when $y = 0 \quad \therefore \quad k = x^n(a-x)^n$ $\Rightarrow$ need 4 intersection points.

$\Rightarrow$ can only happen if n is even



From (d) $\therefore$ k must be even and

$$0 < k < \left(\frac{a}{2}\right)^{2n} \Rightarrow A1$$

A1

2 marks for valid method

2. [Maximum mark: 24]

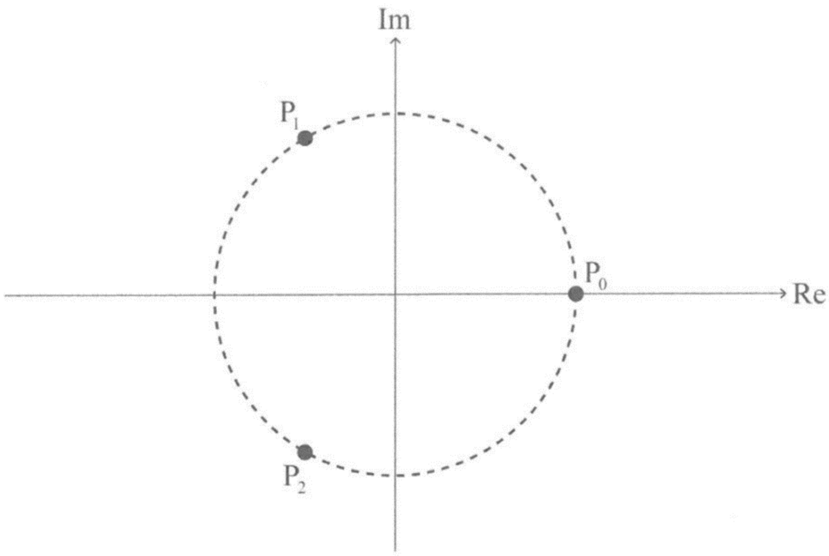
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1 and ω . On an Argand diagram, the root 1 can be represented by a point P_0 , P_1 , P_2 , …, P_{n-1} , respectively, on an Argand diagram.

For example, the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1 and ω . On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1 .

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1 , ω and ω^2 . On the following Argand diagram, the points P_0 , P_1 and P_2 lie on a circle of radius 1 unit with centre O $(0, 0)$.



(a) (i) Show the $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

a) i) $(\omega - 1)(\omega^2 + \omega + 1)$ M1 A1

$= \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1 = \omega^3 - 1$

ii) ω is a root of the equation $z^3 = 1$
sub $z = \omega$:

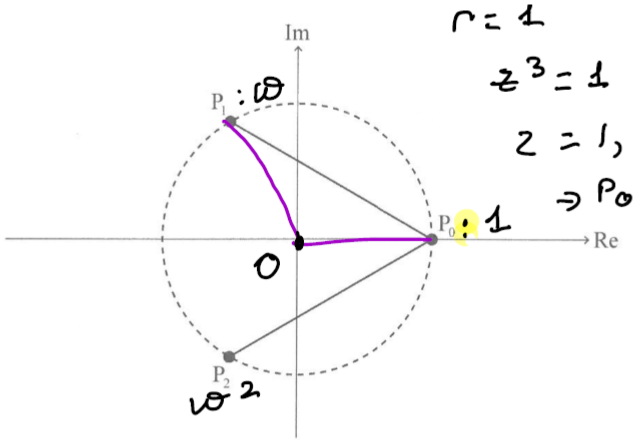
$\therefore \omega^3 = 1$ M1

$\omega^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$ but $\omega \neq 1$ as R1
1, ω and ω^2 are distinct roots $\therefore$ $\omega^2 + \omega + 1 = 0$

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



$r = 1$

$z^3 = 1$

$z = 1, \omega, \omega^2$

$\Rightarrow P_0, P_1, P_2$

P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 and ω^3 .

(c) By factorising $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(b) $\overrightarrow{P_0P_1} = \overrightarrow{P_0O} + \overrightarrow{OP_1}$

$= -1 + \omega \Rightarrow P_0P_1 = |-1+\omega|$ M1

$\overrightarrow{P_0P_2} = \overrightarrow{P_0O} + \overrightarrow{OP_2}$

$= -1 + \omega^2 \Rightarrow P_0P_2 = |-1+\omega^2|$

$|\overrightarrow{P_0P_1}||\overrightarrow{P_0P_2}| = |-1+\omega||-1+\omega^2|$

$= |(\omega-1)||(\omega^2-1)|$

$= |(\omega-1)(\omega^2-1)|$ M1

$= |\omega^3-\omega^2-\omega+1|$

$= |\omega^3-(\omega^2+\omega)+1|$

$= |\omega^3+1+1| = |\omega^3+2| = 3$

From (a)

$\omega^2+\omega+1=0$

$\therefore \omega^2+\omega=-1$

R1

as $z = \omega \Rightarrow z^3-1=0$

$\therefore \omega^3 = 1$

c) $z^4 - 1 = (z^2-1)(z^2+1)$ M1

$= (z-1)(z+1)(z^2+1)$

$= (z-1)(z^3+z^2+z+1)$ A1

Roots of $z^4 = 1$ are $1, \omega, \omega^2, \omega^3$.

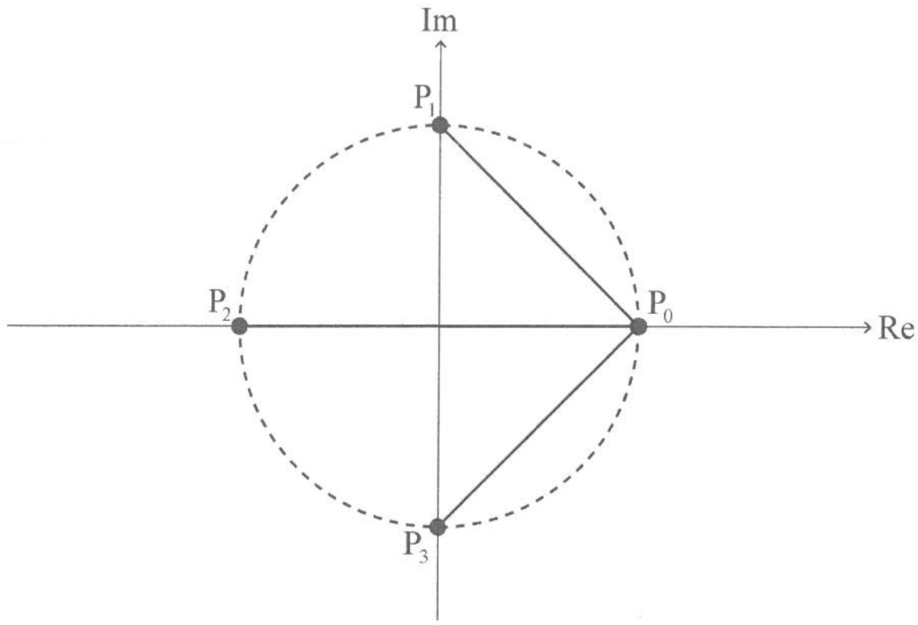
sub $z = \omega$ $\therefore$ $\omega^4 - 1 = 0$

$(\omega-1)(\omega^3+\omega^2+\omega+1) = 0$

as $z = \omega$, $\omega \neq 1$ $\therefore$ $\omega^3+\omega^2+\omega+1=0$

(Question 2 continued)

On the following Argand diagram, the points P_0 , P_1 , P_2 and P_3 lie on a circle of radius 1 unit with centre O $(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1 , ω , ω^2 , ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 , …, ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0 , P_1 , P_2 , …, P_{n-1} respectively where $[P_0P_1]$, $[P_0P_2]$, …, $[P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre O $(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times \ldots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω . [2]

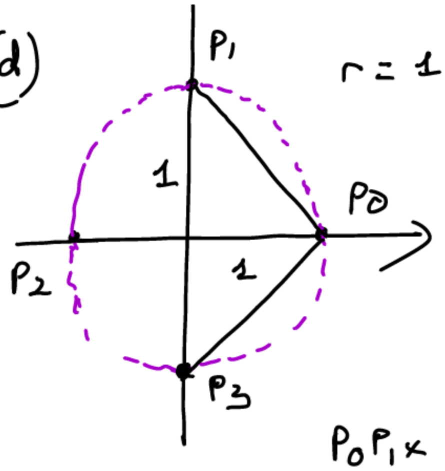
(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω . [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \cdots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

(d)



$r = 1.$

M1

$P_0P_1 = \sqrt{1^2+1^2} = \sqrt{2}$ (Pythagoras theorem

$P_0P_2 = 2$ M1

M1

$P_0P_3 = \sqrt{1^2+1^2} = \sqrt{2}$ (Pythagoras theorem)

$P_0P_1 \times P_0P_2 \times P_0P_3 = \sqrt{2} \times 2 \times \sqrt{2}$

$= 4$ A1

e) $n = 5$ $\therefore z^5 = 1$ has roots: $1, \omega, \omega^2, \omega^3, \omega^4$.

$P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$

in general case $n \geqslant 2$

$\therefore z^n = 1$ roots: $1, \omega, \omega^2 \ldots \omega^{n-1}$

$\therefore P_0P_1 \times P_0P_2 \ldots P_0P_{n-1} = n$ A1.

f) i) $P_0P_1 = |1-\omega| = |\overrightarrow{P_0P_1}| = |\overrightarrow{P_1P_0}|$

$P_0P_2 = |1-\omega^2|$ A1 $P_0P_3 = |1-\omega^3|$ A1

ii) $P_0P_{n-1} = |1-\omega^{n-1}|$ A1

g) i) $\underbrace{z^n - 1}_{\downarrow} = (z-1)(z^{n-1} + z^{n-2} + \dots + z + 1)$

$z^n = 1 \quad \therefore$ Roots: $1, \omega, \omega^2 \dots \omega^{n-1}$ R1

M1

$$z^n - 1 = 0$$
$$= (z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$$
$$= (z-1)(z^{n-1} + z^{n-2} \dots + z + 1)$$

A1

$$\therefore z^{n-1} + z^{n-2} + \dots + z + 1 = (z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$$

ii) from g(i)

$$z^{n-1} + z^{n-2} + \dots + z + 1 = (z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$$

M1

$$\therefore |z^{n-1} + z^{n-2} + \dots z + 1| = |(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})|$$

From (f) $P_0P_1 = 1-\omega \quad P_0P_2 = 1-\omega^2 \dots$

Substitute $z = 1 \therefore$ ⇒ M1

$$|\underbrace{1^{n-1} + 1^{n-2} + \dots 1 + 1}_{n \text{ numbers of } 1}| = |(1-\omega)(1-\omega^2)\dots(1-\omega^{n-1})|$$

⇒ R1

A1

$$n = |(1-\omega)(1-\omega^2)\dots(1-\omega^{n-1})|$$

$$n = P_0P_1 \times P_0P_2 \times \dots P_0P_{n-1}$$