Mathematics: analysis and approaches
Higher level
Paper 3

~~TZ0~~ TZ1

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

2221–7108

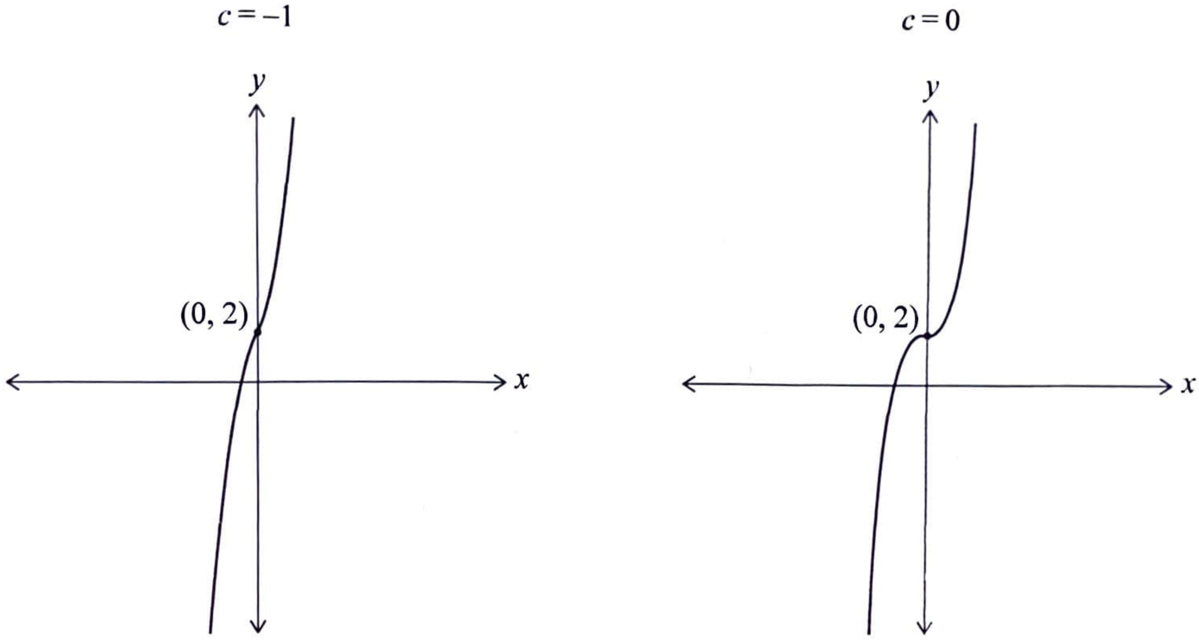
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c,\ d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

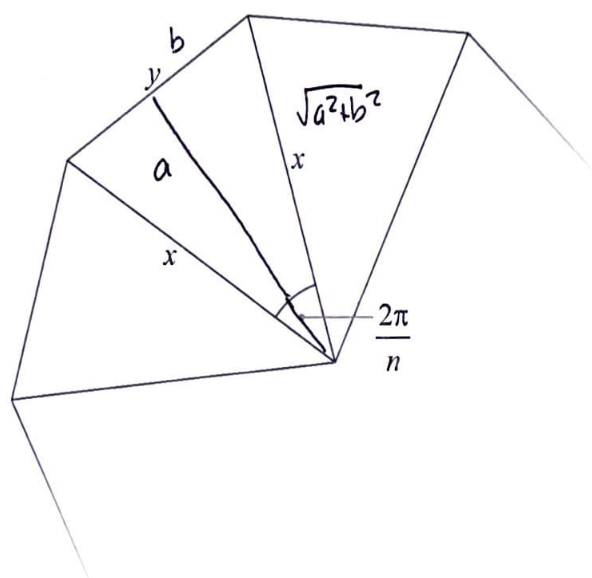
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2+b^2}$, where $a \geq b$, such that $A = P$.

(f) Show that $a=\frac{8}{b-4}+4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a,\ b,\ A,\ P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References:

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①

$\frac{44}{55} = 0.8$

Q1: $\frac{24}{27}$ Q2: $\frac{30}{28}$

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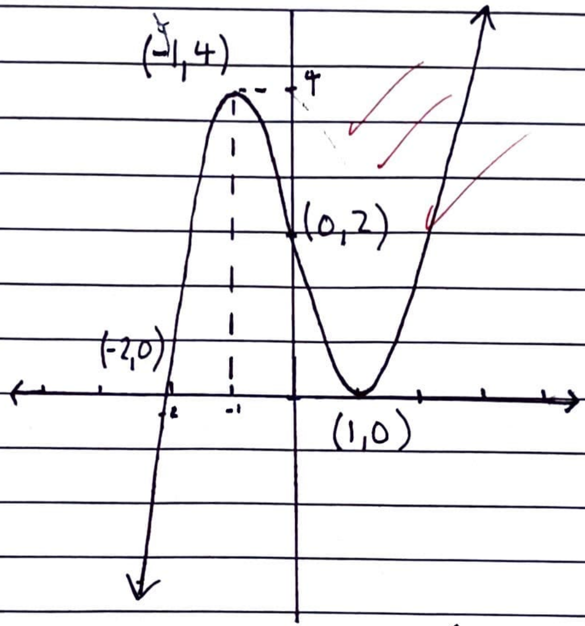
1 2 3 4 5 6 7 8 9 10

1

a) i) $f(x) = x^3 - 3x + 2$



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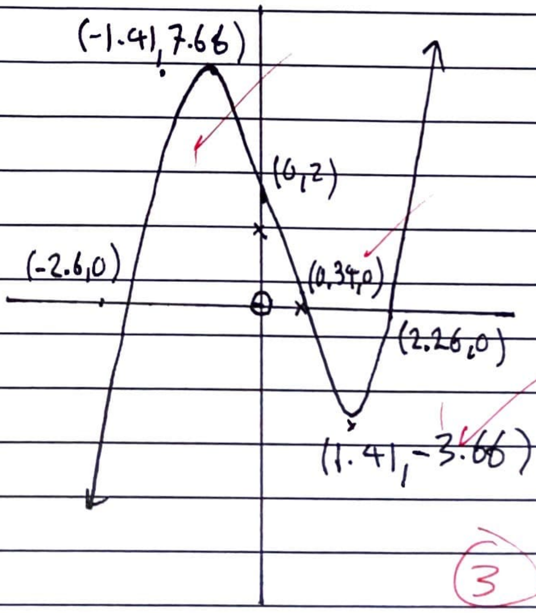
ii) $f(x) = x^3 - 6x + 2$



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b) $f(x) = x^3 - 3cx + 2$

$\therefore f'(x) = 3x^2 - 3c$ (1)

c) i) ~~$f'(x) = 0 = 3x^2 - c$~~

~~$\therefore x^2 = c$~~

~~$\therefore x = \pm\sqrt{c}$~~

~~$\therefore c = x^2$~~

~~$c \geq 0$~~ $c = 0$ (1)

ii) $c > 0$ $3x^2 - 3c = 0$ $c > 0$ (1)

iii) $c < 0$ (1)

d) i) $f'(x) = 3x^2 - 3c = 0$

$\therefore x^2 = c$

$\therefore x = \pm\sqrt{c}$

When $x = -\sqrt{c}$, $f(-\sqrt{c}) = (-\sqrt{c})^3 + 3(\sqrt{c})c + 2$

$= -\sqrt{c^3} + 3c\sqrt{c} + 2$

$= -c^{3/2} + 3c^{3/2} + 2$

$\therefore y = 2c^{3/2} + 2$ (3)

{NOTE: THIS IS MAX - DETERMINED BY G.D.C}

ii) When $x = \sqrt{3}$, $f(\sqrt{c}) = (\sqrt{c})^3 - 3(\sqrt{c})c + 2$

$= c^{3/2} - 3c^{3/2} + 2$

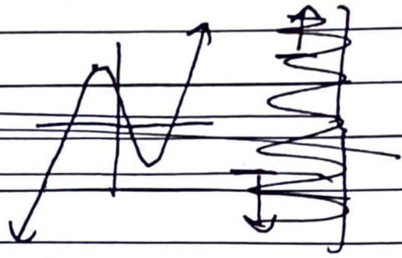
$= -2c^{3/2} + 2$ (1)

{NOTE: MIN determined graphically}

8

e) When $c>0$, there is one local min and one local max.

~~i) 1 intercept~~ :



∴

~~$f(x) = x^3 - 3cx + 2$~~

2

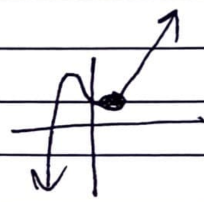
e) i) one intercept ∴ local min > 0

∴ $-2c^{3/2} + 2 > 0$

∴ $2c^{3/2} < 2$

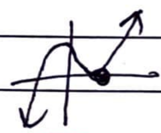
∴ $c^{3/2} < 1$

∴ $c < 1$



∴ $0 < c < 1$ ~~{note: this includes touching, but not crossing the x-axis}~~

ii) two intercepts : local min $= 0$



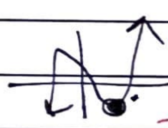
∴ $-2c^{3/2} + 2 = 0$

2

∴ $2c^{3/2} = 2$

∴ $c = 1$

iii) three intercepts: local min < 0



∴ ~~$c > 0$~~ $c > 1$

2

6

f) $g(x) = x^3 - 3cx + d$ $\quad x \in \mathbb{R}$

$c, d \in \mathbb{R}$

$\Rightarrow$ $\therefore$ $g'(x) = 3x^2 - 3c$ {same as from part b}

$\Rightarrow$ Finding y-coord of local min:

~~$y = 2c^{3/2}$~~ $\Rightarrow x = \pm\sqrt{c}$

$\therefore g(\sqrt{c}) = (\sqrt{c})^3 - 3\sqrt{c}(\cancel{c}) + d$

$= c^{3/2} - 3c^{3/2} + d$

$= -2c^{3/2} + d$

$\Rightarrow$ one x-intercept occurs when $g(\sqrt{c}) > 0$

(4)

$\therefore -2c^{3/2} + d > 0$

$\therefore 2c^{3/2} < d$

$\therefore d > 2c^{3/2}$, $c > 0$

OR $c \leq 0$ for all d.

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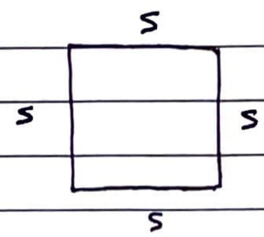
1 2 3 4 5 6 7 8 9 10

2

a)



$\therefore 4s = s^2$

$\therefore s^2 - 4s = 0$

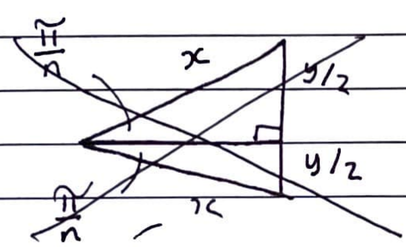
$\therefore s(s-4) = 0$

$\therefore \cancel{s=0},\ s = 4$

$\therefore s = 4 \quad \{s > 0\}$

3

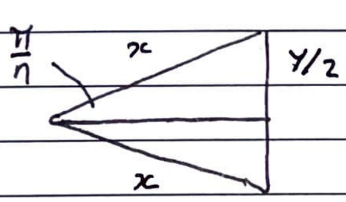
b)



$A_T = \frac{1}{2}(x)(x)\sin(2\pi/n)$ 1

$\therefore A_T = \frac{1}{2}x^2 \sin(2\pi/n)$

c)



$\sin\frac{\pi}{n} = \frac{y/2}{x}$

$\therefore y/2 = x\sin \pi/n$

$\therefore y = 2x \sin \pi/n$

2

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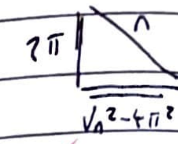
d)



$$A = nA_T$$
$$= n \times \tfrac{1}{2}x^2 \sin\left(\tfrac{2\pi}{n}\right) \quad ..(1)$$

$$P = n \times y$$
$$= 2nx \sin \pi/n \quad ..(2)$$

If $A = P$, then

$$\tfrac{1}{2}nx^2 \sin\left(\tfrac{2\pi}{n}\right) = 2nx\sin\left(\pi/n\right)$$
$$\therefore\ x\sin(2\pi/n) = 4\sin(\pi/n)$$
$$\therefore\ x = 4\frac{\sin(\pi/n)}{\sin(2\pi/n)}$$

$$\cancel{\therefore A = \tfrac{1}{2}n\left(4\frac{\sin(\pi/n)}{\sin(2\pi/n)}\right)^2 \sin\left(\tfrac{2\pi}{n}\right)}$$
$$\cancel{= \tfrac{1}{2}\times 16n \times \frac{\sin^2(\pi/n)}{\sin^2(2\pi/n)}\sin(2\pi/n)}$$
$$\cancel{= 8n\frac{\sin^2(\pi/n)}{\sin(2\pi/n)}}$$

$$x = 4\frac{\sin(\pi/n)}{2\sin \pi/n \cos \pi/n}$$

$$= 2\tan\frac{\pi}{n} \times \frac{1}{\sin \pi/n} \quad ..(3)$$

$$(3) \to (2):\ P = 2n\left(\frac{2\tan \pi/n}{\sin \pi/n}\right)\sin \pi/n$$
$$= 4n\tan \pi/n$$

$$\therefore A = P = 4n\tan \pi/n$$

7

7

e) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

i) $4n\tan\frac{\pi}{n} = 4n\left[\frac{\pi/n}{} + \frac{(\pi/n)^3}{3} + \frac{2(\pi/n)^5}{15}\right]$

$= 4n\left(\frac{\pi}{n} + \frac{(\pi/n)^3}{3} + \frac{2(\pi/n)^5}{15} + \dots\right)$

$= 4n\left(\frac{\pi}{n} + \frac{\pi^3}{3n^3} + \frac{2\pi^5}{15n^5} + \dots\right)$

$= 4\pi + \frac{4\pi^3}{3n^2} + \frac{8\pi^5}{15n^4} + \dots$

$\therefore \lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(4\pi + \frac{4\pi^3}{3n^2} + \frac{8\pi^5}{15n^4} + \dots\right)$

$= 4\pi$

3

ii) As $n \to \infty$, $y \to 0$

↳ the perimeter / area converges to 4π

↳ the polygon approaches a circle with circumference 4π units and area 4π units^2

2

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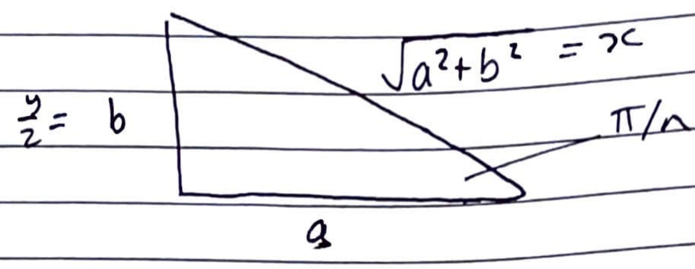
δ)



$\Rightarrow \quad P = a+b+\sqrt{a^2+b^2}$

$\Rightarrow \quad A = \frac{1}{2}ab$

② ③

If $\quad P = A, \quad$ then

$\cancel{\frac{1}{2}ab = a+b+\sqrt{a^2+b^2}}$

$\therefore \cancel{\frac{1}{4}a^2b^2 = a^2+b^2+a^2+b^2}$

$\therefore \cancel{a^2b^2 = 4a^2+4a^2+4b^2+4b^2}$

$\cancel{\frac{1}{2}ab =}$

$\cancel{P = x+\frac{y}{2}+a}$

$a+b+\sqrt{a^2+b^2} = \frac{1}{2}ab$

$\therefore \quad a^2+b^2 = \left(\frac{1}{2}ab-(a+b)\right)$

$\therefore \quad a^2+b^2 = \frac{1}{4}a^2b^2 - ab(a+b) + (a+b)^2$

$\therefore \quad \cancel{a^2+b^2} = \frac{1}{4}a^2b^2 - a^2b - ab^2 + \cancel{a^2} + 2ab + \cancel{b^2}$

$\therefore \quad 0 = \frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab$

$= ab\left(\frac{1}{4}ab - a - b + 2\right) = 0$

$\therefore \quad a\left(\frac{1}{4}b-1\right) = b-2$

$\therefore \quad a(b-4) = 4b-8$

$a = \frac{4b-8}{b-4}$

$= \frac{4b-16}{b-4} + \frac{8}{b-4}$

3/

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g) i) $a = \frac{8}{b-4} + 4$

$\therefore ab - 4a = 8 + 4$

$\therefore ab - 4a = 12$

$\therefore 4a = ab - 12$

$\therefore 4 = b - 12/a$

$\therefore$

$\therefore b = 8, a = 3$ .. ~~(2)~~ .. (1) $\sqrt{a^2+b^2} = \sqrt{73}$

OR

$b = 7, a = 4$ .. (2) $\sqrt{a^2+b^2} = \sqrt{65}$

g ii) $P = a + b + \sqrt{a^2+b^2}$

$A = \frac{1}{2}ab$

for (1): $P = 3 + 8 + \sqrt{3^2+8^2}$

$= 11 + \sqrt{73}$ units

$A = \frac{1}{2}(3)(8)$

$= 12$ units^2

for (2): $P = 4 + 7 + \sqrt{4^2+7^2}$

$= 11 + \sqrt{65}$ units

$A = \frac{1}{2}(4)(7)$

$= 14$ units^2