Markscheme

Additional Practice

Counting Principles (Calculator)

ID: 4005

Mathematics: analysis and approaches

Higher level

1. There is one way of placing the mathematics books on the shelf. This leaves seven gaps into which the science books can be placed. A1 R1

The total number of ways is therefore ${}^{7}C_{3} = 35$. M1A1

2. There are $6!$ ways of arranging the science books. A1

This creates 7 spaces into which we can place the math books. A1

There are ${}^{7}P_{4}$ ways of placing the math books in these spaces. A1

So the total number of ways is $6! \times {}^{7}P_{4} = 604800$. M1A1

3. For the two friends to be able to sit together the first six people must leave one of the 4 sets of seats free. A1

The six people can sit in the remaining seats in 6! different ways. A1

The two friends can sit in the empty set of seats in 2 ways. A1

Multiply these values together to determine the total number of ways in which the two friends can sit together. M1

$$4 \times 6! \times 2 = 5760$$

The total number of ways in which all eight people can sit is $8! = 40320$. A1

So the probability of the two friends being able to sit together is

$$\frac{5760}{40320} \approx 0.143$$

4. There are $5! = 120$ ways for the grade 10 students to stand in a line. A1

This creates six spaces for the grade 11 students. R1

There are $6! = 720$ ways the grade 11 students can stand in these spaces. A1

The probability is therefore

$$\frac{120 \times 720}{11!} = \frac{1}{462} \approx 0.00216$$ M1A1

5. The mathematics books can be placed in $6! = 720$ ways. A1

These leaves 7 gaps into which the science books can be placed. A1

The science books can be placed in $3! = 6$ ways. A1

So the probability is

$$\frac{720 \times 7 \times 6}{9!} = \frac{1}{12}$$ M1A1

6. There are 4! to arrange the mathematics books. A1

There are 3! ways to arrange the science books. A1

There are 2! ways to arrange the English books. A1

There is only 1 dictionary.

There are 4! ways to arrange the book types. A1

Multiply all of these together M1

$$4! \times 3! \times 2! \times 1 \times 4! = 6912$$ A1

7. (a)

(i) 2×7 A1

(ii) $2 \times 3 \times 3$ A1

(iii) $2 \times 3 \times 5$ A1

(b) $2 \times 3 \times 3$ A1

$1 \times 2 \times 9$

$1 \times 3 \times 6$ A1

$1 \times 1 \times 18$

So there are four in total. A1

8. The five Mathematics books can be placed on the shelf in $5! = 120$ ways. A1

After the Mathematics books have been placed there are six spaces where we can place a single English book. These are shown by the vertical lines below. R1

$$| \ \mathrm{M} \ | \ \mathrm{M} \ | \ \mathrm{M} \ | \ \mathrm{M} \ | \ \mathrm{M} \ |$$

There are ${}_6P_3 = 120$ ways in which we can insert the English books into these spaces. A1

So the total number of ways of placing all the books so that no English books are next to each other is $120 \times 120 = 14400$ ways. A1

The total number of ways of placing the books in any order is $8! = 40320$ ways. A1

So the probability of placing them so that no English books are together is

$$\frac{14400}{40320} \approx 0.357$$ A1

9. (a) There are 3 different rows at which they can sit. A1

On a row there are $3 \times 2 = 6$ ways in which the two friends can sit. A1

There are 7! ways in which the other people can sit. A1

The total is therefore 90720. A1

(b) $9! - 90720 = 272160$ M1A1

10. (a) There are ${}^6C_2$ ways of choosing both two vertical lines and two horizontal lines. R1

The total number of rectangles is therefore

$${}^6C_2 \times {}^6C_2 = 225$$ M1A1

(b) We have

$${}^{n+1}C_2 \times {}^{n+1}C_2 = \frac{(n+1)!}{2!(n-1)!} \times \frac{(n+1)!}{2!(n-1)!} = \frac{n^2(n+1)^2}{4}$$ M1A1A1

11. (a) There are 2 ways in which they can stand in order of height: tallest to shortest or shortest to tallest. A1

Determine the number of ways that the four children can stand randomly in a line. e.g. by listing the sample space, or calculating 4! This will give a value of 24. M1 A1

So the probability of them standing in order of height is $\frac{2}{24}=\frac{1}{12}$. A1

(b) Subtract the answer to part (a) from 1. M1

$$1-\frac{1}{12}=\frac{11}{12}$$ A1

12. There are $5! = 120$ ways to arrange the grade 11 students. A1

There are $3! = 6$ ways to arrange the grade 12 students. A1

There are 2 ways to arrange the grades. A1

There are $8! = 40320$ ways to arrange the students. A1

The probability is therefore

$$\frac{120 \times 6 \times 2}{40320} = 0.0357$$ M1A1

13. The number of ways of placing the mathematics books together is $5!$ A1

The number of ways of placing the science books together is $3!$ A1

The number of ways of ordering the two subjects is $2!$. A1

The number of ways of placing the books in any order is $8!$ A1

The probability is therefore

$$\frac{5!3!2!}{8!} = \frac{1}{28} \approx 0.0357$$ M1A1

14. If there are 3 calculus questions the total number of tests is

$$^{5}C_{3} \times {}^{8}C_{7} = 80$$ M1A1

If there are 4 calculus questions the total number of tests is

$$^{5}C_{4} \times {}^{8}C_{6} = 140$$ M1A1

If there are 5 calculus questions the total number of tests is

$$^{5}C_{5} \times {}^{8}C_{5} = 56$$ M1A1

The total number of tests is therefore 276. A1

15. (a)

(i) $\dfrac{200}{1000} = \dfrac{1}{5}$ A1

(ii) Since

$$\frac{999}{5} = 199.8$$ M1

The probability is

$$\frac{199}{1000}$$ A1

(b) The probability the first 50 quesses are all incorrect is $\left(\dfrac{999}{1000}\right)^{50}$. A1

So the probability one of them is correct is

$$1 - \left(\frac{999}{1000}\right)^{50} = 0.0488$$ M1A1

(c) The final digit has to be 5 or 0. R1

If the final digit is 5 then there are $9 \times 8 = 72$ choices for the first two. A1

If the final digit is 0 then there are $9 \times 8 = 72$ choices for the first two. A1

The total number of ways is therefore 144. A1

16. (a) 1111, 2222, 3333, 4444, 5555, 6666 A1A1

(b) $6 \times \left(\frac{1}{6}\right)^4 = \frac{1}{216}$ M1A1

(c)

(i) Use binompdf with trials: 4, *p*: 1/6, *x*-value: 3. M1

The probability is therefore 0.015432. A1

(ii) $\frac{{}^4C_2}{6^4} = 0.0046297$ M1A1

(d) The probability of getting three of the same value is

$$6 \times 0.015432 = 0.092592$$ A1

The probability of getting two lots of two values is

$${}^6C_2 \times 0.0046297 = 0.069444$$ A1

The overall probability is therefore

$$0.092592 + 0.069444 = 0.162$$ M1A1

(e)

(i) $\frac{6 \times 5 \times 4 \times 3}{6^4} = 0.278$ M1A1

(ii) $1 - 1/216 - 0.162 - 0.278 = 0.555$ M1A1

17. (a) If there are no restrictions on the values of a and b then there are ${}^{10}\mathrm{P}_3 = 720$ ways of choosing the values of a, b and c. M1A1

We can only accept half of these as the other have will have $a > b$. So the total number of ways is 360. A1

(b)

(i) $x = c$ A1

(ii) We have

$$x^2 - (a+b)x + ab = (x-c)(x+c-a-b) + c^2 - c(a+b) + ab$$ M1

So

$$\frac{x^2 - (a+b)x + ab}{x-c} = x + c - a - b + \frac{c^2 - c(a+b) + ab}{x-c}$$ A1

The asympotote is therefore

$$y = x + c - a - b$$ A1

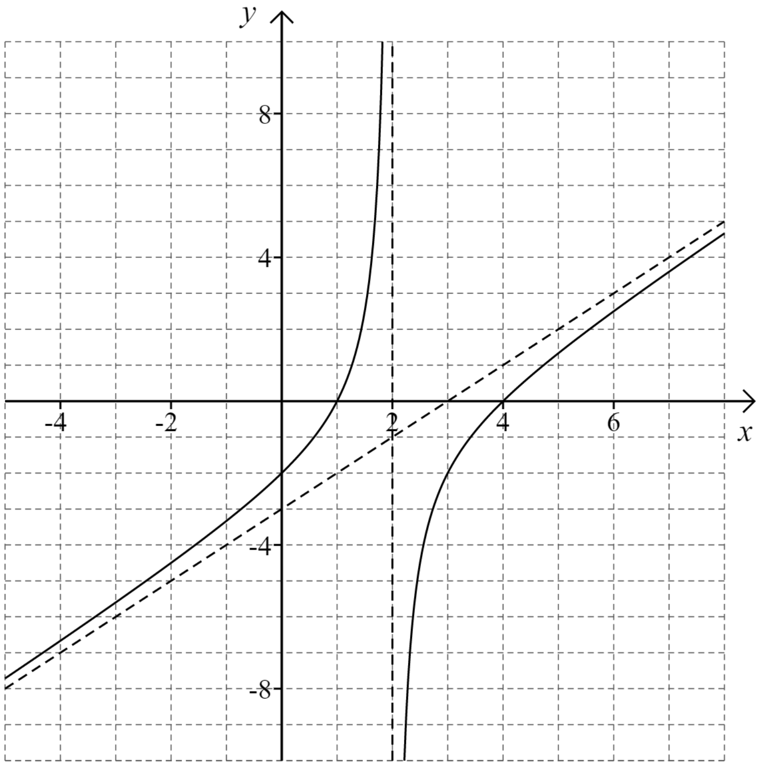
(c) The x-intercepts are $(a,0)$ and $(b,0)$. A1

The y-intercept is $\left(0, -\frac{ab}{c}\right)$. A1

(d)

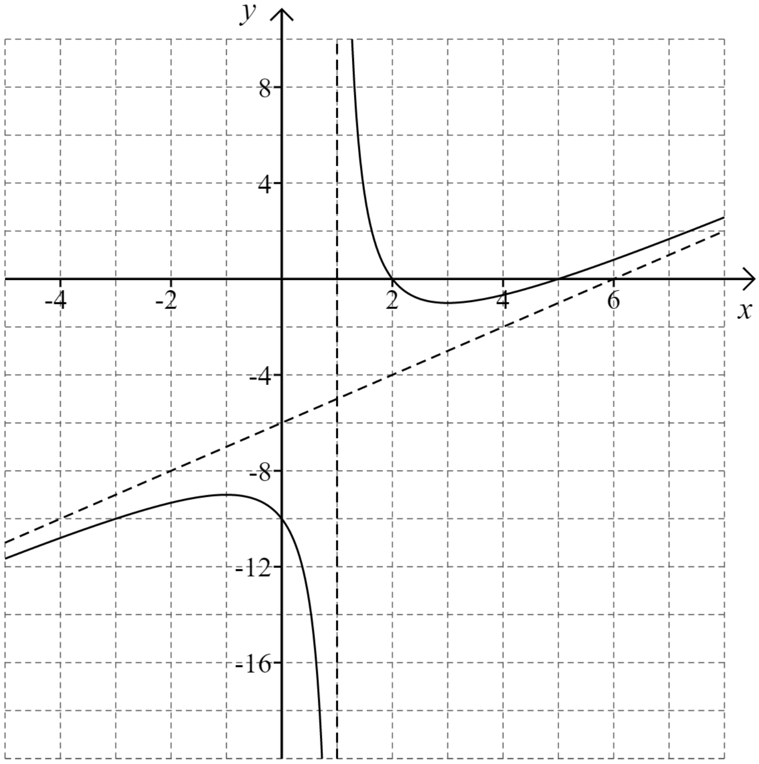
(i) The asymptotes and intercepts are consistent with parts (b) and (c). A1

The shape is approximately correct. A1



(ii) The asymptotes and intercepts are consistent with parts (b) and (c). A1

The shape is approximately correct. A1



(e) $a < c < b$ A1

(f) The number of functions with $a < c < b$ is equal to ${}^{10}C_3 = 120$. M1A1

The probability is therefore $\dfrac{120}{360} = \dfrac{1}{3}$. A1

18. (a) If there are no restrictions on the value of b and c then there is ${}^{10}\mathrm{P}_3 = 720$ ways of choosing their values. M1A1

However we can only accept half of these as the other half would have $b > c$. So the total number of ways is 360. A1

(b)

(i) $x = b$ and $x = c$ A1A1

(ii) We have

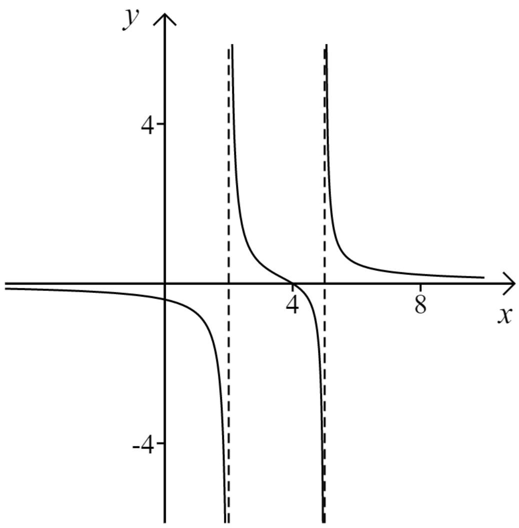
$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^{-1} - ax^{-2}}{(1 - bx^{-1})(1 - cx^{-1})} = 0 \quad \text{M1}$$

So the equation is $y = 0$. A1

(c) $(a,0)$ A1

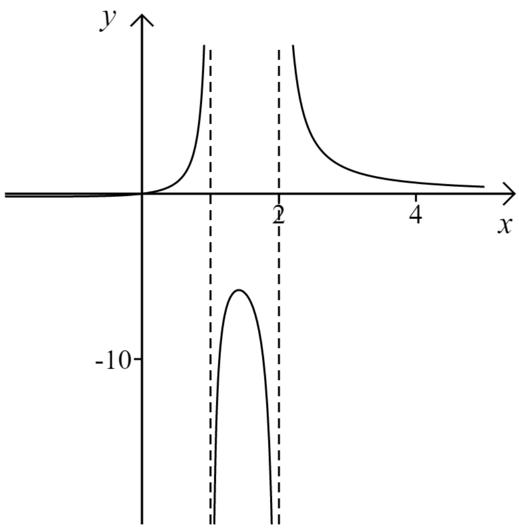
(d) The equation can be rewritten as a quadratic so there can only be a maximum of 2 solutions. A1

(e) If $b < a < c$ then the graph will look something similar to A1



The domain must be $\mathbb{R}$ on the interval $[b,c]$ as it crosses the x-axis and has a vertical asymptote on either side. R1

Otherwise we will have something similar to A1



There is no way the the part of the function on the interval $[b,c]$ can be high enough for the range of the function to be $\mathbb{R}$ because this would mean there would be more than two solutions (with a possible repeated solution) to the equation $f(x) = k$. R1

(f) When $b = 1$ and $c = 10$ there are 8 ways the range can be $\mathbb{R}$. A1

When $c - b = 8$ there are $2 \times 7 = 14$ ways the range can be $\mathbb{R}$. A1

When $c - b = 7$ there are $3 \times 6 = 18$ ways the range can be $\mathbb{R}$ etc. A1

The probability is therefore

$$\frac{1 \times 8 + 2 \times 7 + 3 \times 6 + 4 \times 5 + 5 \times 4 + 6 \times 3 + 7 \times 2 + 8 \times 1}{360} = \frac{1}{3}$$ M1A1