Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3006

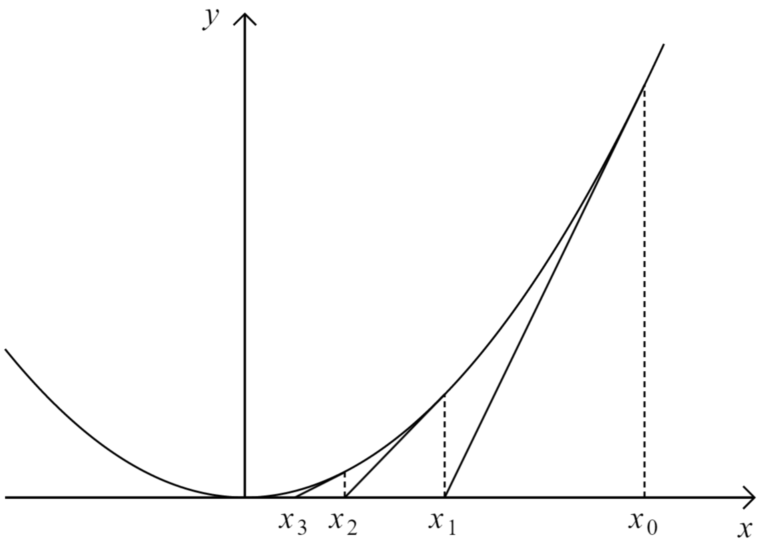
Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[48 marks]**.

1. [Maximum points: 27]

In this problem you will investigate the Newton-Raphson method of estimating the roots of functions.

Let $f(x) = x^2$. The diagram below shows the graph of $y = f(x)$ and the tangent lines to the graph when x is equal to x_0, x_1 and x_2.



Let $x_0 = 1$.

(a) By considering the equations of the tangent lines find the values of x_1, x_2 and x_3. [7]

(b) Show that x_0, x_1, x_2, x_3 is a geometric sequence. [2]

The process is continued to infinity.

(c) Prove that $x_{n+1} = \dfrac{x_n}{2}$ where $n \in \mathbb{N}$. [3]

(d) Hypothesise the value of x_n in terms of n. [1]

(e) Prove your answer to part (d) by induction. [6]

The same method is used to estimate the value of $\sqrt{2}$ using $x_0 = 2$.

(f) Write down a quadratic function with roots equal to $\pm\sqrt{2}$. [1]

(g) For the function found in part (f) prove that $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$. [3]

(h) Find x_3 writing your answer as a fraction. [3]

(i) Comment on the accuracy of the estimation. [1]

2. [Maximum points: 21]

In this problem you will prove that e is an irrational number using a method discovered by Joseph Fourier.

(a) Use the Maclaurin series for e^x to show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. [2]

Assume that e is rational. This means it can be written in the form $e = \frac{p}{q}$ where $p, q \in \mathbb{Z}$.

Let $x = q!\left(e - \sum_{n=0}^{q} \frac{1}{n!}\right)$.

(b) If x is written as $x = \sum_{n=f(q)}^{\infty} \frac{q!}{n!}$ find $f(q)$. [3]

(c) Hence write down whether x is positive or negative. [1]

(d) Show that $x = p(q-1)! - \sum_{n=0}^{q} \frac{q!}{n!}$ [2]

(e) Explain why part (d) implies that x must be an integer. [2]

(f) If $n \geq q+1$ [3]

(i) show that $\frac{q!}{n!} \leq \frac{1}{(q+1)^{n-q}}$

(ii) determine the relationship between n and q for the inequality in part (i) to be strictly less than, and not less than or equal to

(g) Hence use parts (b) and (f) to show that $x < 1$. [5]

(h) Complete the proof to show that e is irrational. [3]

1. (a) We have

$$f'(x) = 2x \qquad \text{A1}$$

So the equation of the first tangent line is

$$y - 1 = 2(x - 1) \qquad \text{M1}$$

The value of x_1 is therefore 0.5. A1

The equation of the second tangent line is

$$y - 0.25 = 1 \times (x - 0.5) \qquad \text{M1}$$

The value of x_2 is therefore 0.25. A1

The equation of the third tangent line is

$$y - 0.0625 = 0.5(x - 0.25) \qquad \text{M1}$$

The value of x_3 is therefore 0.125. A1

(b) $\frac{0.125}{0.25} = \frac{0.25}{0.5} = \frac{0.5}{1} = 0.5.$ M1A1

(c) The value of x_{n+1} is the x-intercept of the line $y - f(x_n) = f'(x_n)(x - x_n)$. M1

So we have

$$0 - x_n{}^2 = 2x_n(x_{n+1} - x_n) \qquad \text{M1}$$

Giving

$$x_{n+1} = \frac{x_n}{2} \qquad \text{A1}$$

(d) $x_n = 0.5^n$ A1

(e) When $n = 0$ we have $0.5^0 = 1$. So it is true for $n = 1$. A1

Assume it is true for $n = k$. So $x_k = 0.5^k$. M1

When $x = k + 1$ we have

$$x_{k+1} = \frac{x_k}{2} = \frac{0.5^k}{2} = 0.5^{k+1} \qquad \text{M1A1A1}$$

So it is true for $n = k + 1$.

By the principle of mathemtical induction it must be true for all natural numbers n. R1

(f) $y = x^2 - 2$ A1

(g) The equation of the tangent line is $y - f(x_n) = f'(x_n)(x - x_n)$. M1

So we have

$$0 - x_n^2 + 2 = 2x_n(x_{n+1} - x_n) \qquad \text{M1}$$

Giving

$$x_{n+1} = x_n + \frac{2 - x_n^2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n} \qquad \text{A1}$$

(h) We have

$$x_1 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} \qquad \text{A1}$$

$$x_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \qquad \text{A1}$$

$$x_3 = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} \qquad \text{A1}$$

(i) The estimation is accurate to 5 decimal places. A1

2. (a) We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{M1}$$

So

$$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} \qquad \text{A1}$$

(b) We have

$$x = q!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{q} \frac{1}{n!}\right) \qquad \text{M1}$$

So

$$x = q! \sum_{n=q+1}^{\infty} \frac{1}{n!} = \sum_{n=q+1}^{\infty} \frac{q!}{n!} \qquad \text{A1}$$

Therefore $f(q) = q + 1$. A1

(c) Positive A1

(d) We have

$$x = q!\left(\frac{p}{q} - \sum_{n=0}^{q} \frac{1}{n!}\right) = p(q-1)! - \sum_{n=0}^{q} \frac{q!}{n!} \qquad \text{M1A1}$$

(e) The $p(q-1)!$ term is an integer. A1

Also $\frac{q!}{n!} = (n+1)(n+2)\ldots(q-1)q$ which is an integer. A1

(f)

(i) We have

$$\frac{q!}{n!} = \frac{1}{(q+1)(q+2)(q+3)\ldots n!} \le \frac{1}{(q+1)^{n-q}} \qquad \text{M1A1}$$

(ii) $n \ge q + 2$ A1

(g) We have

$$x = \sum_{n=q+1}^{\infty} \frac{q!}{n!} < \sum_{n=q+1}^{\infty} \frac{1}{(q+1)^{n-q}} \qquad \text{A1}$$

This is an infinite geometric series with first term and common ratio $\frac{1}{q+1}$. R1

Its value is therefore

$$\frac{\frac{1}{q+1}}{1-\frac{1}{q+1}} = \frac{1}{q} \qquad \text{M1A1}$$

So we have

$$x < \frac{1}{q} \leq 1 \qquad \text{A1}$$

(h) We have shown that x must be an integer and $0 < x < 1$. This is a contradiction as there is no integer between 0 and 1. So the original assumption that e is rational must be false. So e is irrational. A1 R1 A1