Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3001

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

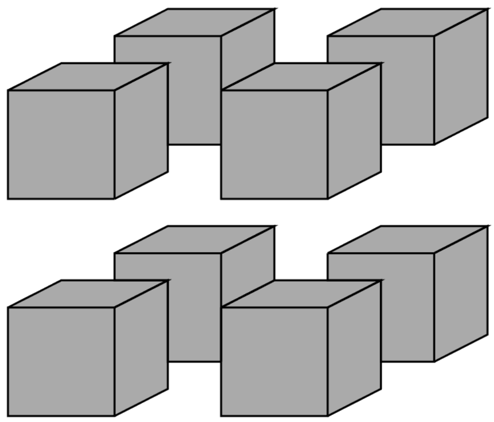
1. [Maximum points: 30]

In this problem you will investigate patterns formed by dividing large cubes into smaller cubes.

A solid cube has sides of length 12 cm.

(a) Show that the surface area of the cube is equal to 864 cm^2. [1]

The cube is divided into eight identical smaller cubes, as shown in the diagram below.



(b) For **one** of the smaller cubes find [2]

(i) the length of each side

(ii) its surface area

(c) Hence show that the total surface area of all the smaller cubes is equal to 1728 cm^2. [1]

The table below shows the total surface area when the original cube is divided into various amounts of smaller cubes.

Step	1	2	3	4
Total Cubes	1	8	27	A
Total Surface Area	864	1728	B	C

(d) Find the value of [4]

(i) A

(ii) B

(e) Show that the value of C is equal to 3456. [3]

(f) Decsribe the type of sequence formed in the bottom row of the table. [1]

(g) Prove that if the process is continued then the bottom row will always follow the type [4] of sequence described in part (f).

A sugar cube is submerged in water. It disolves at a rate proportional to the surface area of the cube and remains cubical at all times. Let the length of one side of the cube at time t be equal to x.

(h) In terms of x write down an expression for [2]

(i) the volume V of the cube

(ii) the surface area A

(i) Show that $\frac{dx}{dt}$ is constant. [5]

A sugar cube with sides of length 1 cm is submerged in water. One minute later the sides of the cube are 0.8 cm in length.

(j) Find the total amount of time it will take for the cube to disolve. [2]

An identical sugar cube is divided into 1000 smaller cubes identical to each other in size.

(k) Find the total surface area of all 1000 cubes. [3]

(l) These cubes are then all submerged in water at the same time. Find the total amount of time it will take for them to disolve. [2]

2. [Maximum points: 25]

In this problem you will investigate the length of a curve formed by a hyperbolic function.

(a) Use L'Hopital's rule to evaluate $\lim_{x \to \infty} x(1 - e^{2/x})$. [4]

The *hyperbolic functions* are defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(b) Show that $\cosh^2 x - \sinh^2 x = 1$. [2]

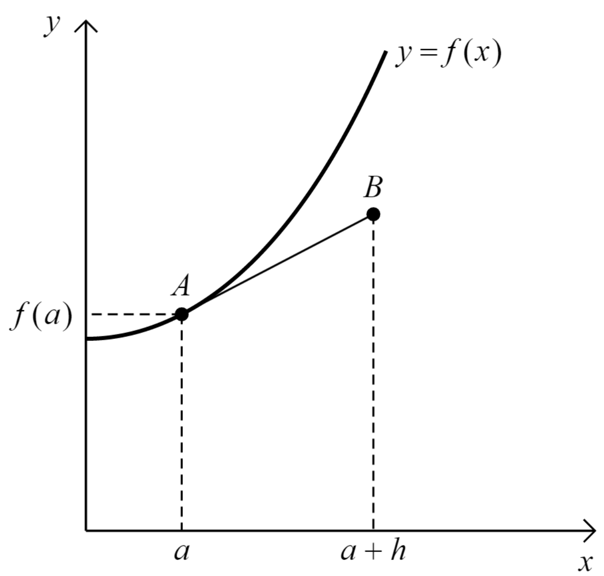
Let $f(x) = \cosh x$ and $g(x) = \sinh x$.

(c) Find [4]

(i) $f'(x)$ in terms of $g(x)$

(ii) $g'(x)$ in terms of $f(x)$

Consider the part of the graph of $y = f(x)$ on the interval $[a, a+h]$ as shown below.



Line AB is tangential to the graph at the point with an x-coordinate of a.

(d) Show that the y-coordinate of point B is equal to $f(a) + h\,g(a)$. [2]

(e) Find the length of AB in terms of $f(a)$. [4]

(f) Show that the length L of the curve of $y = f(x)$ on the interval $[0,2]$ is equal to [4]

$$L = \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{e^{2k/n} + e^{-2k/n}}{n}$$

(g) By considering geometric series find the exact value of L writing your answer in terms of $g(2)$. [5]

1. (a) $6 \times 12 \times 12 = 864$ A1

(b)

(i) 6 cm A1

(ii) $6 \times 6 \times 6 = 216$ A1

(c) $6 \times 216 = 1728$ A1

(d)

(i) $4^3 = 64$ M1A1

(ii) $27 \times 6 \times (12 \div 3)^2 = 2592$ M1A1

(e) The length of one side of a small cube is 3 cm. A1

The total surface area is therefore $64 \times 6 \times 3^2 = 3456$. M1A1

(f) Arithmetic A1

(g) In step n we have n^3 smaller cubes. A1

The length of each side of a smaller cube is $\frac{12}{n}$. A1

The total surface area is therefore

$$n^3 \times 6 \times \left(\frac{12}{n}\right)^2 = 864n$$ M1A1

(h)

(i) $V = x^3$ A1

(ii) $A = 6x^2$ A1

(i) Use implicit differentiation M1

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$ A1

Since $\frac{dV}{dt} = 6kx^2 = Kx^2$ where $k, K \in \mathbb{R}$ we have M1

$$Kx^2 = 3x^2 \frac{dx}{dt}$$ A1

So

$$\frac{dx}{dt} = \frac{K}{3}$$ A1

(j) $\frac{1}{1-0.8} = 5$ minutes M1A1

(k)	The surface area of the cube before being divided is 6 cm^2.	A1
	If we divide as in part (d) the surface area will increase by 6 after each step.	M1
	At step 10 the surface area will therefore be 60 cm^2.	A1
(l)	The length of the side of each cube is 0.1 cm.	M1
	It will therefore take 30 seconds for all the sugar to disolve.	A1

2. (a) Rewrite

$$\lim_{x \to \infty} \frac{1 - e^{2/x}}{1/x} \qquad \text{A1}$$

Use l'Hopital's rule

$$\lim_{x \to \infty} \frac{\frac{2}{x^2} e^{2/x}}{-\frac{1}{x^2}} = -2 \times 1 = -2 \qquad \text{M1A1A1}$$

(b) We have $$\frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1 \qquad \text{M1A1}$$

(c)

(i) $f'(x) = \dfrac{e^x - e^{-x}}{2} = g(x)$ M1A1

(ii) $g'(x) = \dfrac{e^x + e^{-x}}{2} = f(x)$ M1A1

(d) We have

$$\frac{y - f(a)}{a + h - a} = f'(a) \qquad \text{M1}$$

Since $f'(a) = g(a)$ this gives

$$y = f(a) + h\,g(a) \qquad \text{A1}$$

(e) Use the Pythagorean theorem M1

$$\sqrt{(a + h - a)^2 + \left(f(a) + h\,g(a) - f(a)\right)^2} = \sqrt{h^2 + h^2\left(g(a)\right)^2} \qquad \text{A1}$$

This is equal to

$$h\sqrt{1 + \left(g(a)\right)^2} = h\sqrt{\left(f(a)\right)^2} = h f(a) \qquad \text{M1A1}$$

(f) We can approximate the length of the curve by dividing into intervals and find the length of each tangent line like in part (d). R1

This gives

$$L \approx \sum_{k=0}^{n-1} \frac{2f(2k/n)}{n}$$ A1

As the intervals become smaller the approximation becomes more accurate so R1

$$L = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{2f(2k/n)}{n} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{e^{2k/n} + e^{-2k/n}}{n}$$ A1

(g) Use the geometric series formula M1

$$L = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1(1-e^2)}{1-e^{2/n}} + \frac{1(1-e^{-2})}{1-e^{-2/n}}\right) = \lim_{n\to\infty} \frac{1-e^2-e^{2/n}+e^{2/n-2}}{n(1-e^{2/n})}$$ A1A1

Using part (a) this is equal to

$$\frac{1-e^2-1+e^{-2}}{-2} = \frac{e^2-e^{-2}}{2} = g(2)$$ M1A1