# Mathematics: analysis and approaches

# Higher level

# Paper 3

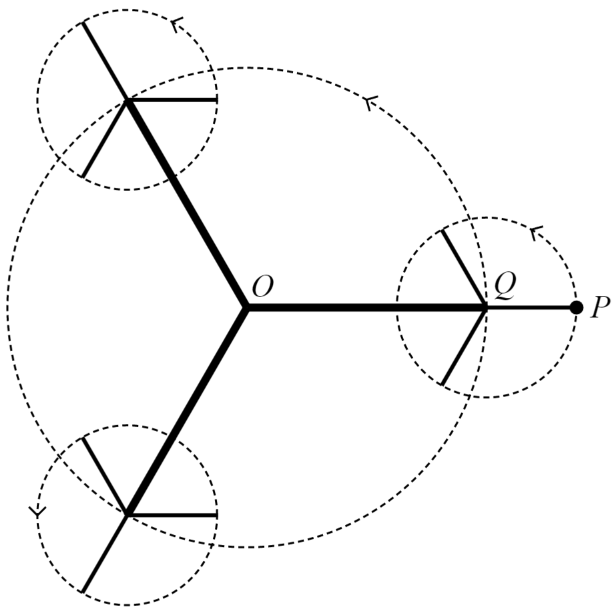
ID: 3003

**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[53 marks]**.

**1.** [Maximum points: 27]

A fairground ride consists of three equally spaced arms 8 m in length. These three arms make one anti-clockwise revolution every $2\pi$ seconds about point $O$. At the end of each arm are three smaller arms 3 m in length. These three arms make one anti-clockwise revolution every $\pi/2$ seconds about the endpoints of the longer arms. This is shown in the diagram below showing the view from above, where $OQ = 8$ m and $PQ = 3$ m.




A rider sits at point $P$ with initial coordinates $(11,0)$ relative to point $O$.

(a) Find the position vector of point $Q$ after $t$ seconds. [2]

(b) Hence show that the position vector of point $P$ after $t$ seconds is given by [4]

$$\overrightarrow{OP} = \begin{pmatrix} 8\cos t + 3\cos 4t \\ 8\sin t + 3\sin 4t \end{pmatrix}$$

Let $T$ represent the smallest value of $t$ for which point $P$ is moving directly towards point $O$.

(c) Show that [6]

$$\frac{8\cos T + 12\cos 4T}{8\sin T + 12\sin 4T} = -\frac{8\sin T + 3\sin 4T}{8\cos T + 3\cos 4T}$$

(d) Solve the equation in part (c) to find the exact value of $T$, writing your answer in the form $T = b \cdot \arccos(a)$ where $a$ and $b$ are rational numbers to be determined. [7]

(e) If $D = \left|\overrightarrow{OP}\right|$ show that $D^2 = 73 + 48\cos 3t$. [4]

(f) For the value of $T$ in part (d) find the rate at which $\left|\overrightarrow{OP}\right|$ is changing. [4]

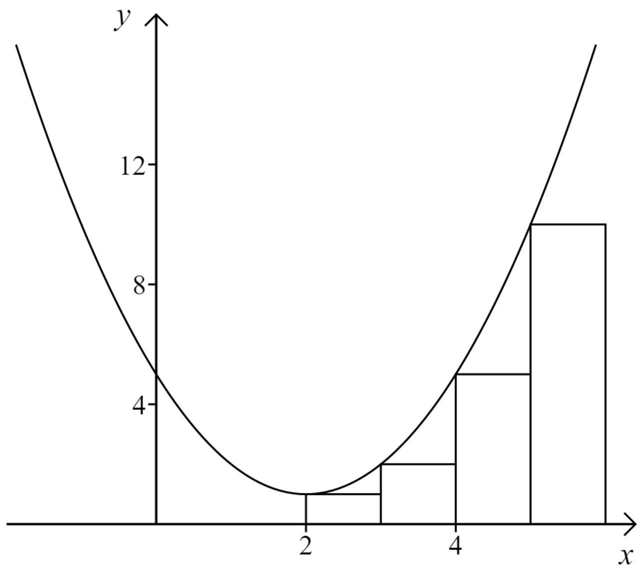
**2.** [Maximum points: 26]

*In this problem you will investigate the area between a parabola and the x-axis by dividing the area into rectangles of equal width.*

(a) Write down an expression for the value of $\sum_{k=1}^{n} k$ in terms of $n$. [2]

(b) Prove by induction $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. [9]

Let $f(x) = x^2 - 4x + 5$. The diagram below shows the graph of $y = f(x)$. Rectangles of width 1 are drawn between the graph and the $x$-axis from $x = 2$ to $x = 6$.




(c) Find the total area of the rectangles. [2]

Suppose $n$ rectangles of equal width are now drawn between the graph of $y = f(x)$ and the $x$-axis from $x = 2$ to $x = 6$.

(d) Write down an expression for the width of each rectangle in terms of $n$. [1]

(e) Show that the total area $A$ of all the rectangles is equal given by [3]

$$A = \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16(k-1)^2}{n^2} + 1\right)$$

(f) Hence use parts (a) and (b) to determine an expression for $A$ without using sigma notation. [4]

(g) Evaluate $\lim_{n \to \infty} A$. [2]

(h) Verify your answer to part (g) by evaluating an appropriate definite integral. [3]