Mathematics: analysis and approaches
Higher level
Paper 3

ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Friday, August 27^{th} (morning)

1 hours

Candidate number

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where *a* = 2.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+, n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$
- the even values $n = 2$ and $n = 4$

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+, n > 1$.

(c) Show that $f_n{}'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(f) Hence, or otherwise, show that $f_n{}'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n{}'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$ [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$ [2]

Consider the graph of $y = x^n(a-x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a-x)^n = k$ has four solutions for x. [5]

2. [Maximum mark: 24]

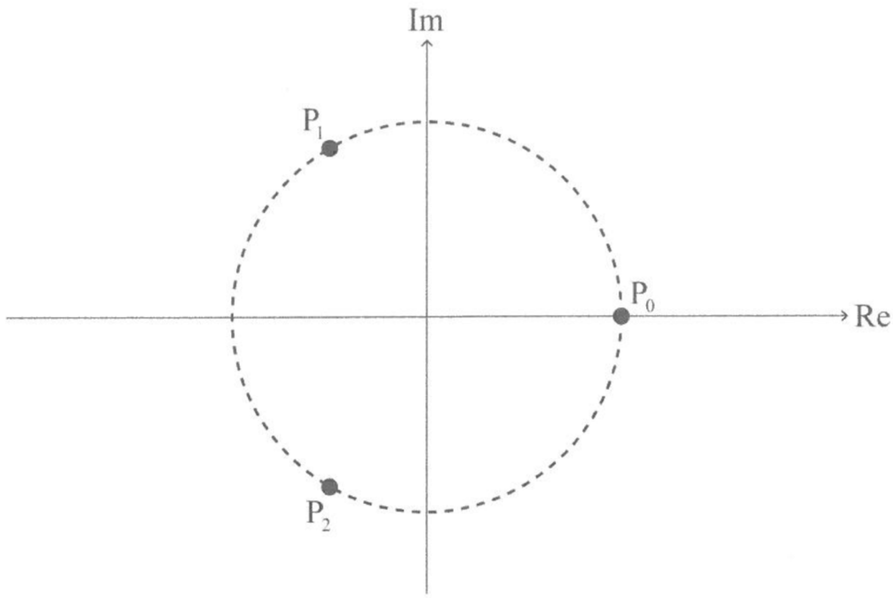
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega,\ \omega^2, \ldots\ \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point P_0 , P_1 , P_2 , …, P_{n-1} , respectively on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω . On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1 .

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1 , ω and ω^2 . On the following Argand diagram, the points P_0 , P_1 and P_2 lie on a circle of radius 1 unit with centre O $(0, 0)$.



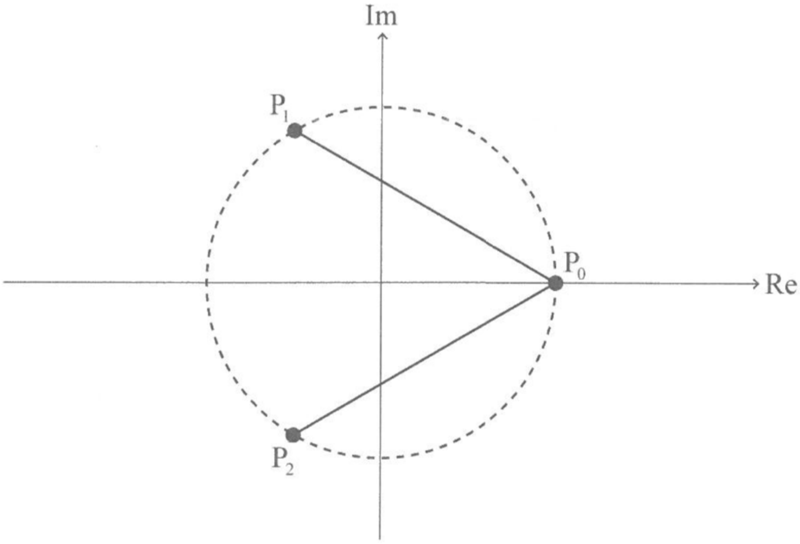
(a) (i) Show the $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

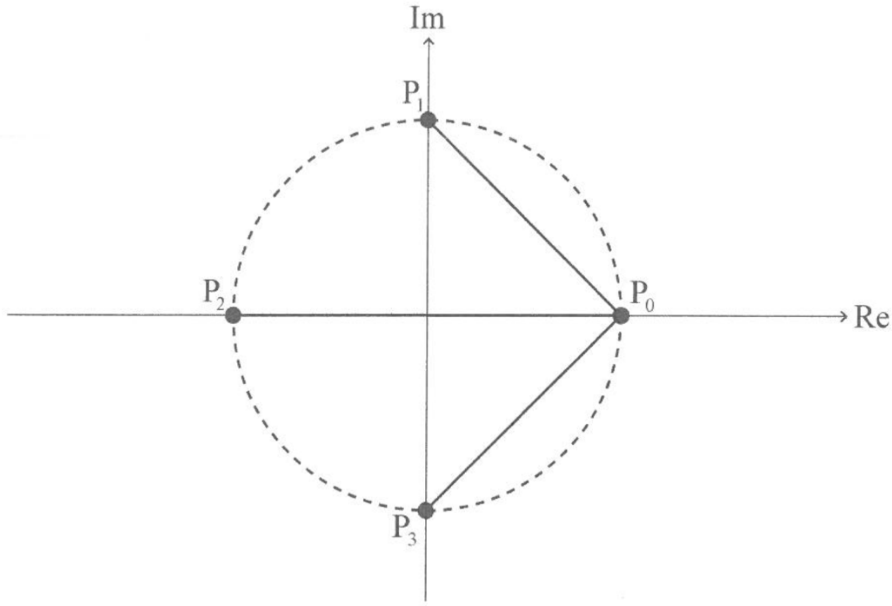
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 and ω^3 .

(c) By factorising $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0 , P_1 , P_2 and P_3 lie on a circle of radius 1 unit with centre O $(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1 , ω , ω^2 , ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 , …, ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0 , P_1 , P_2 , …, P_{n-1} respectively where $[P_0P_1]$, $[P_0P_2]$, …, $[P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre O $(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω . [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω . [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \cdots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

<u>End of paper 3</u>