Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3007

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[48 marks]**.

1. [Maximum points: 25]

In this problem you will investigate values of the form $\cos(m\arctan n)$ *and* $\sin(m\arctan n)$ where $m, n \in \mathbb{Z}$.

(a) Sketch the graph of $y = \arctan x$. [3]

Let $z = 1 - 2i$ and $\arg z = \theta$ where $0 < \theta < 2\pi$.

(b) Use binomial expansion to expand and simplify z^4. [3]

(c) Find the exact value of θ. [2]

(d) Show that $z^4 = 25\left(\cos(4\arctan(-2)) + i\sin(4\arctan(-2))\right)$. [3]

(e) Hence find the exact values of $\cos(4\arctan(-2))$ and $\sin(4\arctan(-2))$. [3]

(f) Use a similar method to find the exact values of $\cos(8\arctan 3)$ and $\sin(8\arctan 3)$. [5]

(g) Prove that $\cos\left((2n+1)\arctan c\right)$ and $\sin\left((2n+1)\arctan c\right)$ for $n \in \mathbb{N}$ and $c \in \mathbb{Z}$ is always irrational. [6]

2. [Maximum points: 23]

In this problem you will investigate the Maclaurin series of functions into which complex values of x are substituted.

The Maclaurin series of $\sin x$, $\cos x$ and e^x allow us to substitute complex values for x. For example

$$\sin i = i - \frac{i^3}{3!} + \frac{i^5}{5!} - \frac{i^7}{7!} + \cdots = i + \frac{i}{3!} + \frac{i}{5!} + \frac{i}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{i}{(2n+1)!}$$

(a) Find the first four terms of the Maclaurin series of $\cos i$. [3]

(b) Write the Maclaurin series of $\cos i$ using sigma notation. [2]

(c) By considering Maclaurin series show that $e^{ix} = \cos x + i \sin x$. [6]

(d) Prove Euler's identity $e^{i\pi} + 1 = 0$. [3]

(e) Find e^{-ix} in terms of $\cos x$, $\sin x$ and i. [3]

(f) Hence find expressions for the following in terms of e^{ix}, e^{-ix} and i. [6]

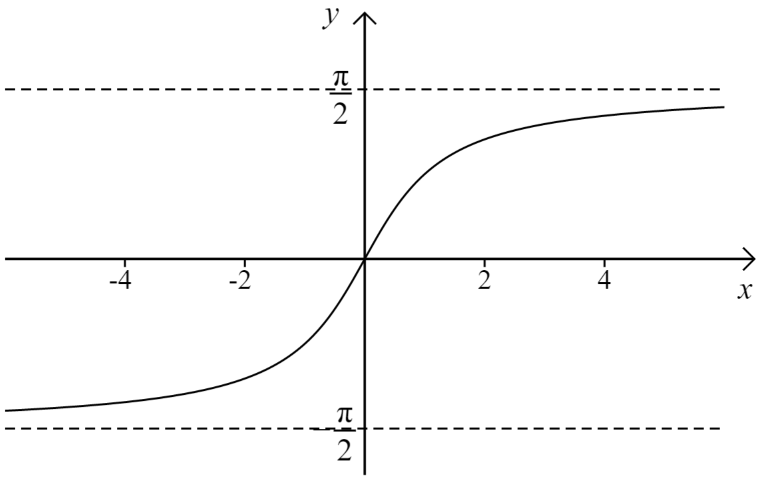
(i) $\sin x$

(ii) $\cos x$

1. (a) The domain is $\mathbb{R}$ and the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. A1

There are horizontal asymptotes as $y = \pm\frac{\pi}{2}$. A1

The shape is approximately correct. A1



(b) We have

$$(1-2i)^4 = 1 + 4(-2i) + 6(-2i)^2 + 4(-2i)^3 + (-2i)^4$$ M1

This is equal to

$$1 - 8i - 24 + 32i + 16 = -7 + 24i$$ A1A1

(c) We have

$$\arg z = \arctan(-2/1) = \arctan(-2)$$ M1

Since this is negative

$$\theta = 2\pi + \arctan(-2)$$ A1

(d) We have

$$z^4 = 5^2\left(\cos(2\pi + \arctan(-2)) + i\sin(2\pi + \arcsin(-2))\right)^4$$ A1

Use De Moivre's theorem

$$25\left(\cos(8\pi + 4\arctan(-2)) + i\sin(8\pi + 4\arctan(-2))\right)$$ M1

Simplify

$$25\left(\cos(4\arctan(-2)) + i\sin(4\arctan(-2))\right)$$ A1

(e) We have

$$25\cos\left(4\arctan(-2)\right) = -7 \quad \text{and} \quad 25\sin\left(4\arctan(-2)\right) = 24 \qquad \text{M1}$$

So

$$\cos\left(4\arctan(-2)\right) = -\frac{7}{25} \qquad \text{A1}$$

And

$$\sin\left(4\arctan(-2)\right) = \frac{24}{25} \qquad \text{A1}$$

(f) We have $(1+3i)^8 = -8432 - 5376i$. A1

Also

$$(1+3i)^8 = 10^4\left(\sin(\arctan 3) + i\sin(\arctan 3)\right)^8 \qquad \text{M1}$$

This is equal to

$$10^4\left(\cos(8\arctan 3) + i\sin(8\arctan 3)\right) \qquad \text{A1}$$

So

$$\cos(8\arctan 3) = -\frac{8432}{10000} = -\frac{527}{625} \qquad \text{A1}$$

And

$$\sin(8\arctan 3) = -\frac{5376}{10000} = -\frac{336}{625} \qquad \text{A1}$$

(g) Let $z = 1 + ci$ therefore $\arg z = c$ and $|z| = (1+c^2)^{1/2} \in \mathbb{Q}'$. A1

We therefore have $\operatorname{Re}(z^{2n+1}) \in \mathbb{Z}$ and $\operatorname{Im}(z^{2n+1}) \in \mathbb{Z}$. A1

Also $z^{2n+1} = (1+c^2)^n \cdot (1+c^2)^{1/2} \cdot \left(\cos(\arctan c) + i\sin(\arctan c)\right)^{2n+1}$ M1

This is equal to

$$(1+c^2)^n \cdot (1+c^2)^{1/2} \cdot \left(\cos\left((2n+1)\arctan c\right) + i\sin\left((2n+1)\arctan c\right)\right) \qquad \text{A1}$$

So

$$\cos\left((2n+1)\arctan c\right) = \frac{\operatorname{Re}(z^{2n+1})}{(1+c^2)^n} \cdot \frac{1}{(1+c^2)^{1/2}} \in \mathbb{Q}' \qquad \text{A1}$$

And

$$\sin\left((2n+1)\arctan c\right) = \frac{\operatorname{Im}(z^{2n+1})}{(1+c^2)^n} \cdot \frac{1}{(1+c^2)^{1/2}} \in \mathbb{Q}' \qquad \text{A1}$$

2. (a) We have

$$\cos i = 1 - \frac{i^2}{2!} + \frac{i^4}{4!} - \frac{i^6}{6!} + \cdots = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$$ M1A1A1

(b) $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$ A1A1

(c) We have

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots$$ M1A1A1

Rewrite using sigma notation M1

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + i \sin x$$ A1A1

(d) We have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$ M1A1

So

$$e^{i\pi} + 1 = 0$$ A1

(e) We have

$$e^{-ix} = \cos(-x) + i \sin(-x)$$ M1

This gives

$$e^{-ix} = \cos x - i \sin x$$ A1A1

(f)

(i) We have

$$e^{ix} = \cos x + i \sin x$$

and

$$e^{-ix} = \cos x - i \sin x$$

Eliminate $\cos x$ e.g. M1

$$e^{ix} - e^{-ix} = 2i \sin x$$ A1

So

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$ A1

(ii) We have

$$e^{ix} = \cos x + i \sin x$$

and

$$e^{-ix} = \cos x - i \sin x$$

Eliminate $\sin x$ e.g. M1

$$e^{ix} + e^{-ix} = 2\cos x$$ A1

So

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$ A1