Mathematics: analysis and approaches

Higher level

Additional Practice

Counting Principles (Non-Calculator)

ID: 4003

Instructions to candidates

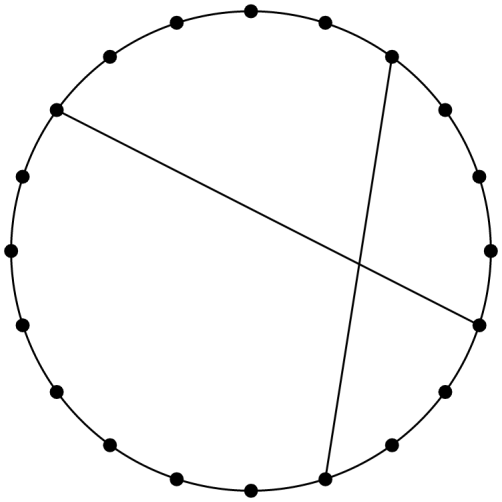
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[93 marks]**.

1. [Maximum points: 4]

Three integers are randomly chosen from 1 to 5 with repetition not allowed. Find the probability that the integers chosen can represent the lengths of each side of a triangle.

2. [Maximum points: 4]

The circle below has 20 points on its circumference. Two points are randomly chosen and connected with a straight line. Another two of the remaining 18 points are then randomly chosen and also connected with a straight line.



Find the probability that the two lines intersect.

3. [Maximum points: 4]

Five friends stand randomly in a line. Determine the probability that

(a) the youngest person is the first person in the line [2]

(b) they are stood in order of age (ascending or descending) [2]

4. [Maximum points: 5]

Three single digit integers are randomly chosen one-by-one with repetition allowed. Calculate the probability that the three digits are increasing in size, for example 2, then 5, then 9.

5. [Maximum points: 5]

Mary wishes to watch two movies. She has a choice of 5 horror movies, 6 comedies and 3 dramas. Find the total combinations of two movies she can watch if both movies are from different genres.

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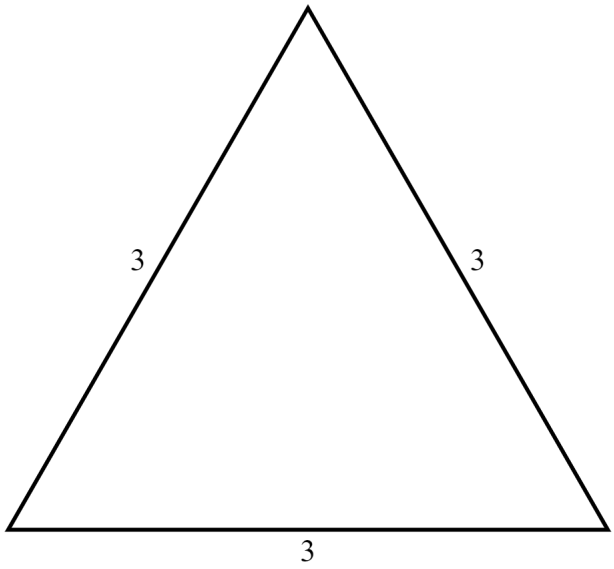
6. [Maximum points: 5]

The diagram below shows an equilateral triangle with sides of length 3.



(a) Divide the triangle into 9 smaller identical equilateral triangles. [2]

(b) Explain why if we randomly draw 10 points inside the triangle then at least two of them must be no more than a distance of 1 apart. [3]

7. [Maximum points: 6]

Let $A = \{\text{factors of } 6\}$.

(a) List the members of A. [1]

Let $B = \{x^2 + bx + c = 0 \mid b, c \in A,\ b \neq c\}$.

(b) Determine how many elements are in set B. [2]

Let $C = \{\text{equations with rational roots}\}$.

(c) Describe the members of set $B \cap C$. [1]

(d) Find all members of $B \cap C$. [2]

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8. [Maximum points: 6]

Let $N, n, r \in \mathbb{N}$, $n \leq N$ and $r \leq N$.

(a) Show that ${}^{N}\mathrm{C}_{r} \cdot {}^{N-r}\mathrm{C}_{n-r} = \dfrac{N!}{r!(n-r)!(N-n)!}$. [2]

(b) Find an expression for $\sum_{r=0}^{n} {}^{n}\mathrm{C}_{r}$ in terms of only n. [2]

(c) Hence prove that $\sum_{r=0}^{n} {}^{N}\mathrm{C}_{r} \cdot {}^{N-r}\mathrm{C}_{n-r} = \dfrac{2^{n} \cdot {}^{N}\mathrm{P}_{n}}{n!}$. [2]

9. [Maximum points: 6]

Rod, Jane and Freddie stand randomly in a line with three other friends.

(a) Find the number of ways the six people can stand in a line. [2]

(b) Find the probability that Rod is between Jane and Freddie, either directly or indirectly. [2]

(c) Hence find the total number of ways the six people can stand in a line so that Rod is between Jane and Freddie, either directly or indirectly. [2]

10. [Maximum points: 6]

Let P_n represent the product of the first n prime numbers.

(a) Prove that $\sum_{k=0}^{n} {}^{n}\mathrm{C}_{k} = 2^{n}$. [2]

(b) Hence determine an expression for the total number of factors of P_n. [4]

11. [Maximum points: 7]

Five mathematics books and three science books are placed randomly on a shelf. Calculate the probability that no science books are placed next to each other.

12. [Maximum points: 7]

A random three digit code is generated using the integers 0 to 9 with no repetition. Codes beginning with zero are read as two digit numbers. For example 073 is read as 73. Find the total number of ways of generating

(a) a multiple of 5 [3]

(b) a multiple of 4 [4]

13. [Maximum points: 9]

Four runners have a race. All the runners complete the race. Determine the total number of ways they could finish the race, including the cases where two or more runners finish at exactly the same time.

14. [Maximum points: 19]

Let a, b and c represent the lengths of the three sides of a triangle where c is the length of the longest side.

(a) Consider the inequality $a + b \blacksquare c$. Determine which of the symbols $<, >, \leq$ or $\geq$ we must replace $\blacksquare$ with. [1]

Let $X = \{x \mid 1 \leq x \leq 10,\ x \in \mathbb{N}\}$.

(b) List the elements of X. [1]

Three numbers are chosen from set X with no repetition.

(c) Find the total number of ways of choosing the three numbers. The order of the numbers does not matter. [2]

Let set Y represent the set of all non-congruent triangles that can be formed by choosing three numbers from set X with no repetition and using these for the lengths of the sides of the triangle.

(d) Explain why no triangles can have a side of length 1. [2]

(e) Show that set Y contains [4]

(i) 7 triangles with a shortest side of length 2

(ii) $6 + 5 = 11$ triangles with a shortest side of length 3

(f) Find the total number of triangles in set Y. [6]

(g) Hence find the probability that the three numbers chosen from set X can be used to construct a triangle. [1]

(h) Given that the three numbers chosen from set X can be used to construct a triangle find the probability that it is a right-angled triangle. [2]