Mathematics: analysis and approaches

Higher level

Paper 3

ID: 0007

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[48 marks]**.

1. [Maximum points: 25]

In this problem you will investigate values of the form $\cos(m \arctan n)$ *and* $\sin(m \arctan n)$ where $m, n \in \mathbb{Z}$.

(a) Sketch the graph of $y = \arctan x$. [3]

Let $z = 1 - 2i$ and $\arg z = \theta$ where $0 < \theta < 2\pi$.

(b) Use binomial expansion to expand and simplify z^4. [3]

(c) Find the exact value of θ. [2]

(d) Show that $z^4 = 25\left(\cos\left(4\arctan(-2)\right) + i\sin\left(4\arctan(-2)\right)\right)$. [3]

(e) Hence find the exact values of $\cos\left(4\arctan(-2)\right)$ and $\sin\left(4\arctan(-2)\right)$. [3]

(f) Use a similar method to find the exact values of $\cos(8\arctan 3)$ and $\sin(8\arctan 3)$. [5]

(g) Prove that $\cos\left((2n+1)\arctan c\right)$ and $\sin\left((2n+1)\arctan c\right)$ for $n \in \mathbb{N}$ and $c \in \mathbb{Z}$ is always irrational. [6]

2. [Maximum points: 23]

In this problem you will investigate the Maclaurin series of functions into which complex values of x are substituted.

The Maclaurin series of $\sin x$, $\cos x$ and e^x allow us to substitute complex values for x. For example

$$\sin i = i - \frac{i^3}{3!} + \frac{i^5}{5!} - \frac{i^7}{7!} + \cdots = i + \frac{i}{3!} + \frac{i}{5!} + \frac{i}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{i}{(2n+1)!}$$

(a) Find the first four terms of the Maclaurin series of $\cos i$. [3]

(b) Write the Maclaurin series of $\cos i$ using sigma notation. [2]

(c) By considering Maclaurin series show that $e^{ix} = \cos x + i \sin x$. [6]

(d) Prove Euler's identity $e^{i\pi} + 1 = 0$. [3]

(e) Find e^{-ix} in terms of $\cos x$, $\sin x$ and i. [3]

(f) Hence find expressions for the following in terms of e^{ix}, e^{-ix} and i. [6]

(i) $\sin x$

(ii) $\cos x$

ANSWER BOOKLET
LIVRET DE RÉPONSES ①
CUADERNILLO DE RESPUESTAS

$\frac{34}{48} = 71\%$

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

4 PAGES / PÁGINAS

Q1: $\frac{17}{25}$ Q2: $\frac{17}{23}$

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno:

0	0	0	7	P	3	-	M	A	H	L

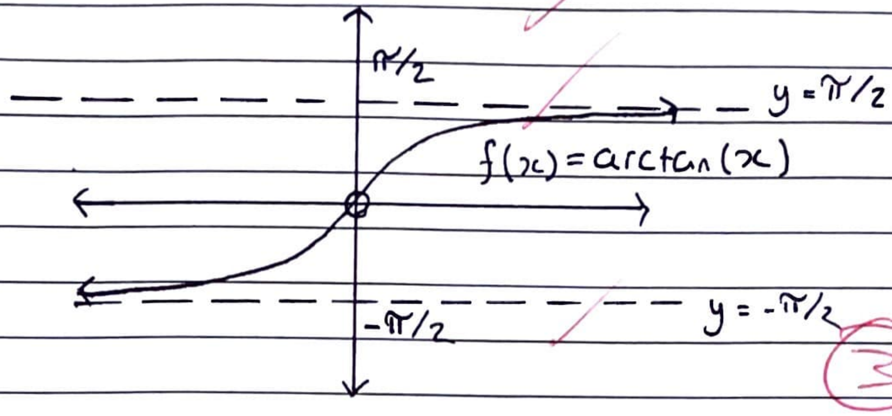
Candidate name: / Nom du candidat: / Nombre del alumno:

Please write question numbers in the following format: / Veuillez numéroter les questions en utilisant la présentation suivante: / Sírvase escribir los números de las preguntas en el siguiente formato:

1 2 3 4 5 6 7 8 9 10

1

(a)



3

(b) $z = 1 + 2i$ $1-2i$

$\therefore z^4 = (1+2i)^4$

$= 1 + 4(2i) + 6(2i)^2 + 4(2i)^3 + (2i)^4$

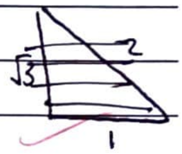
$= 1 + 8i - 24 - 32i + 16$

$= 1 - 24 + 16 + 8i - 32i$

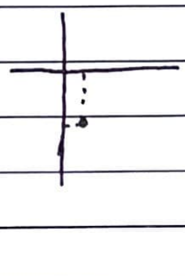
$= -7 - 24i$

FCF 2

(c) $z = 1 + 2i \longrightarrow \arg(z) = \tan^{-1}(2/1)$



$= \arctan(2)$



$\therefore \theta = -\arctan(2)$

1

6

(d) $z^4 = (1-2i)^4$

$\rightarrow |z| = \sqrt{1+4}$
$= \sqrt{5}$

$\rightarrow \arg(z) = \arctan(-2)$ {from (c)}

$\therefore z^4 = (\sqrt{5}\,\text{cis}(\arctan(-2)))^4$ 3
$= 25\,\text{cis}(\cancel{\arctan}\; 4\arctan(-2))$
$= 25[\cos(4\arctan(-2)) + i\sin(4\arctan(-2))]$

(e) $-7-24i = 25\cos(4\arctan(-2)) + 25i\sin(4\arctan(-2))$

$\rightarrow 25\cos(4\arctan(-2)) = -7$ ECF
$\therefore \cos(4\arctan(-2)) = -7/25$

$\rightarrow 25\cancel{i}\sin(4\arctan(-2)) = -24$
$\therefore \sin(4\arctan(-2)) = -24/25$ 2

(f) let $\omega = 1+3i$
$\rightarrow |\omega| = \sqrt{10}$
$\rightarrow \arg(\omega) = \cancel{\theta}\; \arctan(3)$

hence, $\omega^{\cancel{4}\,8} = ((1+3i)^4)^2$
$= (1 + 4(3i) + 6(3i)^2 + 4(3i)^3 + (3i)^4)^2$
$= (1 + 12i - 54 - 108i + 81)^2$
$= (28 - 96i)^2$
$= 784 - 5376i - 9216$
~~$\omega^4 = (\sqrt{10}\,\text{cis}$~~
$= -8432 - 5376i$

5/

$\rightarrow \omega^8 = (\sqrt{10} \operatorname{cis}(\arctan 3))^8$

$= 10000 \operatorname{cis}(8\arctan 3)$

· Equating Re and Im :

$\cancel{10000} \cos(8\arctan 3) = -\frac{8432}{10000}$

$= -527/625$

$10000 \sin(8\arctan 3) = -5376$

$\therefore \sin(8\arctan 3) = -336/625$

5

(g) $\cos((2n+1)\arctan c) = \frac{a}{b}$ $\quad a,b \in \mathbb{Z}$, $b \neq 0$

$\therefore \cancel{(2n+1)\arctan c = \arccos(a/b)}$

$\therefore a = \cos((2n+1)\arctan c)\, b$

1

6/

ANSWER BOOKLET
LIVRET DE RÉPONSES
CUADERNILLO DE RESPUESTAS

2

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno:

0	0	0	7	f	3	M	A	H	L

Candidate name: / Nom du candidat: / Nombre del alumno:

Please write question numbers in the following format: / Veuillez numéroter les questions en utilisant la présentation suivante: / Sírvase escribir los números de las preguntas en el siguiente formato:

1 2 3 4 5 6 7 8 9 10

2

(a) $\cos i = 1 - \frac{i^2}{2!} + \frac{i^4}{4!} - \frac{i^6}{6!}$

$= 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!}$

3

~~(b)~~

(b) $\cos i = \cancel{\sum_{n=0}^{\infty} \frac{1}{(2(n+1))!}} \quad \sum_{n=1}^{\infty} \frac{1}{(2n)!}$

2

(c) $\cos x + i\sin x$

$= 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + i\left(i - \frac{i^3}{3!} + \frac{i^5}{5!} - \frac{i^7}{7!}\right)$

$= 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} - 1 - \frac{1}{3!} - \frac{1}{5!} - \frac{1}{7!}$

$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots$

$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots$

$= 1 +$

$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!}$

$= 1 + ix - \frac{x^2}{2!} + i\frac{x^3}{3!} + \dots$

$= \cos x + i\sin x$

5/

(d) $e^{i\pi} + 1 = \cos\pi + i\sin\pi + 1$

$= -1 + 0 + 1$

$= 0$

3

(e) $e^{-ix} = \cos x - i\sin x$

3

(f) (i) $\cos x + i\sin x = e^{ix}$

$\therefore \cos x + i\sin x - (\cos x - i\sin x) = e^{ix} - e^{-ix}$

$\therefore 2i\sin x = e^{ix} - e^{-ix}$

$\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

(ii) $\cos x + i\sin x + \cos x - i\sin x = e^{ix} + e^{-ix}$

$\therefore 2\cos x = e^{ix} + e^{-ix}$

$\therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

6

12/