Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3005

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[57 marks]**.

1. [Maximum points: 28]

In this problem you will investigate a Bernoulli differential equation.

Let $x\dfrac{dy}{dx} = y(1 - xy)$ where $y(0.2) = 0.4$.

(a) Use Euler's method with a step length of 0.1 to estimate the value of $y(0.5)$. [6]

(b) Show that $\dfrac{d^2y}{dx^2} = \dfrac{y^2(2xy - 3)}{x}$. [5]

(c) When $0 < x < 1$ and $0 < y < 1$ determine whether $\dfrac{d^2y}{dx^2}$ is positive or negative. [2]

(d) Hence determine whether your answer in part (a) is an overestimate or an underestimate. [2]

A *Bernoulli differential equation* is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $n \in \mathbb{R}$. A Bernoulli differential equation can be transformed into a linear differential equation using the substitution $u = y^{1-n}$.

(e) Show that the original differential equation used in part (a) is a Bernoulli differential equation. [2]

(f) Use the substitution $u = \dfrac{1}{y}$ to show that $\dfrac{du}{dx} + \dfrac{u}{x} = 1$. [3]

(g) Hence find the particular solution to the original equation. [6]

(h) Find that actual value of $y(0.5)$. [2]

2. [Maximum points: 29]

In this problem you will investigate properties of polynomials with coefficients which form a geometric sequence.

(a) Use compound angle identities to prove for $z, w \in \mathbb{C}$ then $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$. [5]

Let $f(x) = x^3 + 2x^2 + 4x + 8$.

(b) Given that $f(-2) = 0$ factorise $f(x)$. [2]

(c) Hence find all roots of $f(x)$. [2]

(d) Show that the roots of $f(x)$ form a geometric series with a complex common ratio and find the value of this ratio. [2]

(e) Follow similar steps to show that the roots of $27x^3 - 9x^2 + 3x - 1$ also form a geometric sequence with a complex common ratio. [6]

Consider the polynomial

$$g(x) = 1 + rx + r^2x^2 + \cdots + r^n x^n = \sum_{k=0}^{n} r^k x^k$$

where $r \in \mathbb{R}$ and $r \neq 0$.

(f) Find $(rx - 1)g(x)$. [2]

(g) By solving the equation $(rx - 1)g(x) = 0$ find all roots of $g(x)$. [3]

(h) Hence show that the roots of $g(x)$ form a geometric sequence with a complex common ratio and find the value of this ratio. [4]

(i) In the case when $n = 7$ and $r > 0$ illustrate the roots of $g(x)$ on an Argand diagram. [3]

1. (a) Use $x_{n+1} = x_n + 0.1$. M1

Use $y_{n+1} = y_n + 0.1 \times \dfrac{y_n(1 - x_n y_n)}{x_n}$ M1

n	x_n	y_n	
0	0.2	0.4	A1
1	0.3	0.584	A1
2	0.4	0.7446	A1
3	0.5	0.875	A1

So $y(0.5) \approx 0.875$.

(b) Use implicit differentiation M1

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{dx}(1 - xy) - y\left(y + x\frac{dy}{dx}\right)$$ A1

So

$$x\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 - xy - 1 - xy) - y^2 = -2xy\frac{dy}{dx} - y^2$$ A1

Giving

$$\frac{d^2y}{dx^2} = \frac{-2y^2(1 - xy) - y^2}{x} = \frac{y^2(-2 + 2xy - 1)}{x} = \frac{y^2(2xy - 3)}{x}$$ M1A1

(c) Since $0 < x, y < 1$ we must have $xy < 1$ making $2xy - 3 < 0$. R1

So $\dfrac{d^2y}{dx^2}$ is negative. A1

(d) Since the second derivative is negative the graph is concave downwards. So the estimate is an overestimate. R1 A1

(e) We have

$$\frac{dy}{dx} = \frac{y - xy^2}{x}$$ M1

Giving

$$\frac{dy}{dx} - \frac{y}{x} = -y^2$$ A1

(f) We have $\frac{du}{dx} = -y^{-2}\frac{dy}{dx} = -u^2\frac{dy}{dx}$ so the equation becomes M1

$$-\frac{x}{u^2}\cdot\frac{du}{dx} = u^{-1}\left(1-\frac{x}{u}\right)$$ A1

This rearranges to

$$x\frac{du}{dx} = x - u$$

Giving

$$\frac{du}{dx} + \frac{u}{x} = 1$$ A1

(g) Use the integrating factor $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. M1

So we have

$$ux = \int x\,dx = \frac{x^2}{2} + A$$ A1

Therefore

$$\frac{x}{y} = \frac{x^2 + C}{2}$$ M1

Giving

$$y = \frac{2x}{x^2 + C}$$ A1

When $x = 0.2$ then $y = 0.4$. So

$$0.4 = \frac{0.4}{0.2^2 + C}$$ M1

Giving $C = 0.96$. A1

So the equation is

$$y = \frac{2x}{x^2 + 0.96}$$

(h) $\frac{1}{0.5^2 + 0.96} = 0.826$ M1A1

2. (a) Let $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\beta + i\sin\beta)$. A1

So

$$\frac{z}{w} = \frac{|z|(\cos\theta + i\sin\theta)}{|w|(\cos\beta + i\sin\beta)} = \frac{|z|(\cos\theta + i\sin\theta)(\cos\beta - i\sin\beta)}{|w|}$$ M1

Expand

$$\frac{|z|\left(\cos\theta\cos\beta + \sin\theta\sin\beta + i(\sin\theta\cos\beta - \sin\beta\cos\theta)\right)}{|w|}$$ A1

Use compound angle identities to rewrite

$$\frac{|z|\left(\cos(\theta-\beta) + i\sin(\theta-\beta)\right)}{|w|}$$ M1

So

$$\arg(z/w) = \theta - \beta$$ A1

(b) Use the factor theorem M1

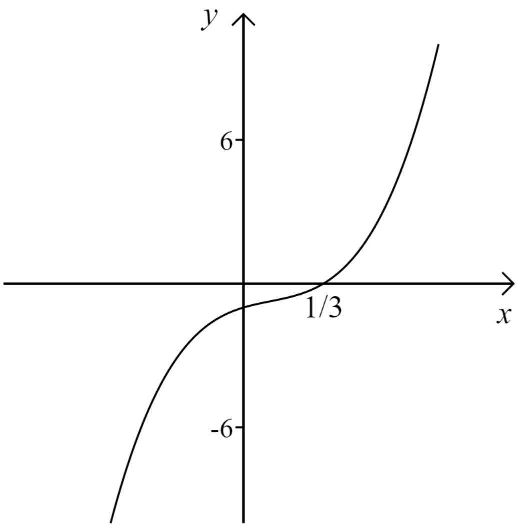
$$x^3 + 2x^2 + 4x + 8 = (x+2)(x^2+4)$$ A1

(c) The roots are -2 and $\pm 2i$. A1A1

(d) We have $2i \times i = -2$ and $-2 \times i = -2i$. A1

So the common ratio is i. A1

(e) Use a GDC to find the real root. M1



So $x = 1/3$. A1

Use the factor theorem M1

$$27x^3 - 9x^2 + 3x - 1 = (3x-1)(9x^2+1)$$ A1

So the other roots are $\pm\dfrac{i}{3}$. A1

The sequence $-\dfrac{i}{3}, \dfrac{1}{3}, \dfrac{i}{3}$ is a geometric sequence with common ratio i. A1

(f) Expand

$$rx - 1 + r^2x^2 - rx + r^3x^3 - r^2x^2 + \cdots + r^{n+1}x^{n+1} - r^nx^n$$ M1

Simplify

$$r^{n+1}x^{n+1} - 1$$ A1

(g) We have

$$x^{n+1} = \frac{1}{r^{n+1}}\left(\cos(2k\pi) + i\sin(2k\pi)\right)$$ A1

where $k \in \mathbb{Z}$.

Use De Moivre's theorem M1

$$x = \frac{1}{r}\left(\cos\left(\frac{2k\pi}{n+1}\right) + i\sin\left(\frac{2k\pi}{n+1}\right)\right)$$

for $k = 0$ to n

So the roots of $g(x)$ are

$$x = \frac{1}{r}\left(\cos\left(\frac{2k\pi}{n+1}\right) + i\sin\left(\frac{2k\pi}{n+1}\right)\right)$$

for $k = 1$ to n. A1

(h) If the k^{th} root/term of the sequence is

$$\frac{1}{r}\left(\cos\left(\frac{2k\pi}{n+1}\right)+i\sin\left(\frac{2k\pi}{n+1}\right)\right)$$ A1

So the common ratio is

$$\frac{\frac{1}{r}\left(\cos\left(\frac{2(k+1)\pi}{n+1}\right)+i\sin\left(\frac{2(k+1)\pi}{n+1}\right)\right)}{\frac{1}{r}\left(\cos\left(\frac{2k\pi}{n+1}\right)+i\sin\left(\frac{2k\pi}{n+1}\right)\right)}$$ M1

This is equal to

$$\cos\left(\frac{2(k+1)\pi-2k\pi}{n+1}\right)+i\sin\left(\frac{2(k+1)\pi-2k\pi}{n+1}\right)=\cos\left(\frac{2\pi}{n+1}\right)+i\sin\left(\frac{2\pi}{n+1}\right)$$ A1A1

(i) Draw a circle with radius $\frac{1}{r}$. A1

Draw seven of the eight vertices of a regular octagaon on this circle. A1

Exclude the vertex which is a positive real number. A1