24/08/22

Mathematics: analysis and approaches
Higher level
Paper 1

Monday, August 23rd (afternoon). 2021 MOCK

ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

2 hours

Candidate number

21	T	R	L	P	1	M	A	H	L

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is **not allowed** for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all the questions in the answer booklet provided. Fill in your session number on the front of the answer booklet and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

$\frac{77}{110} = 70\%$

2 2:00:00

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Consider two consecutive positive integers, n and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$$

$$= 2n+1$$

$$= (n) + (n+1)$$

L3

→ difference of squares equal

3/

2. [Maximum

2. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

$2\cos^2 x + 5\sin x = 4$

$\therefore 2(1-\sin^2 x) + 5\sin x = 4$

$\therefore 2 - 2\sin^2 x + 5\sin x - 4 = 0$

$\therefore -2\sin^2 x + 5\sin x - 2 = 0$

$\therefore 2\sin^2 x - 5\sin x + 2 = 0$

$\therefore 2\sin^2 x - 4\sin x - \sin x + 2 = 0$

$\therefore 2\sin x(\sin x - 2) - (\sin x - 2) = 0$

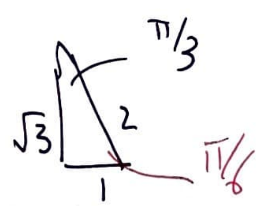
$\therefore (2\sin x - 1)(\sin x - 2) = 0$

$2\sin x = 1$ $\sin x = 2$

$\therefore \sin x = 1/2$ DNE $\{|\sin x| \le 1\}$

$\therefore x = \pi/3 , \pi - \pi/3$ $\{Q1, Q2\}$

$\therefore x = \pi/3 , 2\pi/3$



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3. [Maximum mark: 5]

In the expression of $(x+k)^7$, where $k \in \mathbb{R}$, the coeefficient of the term in x^5 is 63 .

Find the possible values of k .

$T_{n+1} = {}^7C_n x^{7-n} k^n$

$\therefore x^5 = x^{7-n}$

$\therefore n = 7-5$

$\therefore n = 2$

$\therefore T_3 = {}^7C_2 \, x^5 k^2$

$= \frac{7!}{5!\,2!} k^2 x^5$

$= \frac{7 \times 6}{2} k^2 x^5$

$= 21 k^2 x^5$

$\therefore 21k^2 = 63$

$\therefore k^2 = 3$

$\therefore k = \pm\sqrt{3}$

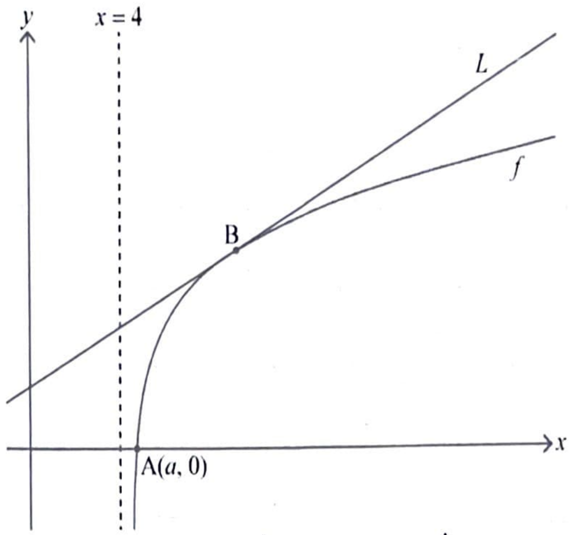
5/

4. [Maximum

4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



$\frac{dy}{dx} = f'(x) = \frac{1}{3}$

(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

a) $f(x) = \ln(x^2-16) = 0$

$\therefore x^2 - 16 = 1 \quad \{\ln(1) = 0\}$

$\therefore x = \pm\sqrt{17}$

$\therefore a = \pm\sqrt{17}$

$\therefore a = \sqrt{17} \quad \{x, a > 4\}$

b) $f'(x) = 2x\left(\frac{1}{x^2-16}\right) = \frac{1}{3}$

$\therefore 3(2x) = x^2 - 16$

$\therefore 0 = x^2 - 6x - 16$

$\therefore x^2 - 8x + 2x - 16 = 0$

$\therefore (x-8)(x+2) = 0$

$\therefore x = 8 \quad \{x > 4\}$

x-coord of B

9/

5. [Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

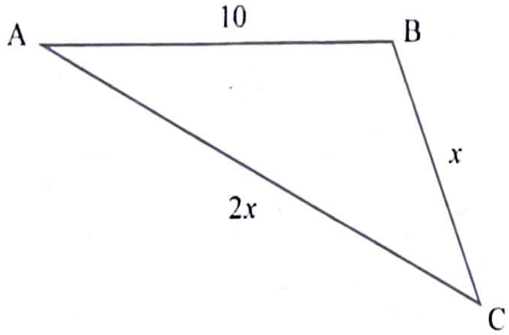
6. [Maximum mark: 7]

The follo

6. [Maximum mark: 7]

The following diagram shows triangle ABC, with $AB = 10$, $BC = x$ and $AC = 2x$.

diagram not to scale



Given that $\cos C = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

$\rightarrow \quad A = \frac{1}{2} ab \cos C$

$10^2 = 4x^2 + x^2 - 2(2x)(x) \cos C$

$\therefore 100 = 4x^2 + x^2 - 4x^2 \cancel{\cos C}\ 3/4$

$\therefore 100 = 5x^2 - 3x^2$

$\therefore x^2 = 50$

$\therefore x = \pm\sqrt{50} = \sqrt{50}$

$\therefore A = \frac{1}{2}(\sqrt{50})(2\sqrt{50}) \cos C$ sin C

$= \cancel{\frac{50}{4}} \quad \frac{3\sqrt{2500}}{4}$

$=$

4/

7. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

$\alpha\beta = -\frac{3k}{1} = -\frac{k^2}{4}$

$\therefore \quad 12k = k^2$

$\therefore k^2 - 12k = 0$

$\therefore k(k-12) = 0$

$\therefore \quad k=0, k=12$

$\therefore \quad k=12 \quad \{k>0\}$

$\alpha\beta + \alpha + \beta = k$

$\therefore (\alpha+\beta)(\alpha\beta) = -3k$

$\therefore \quad k(-k^2/4) = -3k$

$\therefore \quad -k^3 = -12k$

$\therefore \quad k^3 - 12k = 0$

$\therefore \quad k(k^2-12) = 0$

$\therefore \quad k=0, \quad k = \pm\sqrt{12}$

$\therefore k = \sqrt{12}$

3/

8. [Maximum mark: 8]

The lines

8. [Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1: r_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2: r_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

(b) Find the minimum distance between l_1 and l_2. [5]

a) equating x, y and z:

(1) $3+2\lambda = 2+\mu \rightarrow \mu = 2\lambda - 2$

(2) $2-2\lambda = -\mu \rightarrow \mu = 2\lambda - 2$

(3) $-1+2\lambda = 4+\mu \rightarrow \mu = 2\lambda - 5$

$\therefore$ there are no solutions for λ, as 2λ exists in (1), (2), (3).

b) $\vec{b} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+2 \\ 2-2 \\ -2+2 \end{pmatrix}$

$\therefore \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow l_1$ and l_2 are parallel.

3 4

Do not write solutions

9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} dx$.

$$\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} dx = \int \frac{u \cos x}{u^2 - u - 2} dx \quad \left\{ \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right\}$$

$$= \int \frac{u}{u^2 - u - 2} du$$

$$= \int \frac{u}{(u-2)(u+1)} du$$

Partial fractions:

$$\frac{u}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1}$$

$$u = A(u+1) + B(u-2)$$

sub $u = -1$: $-1 = -3B$

$\therefore B = 1/3$

sub $u = 2$: $2 = 3A$

$\therefore A = 2/3$

$$= \frac{2/3}{u-2} + \frac{1/3}{u+1}$$

$$\therefore = \int \frac{2}{3u-6} du + \int \frac{1}{3u+3} du + C$$

$$= \cancel{2\ln|3u-6| + \ln|3u+3|}$$

$$= \frac{2}{3}\int \frac{1}{u-2} du + \frac{1}{3}\int \frac{1}{u+1} du + C$$

$$= \frac{2}{3}\ln|u-2| + \frac{1}{3}\ln|u+1| + C$$

$$= \frac{2}{3}\ln|\sin x - 2| + \frac{1}{3}\ln|\sin x + 1| + C$$

$$= \frac{2}{3}\ln|\sin x - 2| + \frac{1}{3}\ln(\sin x + 1) + C \quad \{|\sin x| \leq 1\}$$

$$D: \{x \mid x \neq 2k\pi + \frac{3\pi}{2}\}$$

6

Do **not** write solutions on this page.

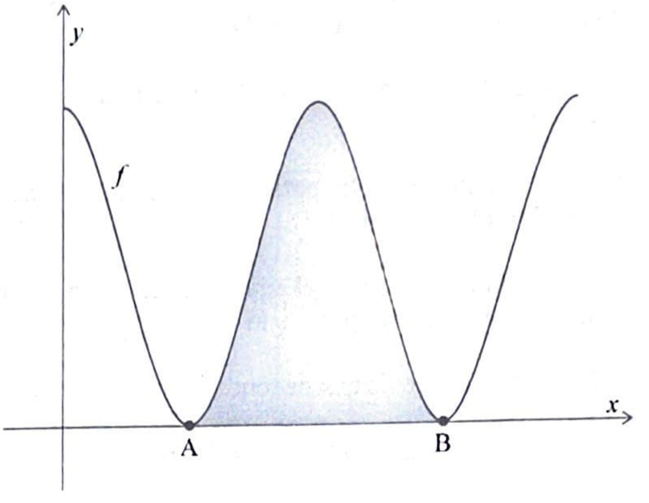
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \leq x \leq 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

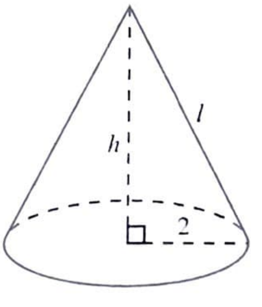
(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π . [5]

The right cone in the following diagram has a total surface area of 12π , equal to the shaded area in the previous diagram.

The cone has a base radius of 2 , height h , and slant height l .

diagram not to scale



(c) Find the value of l . [3]

(d) Hence, find the volume of the cone. [4]

11. [Maximum mark: 20]

The acceleration, $a\text{ms}^{-2}$, of a particle moving in a horizontal line t seconds, $t \geq 0$, is given by $a = -(1 + v)$ where $v\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)e^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle the returns to O.

Let s metres represent the particle's displacement from O and s_{max} its maximum displacement from O.

(i) Show that the time T taken for the particle to reach s_{max} satisfies the equation $e^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for s_{max} in terms of v_0 . [7]

Let $v(T - k)$ represent the particle's velocity k seconds before it reaches s_{max} , where

$$v(T - k) = (1 + v_0)e^{-(T-k)} - 1 .$$

(c) By using the result to part (b) (i), show that $v(T - k) = e^k - 1$. [2]

Similarly, let $v(T + k)$ represent the particle's velocity k seconds after it reaches s_{max} .

(d) Deduce a similar expression for $v(T + k)$ in terms of k . [2]

(e) Hence, show that $v(T - k) + v(T + k) \geq 0$. [3]

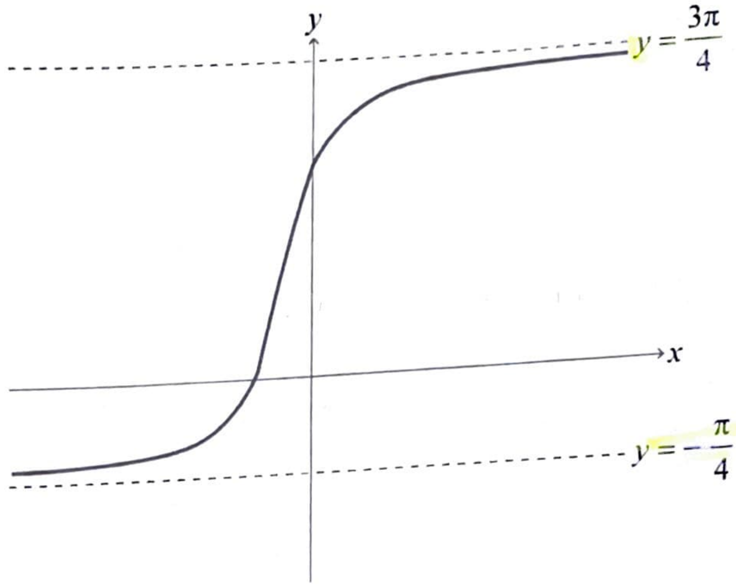
Do **not** write solutions

12. [Maxim

Do **not** write solutions on this page.

12. [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x + 1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x + 1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x + 1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that for $n \in \mathbb{Z}^+$: [9]

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$$

$2k(k^2+2k+1) + k + 1$

$= 2k^3 + 4k^2 + 2k + \cancel{k} + 4$

$= 2k^3 + 6k + k + 1$

$2k^2(k+3\cancel{k}) + 5k + 1$

$2(k^3+3k^2+3k+1) - k$

$= 2k^3 + \cancel{6}5k^2 + 5k + 1$

<u>**End of paper 1**</u>

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1 2 3 4 5 6 7 8 9 10

10

a) $f(x) = 6 + 6\cos x = 0$

$\therefore \cos x = -1$

$\therefore x = \pi, 3\pi$

3

b) $\int_{\pi}^{3\pi} (6+6\cos x)\,dx = \left[6x + 6\sin x\right]_{\pi}^{3\pi}$

$= (6.3\pi + 6\sin(3\pi)) - (6.\pi + 6\sin\pi)$

$= 18\pi - 6\pi$

$= 12\pi$

5

c)

8/

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a) $\frac{dv}{dt} = -(1+v)$

$\therefore \frac{1}{1+v}\frac{dv}{dt} = -1$

$\therefore \ln|1+v| = -t + c$

$\therefore 1+v = Ae^{-t}$ $\{A = \pm e^{c}\}$

$\therefore v = Ae^{-t} - 1$

substitute $t=0$, $v=v_0$.

$\therefore A = v_0 + 1$

hence, $v(t) = (v_0+1)e^{-t} - 1$

6

6

bi) S_{MAX} occurs at $v(t) = 0$.

$\therefore (v_0+1)e^{-t} - 1 = 0$

$\therefore e^{t} = v_0 + 1$

$\therefore$ at time T, $e^{T} = v_0 + 1$

2

bii) if $v(t) = (v_0+1)e^{-t} - 1$, then

$\frac{ds}{dt} = \int ((v_0+1)e^{-t} - 1)\,dt$

$= \int v_0 e^{-t}\,dt + \int e^{-t}\,dt + \int (-1)\,dt$

$\therefore s(t) = -v_0 e^{-t} - e^{-t} - t + c$

$\rightarrow$ for S_{MAX}, sub $e^{T} = v_0 + 1$ and

$T = \ln|v_0+1|$:

$S_{MAX} = -\frac{v_0}{v_0+1} - \frac{1}{v_0+1} - \ln|v_0+1| + c$

2

c) $v(T-k) = \cancel{(1+v_0)e^{-T} \times (1+v_0)e^{k} - 1}$

$= \frac{1+v_0}{e^{T-k}} - 1$

$= e^{k}\left(\frac{1+v_0}{e^{T}}\right) - 1$

$= e^{k}\left(\frac{1+v_0}{1+v_0}\right) - 1$

$= e^{k} - 1$

2

6/

d) $v(T+k) = (1+v_0)e^{-(T+k)} - 1$

$= \dfrac{(1+v_0)}{e^T e^k} - 1$

$= e^{-k} - 1$

2

e) $v(T-k) + v(T+k) = e^k - 1 + (e^{-k} - 1)$

$= e^k + e^{-k} - 2$

$= e^k + \dfrac{1}{e^k} - 2$

$\geq 0 \quad \{e^k \geq 1,\}$

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1 2 3 4 5 6 7 8 9 10

a) ~~$y = \arctan(2x+1) + \pi/4$~~

~~horizontal~~

$y = \arctan x$

horizontal stretch by a scale factor of $1/2$

$y = \arctan(2x)$

Translation through $\begin{pmatrix} -1/2 \\ \pi/4 \end{pmatrix}$

$y = \arctan(2(x + 1/2)) + \pi/4$

$= \arctan(2x+1) + \pi/4$

3

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b) $\tan(\arctan p + \arctan q) = \dfrac{p+q}{1-pq}$

$\therefore \text{LHS} = \tan(p+q)$ ×

$= \dfrac{p+q}{1-pq}$

$= \text{RHS}$

3 2

c) $\arctan p + \arctan q = \arctan\left(\dfrac{p+q}{1-pq}\right)$

let $p = 2x+1$ and q

$\arctan(2x+1) = \arctan\left(\dfrac{x}{x+1}\right) + \pi/4$

$\therefore 2x+1 = \dfrac{x}{x+1} + \tan(\pi/4)$

$\text{RHS} = \dfrac{x}{x+1} + 1$

$= \dfrac{x}{x+1} + \dfrac{x+1}{x+1}$

$= \dfrac{2x+1}{x+1}$

2/

c) $\arctan\left(\frac{x}{x+1}\right) + \pi/4 = \arctan\left(\frac{x}{x+1}\right) + \arctan(1)$

$$= \arctan\left(\frac{1 + \frac{x}{x+1}}{1 - (1)\left(\frac{x}{x+1}\right)}\right)$$

$$= \arctan\left(\frac{x+1+x}{x+1-x}\right)$$

$$= \arctan\left(\frac{2x+1}{1}\right)$$

$$= \arctan(2x+1)$$

3

d) $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$

1) prove for $n=1$:

$LHS = \arctan(1/2)$

$RHS = \arctan(1/1+1)$

$= \arctan(1/2)$

$= LHS$

$\therefore$ true for $n=1$

2) assume $n=k$: $\sum_{r=1}^{k} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{k}{k+1}\right)$

3) prove for $n=k+1$:

$LHS = \sum_{r=1}^{k+1} \arctan\left(\frac{1}{2r^2}\right) = \sum_{r=1}^{k} \arctan\left(\frac{1}{2r^2}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$

$= \arctan\left(\frac{k}{k+1}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$

$\cancel{= \arctan\left(\frac{k}{k+1}\right) + \arctan\left(\frac{1}{2(k^2+2k+1)}\right)}$

$= \arctan\left(\frac{\frac{k}{k+1} + \frac{1}{2(k+1)^2}}{1 - \left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$

$= \arctan\left(\frac{k(k+1)^2 + \frac{1}{2}(k+1)}{1 - k/2}\right)$ 6

$= \arctan\left(\frac{2k(k+1)^2 + (k+1)}{2-k}\right)$

$= \arctan\left(\frac{(k+1)(1+2k(k+1))}{}\right)$

6/

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1 2 3 4 5 6 7 8 9 10

3) Prove for $n=k+1$:

CONTINUED FROM PREVIOUS BOOKLET

$$LHS = \sum_{r=1}^{k+1} \arctan\left(\frac{1}{2r^2}\right)$$

$$= \sum_{r=1}^{k} \arctan\left(\frac{1}{2r^2}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$$

$$= \arctan\left(\frac{k}{k+1}\right) + \arctan\left(\frac{1}{2(k+1)^2}\right)$$ 1

$$= \arctan\left(\frac{\frac{k}{k+1} + \frac{1}{2(k+1)^2}}{1-\left(\frac{k}{2(k+1)^3}\right)}\right)$$

$$= \arctan\left(\frac{2k(k+1)^2 + (k+1)}{2(k+1)^3 - k}\right)$$

$$= \arctan\left(\frac{(k+1)\left[1 + 2k(k+1)\right]}{2(k^3+3k+3k^2+1) - k}\right)$$

$$= \arctan\left(\frac{(k+1)\left[1+2k(k+1)\right]}{2k^3+5k+6k^2+1}\right)$$

$$= \arctan\left(\frac{\quad}{\quad}\right)$$ ✗

1

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