24/08/22

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

21	M	TZ	1	P	1	M	A	H	L

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

110 − 5 − 16

68/89 = 76%

12 pages

2221–7106

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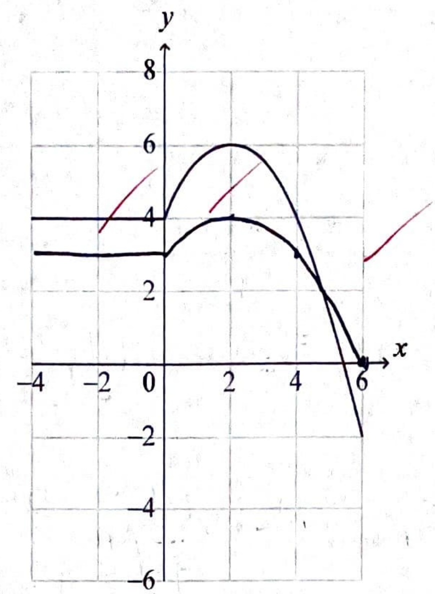
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

ai) $f(2) = 6$

aii) $f(f(2)) = f(6)$

$= -2$

b) ~~g(x)~~ =

5/

2. [Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d.

$u_8 = u_1 + 7d = 8 \quad \ldots (1)$

$S_8 = \frac{8}{2}(u_1 + u_8)$

$= 4(u_1 + u_1 + 7d) = 8$

$\therefore 4(2u_1 + 7d) = 8$

$\therefore 2u_1 + 7d = 4 \quad \ldots (2)$

$\frac{8}{2}(2u_1 + 7d)$

$= 4(2u_1 + 7d)$

$\cancel{2}u_1 + 7d = \cancel{4}\ 8$

$(2u_1 + 7d = 4) -$

$-u_1 + 0d = 4$

$\therefore$ ~~$u_1 = 4$~~ $u_1 = -4$

if $u_1 = -4$, $-4 + 7d = 8$

$\therefore 7d = 12$

$\therefore d = 12/7$

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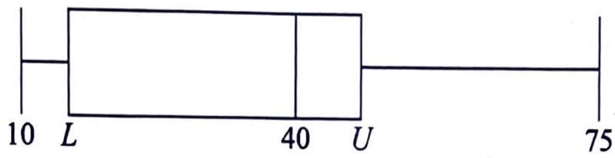
3. ~~[Maximum mark: 5]~~

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where L and U are the lower and upper quartiles respectively.

diagram not to scale



The interquartile range is 20 grams and there are no outliers in the results.

(a) Find the minimum possible value of U. [3]

(b) Hence, find the minimum possible value of L. [2]

— 5

4. [Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h,\ k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \frac{e+6}{2}$. [3]

(c) Hence, show that $k = e + \frac{e^2}{4}$. [3]

a) $f'(x) = -2(x-h)\ \cancel{k}$ ✓

b) $g'(x) = e^{x-2}$ ✓

$g'(3) = f'(3)$

$e^{3-2} = -2(3-h)$ ✓

$e = -6 + 2h$ ✓

$h = (e+6)/2$

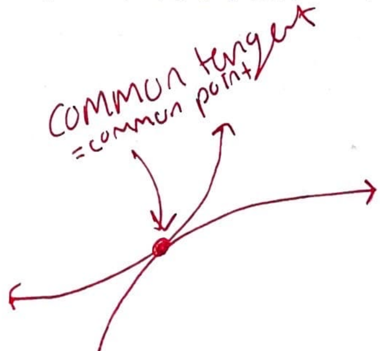
c) $\cancel{f'(3)}$

$f(x) = -\left(x - \frac{e+6}{2}\right)^2 + 2k$ ✗

$= -\left(x^2 - x(e+6) + \left(\frac{e+6}{2}\right)^2\right) + 2k$

$= -x^2 + xe + 6x + \frac{e^2+12e+36}{4} + 2k$

at



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5. [Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

a) $\cancel{\sin 2x + \cos 2x = 2\sin x\cos x - (1 - 2\sin^2 x) - 1}$

$\cancel{= 2\sin x \cos x}$

$\sin 2x + \cos 2x - 1 = 2\sin x\cos x + (1 - 2\sin^2 x) - 1$

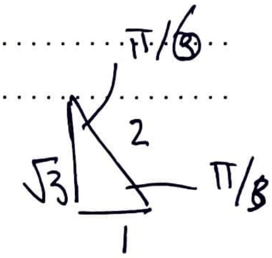
$= 2\sin x\cos x - 2\sin^2 x$

$= 2\sin x(\cos x - \sin x)$

b) $2\sin x(\cos x - \sin x) + \cos x - \sin x = 0$

$\cancel{\therefore 2(\cos x - \sin x) + \cot x - 1 = 0}$

$\therefore (2\sin x + 1)(\cos x - \sin x) = 0$



↳ $\sin x = -1/2$

$\therefore x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$= 7\pi/6, 11\pi/6$

↳ $\cos x = \sin x$

$\therefore \tan x = 1$

$\therefore x = \pi/4, \pi + \pi/4$

$= \pi/4, 5\pi/4$

$\therefore x = \frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}$

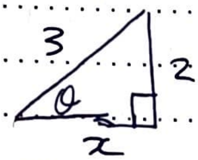
8

Q2, Q3

6. [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$.

$\frac{1}{\sin\theta} = \frac{3}{2} \longrightarrow \sin\theta = \frac{2}{3}$



$\therefore x = \sqrt{3^2 - 2^2}$

$= \sqrt{9-4}$

$= \sqrt{5}$

$\therefore \cos\theta = -\sqrt{5}/3$ {Q2, Q3}

hence, $\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{\sqrt{5}/3}{2/3}$

$= -\frac{\sqrt{5}}{2}$

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7. [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + bi$ and $b + ai$, where $a, b \in \mathbb{Z}$.

Find the possible values of a.

$(a+bi)(a-bi)(b+ai)(b-ai) = \text{PRODUCT}$

$= 400$

$\therefore (a^2-(bi)^2)(b^2-(ai)^2) = 400$

$\therefore (a^2+b^2)(b^2+a^2) = 400$

$\therefore b^2+a^2 = \pm 20 \quad \{b^2+a^2 > 0\}$

$\therefore a^2 = \cancel{\pm} 20 - b^2 \quad \dots (1)$

$a+bi+a-bi+b+ai+b-ai = \text{SUM}$

$\therefore 2a+2b = -4$

$\therefore a+b = -2$

$\therefore a = -2-b \quad \dots (2)$

$(2) \to (1)$: $(-2-b)^2 = \cancel{\pm} 20 - b^2$

$\therefore 4-2(-2)(b)+b^2 = \cancel{\pm} 20 - b^2$

$\therefore 4+4b+b^2 = \cancel{\pm} 20 - b^2$

$\therefore 2b^2+4b-16 = 0$

$\therefore b^2+2b-8 = 0$

$\therefore (b+4)(b-2) = 0$

$\therefore b = -4, \quad b = 2$

hence, $a = -2-(-4)$, $a = -2-2$

$= 2$ $\qquad = -4$

8

[Maximum mark: 5]

9.

8. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$.

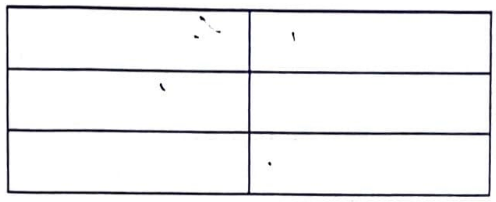
$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right) = \lim_{x\to 0}\left(\frac{2\tan 3x\left(\frac{1}{1+4x^2}\right) + 3\arctan 2x \sec^2 3x}{\tan^2 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{2\tan 3x}{1+4x^2} + \frac{3\arctan 2x}{\cos^2 3x}}{\tan^2 3x}\right)$$

9. [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.

${}^6C_1$



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

a) ${}^6C_1 \times \cancel{5! \times 4! \times 3! \times 2! \times 1!}$ ${}^5C_1 \times 4!$

$= \frac{6!}{5!} \times \frac{5!}{4!} \times 4! = 6 \times 5!$ $6\times5\times6\times6\times6$

b) ~~${}^6C_1$~~

$4 \quad {}^4C_1 \times {}^3C_1 \times \cancel{3!}\ 4! + {}^2C_1 \times {}^2C_1 \times 4!$

$= \cancel{4\times3\times3!}$

$= 4!(4\times3+2\times2)$

$= 4!(16)$

$= 4\times3\times2\times16$

$= 24\times16$

$= 384$

24
16
144
240
384

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	p	p	$\frac{1}{2}p$

(a) Find the value of p. [2]

(b) Hence, find the value of $\mathrm{E}(X)$. [2]

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$\mathrm{P}(Y=y)$	q	q	q	r

(c) (i) State the range of possible values of r.

(ii) Hence, find the range of possible values of q. [3]

(d) Hence, find the range of possible values for $\mathrm{E}(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $\mathrm{E}(Y)$. [6]

Do **not** write solutions on this page.

11. [Maximum mark: 19]

$3-z = -(z-3)$
$= -(z-(3))$

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3-z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is 45°. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

12. [Maximum mark: 20]

$f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^k \frac{(2k-1)!}{(k-1)!} (1+x)^{\frac{1}{2}-k}$

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$

for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = e^{mx}$, $m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

(c) Find the possible values of m. [8]

References:

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1 2 3 4 5 6 7 8 9 10

11

ai) $\frac{-1+1}{2} = 0 \quad \ldots (1) \quad \left\{\frac{x+1}{2}\right\}$

$0 = 0 \quad \ldots (2) \quad \{y\}$

$3 - 3 = 0 \quad \ldots (3) \quad \{3 - z\}$

as $(1) = (2) = (3)$, the point $(-1, 0, 3)$ lies on L_1.

aii) ~~$\frac{x+1}{2} = \frac{y+0}{1} = \frac{z-3}{-1}$~~

~~$\therefore x =$~~

$\frac{x-(-1)}{2} = \frac{y-0}{1} = \frac{z-(+3)}{-1}$

$\therefore x = -1 + 2\lambda$

$y = 0 + \lambda$

$z = +3 - \lambda$

$\therefore r = \begin{pmatrix} -1 \\ 0 \\ +3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \lambda$

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b) $\cos 45 = \frac{1}{\sqrt{2}}$

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$$

$$\therefore \frac{1}{\sqrt{2}} = \frac{|v_1 \cdot v_2|}{|v_1||v_2|}$$

$$= \frac{|2a+1+1|}{\sqrt{4+1+1}\sqrt{a^2+1+1}}$$

$$= \frac{|2a+2|}{\sqrt{6}\sqrt{a^2+2}}$$

$$\therefore \frac{1}{\sqrt{2}} = \frac{|2a+2|}{\sqrt{6}\sqrt{a^2+2}}$$

$$\therefore \frac{\sqrt{2}}{2} = \cancel{\frac{2a+2}{\sqrt{6a^2+12}}} \quad \frac{2a+2}{\sqrt{6}\sqrt{a^2+2}}$$

$$\therefore \cancel{\tfrac{1}{2}}\ 1 = \frac{a+1}{\sqrt{3a^2+6}}$$

$$\therefore \sqrt{3a^2+6} = a+1$$

$$\therefore 3a^2+6 = a^2+2a+1$$

$$\therefore 2a^2-2a+5 = 0$$

$$\therefore a = \frac{2 \pm \sqrt{4-4}}{}$$

$$\frac{1}{\sqrt{2}} = \frac{|2a+2|}{\sqrt{6}\sqrt{a^2+2}}$$

$$\therefore \frac{\sqrt{6}}{\sqrt{2}} = \frac{|2a+2|}{\sqrt{a^2+2}}$$

$$\therefore \sqrt{3} = \frac{|2a+2|}{\sqrt{a^2+2}}$$

$$\sqrt{3}\sqrt{a^2+2} = |2a+2|$$

$$\therefore (3)(a^2+2) = (2a+2)^2$$

$$\therefore 3a^2+6 = 4a^2+8a+4$$

$$\therefore 0 = a^2+8a-2$$

$$\therefore a = \frac{-8 \pm \sqrt{64+8}}{2}$$

$$= \frac{-8 \pm \sqrt{72}}{2}$$

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1 2 3 4 5 6 7 8 9 10

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a) $f(x) = (1+x)^{1/2}$

$\therefore f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$\therefore f''(x) = (\frac{1}{2})(-\frac{1}{2})(1+x)^{-3/2}$

$= -\frac{1}{4(1+x)^{3/2}}$

$= \frac{-1}{4\sqrt{(1+x)^3}}$

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b) ~~Pr~~ $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!} (1+x)^{\frac{1}{2}-n}$

Step 1: Prove for $n=2$:

$LHS = f^{(2)}(x)$

$= -\frac{1}{4\sqrt{(1+x)^3}}$

$RHS = \left(-\frac{1}{4}\right)^{2-1} \frac{(4-3)!}{(2-2)!} (1+x)^{\frac{1}{2}-2}$

$= \left(-\frac{1}{4}\right)(1+x)^{-3/2}$

$= -\frac{1}{4\sqrt{(1+x)^3}}$

$= LHS$

$\therefore$ true for $n=1$

Step 2: assume true for $n=k$:

$f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!} (1+x)^{\frac{1}{2}-k}$

Step 3: prove true for $n=k+1$.

$f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x)$

$= \frac{d}{dx}\left[\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!} (1+x)^{\frac{1}{2}-k}\right]$

$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \cdot \frac{(2k-3)!}{(k-2)!} (1+x)^{-\frac{1}{2}-k}$

$= \left(\frac{1-2k}{2}\right)\left(-\frac{1}{4}\right)^{k}\left(-\frac{4}{1}\right) \frac{(2k-3)!}{(k-2)!} (1+x)^{\frac{1}{2}-(k+1)}$

$= \left(-\frac{1}{4}\right)^{k}\left(\frac{-4(1-2k)(2k-3)!}{2(k-2)!}\right)(1+x)^{\frac{1}{2}-(k+1)}$

$$2(k+1)-3 = 2k+2-3 = 2k+1-1$$

$$k-2 = k-1$$

$$\cancel{= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{(-2+4k)(2k-3)!}{(k-2)!}\right)(1+x)^{\frac{1}{2}-(k+1)}}$$

$$\cancel{= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{(-2+4k)(2k-3)(2k-2)(2k-1)!}{(k-2)(k-1)!}\right)(1+x)^{\frac{1}{2}-(k+1)}}$$

$$\cancel{= \left(-\tfrac{1}{4}\right)^{k}\Big(}$$

$$\cancel{= \frac{2(2k-1)(2k-3)!}{(k-2)(k-1)!}}$$

$$\cancel{= \frac{(4k-2)(2k-3)!}{(k-2)(k-1)!}}$$

$$\cancel{= 2(2k-1)(2k-3)!\,2(k-2)(2k-}$$

$$= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{-4(1-2k)(2k-3)!}{2(k-2)!}\right)(1+x)^{\frac{1}{2}-(k+1)}$$

$$= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{+2(2k-1)(2k-2)(2k-3)!}{(k-2)!(2k-2)}\right)(1+x)^{\frac{1}{2}-(k+1)}$$

$$= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{2(2k-1)!}{(k-1)!\times 2}\right)(1+x)^{\frac{1}{2}-(k+1)}$$

$$= \left(-\tfrac{1}{4}\right)^{k}\left(\frac{(2(k+1)-3)!}{((k+1)-1)!}\right)(1+x)^{\frac{1}{2}-(k+1)}$$

$\therefore$ true for P_{k+1} when P_k is true

Step 4: as P_2 is true and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq 2, n \in \mathbb{Z}$, by mathematical induction

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c) $h(x) = f(x) \times g(x)$, $g(x) = e^{mx}$

$\therefore h'(x) = f(x)g'(x) + f'(x)g(x)$

$\therefore h''(x) = f(x)g''(x) + \cancel{g}f'(x)g'(x) + f'(x)g'(x) + g(x)f''(x)$

$g(x) = e^{mx}$ $\quad g(0) = 1$
$g'(x) = me^{mx}$ $\quad g'(0) = m$
$g''(x) = m^2e^{mx}$ $\quad g''(0) = m^2$

$f(x) = \sqrt{1+x}$ $\quad f(0) = 1$
$f'(x) = \frac{1}{2\sqrt{1+x}}$ $\quad f'(0) = \frac{1}{2}$
$f''(x) = \frac{-1}{4\sqrt{1+x}}$ $\quad f''(0) = -1/4$
~~$f'''(x) = \frac{1}{16}\left(\frac{6}{1}\right)(1+x)^{\frac{1}{2}-3}$ $\quad f'''(0) = 3/8$~~

$\therefore h(0) = 1$
$h'(0) = 1m + \frac{1}{2} = m + 1/2$
$h''(0) = m^2 + \frac{1}{2}m \times 2 + (-1/4) = m^2 + m - 1/4$

$\therefore M = 1 + (m + \frac{1}{2})x + \frac{x^2}{2}(m^2 + m - 1/4)$

$\therefore \frac{1}{2}(m^2 + m - 1/4) = 7/4$
$\therefore \cancel{2}\, 2(m^2 + m - 1/4) = 7$
$\therefore 2m^2 + 2m - 1/2 = 7$
$\therefore 4m^2 + 4m - 1 = 14$
$\therefore 4m^2 + 4m - 15 = 0$
$\therefore 4m^2 + 10m - 6m - 15 = 0$
$\therefore$

~~$2m(m-3)$~~
$2m(2m-3) + 5(2m-3) = 0$
$\therefore (2m+5)(2m-3) = 0$
$\therefore m = 3/2$
OR $m = -5/2$

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