Markscheme

ID: 3015

Mathematics: analysis and approaches

Higher level

1. (a) We have

$$\sum_{r=1}^{n-1} r^3 < \int_0^n x^3\,dx < \sum_{r=1}^{n} r^3$$ A1

Integrate

$$\sum_{r=1}^{n-1} r^3 < \frac{n^4}{4} < \sum_{r=1}^{n} r^3$$ M1

Therefore

$$-n^3 + \sum_{r=1}^{n-1} r^3 < \frac{n^4}{4}$$ A1

So

$$\sum_{r=1}^{n} r^3 < \frac{n^4 + 4n^3}{4}$$ A1

(b)

(i) $n^p \times n = n^{p+1}$ M1A1

(ii) $n + 1$ A1A1

(c) The overall area of the diagram is

$$(n+1)\sum_{r=1}^{n} r^p$$ A1

The area of the shaded rectangles is

$$\sum_{r=1}^{n} r^{p+1}$$ A1

The area of the rth unshaded rectangle is

$$1 \times (1^p + 2^p + \cdots + r^p) = \sum_{m=1}^{r} m^r$$ A1

Therefore

$$\sum_{r=1}^{n} r^{p+1} = (n+1)\sum_{r=1}^{n} r^p - \sum_{r=1}^{n}\left(\sum_{m=1}^{r} m^p\right)$$ A1

(d) We have

$$\sum_{r=1}^{n} r^2 = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n}\left(\sum_{m=1}^{r} m\right) \qquad \text{A1}$$

Use the arithmetic series formula M1

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)^2}{2} - \sum_{r=1}^{n}\frac{r(r+1)}{2} \qquad \text{A1}$$

Expand, rearrange and simplify

$$\frac{3}{2}\sum_{r=1}^{n} r^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)}{4} \qquad \text{M1}$$

So

$$\sum_{r=1}^{n} r^2 = \frac{2}{3} \times \frac{2n^3+3n^2+n}{4} = \frac{n(n+1)(2n+1)}{6} \qquad \text{A1A1}$$

(e) We have

$$\sum_{r=1}^{n} r^3 = (n+1)\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n}\left(\sum_{m=1}^{r} m^2\right) \qquad \text{A1}$$

Use the formula from part (c) M1

$$\sum_{r=1}^{n} r^3 = \frac{n(n+1)^2(2n+1)}{6} - \sum_{r=1}^{n}\frac{r(r+1)(2r+1)}{6} \qquad \text{A1}$$

Expand, rearrange and simplify

$$\frac{4}{3}\sum_{r=1}^{n} r^3 = \frac{n(n+1)^2(2n+1)}{6} - \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{6}\cdot\frac{n(n+1)}{2} \qquad \text{M1}$$

So

$$\sum_{r=1}^{n} r^3 = \frac{3}{4}\cdot\frac{n(n+1)\left(2(n+1)(2n+1)-(2n+1)-1\right)}{12} = \frac{n(n+1)(4n^2+4n)}{16} \qquad \text{A1A1}$$

Factorise

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \qquad \text{A1}$$

2. (a)

(i) $1 + x + x^2 + x^3$ A1

(ii) $1 + 2x + 3x^2 + 4x^3$ A1A1

(b) The function $f(x)$ is an infinite geometric series with first term 1 and common difference x. R1

Its value is therefore $\dfrac{1}{1-x}$. A1

(c) Use the chain rule. M1

$$f'(x) = (-1)\left(-\frac{1}{(1-x)^2}\right) = \frac{1}{(1-x)^2}$$ A1A1

(d) We have

$$a = \frac{1}{6}$$ A1

$$b = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$ M1A1

(e) This is an infinite geometric series with first term 1/6 and common ratio 5/6. R1

Its value is therefore

$$\frac{1/6}{1-5/6} = \frac{1/6}{1/6} = 1$$ M1A1

(f) We have

$$E(X) = \frac{1}{6}\left(1 + 2 \times \frac{5}{6} + 3 \times \left(\frac{5}{6}\right)^2 + 4 \times \left(\frac{5}{6}\right)^3 + \cdots\right)$$ A1

Use the formula from part (c) to evaluate. M1

$$E(X) = \frac{1}{6} \times \frac{1}{(1-5/6)^2} = 6$$ A1A1

(g) $2 + 6x + 12x^2 + 20x^3$ A1A1

(h) We have

$$xf''(x) + f'(x) = x\sum_{k=1}^{\infty}(k+1)(k)x^{k-1} + \sum_{k=1}^{\infty}kx^{k-1}$$ M1

This is equal to

$$1 + \sum_{k=1}^{\infty}\left(x^k(k+1)k + x^k(k+1)\right)$$ A1

Factorise and rewrite

$$1 + \sum_{k=1}^{\infty}(k+1)^2x^k = \sum_{k=1}^{\infty}k^2x^{k-1}$$ A1

(i) Use the chain rule M1

$$f''(x) = (-2)(-1)\left(\frac{1}{(1-x)^3}\right) = \frac{2}{(1-x)^3}$$ A1A1

(j) We have

$$E(X^2) = \frac{1}{6}\left(\sum_{k=0}^{\infty}(k+1)^2 x^k\right) = \frac{1}{6}\left(\frac{5}{6} \cdot \frac{2}{(1-5/6)^3} + \frac{1}{(1-5/6)^2}\right) = 66$$ M1A1

So

$$\mathrm{Var}(X) = 66 - 6^2 = 30$$ M1A1