Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Thursday 4 May 2017 (afternoon)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[100 marks]**.

2217–7205

12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Find the term independent of x in the binomial expansion of $\left(2x^2 + \frac{1}{2x^3}\right)^{10}$.

$T_{r+1} = \binom{n}{r}(2x^2)^{n-r}\left(\frac{1}{2x^3}\right)^r = \binom{10}{r}\cancel{(2^{10-r})(2^{-r})}(x^{20-2r})(x^{-3r})$

$= \binom{10}{r}(2^{10-2r})(x^{20-5r})$

$\Rightarrow$ for independent x term, $r = 4$

$\therefore T_5 = \binom{10}{4}(2^2)(x^{20-20})$

$= \frac{10!}{4!6!} \times 4$

$\cancel{= \frac{10\times9\times8\times7\times4}{4\times3\times2\times1}}$

$\cancel{= \frac{90\times56}{3\times2}}$

$= \frac{10\times9\times8\times7}{3!}$

$= 10\times3\times4\times7$

$= 840$

5

2. [Maximum mark: 6]

The function f is defined by $f(x) = 2x^3 + 5$, $-2 \le x \le 2$.

(a) Write down the range of f. [2]

(b) Find an expression for $f^{-1}(x)$. [2]

(c) Write down the domain and range of f^{-1}. [2]

(a) $f(x) = 2x^3 + 5$, $-2 \le x \le 2$

$\therefore$ ~~Range: $y \in \mathbb{R}$~~

$f(-2) = 2(-8) + 5 = -11$

$f(2) = 2(8) + 5 = 21$ 2

$\therefore$ Range: $-11 \le y \le 21$ $\to f(x)$

(b) $y = 2x^3 + 5$

$\therefore x^3 = (y-5)/2$

$\therefore x = \sqrt[3]{\frac{y-5}{2}}$

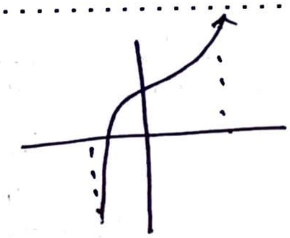
$\therefore f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$ 2

~~(c)~~

$\to f^{-1}(x)$

(c) Range: $-2 \le y \le 2$

Domain: $-11 \le x \le 21$ 2



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3. [Maximum mark: 5]

The 1st, 4th and 8th terms of an arithmetic sequence, with common difference d, $d \neq 0$, are the first three terms of a geometric sequence, with common ratio r. Given that the 1st term of both sequences is 9 find

(a) the value of d; [4]

(b) the value of r. [1]

(a) $u_1 = 9$ ~~$u_2 = 9+d$~~ ~~$u_3 = 9+2d$~~

$u_4 = 9+3d$ $u_8 = 9+7d$

$\therefore \frac{u_8}{u_4} = \frac{u_4}{u_1}$

$\therefore u_8 u_1 = (u_4)^2$

$\therefore (9+7d)(9) = (9+3d)^2$

$\therefore 81 + 63d = 81 + 54d + 9d^2$

$\therefore 0 = 9d^2 - 9d$

$\therefore 9d(d-1) = 0$

$\therefore d = 0$, $\boxed{d=1}$ ← as $d \neq 0$

4

(b) $r = \frac{u_4}{u_1} = \frac{9+3}{9}$

$= \frac{12}{9}$

$\boxed{\therefore r = \frac{4}{3}}$

1

Other method

Simult. $\begin{cases} 9+3d = 9r \\ 9+7d = 9r^2 \end{cases}$

5

4. [Maximum mark: 7]

A particle moves along a straight line. Its displacement, s metres, at time t seconds is given by $s = t + \cos 2t$, $t \geq 0$. The first two times when the particle is at rest are denoted by t_1 and t_2, where $t_1 < t_2$.

(a) Find t_1 and t_2. [5]

(b) Find the displacement of the particle when $t = t_1$. [2]

(a) $s = t + \cos 2t$

$v = 1 - \sin 2t \ (2)$

$= 1 - 2\sin 2t$

$\Rightarrow$ Rest occurs when $1 - 2\sin 2t = 0$

$\therefore 2\sin 2t = 1$

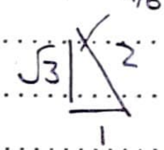
$\therefore \cancel{2t = 0, \pi, 2\pi,}$

$\therefore \cancel{t = 0, \pi/2, \pi, 3\pi/2,}$ $\pi/6$

$\therefore \sin 2t = 1/2$

$\therefore 2t = \pi/6, 5\pi/6$

$\therefore t = \pi/12, 5\pi/12$



$\therefore \cancel{t_1 = 0\ s}$

$\cancel{t_2 = \pi/2\ s}$

$\therefore \boxed{t_1 = \pi/12\ s \quad t_2 = 5\pi/12\ s}$

5

(b) At $t_1 = \pi/12$ s,

$$s = \frac{\pi}{12} + \cos\frac{\pi}{6}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2}$$

$$\therefore \boxed{s = \frac{\pi + 6\sqrt{3}}{12}}$$

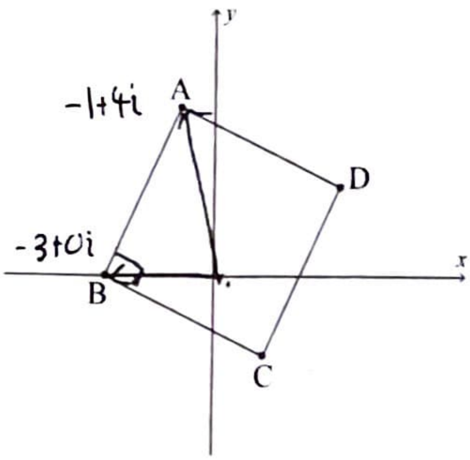
M

2

7

5. [Maximum mark: 4]

In the following Argand diagram the point A represents the complex number $-1+4\mathrm{i}$ and the point B represents the complex number $-3+0\mathrm{i}$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



$A = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ $B = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$\therefore \overrightarrow{AB} = \begin{pmatrix} -3+1 \\ 0-4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$\therefore \overrightarrow{BC} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\Rightarrow C = B + \overrightarrow{BC} = \begin{pmatrix} -3+4 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow D = A + \overrightarrow{BC} = \begin{pmatrix} -1+4 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\therefore C = 1-2i$

$D = 3+2i$

4

6. [Maximum mark: 7]

(a) Using the substitution $x = \tan\theta$ show that $\int_0^1 \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos^2\theta \,\mathrm{d}\theta$. [4]

(b) Hence find the value of $\int_0^1 \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x$. [3]

a) $\int_0^1 \frac{1}{(x^2+1)^2} dx = \int_0^{\pi/4} \frac{1}{(\tan^2\theta+1)^2} \sec^2\theta \, d\theta$

$\left[\begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta \, d\theta \\ \tan(1) = \pi/4 \end{array}\right]$ $= \int_0^{\pi/4} \frac{1}{\sec^4\theta} \sec^2\theta \, d\theta$

$= \int_0^{\pi/4} \frac{1}{\sec^2\theta} d\theta$

$= \int_0^{\pi/4} \cos^2\theta \, d\theta$

4

(b) $\int_0^{\pi/4} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} (\cos 2\theta + 1) \, d\theta$

$= \frac{1}{2}\left[\frac{1}{2}\cancel{\cos}\sin 2\theta + \theta\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4} - \frac{1}{2}\sin 0 - 0\right)$

$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$

$= \frac{1}{4} + \frac{\pi}{8}$

$\boxed{= \frac{2+\pi}{8}}$

3

7

7. [Maximum mark: 7]

(a) The random variable X has the Poisson distribution $\text{Po}(m)$. Given that $\text{P}(X > 0) = \frac{3}{4}$, find the value of m in the form $\ln a$ where a is an integer. [3]

(b) The random variable Y has the Poisson distribution $\text{Po}(2m)$. Find $\text{P}(Y > 1)$ in the form $\frac{b - \ln c}{c}$ where b and c are integers. [4]

8. [Maximum mark: 9]

Prove by mathematical induction that $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$, where $n \in \mathbb{Z}, n \geq 3$.

$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$

$\Rightarrow$ Prove for $n=3$: $LHS=\binom{2}{2}=1$

$RHS=\binom{3}{3}=1$

$=LHS$

$\therefore$ True for $n=3$

$\Rightarrow$ Assume true for $n=k$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$

(inductive hypothesis)

~~$\Rightarrow$ Prove for $n=k+1$:~~

~~$\therefore \binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}=\binom{k+1}{3}$~~

~~$LHS=\binom{k}{3}+\binom{k}{2}$ {by inductive hypothesis}~~

~~$=\frac{k!}{3!(k-3)!}+\frac{k!}{2!(k-2)!}$~~

$=\frac{k!}{3!(k-3)!}+\frac{3k!(k-3)}{3!(k-3)!}$

$=\frac{k!+3k!(k-3)}{3!(k-3)!}$

$=$

continued in answer booklet.

9

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 17]

Consider the function f defined by $f(x) = x^2 - a^2, x \in \mathbb{R}$ where a is a positive constant.

(a) Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

(i) $y = f(x)$;

(ii) $y = \dfrac{1}{f(x)}$;

(iii) $y = \left|\dfrac{1}{f(x)}\right|$. [8]

(b) Find $\int f(x) \cos x \, \mathrm{d}x$. [5]

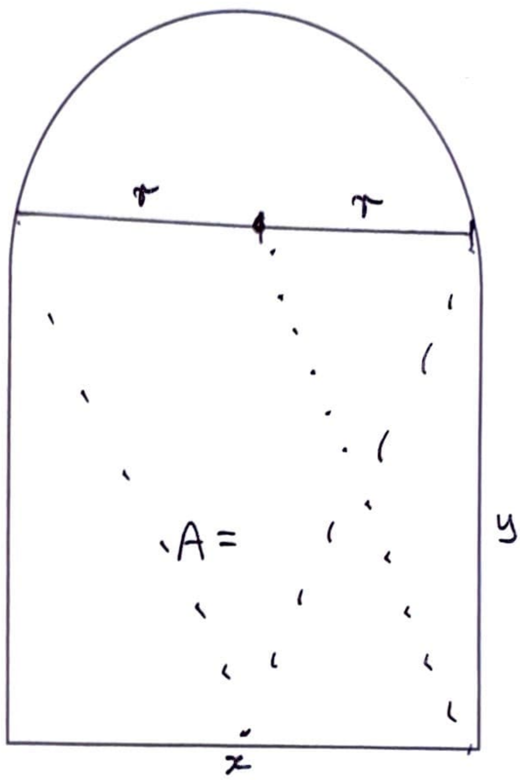
The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

(c) By finding $g'(x)$ explain why g is an increasing function. [4]

Do **not** write solutions on this page.

10. [Maximum mark: 11]

A window is made in the shape of a rectangle with a semicircle of radius r metres on top, as shown in the diagram. The perimeter of the window is a constant P metres.



Perim = P m

(a) (i) Find the area of the window in terms of P and r.

2r

(ii) Find the width of the window in terms of P when the area is a maximum, justifying that this is a maximum. [9]

(b) Show that in this case the height of the rectangle is equal to the radius of the semicircle. [2]

Do **not** write solutions on this page.

11. [Maximum mark: 22]

(a) Solve $2\sin(x+60^\circ) = \cos(x+30^\circ)$, $0^\circ \le x \le 180^\circ$. [5]

(b) Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$. [3]

(c) Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta$, $z \in \mathbb{C}$, $0 \le \theta \le \pi$.

(i) Find the modulus and argument of z in terms of θ. Express each answer in its simplest form.

(ii) Hence find the cube roots of z in modulus-argument form. [14]

$z^3 = 8\,\mathrm{cis}(x + 2k\pi)$

$\therefore z = 2\,\mathrm{cis}\left(\frac{x+2k\pi}{3}\right)$

$= 2\,\mathrm{cis}\left(\frac{x}{3}\right)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$1 - \cos 2\theta = 1 - \cos^2\theta + \sin^2\theta$

$= 2\sin^2\theta$

$\frac{\cos\theta}{\sin\theta} = \cot\theta$ $\quad \cancel{\sqrt{\cos^2}}$ $\quad \frac{\sqrt{\sin^2\theta - 1}}{\sqrt{\cos^2\theta - 1}}$ $\quad \sqrt{\sin^2\theta - 1}\sqrt{\cos^2\theta - 1}$ $\quad \cos^2\theta - 1$

=

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Example / Ejemplo 27 | 2 | 7 | Example / Ejemplo 3 | | 3 |

8

contd. → consider

⇒ Prove for $n=k+1$:

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\dots+\binom{k-1}{2}+\binom{k}{2}=\binom{k+1}{3}$$

$$\therefore LHS=\binom{k}{3}+\binom{k}{2} \quad \{\text{by inductive hypothesis}\}$$

$$=\frac{k!}{3!(k-3)!}+\frac{k!}{2!(k-2)!}$$

$$=\frac{k!(k-2)}{3!(k-2)(k-3)!}+\frac{3k!}{3!(k-2)!}$$

$$=\frac{k!(k-2)}{3!(k-2)!}+\frac{3k!}{3!(k-2)!}$$

$$=\frac{k!(k-2+3)}{3!(k-2)!}$$

$$=\frac{k!(k+1)}{3!(k-2)!}$$

$$=\frac{(k+1)!}{3!(k-2)!}$$

$$=\frac{(k+1)!}{3!((k+1)-3)!}$$

$$=\binom{k+1}{3}=RHS$$

$\therefore$ true for $k+1$

Step 4: as true for $n=3$, and true for $n=k+1$ whenever $n=k$ is true, it is true for all $n \in Z^{+}$, $n \geq 3$ by mathematical induction.

9 $f(x) = x^2 - a^2$ $x \in \mathbb{R}$ $a > 0$

(a)(i) $y = f(x) = x^2 - a^2$ → ~~y-int: $-a^2$~~ ~~x-int~~

→ y-int: $y = -a^2$

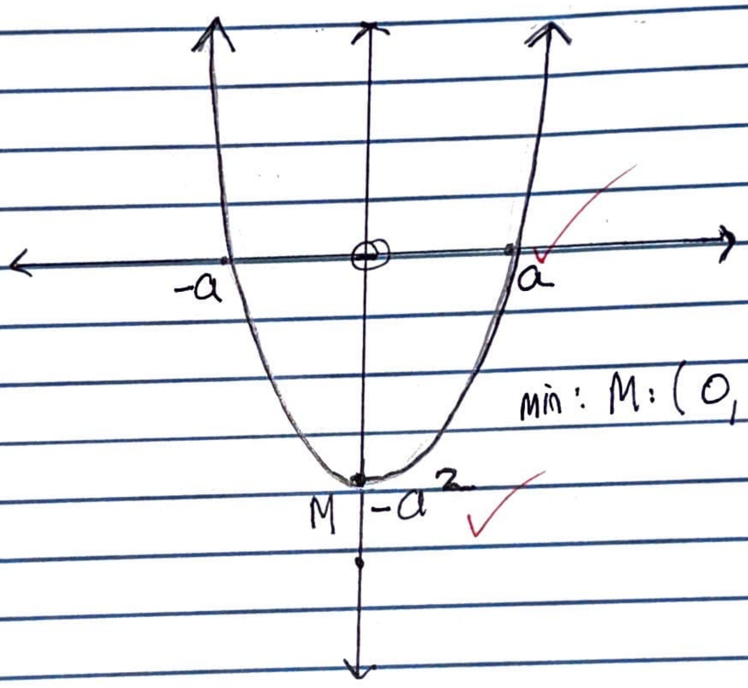
→ x-int: $x^2 - a^2 = 0$

$\therefore x^2 = a^2$

$\therefore x = \pm a$

→ asymptotes: none

→ concave up parabola



min: M: $(0, -a)$

(ii) $y = \frac{1}{x^2 - a^2}$

$f(x) = (x^2+a^2)^{-1}$

$f'(x) = -(x^2+a^2)^{-2}(2x)$

$= -2x/(x^2+a^2)^2 = 0$

$\therefore \quad x = 0$

→ y-int $= -1/a^2$

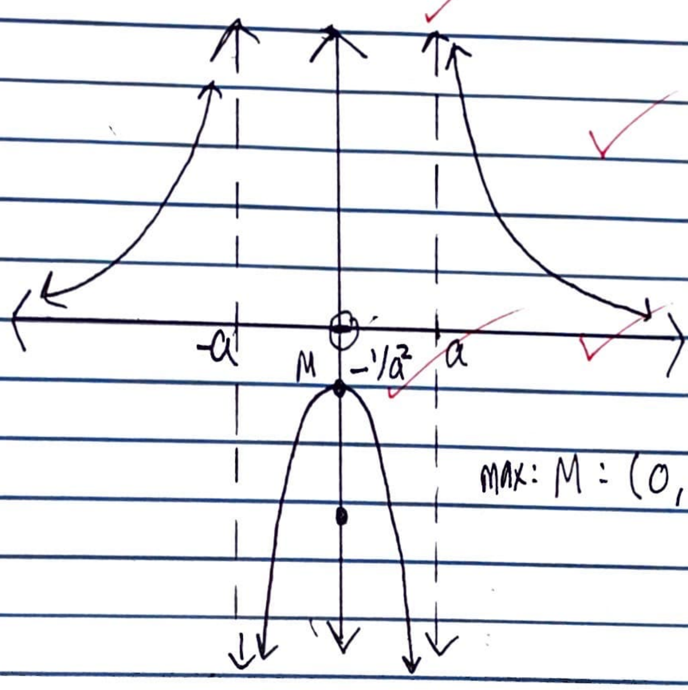
→ x-int : DNE

→ ~~verita~~ vertical asym : $x^2 - a^2 \neq 0$

$\therefore \quad x^2 \neq a^2$

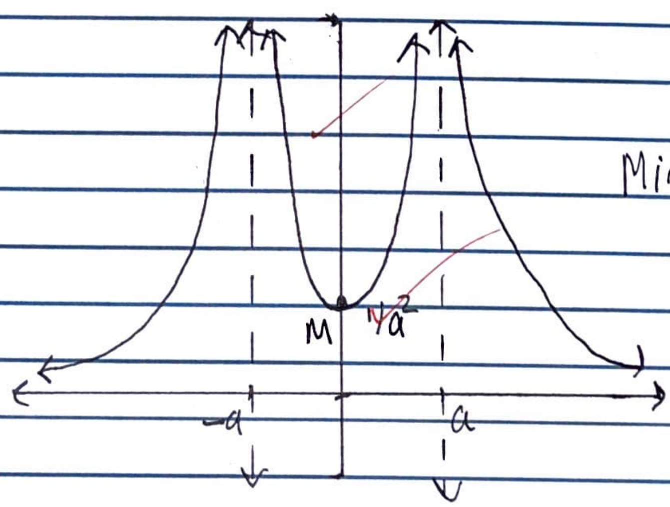
$\therefore \quad x \neq \pm a$

→ horizontal asymptote : DNE



max: M : $(0, -1/a^2)$

(iii)



Min: M: $(0, 1/a^2)$

8

(b) $\int f(x)\cos x\,dx = \int (x^2-a^2)\cos x\,dx$

$= \int x^2\cos x\,dx - a^2\int\cos x\,dx$

Parts: $\boxed{\begin{array}{ll} u=x^2 & du=2x \\ dv=\cos x & v=\sin x \end{array}}$

$= x^2\sin x - \int 2x\sin x\,dx - a^2\sin x + c$

Parts: $\boxed{\begin{array}{ll} u=x & du=1 \\ dv=\sin x & v=-\cos x \end{array}}$

$= x^2\sin x - 2\left(-x\cos x - \int -\cos x\,dx\right) - a^2\sin x + c$

$= x^2\sin x + 2x\cos x - 2\int\cos x\,dx - a^2\sin x + c$

$= x^2\sin x + 2x\cos x - 2\sin x - a^2\sin x + c$

$= (x^2-a^2-2)\sin x + 2x\cos x + c$

5

(c) $\cancel{g(x)} = \cancel{x\sqrt{f(x)}}$ $\cancel{|x|>a}$

$= \cancel{x/f(x)})$

$\cancel{\therefore g'(x) =}$

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Example Ejemplo 27 [27] Example Ejemplo 3 [3]

(c) $g(x) = x\sqrt{f(x)}$

$= g(x)\sqrt{f(x)}$

$= g(x)\,(f(x))^{1/2}$

$\therefore g'(x) = g(x)\left(\tfrac{1}{2}\right)(f(x))^{-1/2}(f'(x)) + (f(x))^{1/2}\, g'(x)$

$= \tfrac{1}{2} g(x) f(x)^{-1/2} f'(x) + f(x)^{1/2} g'(x)$

$= \tfrac{1}{2} x (x^2-a^2)^{-1/2} f'(x) + (x^2-a^2)^{1/2}$

$= \tfrac{1}{2} x (x^2-a^2)^{-1/2} (2x) + (x^2-a^2)^{1/2}$

$= x^2 (x^2-a^2)^{-1/2} + (x^2-a^2)^{1/2}$

$= \dfrac{x^2}{\sqrt{x^2-a^2}} + \sqrt{x^2-a^2}$

~~$\therefore$ As $x^2 > \sqrt{x^2-a^2}$ and x^2 is always increasing~~

~~And $\sqrt{x^2-a^2}$ is always increasing~~

~~$g'(x)$ is always increasing~~

3

As $x > a$ (because of $\sqrt{x^2-a^2}$)

And $(-x)^2 = (x^2)$ {always positive},

$g'(x) > 0$ for all $x > a$

=> explicitly refer to each part of the expression

16

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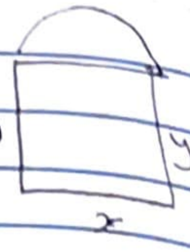
10

(a)(i) $P = x + 2y + \pi r$

$A = xy + \pi r^2/2$



$A = xy + \pi r^2/2$

$\cancel{x + 2y + \pi r = P}$

$\therefore A = 2ry + \pi r^2/2 \qquad \{x = 2r\}$

$\therefore A = 2r\left(\frac{P - x - \pi r}{2}\right) + \frac{\pi r^2}{2}$

$= rP - rx - \pi r^2 + \pi r^2/2$

$= rP - rx - \pi r^2/2$

$= r(P - x - \pi r/2)$

$= Pr - 2r^2 - \pi r^2/2$

(ii) $\cancel{A = r(P - x - \pi r/2)}$

$\cancel{\therefore A = rP - rx - \pi r^2/2}$

$\cancel{\therefore A = \frac{w}{2}P - \frac{w}{2}x - \frac{\pi w^2}{8}}$

$\cancel{\therefore \frac{dA}{dw} = \frac{1}{2}P - \frac{1}{2}x}$

$\therefore = \cancel{Pr - 2r^2 - \frac{\pi r^2}{2}}$

(ii) $A = rP - rx - \pi r^2/2$

$\therefore A = \frac{w}{2}P - \frac{w}{2}x - \frac{\pi w^2}{8}$

$\therefore 8A = 4wP - 4wx - \pi w^2$

$\therefore \pi w^2 + 4wx - 4wP + 8A = 0$

$\therefore$ ~~$\pi w^2 + 4wx + ($~~

$\pi w^2 + 4w^2 - 4wP + 8A = 0$

$4 \times 16 = 8 \ldots 128$

$\therefore (\pi + 4)w^2 - 4Pw + 8A = 0$

$\therefore w = \frac{4P \pm \sqrt{16P^2 - 4(\pi+4)(8A)}}{2\pi + 8}$

$= \frac{4P \pm \sqrt{16P^2 - 32\pi - 128A}}{2\pi + 8}$

$\frac{dA}{dr} = P - 4r - \frac{2}{2} \times \pi r$ ✓

$= P - 4r - \pi r = 0$

$\therefore P - (4 + \pi) r = 0$ ✓

$\therefore r = P / 4 + \pi$

$\therefore$ $\boxed{\text{Width} = \frac{2P}{4+\pi}}$ ✓

$\frac{d^2A}{dr^2} = -4 - \cancel{2}\pi \cancel{P}$ ✓

$= -4 - \cancel{2}\pi \left(\cancel{P/4+\pi}\right) < 0$ ✓

~~$= -4 - 2P\pi / 4 + \pi$~~

$\therefore$ ~~$d^2A/dr^2 < 0$ as $P > 0$~~

9

(b) Unanswered.

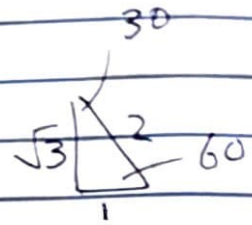
9

11

(a) $2\sin(x+60) = \cos(x+30) \quad 0 \leq x \leq 180$

$\therefore 2\sin(x+60) = \sin(90-x-30)$

$\cancel{(\sin\theta = \cos(90-\theta))}$



$\therefore 2\sin(x+60) = \sin(60-x)$

$2(\sin x\cos 60 + \cos x \sin 60) = \cos x \cos 30 - \sin x \sin 30$

$2\left(\frac{\sin x}{2} + \frac{\sqrt{3}\cos x}{2}\right) = \frac{\sqrt{3}\cos x}{2} - \frac{\sin x}{2}$

$\therefore \sin x + \sqrt{3}\cos x = \frac{1}{2}\sqrt{3}\cos x - \frac{1}{2}\sin x$

$\therefore 2\sin x + 2\sqrt{3}\cos x = \sqrt{3}\cos x - \sin x$

$\therefore 3\sin x + (2\sqrt{3}-\sqrt{3})\cos x = 0$

$\therefore 3\sin x + \sqrt{3}(2-1)\cos x = 0$

$\therefore 3\sin x + \sqrt{3}\cos x = 0$

$\therefore \sqrt{3}\cos x = -3\sin x$

$\therefore \sqrt{3} = -3\tan x$

$\therefore 3\tan x = -\sqrt{3}$

$\therefore \tan x = -\sqrt{3}/3$

$\therefore \tan x = -1/\sqrt{3}$

$\therefore$ acute $x = 30°$

As $\tan x < 0$, $x = 180 - 30$

$\therefore x = 150°$

5

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Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno: ______ - ____

Candidate name: / Nom du candidat: / Nombre del alumno: ______

At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla.

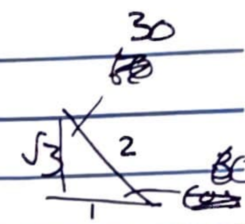
Example Ejemplo 27 [2 7] Example Ejemplo 3 [3]

[1 1]

(b) ~~sin(105) + cos(105)~~ =

~~= sin(105~~

~~= sin(90+1~~



$105° = 60° + 45°$

$\therefore \sin(105°) + \cos(105°) = \sin(60+45) + \cos(60+45)$

$= \sin 60 \cos 45 + \sin 45 \cos 60 + \cos 60 \cos 45 - \sin 60 \sin 45$

$= \cancel{\ldots}$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}} \left(\frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{3}}{2} \right)$

$= 1/\sqrt{2}$

3

04AX01

(c)(i) $z = 1-\cos 2\theta - i\sin 2\theta \qquad z \in \mathbb{C},\ 0 < \theta < \pi$

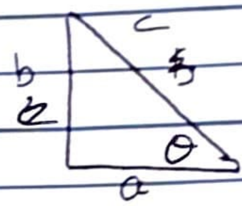
$$\begin{aligned}
|z| &= \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2} \\
&= \sqrt{(1-\cos 2\theta)^2 + (1-\cos^2 2\theta)} \\
&= \sqrt{1-2\cos 2\theta + \cos^2 2\theta + 1 - \cos^2 2\theta} \\
&= \sqrt{2-2\cos 2\theta} \\
&= \sqrt{2}\sqrt{1-\cos 2\theta} \\
&= \sqrt{2}\sqrt{1-2\cos^2\theta+1} \\
&= \sqrt{2}\sqrt{2-2\cos^2\theta} \\
&= 2\sqrt{1-\cancel{\sin 2\theta}\cos^2\theta} \\
&= \cancel{2\cos\theta}\ 2\sin\theta
\end{aligned}$$

$\left\{ \cos 2\theta = 2\cos^2\theta - 1 \right.$

$$\begin{aligned}
\arg(z) &= \tan^{-1}\left(\frac{-\sin 2\theta}{1-\cos 2\theta}\right) \\
&= \tan^{-1}\left(\frac{-\sin 2\theta}{1-2\cos^2\theta+1}\right) \\
&= \tan^{-1}\left(\frac{-2\sin\theta\cos\theta}{2-2\cos^2\theta}\right) \\
&= \tan^{-1}\left(\frac{-\sin\theta\cos\theta}{1-\cos^2\theta}\right) \\
&= \tan^{-1}\left(\frac{-\sin\theta\cos\theta}{\sin^2\theta}\right) \\
&= \tan^{-1}\left(\cancel{-\sin}\ \frac{-\cos\theta}{\sin\theta}\right) \\
&= \tan^{-1}\left(-\cancel{\tan}\ \cot\theta\right) \\
&= \tan^{-1}\left(\frac{-1}{\tan\theta}\right)
\end{aligned}$$

~~(ii)~~

$$= \frac{\cancel{\arcsin(\cos\theta/\sin\theta)}}{\cancel{\arccos(\cos\theta/\sin\theta)}}$$



$$= -\tan^{-1}(-\cot\theta)$$

$$= -\tan^{-1}(-\tan(90 - \theta))$$

$$= -\tan^{-1}(\tan(90-\theta))$$

$$= -(90^\circ - \theta)$$

$$= \theta - 90^\circ$$

$\cot\theta = a/b$

$\sin\theta = b/c$

$\tan\theta = b/a$

$\theta = \tan^{-1} b/a$

(ii) $\cancel{z = 1 - (\cos 2\theta + i\sin 2\theta)}$

$\cancel{= 1 - \text{cis } 2\theta}$

let $z^3 = 2\sin\theta \text{ cis}(\theta - 90^\circ)$

$$\therefore z = \sqrt[3]{2\sin\theta} \text{ cis}\left(\frac{\theta - \frac{\pi}{2} + 2k\pi}{3}\right)$$

$$= \sqrt[3]{2}\sqrt[3]{\sin\theta} \text{ cis}\left(\frac{2\theta - \pi + 4k\pi}{6}\right)$$

$$\therefore z_1 = \sqrt[3]{2\sin\theta} \text{ cis}\left(\frac{2\theta - \pi}{6}\right)$$

$$z_2 = \sqrt[3]{2\sin\theta} \text{ cis}\left(\frac{2\theta + 3\pi}{6}\right)$$

$$z_3 = \sqrt[3]{2\sin\theta} \text{ cis}\left(\frac{2\theta + 7\pi}{6}\right)$$

14

22