Markscheme

Additional Practice

Series and Sequences (Non-Calculator)

ID: 4004

Mathematics: analysis and approaches

Higher level

1. This is an infinite geometric series with first term $-\frac{4}{3}$ and common ratio $-\frac{1}{3}$. A1A1

Use the infintie geometric series formula

$$S_\infty = \frac{-\frac{4}{3}}{1-\left(-\frac{1}{3}\right)} = -1$$ M1A1

2. (a) $\dfrac{1000}{1-0.1}=\dfrac{10000}{9}$ M1A1

(b) $\dfrac{64}{1-(-1/4)}=\dfrac{256}{5}$ M1A1

(c) Does not exist because $r=-1.5$ which is less than -1. A1R1

3. (a)

(i) $\frac{52}{100} = \frac{13}{25}$ M1A1

(ii) $\frac{1}{100}$ A1

(b) Use the infinite geometric series formula

$$x = \frac{\frac{13}{25}}{1 - \frac{1}{100}} = \frac{13}{25} \times \frac{100}{99} = \frac{52}{99}$$ M1A1A1

4. Write $0.\dot{8}\dot{5}$ as an infinite geometric series. M1

$$0.85 + 0.0085 + 0.000085 + \ldots$$

So the first term is $\frac{85}{100}$ and the common ratio is $\frac{1}{100}$. A1A1

Substitute these values into the infinite geometric series formula. M1

$$S_\infty = \frac{\frac{85}{100}}{1 - \frac{1}{100}}$$

Simplify into a proper fraction

$$S_\infty = \frac{85}{99}$$ A1

Therefore it must be rational as it can be written as a fraction. R1

5. (a)

(i) 2 A1

(ii) 60 A1

(b) We have

$$4r^3 + 4r^2 + 4r + 4 = 60$$ M1

So

$$r^3 + r^2 + r - 14 = 0$$ A1

Factorise

$$(r-2)(r^2 + 3r + 7) = 0$$ A1

Take the discriminant of the second factor M1

$$3^2 - 4(1)(7) = -19$$

This is negative so the second factor produces no more solutions. A1

6.

(a) $\dfrac{\sqrt{x^2+x^2}}{2}=\dfrac{x\sqrt{2}}{2}$ M1A1

(b) $\dfrac{\sqrt{\left(\dfrac{x\sqrt{2}}{2}\right)^2+\left(\dfrac{x\sqrt{2}}{2}\right)^2}}{2}=\dfrac{\sqrt{x^2}}{2}=\dfrac{x}{2}$ M1A1

(c) The lengths of the base and height follow a geometric series with $r=\dfrac{\sqrt{2}}{2}$. A1

The area of the largest triangle is $\dfrac{x^2}{2}$. A1

So we have

$$\frac{x^2/2}{1-\left(\sqrt{2}/2\right)^2}=100$$ M1

So $x=10$. A1

7. For $n = 1$ we have

$$\frac{t_1(1 - r^1)}{1 - r} = t_1 \quad \text{M1}$$

So it is true for $n = 1$. A1

Assume it is true for $n = k$. So

$$S_k = \frac{t_1(1 - r^k)}{1 - r} \quad \text{A1}$$

For $n = k + 1$ we have

$$S_{k+1} = t_{k+1} + S_k \quad \text{M1}$$

Using our inductive hypothesis this is equal to

$$t_1 r^k + \frac{t_1(1 - r^k)}{1 - r} \quad \text{A1}$$

Write as one fraction

$$\frac{t_1 r^k(1 - r) + t_1(1 - r^k)}{1 - r} \quad \text{M1}$$

Factorise

$$\frac{t_1(r^k - r^{k+1} + 1 - r^k)}{1 - r} \quad \text{A1}$$

Simplify

$$\frac{t_1(1 - r^{k+1})}{1 - r} \quad \text{A1}$$

So it is true for $n = k + 1$. A1

By the principle of mathematical induction it must be true for all positive integers n. A1

8. (a) This is an infinite geometric series with first term 1 and common ratio $-x$. R1

Its value is therefore $\frac{1}{1+x}$. A1

(b) We have

$$\frac{x+1+x+2}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{1}{x+2}$$ M1A1

(c) The first three terms of $\frac{1}{x+1}$ are

$$1 - x + x^2$$ A1

The first three terms of $\frac{1}{x+2} = (x+2)^{-1}$ are

$$2^{-1}\left(1 - \frac{x}{2} + \frac{(-1)(-2)}{2}\left(\frac{x}{2}\right)^2\right) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$$ M1A1

Adding these together gives

$$\frac{3}{2} - \frac{5x}{4} + \frac{9x^2}{8}$$ A1

(d) The series for $\frac{1}{x+2}$ converges for $\left|\frac{x}{2}\right| < 1$ which gives $|x| < 2$. A1

The series for $\frac{1}{x+1}$ converges for $|x| < 1$.

So the whole series converges for $|x| < 1$. A1

9. (a)

(i) $5+7x+9x^2+11x^3$ A1

(ii) $5x+7x^2+9x^3+11x^4$ A1

(b) We have

$$f(x)-xf(x)=5+2x+2x^2+2x^3+\cdots+2x^{n-1}-(3+2n)x^n \qquad \text{A1}$$

Use the geometric series formula M1

$$f(x)-xf(x)=5-(3+2n)x^n+\frac{2x(1-x^{n-1})}{1-x} \qquad \text{A1}$$

(c) Factorise and rearrange

$$(1-x)f(x)=5-(3+2n)x^n+\frac{2x(1-x^{n-1})}{1-x} \qquad \text{M1}$$

So

$$f(x)=\frac{5-(3+2n)x^n}{1-x}+\frac{2x(1-x^{n-1})}{(1-x)^2} \qquad \text{A1}$$

(d) We have $\lim_{n\to\infty} x^n=0$ so the equation becomes M1

$$f(x)=\frac{5-(3+2n)\times 0}{1-x}+\frac{2x(1-0)}{(1-x)^2}=\frac{5}{1-x}+\frac{2x}{(1-x)^2} \qquad \text{A1A1}$$

(e) Replace x with $1/2$ in the expression from part (d) M1

$$\frac{5}{1-1/2}+\frac{1}{(1-1/2)^2}=10+4=14 \qquad \text{A1A1}$$

10. (a) This is an infinite geometric series with common ratio r^2. R1

Use the infinite geometric series formula to determine the sum. M1

$$S_\infty = \frac{1}{1-\left(-x^2\right)} = \frac{1}{1+x^2}$$ A1

(b) Let $y = \arctan x$. This gives $x = \tan y$. A1

Differentiate

$$\frac{dx}{dy} = \sec^2 y$$ A1

Use the identity $\sin^2 y + \cos^2 y = 1$ to determine the relationship between $\tan y$ and $\sec y$. M1

$$\sin^2 y + \cos^2 y = 1$$

Divide by $\cos^2 y$.

$$\tan^2 y + 1 = \sec^2 y$$ A1

So

$$\frac{dx}{dy} = \tan^2 y + 1$$ A1

Replace $\tan y$ with x and rearrange. M1

$$\frac{dx}{dy} = x^2 + 1$$

So

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$ A1

(c) Replace $\frac{1}{1+x^2}$ with its infinite geometric series and integrate. M1

$$\frac{d}{dx}(\arctan x) = 1 - x^2 + x^4 - x^6 + \ldots$$

So

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$ A1

(d) Since $\arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ replace x with $\frac{\sqrt{3}}{3}$. M1

This gives

$$\frac{\pi}{6} = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3^2 \times 3} + \frac{\sqrt{3}}{3^3 \times 5} - \frac{\sqrt{3}}{3^4 \times 7} + \ldots = \frac{\sqrt{3}}{3} \times \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k \times (2k+1)}$$ A1A1

So

$$\pi = 2\sqrt{3} \times \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k \times (2k+1)}$$ A1

11. (a) Maria's first guess is 4 leaving three numbers from which to guess. Three is odd. A1

Her next guess will leave one number. One is odd. A1

So after each guess there is always an odd number of values remaining.

(b) The next value is

$$2(2(2 \cdot 1 + 1) + 1) + 1 = 15$$ M1A1

The next value is

$$2 \cdot 15 + 1 = 31$$ M1A1

The next value is

$$2 \cdot 31 + 1 = 63$$ M1A1

(c) The value of x must take the form $2^n - 1$ where $n \in \mathbb{Z}^+$. A1A1

(d) When $n = 1$ there are $2^1 - 1 = 1$ values from which to choose. This is odd. A1

So it is true for $n = 1$. A1

Assume it is true for $n = k$. So values of x of the form $2^k - 1$ will always leave an odd number of values left from which to choose. A1

If we begin with $2^{k+1} - 1$ values from which to choose then the first guess will result in the following number of values remaining from which to guess

$$\frac{2^{k+1} - 1 + 1}{2} - 1 = 2^k - 1$$ M1A1

So by our inductive hypothesis it must also be true for $n = k + 1$. A1

By the principle of mathematical induction it must be true for all positive integers n. R1

12. (a)

(i) $1 + x + x^2 + x^3$ A1

(ii) $1 + 2x + 3x^2 + 4x^3$ A1A1

(b) The function $f(x)$ is an infinite geometric series with first term 1 and common difference x. R1

Its value is therefore $\frac{1}{1-x}$. A1

(c) Use the chain rule. M1

$$f'(x) = (-1)\left(-\frac{1}{(1-x)^2}\right) = \frac{1}{(1-x)^2}$$ A1A1

(d) We have

$$a = \frac{1}{6}$$ A1

$$b = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$ M1A1

(e) This is an infinite geometric series with first term 1/6 and common ratio 5/6. R1

Its value is therefore

$$\frac{1/6}{1 - 5/6} = \frac{1/6}{1/6} = 1$$ M1A1

(f) We have

$$E(X) = \frac{1}{6}\left(1 + 2 \times \frac{5}{6} + 3 \times \left(\frac{5}{6}\right)^2 + 4 \times \left(\frac{5}{6}\right)^3 + \cdots\right)$$ A1

Use the formula from part (c) to evaluate. M1

$$E(X) = \frac{1}{6} \times \frac{1}{(1 - 5/6)^2} = 6$$ A1A1

13. (a) Use the infinite geometric series formula M1

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$$ A1

(b)

(i) Use the chain rule M1

$$f'(x) = \frac{1}{(1-x)^2}$$ A1

(ii) Use the chain rule M1

$$f''(x) = \frac{2}{(1-x)^3}$$ A1

(c) We have

$$f'(x) = \sum_{r=0}^{\infty} r x^{r-1} = \sum_{r=1}^{\infty} r x^{r-1} = \frac{1}{(1-x)^2}$$ A1

And

$$f''(x) = \sum_{r=1}^{\infty} r(r-1) x^{r-2} = \sum_{r=2}^{\infty} r(r-1) x^{r-2} = \frac{2}{(1-x)^3}$$ A1

We therefore have

$$\sum_{r=2}^{\infty} r(r-1) x^{r-1} = \sum_{r=2}^{\infty} r^2 x^{r-1} - \sum_{r=2}^{\infty} r x^{r-1} = \frac{2x}{(1-x)^3}$$ M1A1

So

$$\sum_{r=2}^{\infty} r^2 x^{r-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} - 1$$ M1

Giving

$$\sum_{r=1}^{\infty} r^2 x^{r-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} - 1 + 1 = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$ A1

(d) $\frac{1}{6} \times \left(\frac{5}{6}\right)^{n-1}$ A1

(e)

(i) $E(X) = \frac{1}{6}\sum_{r=1}^{\infty} r \times \left(\frac{5}{6}\right)^{r-1} = \frac{1}{6} \times \frac{1}{(1-5/6)^2} = 6$ M1A1A1

(ii) $\operatorname{Var}(X) = \frac{1}{6}\sum_{r=1} r^2\left(\frac{5}{6}\right)^{r-1} = \frac{1}{6} \times \left(\frac{5/3}{(1-5/6)^3} + \frac{1}{(1-5/6)^2}\right) - 36$ M1A1

This is equal to 30. A1