Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3015

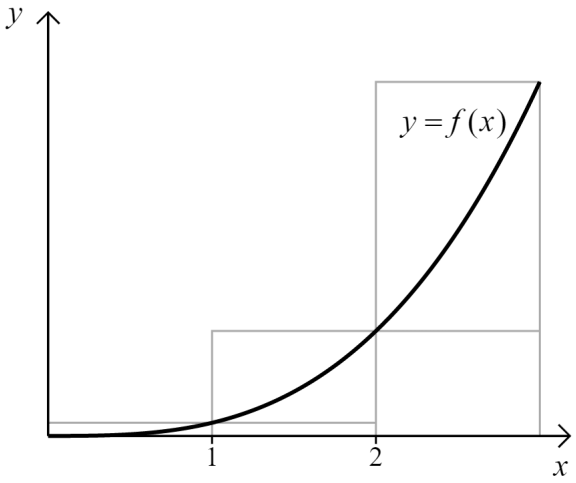
Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

1. [Maximum points: 25]

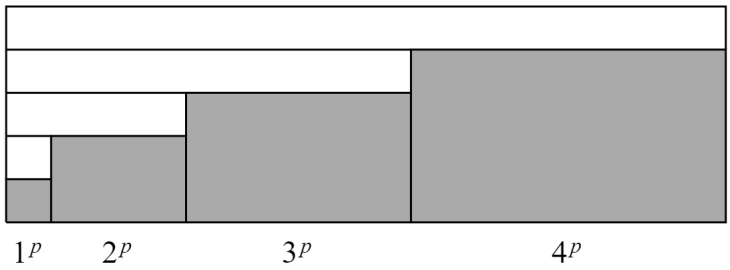
In this problem you will develop formulae for $\sum_{r=1}^{n} r^p$ *where* $p \in \mathbb{Z}^+$.

Let $f(x) = x^3$. The diagram below shows the graph of $y = f(x)$.



(a) Use the diagram to show that $\frac{n^4}{4} < \sum_{r=1}^{n} r^3 < \frac{n^4 + 4n^3}{4}$. [4]

The diagram below shows shaded rectangles of lengths 1^p, 2^p, 3^p and 4^p, where $p \in \mathbb{Z}^+$, and heights 1, 2, 3 and 4.



The pattern is continued until there are n shaded rectangles.

(b) In terms of n find an expression for [4]

(i) the area of the largest shaded rectangle

(ii) the height of the overall diagram (for example when $n = 3$ the height is 4)

(c) Explain why [4]

$$\sum_{r=1}^{p} r^{p+1} = (n+1)\sum_{r=1}^{n} r^p - \sum_{r=1}^{n}\left(\sum_{m=1}^{r} m^p\right)$$

(d) Hence show that [6]

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(e) Hence find an expression for $\sum_{r=1}^{n} r^3$. [7]

2. [Maximum points: 30]

In this problem you will investigate the expected value and variance of a discrete random variable with a geometric probability distribution.

Let $f(x) = \sum_{k=1}^{\infty} x^{k-1}$ where $|x| < 1$.

(a) Write down the first four non-zero terms of [3]

(i) $f(x)$

(ii) $f'(x)$

(b) Explain why $f(x) = \frac{1}{1-x}$. [2]

(c) Hence find an expression for $f'(x)$ writing your answer as a single fraction. [3]

A regular six-sided die is repeatedly rolled. Let the random variable X represent the number of rolls needed until a 6 appears.

(d) The table below shows the values of $P(X = x)$ for the first four values of x. [3]

x	1	2	3	4
$P(X = x)$	a	b	$\frac{25}{216}$	$\frac{125}{1296}$

Find the values of a and b.

(e) Verify that $\sum_{x=1}^{\infty} P(X = x) = 1$. [3]

(f) Find the exact value of $E(X)$. [4]

(g) Write down the first four non-zero terms of $f''(x)$. [2]

(h) Show that [3]

$$xf''(x) + f'(x) = \sum_{k=1}^{\infty} k^2 x^{k-1}$$

(i) Use your answer to part (c) to find an expression for $f''(x)$ writing your answer as a single fraction. [3]

(j) Hence find the exact value of $\text{Var}(X)$. [4]