Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3009

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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1. [Maximum points: 24]

In this question you will investigate the relationship between the definite integral and the area between a curve and the x-axis.

Let $g(x) = 2x - x^2$.

(a) Starting from $g(0) = 0$ use Euler's method to **approximate** the value of $g(4)$ when the step length is equal to [11]

(i) 2

(ii) 1

(iii) 0.5

(b) Calculate the actual value of $g(4)$. [2]

(c) Explain what happens to your estimations in part (a) as the step length decreases. [2]

Based on parts (a) to (c) we can deduce that

$$g(b) = g(a) + \lim_{n \to \infty} \sum_{i=0}^{n} g'(x_i) \cdot \Delta x$$

where n represents the number of steps, Δx represents the step length and $x_i = a + i \cdot \Delta x$.

(d) Let $f(x) = g'(x)$ and $F(x)$ represent the anti-derivative of $f(x)$. [4]

(i) Rewrite this limit equation in terms of f and F.

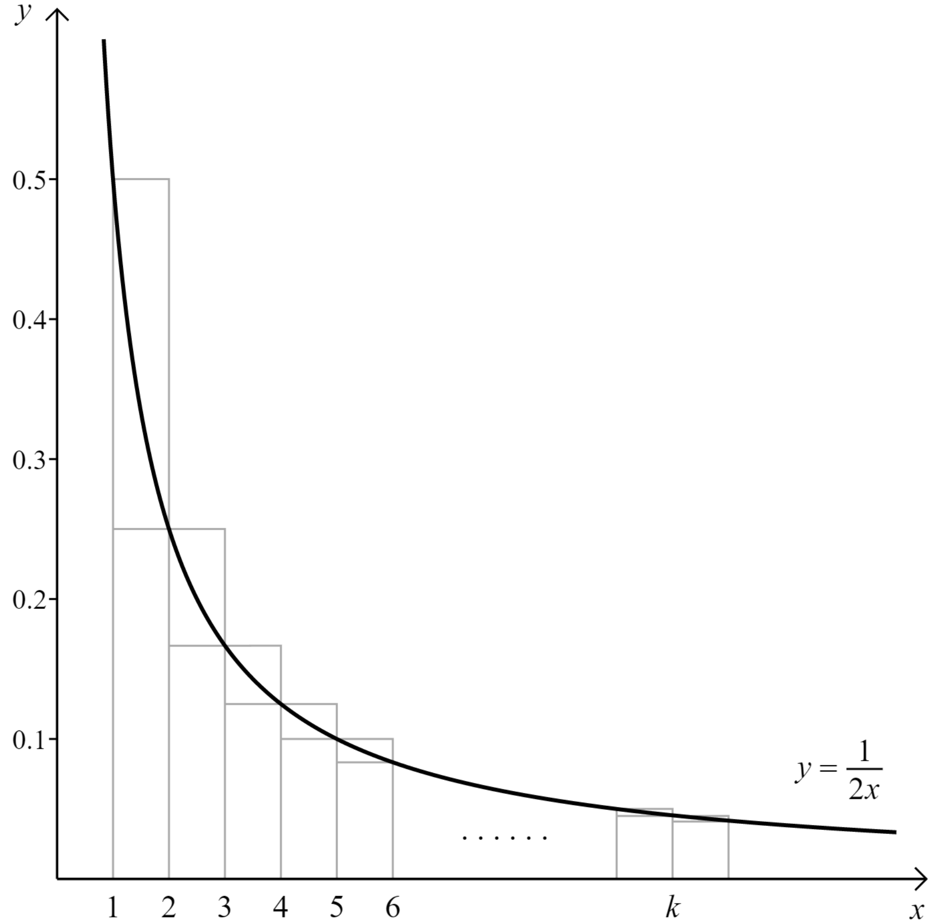
(ii) If $\int_a^b f(x)dx = F(b) - F(a)$ show that $\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \cdot \Delta x$.

(e) If f is non-negative on the interval $[a,b]$ explain, with the help of a diagram, why the area A bound by the function f, the x-axis and the lines $x = a$ and $x = b$ is equal to [5]

$$A = \int_a^b f(x)dx$$

2. [Maximum points: 31]

The diagram below shows the graph of $y = \dfrac{1}{2x}$.



Rectangles of width 1 are added to the graph to estimate the area between the graph and the x-axis.

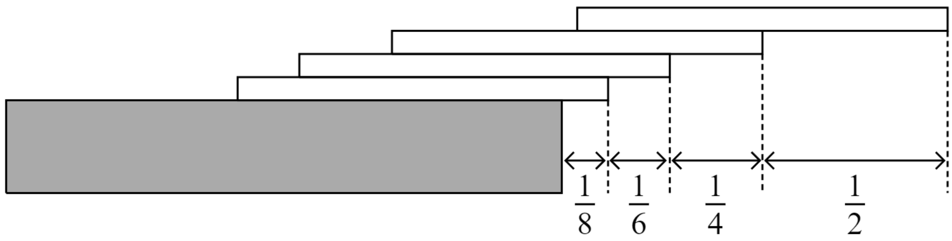
(a) Determine $\int_1^{k+1} \dfrac{1}{2x}\, dx$ where k is a positive integer greater than 1. [3]

(b) Show that [4]

$$\frac{\ln(k+1)}{2} < \sum_{n=1}^{k} \frac{1}{2n} < \frac{(k+1)\big(1+\ln(k+1)\big)-1}{2(k+1)}$$

(c) Explain why $\lim_{k \to \infty} \sum_{n=1}^{k} \dfrac{1}{2n}$ diverges. [2]

Four identical cards of length 1 are placed on top of each other on the edge of a desk with each card offset from the card (or table) below as shown in the following diagram.



The x-coordinate of the centre of mass of the four cards is defined as the mean of the x-coordinates of the midpoints of all four cards.

(d) Show that the x-coordinate of the center of mass of the four cards is equal to the x-coordinate of the right-edge of the table. [6]

More cards are placed on top of each other. The offset of each card, starting with the top card, from the next card (or table) below follows the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \ldots$$

The cards will only fall if the x-coordinate of the centre of mass of the top n cards is to the right of the x-coordinate of the right-edge of the next card (or table) below.

(e) Prove by induction that the x-coordinate of the centre of mass of the top n cards is equal to the x-coordinate of the right-edge of the next card below. [9]

(f) Explain why it is possible for the cards to span any horizontal distance as long as we use enough cards. [2]

(g) Determine an upper limit for the number of cards needed to span a horizontal distance of 5 from the right-edge of the table. [3]

(h) For the number of cards you found in part (g) calculate an upper limit for the horizontal distance they span. [2]

1. (a)

(i) Differentiate $g(x)$ and use it to complete the table below. M1

$$g'(x) = 2 - 2x$$

so

x_i	y_i	$g'(x_i)$	
0	0	2	A1
2	4	-2	
4	0		A1

(ii)

x_i	y_i	$g'(x_i)$	
0	0	2	
1	2	0	A1
2	2	-2	
3	0	-4	A1
4	-4		A1

(iii)

x_i	y_i	$g'(x_i)$	
0	0	2	
0.5	1	1	A1
1	0.5	0	
1.5	0.5	-1	A1
2	0	-2	
2.5	-1	-3	A1
3	-2.5	-4	
3.5	-4.5	-5	A1
4	-7		A1

(b) $g(4) = 8 - 16 = -8$ M1A1

(c) As the step length gets smaller that estimation of $g(4)$ gets more accurate. A1A1

(d)

(i) Replace $g'(x)$ with $f(x)$ and $g(x)$ with $F(x)$. M1

So we have

$$F(b) = F(a) + \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \cdot \Delta x$$ A1

(ii) Rearrange and then replace $F(b) - F(a)$ with $\int_a^b f(x)dx$. M1

$$F(b) - F(a) = \lim_{n \to \infty} \sum_{i-0}^{n} f(x_i) \cdot \Delta x$$

So

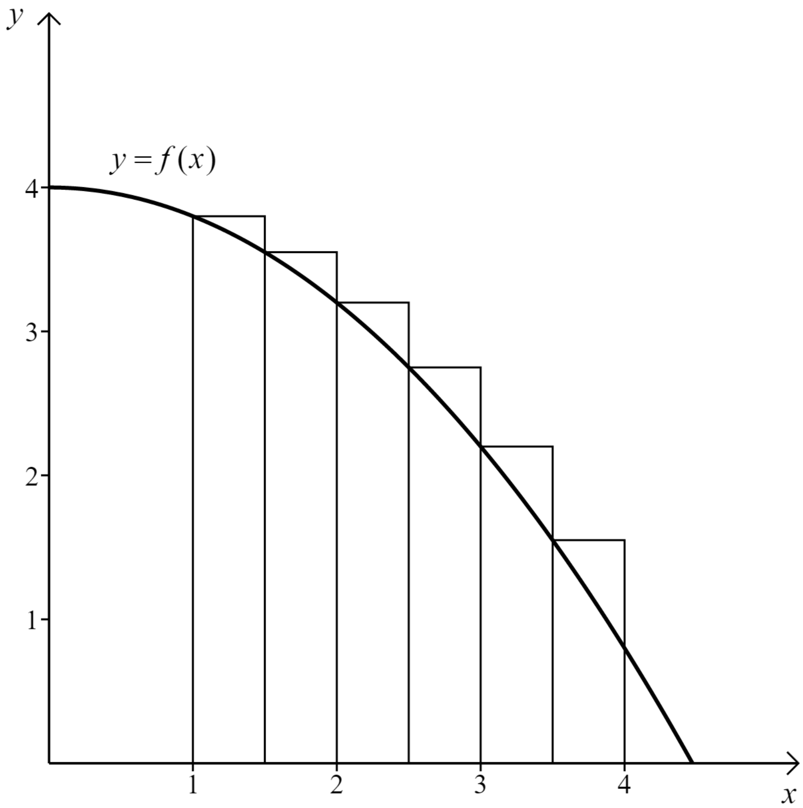
$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \cdot \Delta x$$ A1

(e) We can estimate the area bound by the function, the x-axis, and the lines $x = a$ and $x = b$ by dividing the regions into rectangles of equal width. R1

This is demonstrated in the diagram below showing the estimation of the area between the function, the x-axis and the lines $x = 1$ and $x = 4$.

(Draw any non-negative function) A1

(Divide a region into rectangles of equal width) A1



The total area of the rectangles is

$$A = \sum_{i=0}^{6} f(x_i) \cdot \Delta x$$ A1

This will get more accurate as the number of rectangles increases. R1

2. (a) We have

$$\int_1^{k+1} \frac{1}{2x}\,dx = \frac{1}{2}[\ln x]_1^{k+1} = \frac{1}{2}(\ln(k+1) - \ln 1) = \frac{1}{2}\ln(k+1) \qquad \text{M1A1A1}$$

(b) Considering the upper rectangles we have

$$\frac{1}{2}\ln(k+1) < \sum_{n=1}^{k} \frac{1}{2n} \qquad \text{A1}$$

Considering the lower rectangles we have

$$\sum_{n=2}^{k+1} \frac{1}{2n} < \frac{1}{2}\ln(k+1) \qquad \text{M1}$$

So

$$\sum_{n=1}^{k} \frac{1}{2n} < \frac{1}{2}\ln(k+1) + \frac{1}{2\cdot 1} - \frac{1}{2(k+1)} \qquad \text{A1}$$

Rewrite

$$\sum_{n=1}^{k} \frac{1}{2n} < \frac{(k+1)(1+\ln(k+1)) - 1}{2(k+1)} \qquad \text{A1}$$

(c) Since $\lim_{k\to\infty} \frac{1}{2}\ln(k+1)$ diverges and $\sum_{n=1}^{k} \frac{1}{2n} > \frac{1}{2}\ln(k+1)$ then $\lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2n}$ M1

must also diverge. A1

(d) The x-coordinate of the centre of mass of the 1st card is

$$-\frac{1}{2}+\frac{1}{8}$$ A1

The x-coordinate of the centre of mass of the 2nd card is

$$-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}$$ A1

The x-coordinate of the centre of mass of the 3rd card is

$$-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}+\frac{1}{4}$$ A1

The x-coordinate of the centre of mass of the 4th card is

$$-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}+\frac{1}{4}+\frac{1}{2}$$ A1

The mean of these x-coordinates is

$$\frac{-\frac{1}{2}+\frac{1}{8}-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}+\frac{1}{4}-\frac{1}{2}+\frac{1}{8}+\frac{1}{6}+\frac{1}{4}+\frac{1}{2}}{4}=0$$ M1A1

(e) When $n=1$ the x-coordinate of the centre of mass of the top card is in the middle of the card, which is equal to the x-coordinate of the right-edge of the card below. R1 A1

So it is true for $n=1$. A1

Assume it is true for $n=k$. So the x-coordinate of the centre of mass of the top k cards is equal to the x-coordinate of the right-edge of the card below. A1

For $n=k+1$ cards the x-coordinate of the centre of mass, compared to the x-coordinate of the right-edge of the bottom of the $k+1$ cards, is

$$\frac{k\times 0-\frac{1}{2}}{k+1}=-\frac{1}{2(k+1)}$$ M1A1

This is equal to the x-coordinate of the right-edge of the $(k+1)th$ card. R1

So it is true for $n=k+1$. A1

By the principle of mathematical induction it is true for all positive integers n. R1

(f) The horizontal distance the cards span is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$$ A1

By part (c) as the number of cards increases then this value diverges. So it is possible to span any horizontal distance. R1

(g) We need

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2k}=\sum_{n=1}^{k}\frac{1}{2n}>5$$ A1

For the upper limit we have $$\frac{\ln(k+1)}{2}=5$$ M1

The solution to this is $k=\mathrm{e}^{10}-1\approx 22025$. A1

(h) $$\frac{(\mathrm{e}^{10}-1+1)\big(1+\ln(\mathrm{e}^{10}-1)\big)-1}{2(\mathrm{e}^{10}-1+1)}=5.50$$ M1A1