Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Tuesday 10 May 2016 (afternoon)

2 hours

Candidate session number

16	M	TZ	1	P	1	M	A	H	L

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

$\frac{86}{120} = 71.7\,\%$

14 pages

2216–7203

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The fifth term of an arithmetic sequence is equal to 6 and the sum of the first 12 terms is 45. Find the first term and the common difference.

$u_5 = 6 = u_1 + 4d$ ✓ ... (1)

$S_{12} = 45 = \frac{12}{2}(2u_1 + 11d)$
$= 6(2u_1 + 11d)$ ✓
$= 12u_1 + 66d$
$\therefore 45 = 12u_1 + 66d$ ... (2)

$\therefore 72 = 12u_1 + 48d$ ✓
$(45 = 12u_1 + 66d) -$
$27 = -18d$

$\therefore 3 = -2d$
$\therefore \boxed{d = -3/2}$ ✓

using (1) $\rightarrow$ $u_1 = 6 - 4(-3/2)$ ✓
$= 6 + 6$
$\boxed{\therefore u_1 = 12}$ ✓

6

2. [Maximum mark: 4]

At a skiing competition the mean time of the first three skiers is 34.1 seconds. The time for the fourth skier is then recorded and the mean time of the first four skiers is 35.0 seconds. Find the time achieved by the fourth skier.

$\mu_1 = 34.1 = \frac{t_1+t_2+t_3}{3}$

$\therefore t_1+t_2+t_3 = 102.3$

$34 \times 3 = 102.3$

$\mu_2 = \frac{t_1+t_2+t_3+t_4}{4} = 35.0$

$\therefore (t_1+t_2+t_3)+t_4 = 140$

$35 \times 4 = 140$

$\therefore 102.3 + t_4 = 140$

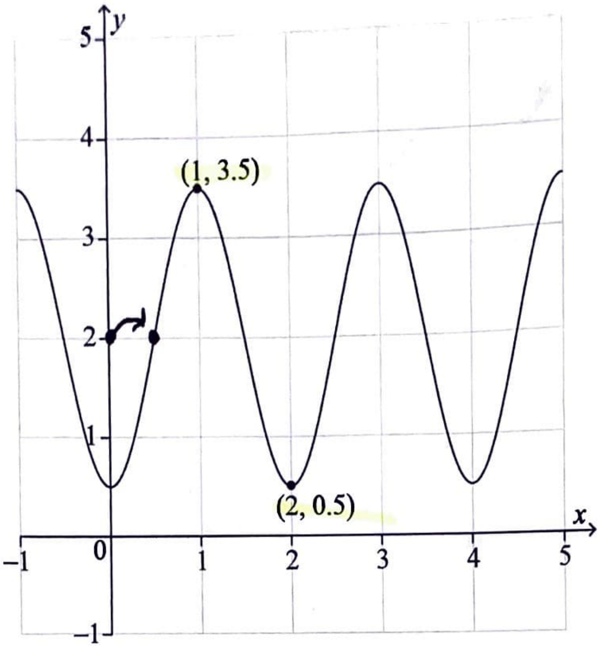
$\therefore t_4 = 140-102.3$

$= \cancel{13}\ 37.7$ s

4

3. [Maximum mark: 6]

The following diagram shows the curve $y = a\sin(b(x+c)) + d$, where a, b, c and d are all positive constants. The curve has a maximum point at $(1, 3.5)$ and a minimum point at $(2, 0.5)$.



(a) Write down the value of a and the value of d. **[2]**

(b) Find the value of b. **[2]**

(c) Find the smallest possible value of c, given $c > 0$. **[2]**

Right shift

(a) a = amplitude $= (\text{MAX} - \text{MIN})/2$ $\boxed{= 3/2}$ ✓
$d = (\text{MAX} + \text{MIN})/2 = 4/2$ $\boxed{= 2}$ ✓ 2

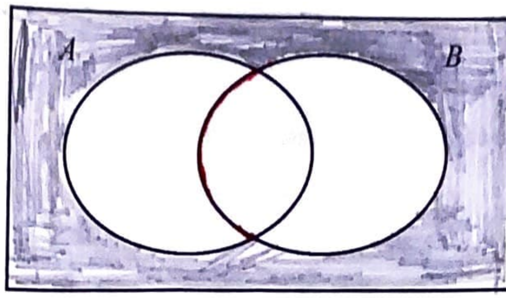
(b) $y = \frac{3}{2}\sin(b(x+c)) + 2 \rightarrow b = 2\pi/T$ ✓
$= 2\pi/2$ ✓
$= \pi$ 2

(c) ~~First max occurs @ $x = 0 + \frac{1}{2}T = 1$~~
~~∴ when $y = 2$, $x = 0 + \frac{1}{4}T = \frac{1}{2}$~~
~~∴ $c = \frac{1}{2}$~~
$c = 3\pi/2$ ✗

4

4. [Maximum mark: 5]

(a) On the Venn diagram shade the region $A' \cap B'$. [1]



Two events A and B are such that $P(A \cap B') = 0.2$ and $P(A \cup B) = 0.9$.

(b) Find $P(A'|B')$. [4]

(b) $P(A'|B') = P(A' \cap B') / P(B')$

$\Rightarrow P(A' \cap B') = P(A') + P(B') - P(A' \cup B')$

$= P(A') + P(B') - (1 - 0.9)$

$\therefore P(A' \cap B') = P(A') + P(B') - 0.1$

2

$\cancel{P(B') = 1 - P(B)}$

$= 1 -$

$P(A'|B') = \dfrac{P(A' \cap B')}{P(B')}$

$P(B') = 1 - P(B) = 1 - (0.9 - 0.2) = 0.3$

$P(A' \cap B') = 1 - 0.9 = 0.1$

$\therefore P(A'|B') = \dfrac{0.1}{0.3} = 1/3$

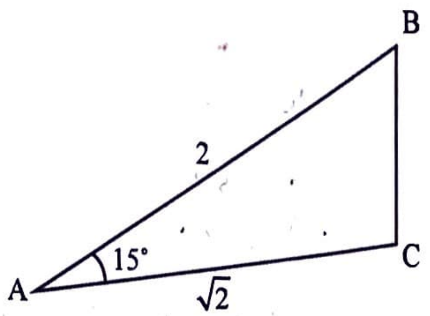
3

5. [Maximum mark: 8]

(a) Expand and simplify $(1-\sqrt{3})^2$. [1]

(b) By writing $15°$ as $60° - 45°$ find the value of $\cos(15°)$. [3]

The following diagram shows the triangle ABC where $AB = 2$, $AC = \sqrt{2}$ and $B\hat{A}C = 15°$.



(c) Find BC in the form $a + \sqrt{b}$ where $a, b \in \mathbb{Z}$. [4]

(a) $(1-\sqrt{3})^2 = 1 - 2\sqrt{3} + 3$
$= 4 - 2\sqrt{3}$ ✓ 1

(b) $\cos(15°) = \cos(60° - 45°)$
$= \cos 60° \cos 45° + \sin 60° \sin 45°$
$= (\frac{1}{2})(\frac{1}{\sqrt{2}}) + (\frac{\sqrt{3}}{2})(\frac{1}{\sqrt{2}})$
$= 1/2\sqrt{2} + \sqrt{3}/2\sqrt{2}$
$= \sqrt{2}/4 + \sqrt{3}\sqrt{2}/4$
$= (\sqrt{2} + \sqrt{6})/4$ ✓ 3

$\sqrt{3}$, 2, 1, 30°, 60°

(c) $[BC]^2 = 2^2 + (\sqrt{2})^2 - 2(2)(\sqrt{2}) \cos 15$ ~~cos 15~~ ~~BC~~
$= 4 + 2 - 4\sqrt{2}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$
$= 6 - \sqrt{2}(\sqrt{2}+\sqrt{6})$ ✓
$= 6 - 2 - \sqrt{12}$
$= \cancel{4 - \sqrt{12}}\ 4 - 2\sqrt{3}$ 2

$\therefore$ ~~$[BC] = \sqrt{4 - 2\sqrt{3}}$~~ $= \pm(1 - \sqrt{3})$
~~$= 2\sqrt{1-\sqrt{6}}$~~ $= -1 + \sqrt{3}$

6

6. [Maximum mark: 4]

Find integer values of m and n for which

$$m - n\log_3 2 = 10\log_9 6$$

$\log_3 3^m - \log_3 2^n = \dfrac{\log_3 6^{10}}{\log_3 9^{10}}$ ✗

$= \dfrac{\log_3 6^{10}}{\log_3 3^{20}}$

$= \dfrac{1}{20}\log_3 6^{10}$

$\therefore \log_3\left(\dfrac{3^m}{2^n}\right) = \log_3 6^{1/2}$

$\Rightarrow \dfrac{3^m}{2^m} = \sqrt{6}$

$\therefore 3^m = 2^m 6^{1/2}$

$\therefore$

$\log_3 3^m - \log_3 2^n = \dfrac{\log_3 6^{10}}{\log_3 3^{20}}$

$\therefore \log_3 3^m - \log_3 2^n = \log_3 6^{1/2}$

$\therefore \dfrac{3^m}{2^n} = \sqrt{6}$

$m - n\log_3 2 = \dfrac{\log_3 6^{10}}{\log_3 3^2}$

$\therefore m - n\log_3 2 = 5\log_3 6$

$\therefore m - n\log_3 2 = 5\log_3 3 + 5\log_3 2$

$\therefore m = 5$ $\cancel{n = 2}$

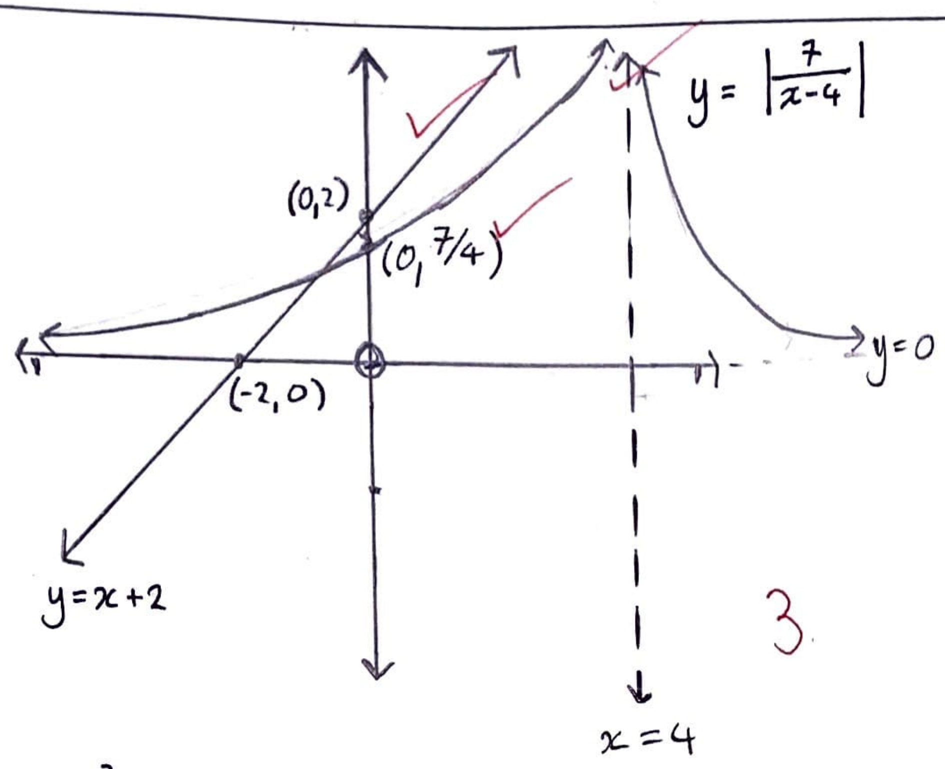
$n = -5$

0

7. [Maximum mark: 8]

(a) Sketch on the same axes the curve $y = \left|\dfrac{7}{x-4}\right|$ and the line $y = x + 2$, clearly indicating any axes intercepts and any asymptotes. [3]

(b) Find the exact solutions to the equation $x + 2 = \left|\dfrac{7}{x-4}\right|$. [5]



(a) $y = \left|\frac{7}{x-4}\right| \rightarrow y = \frac{7}{x-4} \rightarrow$ vert asym : $x = 4$

horiz asym : $y = 0$

y-int : $y = -7/4$

x-int : DNE

(b) 3 intercepts :

$x < 4$: $-x - 2 = \frac{7}{x-4}$

$\therefore (x+2)(x-4) = -7$

$\therefore x^2 - 2x - 8 = -7$

$\therefore x^2 - 2x - 1 = 0$

$\therefore x = \frac{2 \pm \sqrt{4+4}}{2}$

$= 1 \pm \sqrt{2}$

$\boxed{\therefore x = 1 + \sqrt{2},\ x = 1 - \sqrt{2}}$

$x > 4$: $x + 2 = \frac{7}{x-4}$

~~$x^2 - 8x - 8 = 7$~~

$\therefore$ ~~$x^2 - 6x - 15 = 0$~~

$\therefore x^2 - 2x - 15 = 0$ 5

$\therefore (x-5)(x+3) = 0$

$\therefore x = 5, -3$

$\therefore \boxed{x = 5}$ $\{x > 4\}$ 8

8. [Maximum mark: 5]

O, A, B and C are distinct points such that $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

It is given that $\boldsymbol{c}$ is perpendicular to $\overrightarrow{AB}$ and $\boldsymbol{b}$ is perpendicular to $\overrightarrow{AC}$.

Prove that $\boldsymbol{a}$ is perpendicular to $\overrightarrow{BC}$.

$\rightarrow\ c = \overrightarrow{OC} \perp \overrightarrow{AB} \rightarrow \overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

$\rightarrow\ b = \overrightarrow{OB} \perp \overrightarrow{AC} \rightarrow \cancel{OB}\ \overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= \cancel{\ }$

$a = \overrightarrow{OA} \quad b = \overrightarrow{OB} \quad c = \overrightarrow{OC}$

$c \cdot \overrightarrow{AB} = 0$

$\therefore c \cdot (b - a) = 0$

$\therefore c \cdot b = a \cdot c$

$b \cdot \overrightarrow{AC} = 0$

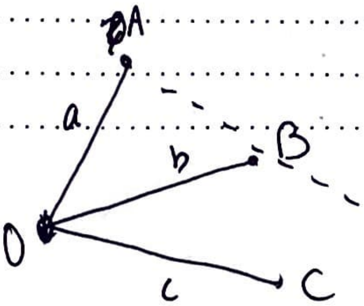
$\therefore b \cdot (c - a) = 0$

$\therefore b \cdot c = a \cdot b$

$\underline{\overrightarrow{AB} = c - b}$

$\therefore a \cdot b = a \cdot c$

$\therefore 0 = a \cdot (c - b)$



$c \cdot \overrightarrow{AB} = 0$ ✓

$b \cdot \overrightarrow{AC} = 0$

$\therefore \cancel{c \cdot AB = b \cdot \overrightarrow{AC}}$

$\therefore$

$c \cdot (AO + OB) = 0$

$\therefore c \cdot (-a) \cdot \overrightarrow{OB} = 0$

$\therefore \overrightarrow{OC} \cdot (OB - a) = 0$

$\therefore \overrightarrow{CO}(a - \overrightarrow{OB}) = 0$

$\therefore \overrightarrow{CB} \cdot a = 0$

$\therefore a \perp \overrightarrow{CO}$.

1

9. [Maximum mark: 7]

A curve is given by the equation $y = \sin(\pi \cos x)$.

Find the coordinates of all the points on the curve for which $\frac{dy}{dx} = 0$, $0 \le x \le \pi$.

$y = \sin(\pi \cos x)$

$\frac{dy}{dx} = \cos(\pi \cos x) \times -\pi \sin x$

$= -\pi \sin x \cos(\pi \cos x) = 0$

$\therefore \sin x \cos(\pi \cos x) = 0$

when $x \neq 0, \pi$, $\quad \cos(\pi \cos x) = 0$

$\therefore \pi \cos x = \pi/2$

$\therefore \cos x = 1/2$

$\therefore \boxed{x = \pi/3}$

and $\boxed{x = 2\pi/3}$

when $x \neq \pi/2$, $\quad \sin x = 0$

$\therefore \boxed{x = 0, \pi/2}$

Hence, $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, 2\pi/2$.

$\therefore$ coordinates: $x = 0$, $y = \sin \pi = 0$ $\quad (0, 0)$

$x = \pi/3$, $y = \sin(\pi \cos \pi/3)$

$= \sin(\pi/2)$

$= 1$ $\quad \rightarrow \left(\frac{\pi}{3}, 1\right)$

$x = \pi/2$, $\sin(\pi \cos \pi/2)$

$= \sin \pi$

$= 0$ $\quad \rightarrow \left(\frac{\pi}{2}, 0\right)$

$x = 3\pi/2$, $\sin(\pi \cos 3\pi/2)$

$= \sin(-\pi/2)$

$= -1$ $\quad \rightarrow \left(\frac{3\pi}{2}, -1\right)$

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10. [Maximum mark: 7]

Find the x-coordinates of all the points on the curve $y = 2x^4 + 6x^3 + \frac{7}{2}x^2 - 5x + \frac{3}{2}$ at which the tangent to the curve is parallel to the tangent at $(-1, 6)$.

$y = 2x^4 + 6x^3 + \frac{7}{2}x^2 - 5x + \frac{3}{2}$

$\therefore \frac{dy}{dx} = 8x^3 + 18x^2 + 7x - 5$

At $x = -1$, $y = 6$,

$dy/dx = 8(-1) + 18 - 7 - 5$
$= -8 - 7 - 5 + 18$
$= 10 - 7 - 5$
$= 3 - 5$
$= -2$

~~Hence, $2x^4 + 6x^3 + \frac{7}{2}x^2 - 5x + \frac{3}{2} = -2$~~

$\therefore 8x^3 + 18x^2 + 7x - 5 = -2$

$\therefore 8x^3 + 18x^2 + 7x - 3 = 0$

$\therefore (x+1)(ax^2 + bx + c) = 0$

$\therefore ax^3 + bx^2 + cx + ax^2 + bx + c = 0$

$\therefore ax^3 + (b + a)x^2 + (b + c)x + c = 0$

$a = 8$
$b = b + 8 = 18 \rightarrow b = 10$
$c = -3$

Hence, $8x^2 + 10x - 3 = 0$

$\therefore x = \frac{-10 \pm \sqrt{100 + 96}}{16}$
$= \frac{-10 \pm 2\sqrt{49}}{16}$
$= \frac{-10 \pm 14}{16}$

$x = \frac{4}{16}, \frac{-24}{16}$

$\boxed{\therefore x = \frac{1}{4}, -\frac{3}{2}}$

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Two planes have equations

$$\Pi_1 : 4x + y + z = 8 \text{ and } \Pi_2 : 4x + 3y - z = 0$$

(a) Find the cosine of the angle between the two planes in the form $\sqrt{\frac{p}{q}}$ where $p, q \in \mathbb{Z}$. [4]

$\sqrt{\frac{9}{13}}$

Let L be the line of intersection of the two planes.

(b) (i) Show that L has direction $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

(ii) Show that the point $\text{A}(1, 0, 4)$ lies on both planes.

(iii) Write down a vector equation of L. [6]

B is the point on Π_1 with coordinates $(a, b, 1)$.

(c) Given the vector $\overrightarrow{\text{AB}}$ is perpendicular to L find the value of a and the value of b. [5]

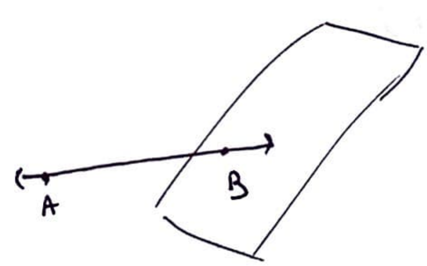
(d) Show that $\text{AB} = 3\sqrt{2}$. [1]

The point P lies on L and $\text{A}\hat{\text{B}}\text{P} = 45°$.

(e) Find the coordinates of the two possible positions of P. [5]

$4 + 3 + 1 = 8$

$B = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = -1 + 6 + 2$



$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

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12. [Maximum mark: 21]

(a) Use de Moivre's theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3$. [2]

(b) Use mathematical induction to prove that

$$(\cos\theta - \mathrm{i}\sin\theta)^n = \cos n\theta - \mathrm{i}\sin n\theta \text{ for } n \in \mathbb{Z}^+.$$ [6]

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(c) Find an expression in terms of θ for $(z)^n + (z^*)^n$, $n \in \mathbb{Z}^+$ where z^* is the complex conjugate of z. [2]

(d) (i) Show that $zz^* = 1$.

(ii) Write down the binomial expansion of $(z + z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [5]

(e) Hence solve $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \le \theta < \pi$. [6]

Do **not** write solutions on this page.

13. [Maximum mark: 18]

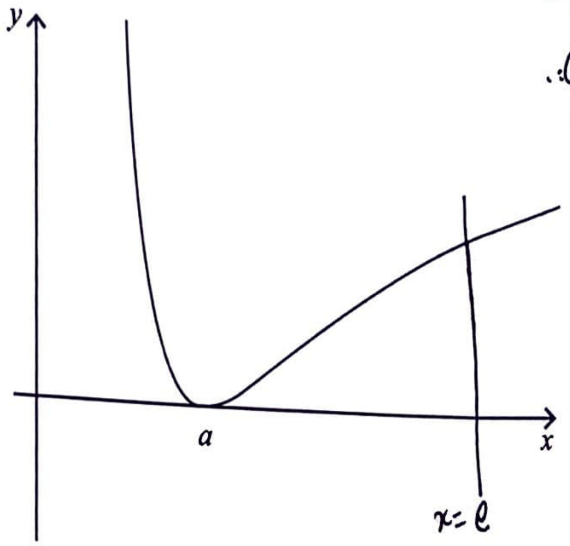
The following diagram shows the graph of $y = \frac{(\ln x)^2}{x}, x > 0$.

$\frac{(\ln x)^2}{x} = 0$

$\therefore (\ln x)^2 = 0$

$\therefore \ln x = 0$

$\therefore x = 1$



(a) Given that the curve passes through the point $(a, 0)$, state the value of a. [1]

The region R is enclosed by the curve, the x-axis and the line $x = \mathrm{e}$.

(b) Use the substitution $u = \ln x$ to find the area of the region R. [5]

Let $I_n = \int_1^{\mathrm{e}} \frac{(\ln x)^n}{x^2} \mathrm{d}x, n \in \mathbb{N}$.

(c) (i) Find the value of I_0.

(ii) Prove that $I_n = -\frac{1}{\mathrm{e}} + nI_{n-1}, n \in \mathbb{Z}^+$.

(iii) Hence find the value of I_1. [7]

(d) Find the volume of the solid formed when the region R is rotated through 2π about the x-axis. [5]

ANSWER BOOKLET
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4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno:

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Example Ejemplo 27 | 2 | 7 | Example Ejemplo 3 | | 3 |

| 1 | 1 |

$\Pi_1 : 4x + y + z = 8 \qquad \Pi_2 : 4x + 3y - z = 0$

(a)

$n_1 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \qquad n_2 = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$\therefore \cos\theta = \dfrac{16+3-1}{\sqrt{16+1+1}\sqrt{16+9+1}}$ ✓

$= \dfrac{18}{\sqrt{18}\sqrt{26}}$

$= \dfrac{18}{3\sqrt{2}\ \sqrt{2}\sqrt{13}}$ ✓

$= \dfrac{18}{6\sqrt{13}}$

$= 3/\sqrt{13}$

$= \sqrt{\dfrac{9}{13}}$ ✓✓ $\left\{ \begin{aligned} p &= 9 \\ q &= 13 \end{aligned} \right\}$

4

04AX01

(b)(i)

$L: \vec{b} = n_1 \times n_2$

$= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1-3 \\ 4+4 \\ 12-4 \end{pmatrix}$

$= \begin{pmatrix} -4 \\ 8 \\ 8 \end{pmatrix}$ (div by 4)

$= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

(ii) $x = 1, \quad y = 0, \quad z = 4$

$\Pi_1 : 4(1) + 0 + 4 = 8$ ✓

$\Pi_2 : 4(1) + 3(0) - 4 = 0$ ✓

(iii) $r = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

6

(c) $B(a, b, 1)$

$\Rightarrow \quad \overrightarrow{AB} \cdot \vec{b} = 0$

$\therefore \begin{pmatrix} a-1 \\ b-0 \\ 1-4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ✓

$\therefore \begin{pmatrix} a-1 \\ b \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ✓

$\therefore -(a-1) + 2b - 6 = 0$

$\therefore -a + 2b - 5 = 0$

$\therefore a = 2b - 5$ ✓ $\quad \ldots (1)$

$\Rightarrow$ as B lies on Π_1, then:

$4a + b + 1 = 8$

$\therefore 4a + b - 7 = 0$ ✓ $\quad \ldots (2)$

$\Rightarrow (1) \to (2): \quad 4(2b-5) + b - 7 = 0$

$\therefore 8b - 20 + b - 7 = 0$

$\therefore 9b = 27$

$\therefore \boxed{b = 3}$ ✓ $\quad \ldots (3)$

$\Rightarrow (3) \to (1): \quad a = 2(3) - 5$

$\therefore \boxed{a = 1}$

5

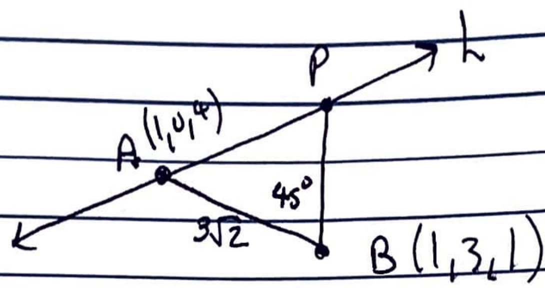
(d) $\vec{AB} = \begin{pmatrix} 1-1 \\ 3-0 \\ 1-4 \end{pmatrix} = \cancel{\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}} \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$

$\therefore |\vec{AB}| = AB = \sqrt{9+9}$

$= \sqrt{18}$ 1

$\therefore AB = 3\sqrt{2}$

(e)



If $\vec{AB} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$ and B is (1,3,1)

let P be (x, y, z)

$\vec{AP} = \begin{pmatrix} x-1 \\ y \\ z-4 \end{pmatrix}$ $\vec{BP} = \begin{pmatrix} x-1 \\ y-3 \\ z-1 \end{pmatrix}$

$\cancel{\vec{AB}}$ $\cos 45 = \dfrac{|\vec{AB} \cdot \vec{BP}|}{|\vec{AB}||\vec{BP}|}$

$\therefore \frac{1}{\sqrt{2}} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-3 \\ z-1 \end{pmatrix} \Big/ 3\sqrt{2} \times \sqrt{(x-1)^2+(y-3)^2+(z-1)^2}$

$= \dfrac{3y-9-3z+3}{\sqrt{x^2-2x+1+y^2-6x+9+z^2-2z+1}}$

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ANSWER BOOKLET
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$\frac{86}{120} = 71.7\%$

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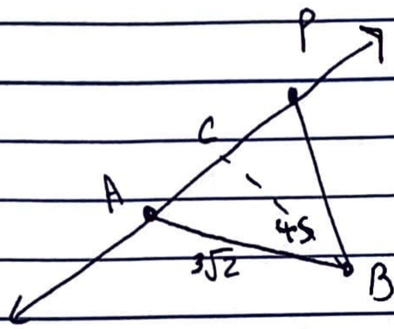
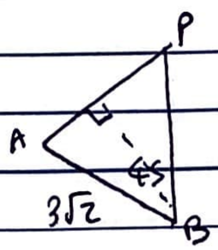
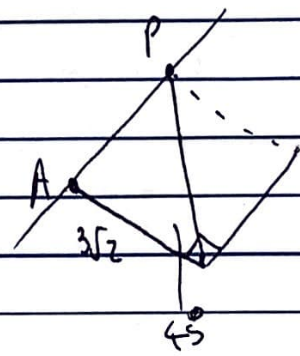
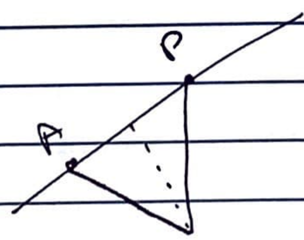
| 1 | 2 |









04AX01

1 2

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 = \left(\text{cis}\frac{\pi}{3}\right)^3$

$= \text{cis}\,\pi$ {DM's}

$= \cos\pi + i\sin\pi$ 2

$= -1$

(b) $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta) \quad n \in \mathbb{Z}^+$

Step 1) Prove for $n=1$:

$LHS = (\cos\theta - i\sin\theta)^1$

$RHS = \cos(\theta) - i\sin(\theta)$

$= LHS.$

$\therefore$ true for $n=1$

Step 2) Assume $n=k$ is true for $k = \mathbb{Z}^+$

$\therefore (\cos\theta - i\sin\theta)^k = \cos k\theta - i\sin k\theta$

Step 3) Consider when $n=k+1$

~~RHS~~ $\cancel{\cos(k+1)\theta - i\sin(k+1)\theta}$.

$(\cos\theta - i\sin\theta)^{k+1} = \cos(k+1)\theta - i\sin(k+1)\theta$

$= \cos(k\theta+\theta) - i\sin(k\theta+\theta)$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta - i(\sin k\theta \cos\theta - \sin\theta\cos k\theta)$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta - i\sin k\theta\cos\theta + i\sin\theta\cos k\theta$

$= \cos\theta(\cos k\theta - i\sin k\theta) - \sin\theta(\sin k\theta - i\cos k\theta)$

$= \cos\theta(\cos k\theta - i\sin k\theta) - i\sin\theta(\cos k\theta - i\sin k\theta)$

$= \cancel{(\cos\theta - i\sin\theta)(\cos k\theta - i\sin k\theta)}$

$= \cancel{(\cos\theta - i\sin\theta)^{k+1}} = LHS.$

~~Step 4) As true for n=1, and true for n=k n=k+1~~

$= (\cos\theta - i\sin\theta)(\cos\theta - i\sin\theta)^k$ (by assumption)

$= (\cos\theta - i\sin\theta)^{k+1}$

$= RHS$

Step 4: as true for $n=1$, and true for $n=k+1$ where $n=k$ is true, then true for all $n \in \mathbb{Z}^+$ by mathematical induction.

6

(c) $z = \cos\theta + i\sin\theta = \text{cis}\,\theta$

$\therefore z^* = \cos\theta - i\sin\theta$

$\therefore z^n + (z^*)^n$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ {DMs}

$= 2\cos n\theta$

2

(d)(i) $zz^* = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$

$= \cos^2\theta - i^2\sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

(ii) $(z+z^*)^3 = z^3 + 3z^2(z^*) + 3z(z^*)^2 + (z^*)^3$

(iii) $(z+z^*)^3 = z^3 + (z^*)^3 + 3z + 3z^*$

$= 2\cos 3\theta + 3(z - z^*)$

~~$= 2\cos 3\theta + 3(\cos^2\theta + 2i\cos\theta\sin\theta$~~

~~$= 2\cos 3\theta + 3(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta - \cos\theta + \sin\theta)$~~

~~$= 2\cos 3\theta$~~

as $(z+z^*)^3 = \cancel{(2\cos\theta)^3}\ (\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)^3$

$= 2^3\cos^3\theta$

$= 8\cos^3\theta$

$8\cos^3\theta = 2\cos 3\theta + 3(\text{cis}\,2\theta$

4

(e) $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$\therefore \cos 3\theta - 2\cos^2\theta + 1 = 0$

$\therefore -2\cos^2\theta + \cos 3\theta + 1 = 0$

$\therefore -2\cos^2\theta + \cos(\theta + 2\theta) + 1 = 0$

$\therefore -2\cos^2\theta + \cos\theta\cos 2\theta - \sin\theta\sin 2\theta + 1 = 0$

$\therefore -2\cos^2\theta + \cos\theta(2\cos^2\theta - 1) - 2\sin^2\theta\cos\theta + 1 = 0$

$\therefore -2\cos^2\theta + 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta + 1 = 0$

$\therefore 2\cos^3\theta - 2\cos^2\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta + 1 = 0$

$\therefore 2\cos^3\theta - \cancel{2\cos^2\theta} - \cos\theta - 2\cos\theta + \cancel{2\cos^3\theta} + 1 = 0$

$\therefore 4\cos^3\theta - 3\cos\theta + 1 = 0$

$\therefore \cos 3\theta + 1 = 0$

$\therefore \cos 3\theta = -1$

$\therefore 3\theta = \pi, 3\pi, \ldots$

$\therefore \theta = \pi/3, \pi, \ldots$

$\boxed{\therefore \theta = \pi/3}$

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ANSWER BOOKLET
LIVRET DE RÉPONSES
CUADERNILLO DE RESPUESTAS

3

International Baccalaureate
Baccalauréat International
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4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno:

16 M TZ 1 P 1 - M A M L

Candidate name: / Nom du candidat: / Nombre del alumno:

At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla

Example Ejemplo 27 | 2 7 | Example Ejemplo 3 | 3 |

13

(a) $y = \frac{(\ln x)^2}{x}$ $x > 0$

$= (\ln x)^2 (x^{-1})$

$\therefore dy/dx = (x^{-1})(2)(\ln x)(1/x) + (\ln x^2)(-1)(x^{-2})$

$= \frac{2\ln x}{x^2} - \frac{\ln x^2}{x^2}$

$= \frac{2\ln x - \ln x^2}{x^2} = 0$

$\therefore 2\ln x - \ln x^2 = 0$

$\therefore x = 1$

Hence, $a = 1$

1

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(b)

$$y = \frac{(\ln x)^2}{x}$$

$$R = \int_1^e y\,dx$$

let $u = \ln x$

$\therefore du = \frac{1}{x}\,dx$

limits : $u = \ln e = 1$

$u = \ln(1) = 0$

$$= \int_1^e \frac{u^2}{x}\,dx$$

$$= \int_0^1 u^2\,du$$

$$= \left[\frac{1}{3}u^3\right]_0^1$$

$$= \frac{1}{3}(1-0)^3$$

$$\therefore R = \frac{1}{3}$$

5

(c)(i)

$$I_0 = \int_1^e \frac{(\ln x)^0}{x^2}\,dx$$

$$= \int_1^e x^{-2}\,dx$$

$$= -\left[x^{-1}\right]_1^e$$

$$= -\left(\frac{1}{e} - 1\right)$$

$$= 1 - \frac{1}{e}$$

(ii)

$$I_{n+1} = \int_1^e \frac{(\ln x)(\ln x)^n}{x^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

See page 4 of booklet.

(iii)

$$I_1 = -\frac{1}{e} + I_0$$
$$= -1/e + (1 - 1/e)$$
$$= -2/e + 1$$

2

(d)

$$\cancel{V = \pi \int_1^e \left(\frac{1}{3}\right)^2 dx}$$

$\cancel{= \pi}$

$$V = \pi \int_1^e \frac{(\ln x)^4}{x^2} dx$$

$$= \pi \int_1^e \left(\frac{\ln x^2}{x}\right)^2 dx$$

$$= \pi \left(\frac{1}{3}\right)^2$$

$$= \pi/3$$

1

$\cancel{= \pi I_4}$

$\cancel{= \pi\left(-\frac{1}{e} + 4 I_3\right)}$

$=$

$\cancel{I_4 = -\frac{1}{e} + 4 I_3}$

$\cancel{= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3 I_2\right)}$

$\cancel{= -\frac{1}{e} - \frac{4}{e} + 12 I_2}$

$\cancel{= -\frac{5}{e} + 12\left(-\frac{1}{e} + 2 I_1\right)}$

$\cancel{= -\frac{5}{e} - 12/e + 24\left(-\frac{1}{e} + I_0\right)}$

see page 4 of booklet.

$\cancel{I_n = \int_1^e \frac{(\ln x)^n}{x^2} dx}$

$\cancel{u = \ln x^n \quad du = n \ln x^{n-1} \times 1/x \quad dv = x^{-2} \quad v = -x^{-1}}$

$\cancel{I_n = \left[-\left(\frac{1}{x}\right)(\ln x)^n\right]_1^e - \int_1^e \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)(n \ln x)^{n-1} dx}$

$\cancel{= -\left(\left(\frac{(\ln e)^n}{e}\right) - 0\right) - n I_{n-1}}$

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(d) $I_4 = -\frac{1}{e} + 4I_3$

$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3I_2\right)$

$= -\frac{1}{e} - \frac{4}{e} + 12\left(-\frac{1}{e} + 2I_1\right)$

$= -5/e - 12/e + 24\left(-\frac{1}{e} + I_0\right)$

$= \cancel{-5/e} - 17/e - 24/e + 24(1 - 1/e)$

$= -41/e + 24 - 24/e$

$= -65/e + 24.$

$\therefore V = \pi\left(-\frac{65}{e} + 24\right)$

(c)(ii) $I_n = \int_1^e \frac{(\ln x)^n}{x^2}\,dx$

$u = (\ln x)^n;\ du = n(\ln x)^{n-1}(1/x)$
$dv = x^{-2};\ v = -x^{-1}$

$= -\left[\frac{(\ln x)^n}{x}\right]_1^e - \int_1^e -\frac{n(\ln x)^{n-1}}{x^2}\,dx$

$= -\left(\frac{1}{e} - 0\right) + n\int_1^e \frac{(\ln x)^{n-1}}{x^2}\,dx$

$= -\frac{1}{e} + nI_{n-1}.$