Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3008

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

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ANSWER BOOKLET ①
LIVRET DE RÉPONSES
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Example Ejemplo 27 [27] Example Ejemplo 3 [3]

1

(a)
$$\vec{AB} = \begin{pmatrix} 3+5t-(-1+3\lambda) \\ 1+3t-(2+\lambda) \\ -1-t-(2-\lambda) \end{pmatrix} = \begin{pmatrix} 4+\cancel{9\lambda}\ 5t-3\lambda \\ -1+3t-\lambda \\ -3-t+\lambda \end{pmatrix}$$

2

(b) $\cancel{\vec{AB}\cdot\vec{r_1}=0} \rightarrow \begin{pmatrix} 4+5t-3\lambda \\ \cancel{-1+3t-\lambda} \\ -3-t+\lambda \end{pmatrix}\begin{pmatrix} -1+3\lambda \\ \cancel{2+\lambda} \\ 2-\lambda \end{pmatrix} = 0$

(i) $\begin{pmatrix} 4+5t-3\lambda \\ -1+3t-\lambda \\ -3-t+\lambda \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 0$

$\cancel{\vec{AB} = \sqrt{(4+5t-3\lambda)^2}}$

(ii) $\begin{pmatrix} 4+5t-3\lambda \\ -1+3t-\lambda \\ -3-t+\lambda \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = 0$

$(4+5t-3\lambda)(-1+3\lambda)+(-1+3t-\lambda)(2+\lambda)+(-3-t+\lambda)(2-\lambda)=0$

$\therefore -4-5t+3\lambda+12\lambda+15\lambda t-9\lambda^2-2-\lambda+6t+3t\lambda-2\lambda-\lambda^2-6-2t+2\lambda+3\lambda+\lambda t-\lambda^2$

$$\begin{pmatrix} -1+3\lambda \\ 2+\lambda \\ 2-\lambda \end{pmatrix} \times \begin{pmatrix} 3+5t \\ 1+3t \\ -1-t \end{pmatrix} = \begin{pmatrix} \cancel{(1+3t)}(2+\lambda)(-1-t)-(1+3t)(2-\lambda) \\ (3+5t)(2-\lambda)-(-1+3\lambda)(-1-t) \\ (-1+3\lambda)(+1+3t)-(3+5t)(2-\lambda) \end{pmatrix}$$

04AX01

(c)

$$\text{linsolve}\left(\begin{cases} 11\cdot\lambda - 19\cdot t = 14 \\ 19\cdot\lambda - 35\cdot t = 20 \end{cases}, \{\lambda, t\}\right) \quad \left\{\frac{55}{12}, \frac{23}{12}\right\}$$

$\therefore \lambda = \frac{55}{12} \approx 4.5833$ (s)

$t = \frac{23}{12} \approx 1.9167$ (s)

3

(d) $|\vec{AB}| = \sqrt{(4+5t-3\lambda)^2 + (-1+3t-\lambda)^2 + (-3-t+\lambda)^2}$

$\Rightarrow$ substituting $t = 23/12$ and $\lambda = 55/12$:

$|\vec{AB}| \approx 0.408248$ km

≈ 0.408 km

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(e)

$$r_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + \lambda \cancel{t} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \Rightarrow A$$

$$r_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \cancel{\lambda} t \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \Rightarrow B$$

$$\therefore \vec{AB} = \begin{pmatrix} 3 + 5\cancel{\lambda}t - (-1 + \lambda U_1) \\ 1 + 3t - (2 + \lambda U_2) \\ -1 - t - (2 + \lambda U_3) \end{pmatrix} \qquad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = k \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 5t + 1 - \cancel{3} 3k\lambda \\ 1 + 3t - 2 - k\lambda \\ -1 - t - 2 + k\lambda \end{pmatrix}$$

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2

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Example Ejemplo 27 | 27 Example Ejemplo 3 | 3

2

(a) $P(X=1)$ is $\{T\}$

$\therefore P(X=1) = 1/2$ 1

~~(b)(i) $P(X\leq 2)$ is $\{T, HT\}$~~

~~$\therefore P(X\leq 2) = \frac{1}{2}\times\frac{1}{2}$~~

~~$= \frac{1}{4}$~~

~~(ii) $P(X\leq 3) = (1/2)^3$~~

~~$= 1/8$~~

~~(b)(i) $P(X=2) = 1/4$ $\{T, HT\}$~~

~~$\therefore E(X\leq 2) = \frac{1}{4} + 2\times\frac{1}{4}$~~

~~$= 3/4$~~

(b)(i) $P(X\leq 2)$ is $\{T, HT\}$ 2

$\therefore P(X\leq 2) = \frac{1}{2}+\left(\frac{1}{2}\right)^2$

$= 3/4$

(ii) $P(X\leq 3) = \{T, HT, HHT\}$

$= \frac{3}{4} + \left(\frac{1}{2}\right)^3$ 2

$= 7/8$

(c) $P(X \leq n) = (2^n - 1)/2^n$

$\Rightarrow$ Prove for $n=1$: $P(X \leq 1) = \frac{2-1}{2}$

$= 1/2$

$=$ part (a)

$\therefore$ true for $n=1$

$\Rightarrow$ Assume true for $n=k$

$\therefore P(X \leq k) = \frac{2^k - 1}{2^k}$ {inductive hypothesis}

$\Rightarrow$ Consider $n = k+1$

$\therefore P(X \leq k+1) = \frac{2^{k+1}-1}{2^{k+1}}$

LHS $= P(X \leq k) + P(X = k+1)$

$= \frac{2^k - 1}{2^k} + \left(\frac{1}{2}\right)^{k+1}$ {by inductive hypothesis}

$= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}}$

$= \frac{2^k - 1}{2^k} + \frac{1}{2 \cdot 2^k}$

$= \frac{2 \cdot 2^k - 2}{2 \cdot 2^k}$

$= 2^k$

$= \frac{2^k - 1}{2^k} + \frac{1}{2 \cdot 2^k}$

$= \frac{2(2^k - 1) + 1}{2 \cdot 2^k}$

$= \frac{2^{k+1} - 2 + 1}{2^{k+1}}$

$= (2^{k+1} - 1)/2^{k+1}$ $\therefore$ true for $n = k+1$

04AX02

⇒ as true for $n=1$ and true for $n=k+1$ whenever $n=k$ is assumed to be true, true for all $n \in \mathbb{Z}^+$ by mathematical induction.

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(d)(i) ~~$P(Y \leq n) =$~~

minimum number of flips = 3 (a+b+c)

∴ number of flips for all 3, a, b, and c.

$$P(Y \leq n) = \left(\frac{2^n - 1}{2^n}\right)^3$$

(ii) $P(Y \leq n+1) = \left(\frac{2^{n+1} - 1}{2^{n+1}}\right)^3$

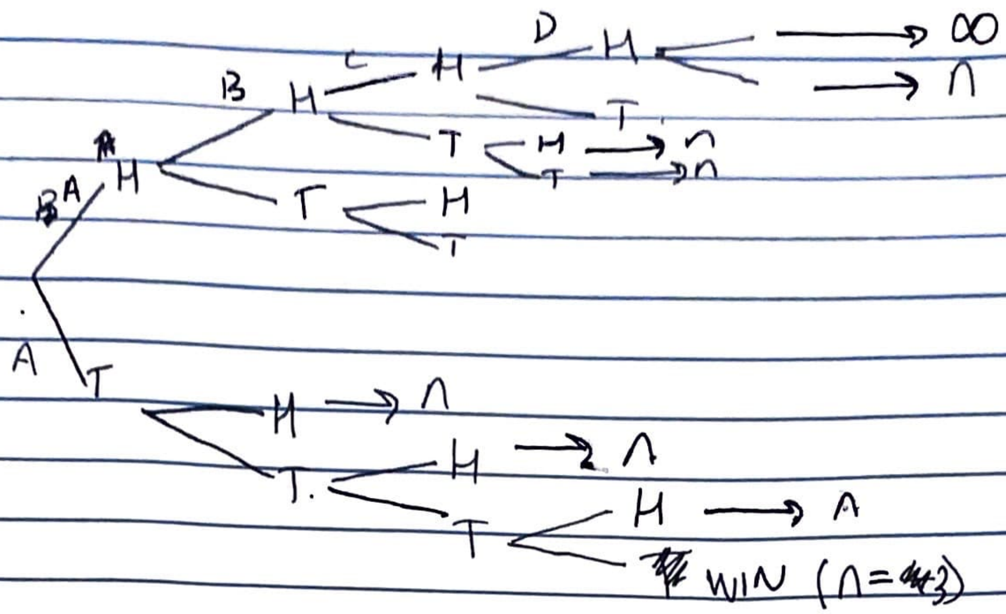
2

(e) $P(Y = n+1) = P(Y \leq n+1) - P(Y \leq n)$

$$= \left(\frac{2^{n+1} - 1}{2^{n+1}}\right)^3 - \left(\frac{2^n - 1}{2^n}\right)^3$$

2

f)



=> Probability that Player D wins equals:

· probability that for each round, n, D throws a head

↳ $\left(\frac{1}{2}\right)^n$

~~PR PROOF~~.

TIMES ← AND · probability ~~that~~ of the maximum value of a, b and c for the n rounds (aforementioned)

=> This ~~&~~ extends to infinity as ~~eo~~ at the end of each decision tree, that same one repeats ~~for~~ infinity times. (see diagram)

$$\therefore P(D \text{ wins}) = \sum_{n=1}^{\infty}\left(\frac{2^n-1}{2^n}\right)^3 \times \left(\frac{1}{2}\right)^n$$

(g)

$$P(1\text{ winning}) = \sum_{n=1}^{\infty}\left(\frac{2^{3n}+3(2^{2n})(-1)+3(2^{n})(1)\cancel{4}-1}{2^{3n}}\right)\left(\frac{1}{2}\right)^{n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{2^{3n}\cancel{4}-3.2^{2n}+3.2^{n}-1}{2^{3n}}\right)\left(\frac{1}{2}\right)^{n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{2^{3n}}{2^{3n}}-\frac{3.2^{2n}}{2^{3n}}+\frac{3.2^{n}}{2^{3n}}-\frac{1}{2^{3n}}\right)\left(\frac{1}{2^{n}}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{2^{3n}}{2^{4n}}-3\frac{2^{2n}}{2^{4n}}+\frac{3.2^{n}}{2^{4n}}\cancel{4}-\frac{1}{2^{4n}}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n}}-\frac{3}{2^{2n}}+\frac{3}{2^{3n}}-\frac{1}{2^{4n}}\right)$$

$$= \sum_{n=1}^{\infty}\left(2^{-n}\right)-3\sum_{n=1}^{\infty}\left(2^{-2n}\right)+3\sum_{n=1}^{\infty}\left(2^{-3n}\right)-\sum_{n=1}^{\infty}\left(2^{-4n}\right)$$

↓	↓	↓	↓
$u_1 = 1/2$	$u_1 = 1/4$	$u_1 = 1/8$	$u_1 = 1/16$
$r = 1/2$	$r = 1/4$	$r = 1/8$	$r = 1/16$

$$= \frac{1/2}{1-1/2} - 3\frac{1/4}{1-1/4} + 3\frac{1/8}{1-1/8} - \frac{1/16}{1-1/16}$$

$$= \frac{1/2}{1/2} - 3\frac{1/4}{3/4} + 3\frac{1/8}{7/8} - \frac{1/16}{15/16}$$

$$= 1 - 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{7}\right) - \left(\frac{1}{15}\right)$$

$$= 1 - 1 + 3/7 - 1/15$$

$$= 3/7 - 1/15$$

$$= 38/105$$

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(h) P(A win) occurs when A ~~is~~ gets 1 more Heads than all the other players.

1

$$\sum_{n=1}^{\infty} P(Y \leq n+1)\left(\frac{1}{2}\right)^n$$

$$\frac{38}{105} \times \frac{1}{2}$$

. . .

A B C D
A B C D A B C .

(f) $\sum_{n=\phi}^{\infty} P(Y \leq n)\left(\frac{1}{2}\right)^{n-1}$

$= \sum_{n=1}^{\infty} \left(\frac{2^n - 1}{2^n}\right)^3 \left(\frac{1}{2}\right)^{n-1}$

$$\sum_{n=0}^{\infty} P(Y = n+1)\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\left(\frac{2^{n+1}-1}{2^{n+1}}\right)^3 - \left(\frac{2^n - 1}{2^n}\right)^3\right)\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{(2^{n+1}-1)^3}{(2^{n+1})^3} - \frac{(2^{n+1}-2)^3}{(2^{n+1})^3}\right)\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{3(2^{n+1})(1) + 3(2^{2n+2})(-1) - 1 - ((3)(-2)(2^{2n+2}) + (3)(4)(2^{n+1}) - 8)}{2^{3n+3}}\right)\left(\frac{1}{2}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{7 + 3(2^{n+1}) - 12(2^{n+1}) - 3(2^{2n+2}) + 6(2^{2n+2})}{2^{3n+3}}\right)\left(\frac{1}{2}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{7 - 9(2^{n+1}) + 3(2^{n+1})}{2^{3n+3}}\right)\left(\frac{1}{2}\right)$$

~~edit~~ $\Rightarrow$ question already completed, but this is correct working

(f) $$\sum_{n=0}^{\infty}\left[\left(\frac{2^{n+1}-1}{2^{n+1}}\right)^3-\left(\frac{2^n-1}{2^n}\right)^3\right]\left(\frac{1}{2}\right)^n$$

$= \cancel{\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)}$

$$= \left(\frac{2^1-1}{2^1}\right)^3\left(\frac{1}{2}\right)^0 - \left(\frac{2^0-1}{2^0}\right)^3\left(\frac{1}{2}\right)^0 + \left(\frac{2^2-1}{2^2}\right)^3\left(\frac{1}{2}\right)^1 - \left(\frac{2^1-1}{2^1}\right)^3\left(\frac{1}{2}\right)^1 + \cdots$$

$$+ \left(\frac{2^3-1}{2^3}\right)^3\left(\frac{1}{2}\right)^2 - \left(\frac{2^2-1}{2^2}\right)^3\left(\frac{1}{2}\right)^2$$

$$= \left(\frac{2^2-1}{2^2}\right)^3\left(\left(\frac{1}{2}\right)^1-\left(\frac{1}{2}\right)^2\right) + \left(\frac{2^1-1}{2^1}\right)^3\left(\left(\frac{1}{2}\right)^0-\left(\frac{1}{2}\right)^1\right) + \cdots$$

$$= \left(\frac{2^2-1}{2^2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{2^1-1}{2^1}\right)\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{2^n-1}{2^n}\right)\left(\frac{1}{2}\right)^n$$

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