Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3011

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Document name: Taskmaker-ID

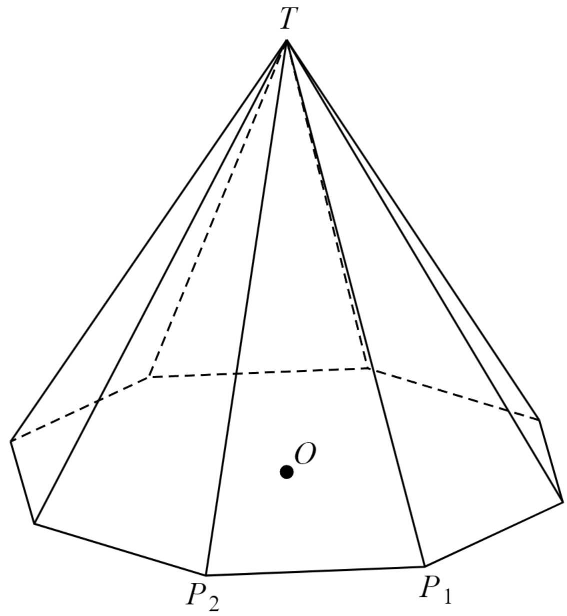
Scanned document name: Taskmaker-ID-MAHL-DD.MM

1. [Maximum points: 25]

In this problem you will investigate the surface area of a regular n-gon pyramid and investigate the case when n tends to infinity.

A regular n-gon pyramid has a base in the shape of a regular polygon with n sides. For example the diagram below shows a regular 8-gon pyramid (a regular octagonal pyramid). The tip T of the pyramid is directly above the centre O of the polygon.

Let $\left|\overrightarrow{OP_1}\right| = \left|\overrightarrow{OP_2}\right| = r$, $\angle P_1OP_2 = \theta$ and $\left|\overrightarrow{OT}\right| = h$.



Let point O represent the origin, the coordinates of P_1 be $(r,0,0)$ and the coordinates of point P_2 be $(r\cos\theta, r\sin\theta, 0)$. Point T has a positive z-coordinate.

(a) Write down the coordinates of point T in terms of h. [1]

(b) Find the following vectors [4]

(i) $\overrightarrow{TP_1}$

(ii) $\overrightarrow{TP_2}$

(c) By considering $\overrightarrow{TP_1} \times \overrightarrow{TP_2}$ show that the area A of $\triangle P_1TP_2$ is given by [5]

$$A = \frac{r\sqrt{2h^2(1-\cos\theta) + r^2\sin^2\theta}}{2}$$

(d) Find the relationship between n and θ. [1]

(e) Show that the surface area S of the pyramid, excluding the base, is equal to [2]

$$S = \frac{\pi r\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}}{\theta}$$

(f) Explain why $\lim_{n\to\infty}\theta = 0$. [1]

Using l'Hopital's rule on the expression from (e) gives

$$\lim_{\theta\to 0} S = \frac{\pi r\sin\theta(h^2+r^2\cos\theta)}{\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}}$$

(g) Explain the problem with trying to evaluate $\lim_{\theta\to 0} S$ using this approach. [2]

(h) By considering $\lim_{\theta\to 0}\frac{2h^2(1-\cos\theta)+r^2\sin^2\theta}{\theta^2}$ evaluate $\lim_{\theta\to 0} S$. [7]

(i) Explain the significance of your answer to part (f). [2]

2. [Maximum points: 30]

In this problem you will investigate the relationship between binomial coefficients and definite integrals.

Let $B(m,n) = \int_0^1 x^m(1-x)^n\,dx$ where $m,n \in \mathbb{N}$.

(a) **Without** using your GDC find the exact value of [6]

(i) $B(0,0)$

(ii) $B(1,0)$

(iii) $B(0,1)$

(b) Find $B(m,0)$ in terms of m. [2]

(c) Use integration by parts to show that $B(m,n) = \dfrac{nB(m+1,n-1)}{m+1}$ when $n > 0$. [5]

(d) Hence **copy and complete** the following table showing the values of $B(4,n)$ for values of n from 0 to 4. [6]

n	0	1	2	3	4
$B(4,n)$					

It is hypothesized that $B(m,n) = \dfrac{1}{(m+n+1)\cdot {}^{m+n}\mathrm{C}_n}$.

(e) Show that the hypothesis is true for $B(m,0)$. [2]

(f) Let k be a specific natural number. Show that if the hypothesis is true for $n = k$ and all natural numbers m, then it must also be true for $n = k+1$ and all natural numbers m. [6]

(g) Hence complete the proof of the hypothesis by induction. [3]

1. (a) $(0,0,h)$ A1

(b)

(i) $\begin{pmatrix} r \\ 0 \\ -h \end{pmatrix}$ A1A1

(ii) $\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ -h \end{pmatrix}$ A1A1

(c) We have

$$\begin{pmatrix} 0(-h)-(-h)r\sin\theta \\ -hr\cos\theta - r(-h) \\ r^2\sin\theta \end{pmatrix} = \begin{pmatrix} hr\sin\theta \\ hr(1-\cos\theta) \\ r^2\sin\theta \end{pmatrix}$$ M1A1

So the area of the triangle is

$$\frac{\sqrt{h^2r^2\sin^2\theta + h^2r^2(1-\cos\theta)^2 + r^4\sin^2\theta}}{2} = \frac{\sqrt{2h^2r^2 - 2h^2r^2\cos\theta + r^4\sin^2\theta}}{2}$$ M1A1

This simplifies to

$$\frac{r\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}}{2}$$ A1

(d) $n = \frac{2\pi}{\theta}$ or $\theta = \frac{2\pi}{n}$ A1

(e) The total area is

$$\frac{2\pi}{\theta} \times \frac{r\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}}{2} = \frac{\pi r\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}}{\theta}$$ M1A1

(f) $\lim\limits_{n\to\infty} \frac{2\pi}{n} = 0$ A1

(g) Since this is still of the form $\frac{0}{0}$ we will need to use l'Hopital's rule again. R1

However, this will happen indefinitely as the $\sqrt{2h^2(1-\cos\theta)+r^2\sin^2\theta}$ will keep creating 0/0. A1

(h) Since the fraction is of the form $\frac{0}{0}$ we can use l'Hopital's rule. R1

This gives

$$\lim_{\theta \to 0} \frac{2h^2 \sin\theta + 2r^2 \sin\theta \cos\theta}{2\theta} = \lim_{\theta \to 0} \frac{2h^2 \sin\theta + r^2 \sin 2\theta}{2\theta}$$ M1A1

This is still of the form $\frac{0}{0}$ se we can use l'Hopital's rule again. R1

This gives

$$\lim_{\theta \to 0} \frac{2h^2 \cos\theta + 2r^2 \cos 2\theta}{2} = h^2 + r^2$$ M1A1

So

$$\lim_{\theta \to 0} S = \pi r \sqrt{h^2 + r^2}$$ A1

(i) This is the surface area of the curved surface of a cone. A1A1

2. (a)

(i) $\int_0^1 dx = [x]_0^1 = 1$ M1A1

(ii) $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ M1A1

(iii) $\int_0^1 1 - x\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ M1A1

(b) $\int_0^1 x^m\,dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$ M1A1

(c) Let $u = (1-x)^n$ and $v' = x^m$. M1

So $u' = -n(1-x)^{n-1}$ and $v = \frac{x^{m+1}}{m+1}$. A1

The integral then becomes

$$B(m,n) = \left[\frac{(1-x)^n x^{m+1}}{m+1}\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}\,dx$$ A1

This is equal to

$$0 - 0 + \frac{nB(m+1,n-1)}{m+1} = \frac{nB(m+1,n-1)}{m+1}$$ M1A1

(d) From part (b) we have

$$B(4,0) = \frac{1}{5}$$ A1

Use part (c) to calculate the rest M1

$$B(3,1) = \frac{1 \times B(4,0)}{4} = \frac{1}{20}$$ A1

$$B(2,2) = \frac{2B(3,1)}{3} = \frac{1}{30}$$ A1

$$B(1,3) = \frac{3B(2,2)}{2} = \frac{1}{20}$$ A1

$$B(0,4) = \frac{4B(1,3)}{1} = \frac{1}{5}$$ A1

(e) From part (b) we have

$$B(m,0) = \frac{1}{m+1}$$ A1

Also

$$\frac{1}{(m+0+1)^m C_0} = \frac{1}{m+1}$$ A1

(f) We assume

$$B(m,k) = \frac{1}{(m+k+1)\cdot {}^{m+k}\mathrm{C}_k} \qquad \text{A1}$$

When $n = k + 1$ we have

$$B(m,k+1) = \frac{(k+1)B(m+1,k)}{m+1} \qquad \text{A1}$$

Using our inductive hypothesis this gives

$$B(m,k+1) = \frac{k+1}{(m+k+2)(m+1)\cdot {}^{m+k+1}\mathrm{C}_k} \qquad \text{M1}$$

This can be written as

$$B(m,k+1) = \frac{(k+1)(m+1)!k!}{(m+k+2)(m+1)(m+k+1)!} = \frac{(k+1)!m!}{(m+k+2)(m+k+1)!} \qquad \text{A1A1}$$

This simplifies to

$$\frac{1}{(m+k+2)\cdot {}^{m+k+1}\mathrm{C}_{k+1}} \qquad \text{A1}$$

So it is true for $n = k + 1$.

(g) In part (e) we have shown it is true when $n = 0$. A1

In part (e) we have shown that if it is true for $n = k$ then it is also true for $n = k + 1$. R1

By the principle of mathematical induction it must be true for all $n \in \mathbb{N}$. R1