Mathematics: analysis and approaches

Higher level

Paper 3

ID: 0003

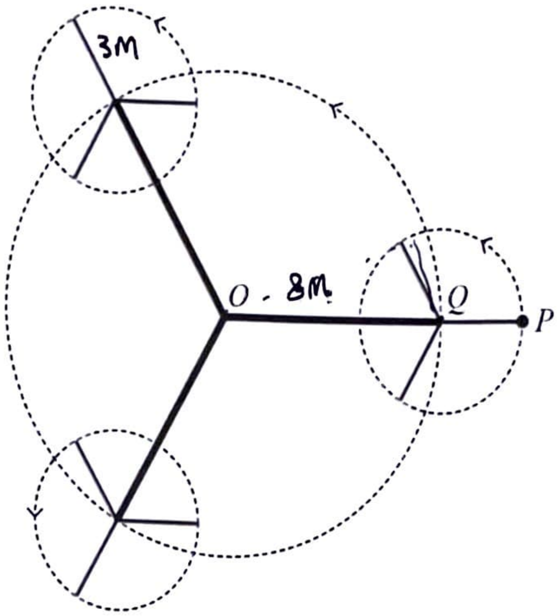
Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[53 marks]**.

Attempt 1: ~~75~~ $\frac{40}{53} \approx 75.85\%$.

1. [Maximum points: 27]

A fairground ride consists of three equally spaced arms 8 m in length. These three arms make one anti-clockwise revolution every 2π seconds about point O. At the end of each arm are three smaller arms 3 m in length. These three arms make one anti-clockwise revolution every $\pi/2$ seconds about the endpoints of the longer arms. This is shown in the diagram below showing the view from above, where $OQ = 8$ m and $PQ = 3$ m.



A rider sits at point P with initial coordinates $(11,0)$ relative to point O.

(a) Find the position vector of point Q after t seconds. [2]

(b) Hence show that the position vector of point P after t seconds is given by [4]

$$\overrightarrow{OP} = \begin{pmatrix} 8\cos t + 3\cos 4t \\ 8\sin t + 3\sin 4t \end{pmatrix}$$

Let T represent the smallest value of t for which point P is moving directly towards point O.

(c) Show that [6]

$$\frac{8\cos T + 12\cos 4T}{8\sin T + 12\sin 4T} = -\frac{8\sin T + 3\sin 4T}{8\cos T + 3\cos 4T}$$

(d) Solve the equation in part (c) to find the exact value of T, writing your answer in the form $T = b \cdot \arccos(a)$ where a and b are rational numbers to be determined. [7]

(e) If $D = |\overrightarrow{OP}|$ show that $D^2 = 73 + 48\cos 3t$. [4]

(f) For the value of T in part (d) find the rate at which $|\overrightarrow{OP}|$ is changing. [4]

2. [Maximum points: 26]

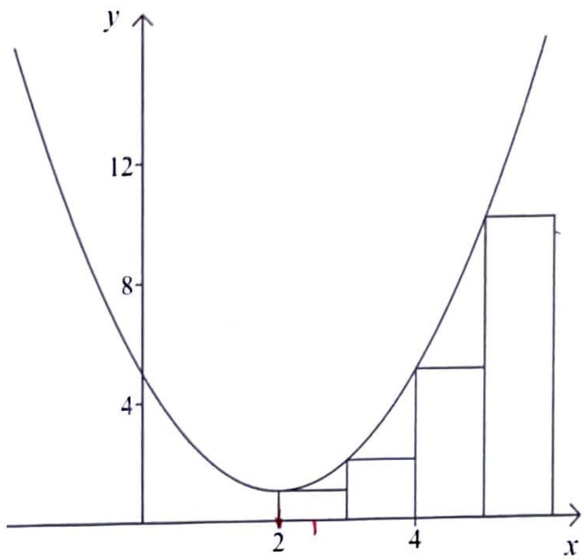
In this problem you will investigate the area between a parabola and the x-axis by dividing the area into rectangles of equal width.

(a) Write down an expression for the value of $\sum_{k=1}^{n} k$ in terms of n. [2]

(b) Prove by induction $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. [9]

Let $f(x) = x^2 - 4x + 5$. The diagram below shows the graph of $y = f(x)$. Rectangles of width 1 are drawn between the graph and the x-axis from $x = 2$ to $x = 6$.

$2(k+1)+1$
$= 2k+3$



$(k+1)(k+2)$
$= k^2+3k+2$

(c) Find the total area of the rectangles. [2]

Suppose n rectangles of equal width are now drawn between the graph of $y = f(x)$ and the x-axis from $x = 2$ to $x = 6$.

(d) Write down an expression for the width of each rectangle in terms of n. [1]

(e) Show that the total area A of all the rectangles is equal given by [3]

$$A = \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16(k-1)^2}{n^2} + 1\right)$$

(f) Hence use parts (a) and (b) to determine an expression for A without using sigma notation. [4]

(g) Evaluate $\lim_{n \to \infty} A$. [2]

(h) Verify your answer to part (g) by evaluating an appropriate definite integral. [3]

~~$k + \frac{3}{2} = 0$~~

$\therefore k = \frac{2}{3}$

$k = -\frac{3}{2}$

$\therefore 2k+3=0$

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$\frac{40}{53} = 76\%$

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Q1: 20/27 Q2: 20/26

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1 2 3 4 5 6 7 8 9 10

1

a) $T = 2\pi$ s

$\therefore f = 1$ Hz

at $t=0$, $Q_y = 8\sin 0$

$Q_x = 8\cos 0$

$\therefore$ ~~$\vec{OQ} = \left(8\sin\theta \right)$~~ $\vec{OQ} = \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix}$ 2

b) $\vec{OP} = \vec{OQ} + \vec{QP}$

$= \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix} + \vec{QP}$

$\vec{QP} = \begin{pmatrix} 3\cos bt \\ 3\sin bt \end{pmatrix} \rightarrow b = \frac{2\pi}{T}$

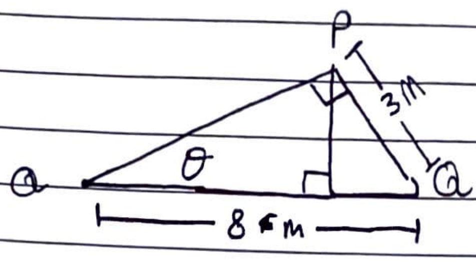
$= 2\pi / \pi/2$

$= 4$

$\therefore \vec{OP} = \begin{pmatrix} 8\cos t + 3\cos 4t \\ 8\sin t + 3\sin 4t \end{pmatrix}$ 4

6

c)



$|OP|$ ~~...~~ will be a minimum when P is moving directly towards

When $\vec{QP} \cdot \vec{OP} = 0$

$$\therefore \begin{pmatrix} 3\cos 4T \\ 3\sin 4T \end{pmatrix} \cdot \begin{pmatrix} 8\cos T + 3\cos 4T \\ 8\sin T + 3\sin 4T \end{pmatrix} = 0$$

$\therefore (3\cos 4T)(8\cos T + 3\cos 4T) = -(3\sin 4T)(8\sin T + 3\sin 4T)$

$\therefore$ ~~3~~ $3\cos 4T \cdot 8\cos T + 3\cos 4T \cdot 3\cos 4T = -3\sin 4T \, 8\sin T - 3\sin 4T \, 3\sin 4T$

$\therefore$ ~~3~~ ~~Gradient~~ $x = 8\cos t + 3\cos 4t$

$y = 8\sin t + 3\sin 4t$

$\therefore dx/dt = -8\sin t - 12\sin 4t$

$\therefore dy/dt = 8\cos t + 12\cos 4t$

d) $\dfrac{8\cos T + 12\cos 4T}{8\sin T + 12\sin 4T} = -\dfrac{8\sin T + 3\sin 4T}{8\cos T + 3\cos 4T}$

$\therefore (8\cos T + 12\cos 4T)(8\cos T + 3\cos 4T) = -(8\sin T + 3\sin 4T)(8\sin T + 12\sin 4T)$

$\therefore 64\cos^2 T + 24\cos T\cos 4T + 96\cos T\cos 4T + 36\cos^2 4T$

$= -64\sin^2 T - 96\sin 4T\sin T - 24\sin T\sin 4T - 36\sin^2 4T$

$\therefore \cancel{32\cos^2 T + 60\cos T\cos 4T + 18\cos^2 4T}$

$\therefore \cancel{(8\cos T + 6\cos 4T)^2 = -(8\sin T + 6\sin 4T)^2}$

$\therefore \cancel{8\cos T + 8\sin T = -6\cos 4T - 6\sin 4T}$

$\therefore$

$\therefore 64(\cos^2 T + \sin^2 T) + 120(\cos T\cos 4T + \sin T\sin 4T) + 36(\cos^2 4T + \sin^2 4T) = 0$

$\therefore 100 + 120\cos(4T - T) = 0$

$\therefore \cos(3T) = -100/120$

$\therefore 3T = \arccos(-5/6)$

$\therefore T = \frac{1}{3}\arccos(-5/6)$

7

7

e) $D = |\overrightarrow{OP}|$

$$= \left\| \begin{pmatrix} 8\cos t + 3\cos 4t \\ 8\sin t + 3\sin 4t \end{pmatrix} \right\|$$

$$= \sqrt{(8\cos t + 3\cos 4t)^2 + (8\sin t + 3\sin 4t)^2}$$

$$= \sqrt{64(\cos^2 t + \sin^2 t) + 9(\cos^2 4t + \sin^2 4t) + 18(\cos t\cos 4t + \sin t \sin 4t)}$$

$$= \sqrt{73 + 48\cos(4t - t)}$$

$$= \sqrt{73 + 48\cos 3t}$$

$$\therefore D^2 = 73 + 48\cos 3t$$

4

f) $\cancel{\frac{\partial D}{\partial t}}$ $D^2 = 73 + 48\cos 3t$

$$\therefore 2D \, \partial D/\partial t = -48\sin(3t) \times 3$$

$$= -144\sin(3t)$$

$$\therefore \partial D/\partial t = -\frac{144}{2D}\sin(3T) \quad \times$$

at $t = T$, $D = \sqrt{73 + 48\cos(\arccos(-5/6))}$

$$= 33$$

$$\therefore \partial D/\partial t = -\frac{144}{33}\sin(\arccos(-5/6))$$

$$\approx -26.5 \text{ ms}^{-1}$$

ECF

3

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2

a) $\sum_{k=1}^{n} k = 1+2+3+\cdots+(n-1)+n$ 2

b) $\sum_{r=1}^{n} r^2 = 1+4+9+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$

→ Step 1: Prove true for $n=1$:

$LHS = 1$

$RHS = 1(2)(3)/6$

$= 1$

$= LHS$ ∴ true for $n=1$

→ Step 2: assume true for $n=k$:

$1+4+9+\cdots+k^2 = k(k+1)(2k+1)/6$

→ Step 3: prove true for $n=k+1$:

$\therefore LHS = 1+4+9+\cdots+k^2+(k+1)^2$ 9

$= k(k+1)(2k+1)/6 + (k+1)^2$ (by assumption)

$= \frac{k(k+1)(2k+1)+6(k^2+2k+1)}{6}$

$= \cancel{k(k+1)}((k^2+k)(2k+1)+6k^2+12k+6)/6$

$= \cancel{(2k^3+k^2+2k^2+k+6k^2+12k+6)/6}$

$= \cancel{((k^2+3k+2)+10k+2k^3+4}$

$= \cancel{k(k+1)(2k+1)+}$

$= (k+1)(k(2k+1)+6(k+1))/6$

$= (k+1)(2k^2+k+6k+6)/6$

$= (k+1)(k+2)(k+3/2)/6$

$= (k+1)(k+2)(2k+3)/6 = RHS$

∴ true by mathematical induction as P_1 and P_{n+1} when P_n.

11

2

(c) $A = \sum_{r=?}^{?} f(r) \times 1$

$= 18 \text{ units}^2$ (using summation on G.D.C)

(d) $n = 4$ when $\Delta x = 6 - 2 = 4$

1

$\therefore \omega = \frac{\Delta x}{n}$

$\therefore \omega = 4/n$

(e) Using part (c):

$$A = \sum_{k=2}^{n+1} \frac{4}{n}$$

$$A = \frac{4}{n} \sum_{k=1}^{n} \left(x^2 - 4x + 5 \right), \text{ where } x = 2 + \frac{(k-1) \times 4}{n}$$

$$= \frac{4}{n} \sum_{k=1}^{n} \left(\left[2 + \frac{4(k-1)}{n} \right]^2 - 4 \left[2 + \frac{4(k-1)}{n} \right] + 5 \right)$$

$$= \frac{4}{n} \sum_{k=1}^{n} \left(4 + \frac{16(k-1)}{n} + \frac{16(k-1)^2}{n^2} - 8 - \frac{16(k-1)}{n} + 5 \right)$$

$$= \frac{4}{n} \sum_{k=1}^{n} \left(\frac{16(k-1)^2}{n^2} + 1 \right)$$

3

(f) $A = \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16(k-1)^2}{n^2}+1\right)$

$= \frac{4}{n}\times\frac{16}{n^2}\times\sum_{k=1}^{n}\left((k-1)^2\right)+1$ 1

$= \frac{64}{n^3}\times\frac{\cancel{(k-1)(k-1+1)(2(k-1)+1)}}{6}+1$

$= \frac{64}{n^3}\times\frac{(n-1)(n)(2n-1)}{6}+1$

$= \cancel{32(1} \quad \frac{32n(n-1)(2n-1)}{3n^3}+3$ ✗

$= 32(n-1)(2n-1)/3n^2 \quad +3$

ECF

(g) $\lim_{n\to\infty} A = \lim_{n\to\infty}\left(\frac{32n(n-1)(2n-1)}{3n^2}+3\right)$

$\cancel{= 0+1}$ $= \lim_{n\to\infty}\left(\frac{32(2n^2-n-2n+1)}{3n^2}\right)+3$

$\cancel{= 1}$

2

$= \lim_{n\to\infty}\left(\frac{64n^2-96n+32}{3n^2}\right)+3$

~~h)~~ $\cancel{\int_2^6 (x^2-4x+5)\,dx =}$

$= \lim_{n\to\infty}\left(\frac{64-96/n+32/n^2}{3}\right)+3$

$= 64/3 + 3$

$= \cancel{22.333}$ 25.333

(a) $\int_2^6 (x^2-4x+5)\,dx = 25.333$ 3

6/