TZ2

Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
## Higher level
## Paper 1

Friday 6 May 2022 (afternoon)

2 hours

Candidate session number

| 22 | M | TZ | 2 | P | 1 | M | A | H | L |
|---|---|---|---|---|---|---|---|---|---|

### Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

$\frac{88}{110} = \cancel{88}\ 80\%$

14 pages

2222–7111

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

The $n^{\text{th}}$ term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, $u_1$. [1]

(b) Given that the $n^{\text{th}}$ term of this sequence is $-33$, find the value of $n$. [2]

(c) Find the common difference, $d$. [2]

(a) $u_n = 15 - 3n$ ~~$u_1 + (n-1)d$~~

$\therefore u_1 = 15 - 3$

$\therefore \boxed{u_1 = 12}$

1

(b) $u_n = 15 - 3n = -33$

$\therefore 3n = 48$

$\therefore \boxed{n = 16}$

2

33
15
48

(c) $u_2 = 15 - 6 = 9$

$u_1 = 12$

$\therefore d = u_2 - u_1$

$= 9 - 12$

$\boxed{\therefore d = -3}$

2

5

**2.** [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, $n$ and $n+1$.

(a) Prove that the sum of these three integers is always divisible by $3$. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by $3$. [4]

(a) $(n-1)+(n)+(n+1) = 3n$

$\Rightarrow 3n/3 = n$, $\cancel{n \in \mathbb{Z}}$ $n \in \mathbb{Z}$

$\therefore$ The sum is always divisible by 3

2

(b) $(n-1)^2 + n^2 + (n+1)^2 = n^2 - 2n + 1 + n^2 + n^2 + 2n + 1$

$= 3n^2 + 2$

$= 3(n^2 + 2/3)$

$\therefore$ $3(n^2+2/3)$ is never divisible by 3 as $n^2+2/3$ ~~always~~ never returns an integer

4

6

**3.** [Maximum mark: 8]

A function $f$ is defined by $f(x)=\frac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

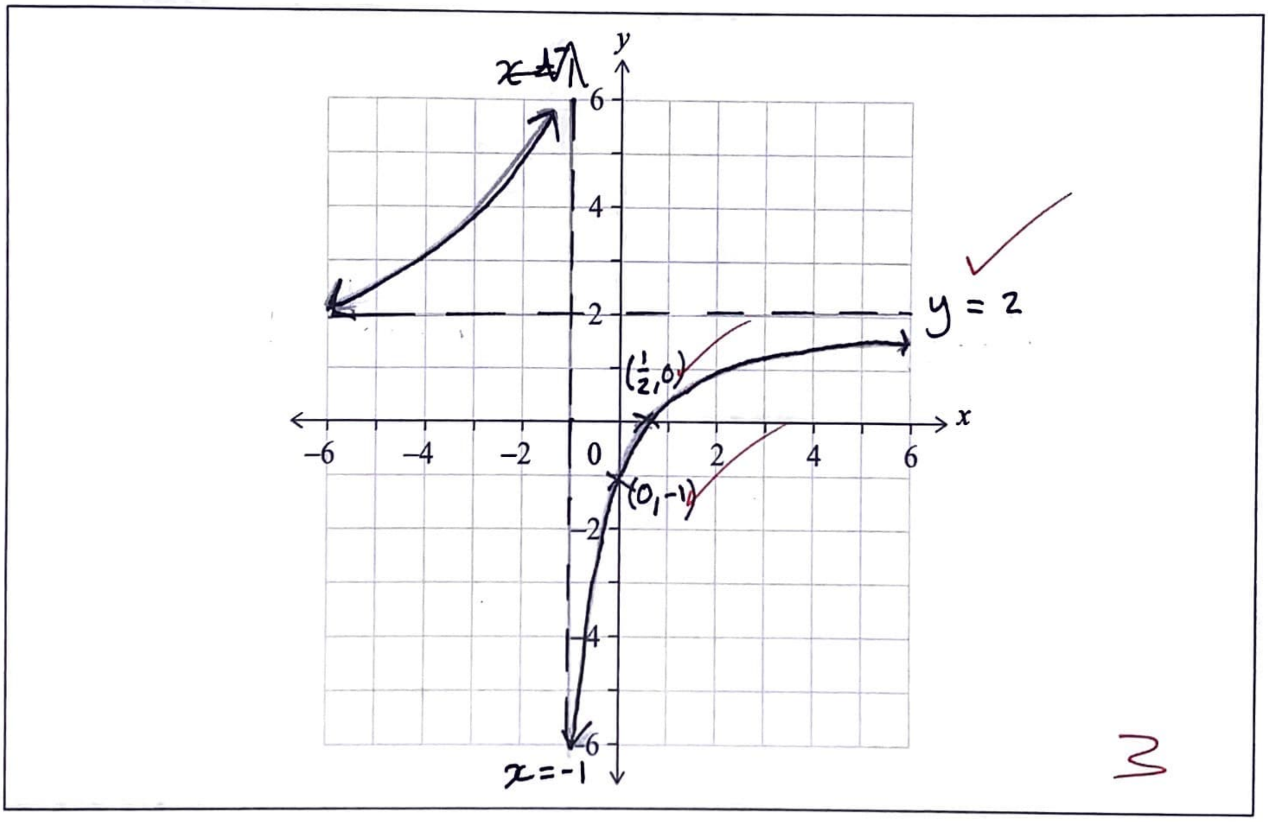
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]




(c) Hence, solve the inequality $0<\frac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\frac{2|x|-1}{|x|+1}<2$. [2]

**(This question continues on the following page)**

**(Question 3 continued)**

$f(x) = \frac{2x-1}{x+1}$  $x \neq -1$

(a)(i) Vertical asymptote : $\boxed{x = -1}$ ✓

(ii) Horizontal asymptote : $\boxed{y = 2}$ ✓  2

(b) Working : $x$-int $\Rightarrow$ $2x-1=0$

$\therefore x = 1/2$

$y$-int $\Rightarrow$ $y = -1/1 = -1$

(c) $0 < f(x) < 2$

$\Rightarrow$ ~~x > 1~~ $\boxed{x > \frac{1}{2}}$ ✓  1

(d) $0 < \frac{2|x|-1}{|x|+1} < 2$ $\Rightarrow$ mirror over $y$-axis

$\boxed{x > \frac{1}{2} \quad x < -5/2}$ ✗

$\cancel{\frac{2x-1}{x+1}}$

6

**4.** [Maximum mark: 5]

Find the least positive value of $x$ for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

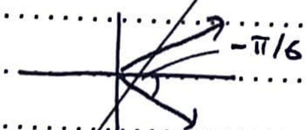
$\cos\left(\frac{x}{2}+\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$


$\therefore \frac{x}{2}+\frac{\pi}{3} = \frac{\pi}{4}$

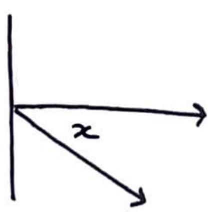
$\therefore 6x+4\pi = 3\pi$

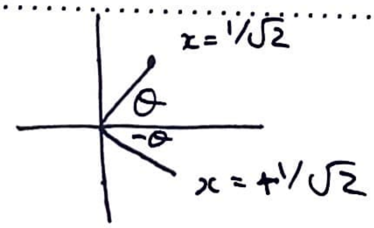
$\Rightarrow 6x = -\pi$

$\therefore x = -\pi/6$




~~AS cos θ = cos(θ+2~~




$\cos\left(\frac{x}{2}+\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$\therefore -\left(\frac{x}{2}+\frac{\pi}{3}\right) = \frac{\pi}{4}$

$\therefore -\frac{x}{2} = \frac{\pi}{4}+\frac{\pi}{3}$

$\therefore -6x = 3\pi+4\pi$

$\therefore x =$

$\frac{x}{2}+\frac{\pi}{3} = 2\pi - \pi/4$

$\therefore 6x+4\pi = 21\pi$

$\therefore x = 17\pi/6$

ANSWER BOOKLET. 5

**5.** [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of $x$. [5]

(a) $(x+1)^7 \Rightarrow$ finding the $x^5$ coefficient.

$\Rightarrow \therefore T_{r+1} = \binom{7}{r}(x^{7-r})(1^r) = \binom{7}{r}(x^{7-r})$

$\Rightarrow \therefore x^{7-r} = x^5$

$\therefore r = 2$

$\Rightarrow T_3 = \binom{7}{2}x^5$

$= \frac{7!}{5!\,2!}x^5$

$= 7\times6/2 \; x^5$

$= 21x^5 \longrightarrow \boxed{b = 21}$

2

(b) $T_3 = \frac{T_2 + T_4}{2}$

$\therefore T_2 + T_4 = 42x^5$

$\therefore \binom{7}{1}x^6 + \binom{7}{3}x^4 = 42x^5$

$\therefore 7!x^2 + \frac{7!}{4!3!}x^0 = \cancel{42x} \; \frac{7!}{5!2!}x \times 2$

$\therefore x^2 + \frac{1}{4!3!} = \frac{1}{5!}x$

$\therefore \cancel{144}x^2 + \cancel{144}\,1 = \frac{4\times3\times2\times1\times3!}{5\times4\times3\times2}x$

$\therefore 144x^2 + 1 = \frac{6}{5}x$

$4\times3\times2 = 24$
$\times 3\times2$
$= 48$
$3$
$144$

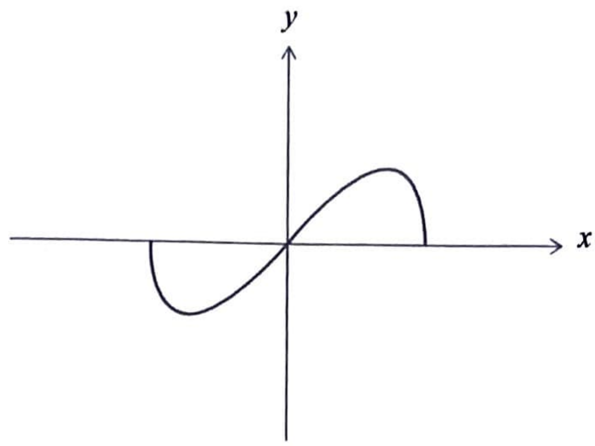
CONTINUED IN ANSWER BOOKLET.

7

**6.** [Maximum mark: 8]

A function $f$ is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.




(a) Show that $f$ is an odd function. [2]

The range of $f$ is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of $a$ and the value of $b$. [6]

(a) ~~$f(-x) = -f(x)$~~

$-f(x) = f(-x)$

$= (-x)\sqrt{1-(-x)^2}$

$= -x\sqrt{1-x^2}$ 2

$= -f(x)$

(b) $f(x) = x\sqrt{1-x^2}$

$\therefore f'(x) = x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) + \sqrt{1-x^2}$

$= \dfrac{-2x^2}{2\sqrt{1-x^2}} + \sqrt{1-x^2} = 0$

$\therefore -x^2 = -(1-x^2)$

$\therefore -x^2 = -1 + x^2$

$\therefore -2x^2 = -1$

$\therefore x^2 = 1/2$

$\therefore x = \pm\sqrt{1/2}$

When $x = \sqrt{1/2}$,

$f(\sqrt{1/2}) = \sqrt{1/2}\sqrt{1-1/2}$

$= 1/2$

When $x = -\sqrt{1/2}$,

$f(-\sqrt{1/2}) = -1/2$

$\boxed{\therefore -1/2 \le y \le 1/2}$

6 8

**7.** [Maximum mark: 6]

By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, dx$ in terms of $n$, where $n$ is a non-zero real number.

$u = \sec x$ ~~$\pi/3 = \sec x = \frac{1}{\cos x}$~~

$\therefore du = \sec x \tan x$ $\therefore$ ~~$\cos x = 3/\pi$~~

$\therefore \int_0^{\pi/3} \sec^n x \tan x \, dx = \int_0^{\pi/3} u^n \tan x \cdot \sec x \tan x \, du$

$= \int u^n \frac{\sin^2 x}{\cos^2 x} \times \frac{1}{\cos x} \, du$

$= \int u^n \sin^2 x \, u^{-3}$

$= \int u^{n+3} \sin^2 x \, du$

$u = \sec x$
$du = \sec x \tan x$

$\Rightarrow \int \sec^n x \tan x \, dx$

$= \int u^n \tan x \sec x \tan x \, du$

$= \int u^n \sec^3 x \sin^2 x \, du$

$= \int u^{n+3}$

COMPLETED IN ANSWER BOOKLET

1

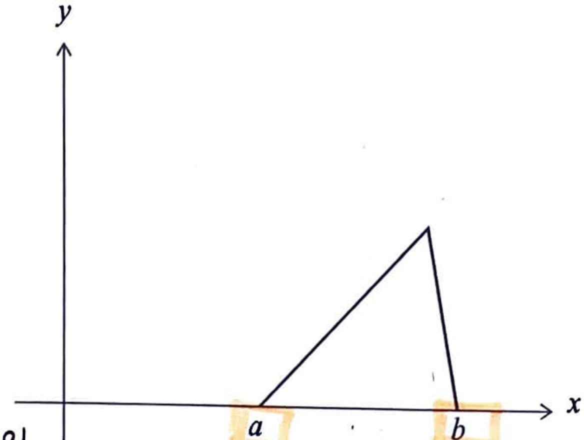
**8.** [Maximum mark: 6]

A continuous random variable $X$ has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\[2mm] \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\,.\\[2mm] 0, & \text{otherwise}\end{cases}$$

$a — c — b$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.




$c \geq$ avg of a&b

Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of $X$ in terms of $a$, $b$ and $c$.

$\int_a^M \frac{2}{(b-a)(c-a)}(x-a)dx = 1/2$

$\therefore \left(\frac{2}{(b-a)(c-a)}\right)\int_a^M (x-a)dx = 1/2$ ✓

$\therefore \left(\frac{2}{(b-a)(c-a)}\right)\left[\frac{1}{2}x^2-ax\right]_a^M = \frac{1}{2}$ ✓✓

$\therefore \frac{1}{2}M^2 - aM - \frac{1}{2}a^2 + a^2 = \frac{(b-a)(c-a)}{4}$

$\therefore 2M^2 - 4aM + 2a^2 = bc - ab - ac + a^2$

$\therefore 2M^2 - 4aM + a^2 + ab + ac - bc = 0$

$\therefore M = \frac{4a + \sqrt{16a^2 - (8a^2+ab+ac-bc)}}{4}$

$= a + \sqrt{8a^2+ab+ac-bc}$

4

28

**9.** [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

$2x^3 + 6x + 1 = 0$

$= (x-\alpha)(x-\beta)(x-\gamma)$

$= (x^2 - \alpha x - \beta x + \alpha\beta)(x-\gamma)$

$= x^3 - \alpha x^2 - \beta x^2 + \alpha\beta x - \gamma x^2 + \alpha\gamma x + \beta\gamma x - \alpha\beta\gamma$

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma$

(1) $\therefore -(\alpha+\beta+\gamma) = 0$

$\therefore \alpha+\beta+\gamma = 0$

(2) $\alpha\beta+\alpha\gamma+\beta\gamma = 6$

(3) $-\alpha\beta\gamma = 1$

$\therefore \alpha\beta\gamma = -1$

(1) $\to$ (2) $\beta = \gamma - \alpha$

$\therefore \alpha(\gamma-\alpha) + \alpha\gamma + (\gamma-\alpha)\gamma = 6$

$\therefore \cancel{\alpha\gamma} - \alpha^2 + \alpha\gamma + \gamma^2 - \cancel{\alpha\gamma} = 6$

$\gamma^2 + \alpha\gamma - \alpha^2 = 6$

(1) $\to$ (3) $\alpha(\gamma-\alpha)(\gamma) = -1$

$\therefore \alpha(\gamma^2 - \alpha\gamma) = -1$

$\therefore \alpha\gamma^2 - \alpha^2\gamma = -1$

$\Rightarrow$ as coefficient of $x^3$ in the expansion be $\neq 2$, the roots CANNOT integers.

0

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A biased four-sided die with faces labelled $1, 2, 3$ and $4$ is rolled and the result recorded. Let $X$ be the result obtained when the die is rolled. The probability distribution for $X$ is given in the following table where $p$ and $q$ are constants.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(X = x)$ | $p$ | 0.3 | $q$ | 0.1 |

For this probability distribution, it is known that $\mathrm{E}(X) = 2$.

(a) Show that $p = 0.4$ and $q = 0.2$. [5]

(b) Find $\mathrm{P}(X > 2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four. → binomial

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair. Both yellow dice have faces labelled $1, 2, 3$ and $4$. Let $S$ represent the sum obtained by rolling the two yellow dice. The probability distribution for $S$ is shown below.

Y } 1,2,3,4
Y
R
R

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\mathrm{P}(S = s)$ | $\frac{1}{16}$ | $\frac{2}{16}$ | $\frac{3}{16}$ | $\frac{4}{16}$ | $\frac{3}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ |

The first red die has faces labelled $1, 2, 2$ and $3$. The second red die has faces labelled $1, a, a$ and $b$, where $a < b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of $b$. [2]

(e) Find the value of $a$, providing evidence for your answer. [2]

Do **not** write solutions on this page.

**11.** [Maximum mark: 20]

A function $f$ is defined by $f(x)=\frac{1}{x^2-2x-3}$, where $x \in \mathbb{R}, x \neq -1, x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function $g$ is defined by $g(x)=\frac{1}{x^2-2x-3}$, where $x \in \mathbb{R}, x > 3$.

(b) The inverse of $g$ is $g^{-1}$.

(i) Show that $g^{-1}(x)=1+\frac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of $g^{-1}$. [7]

A function $h$ is defined by $h(x)=\arctan\frac{x}{2}$, where $x \in \mathbb{R}$.

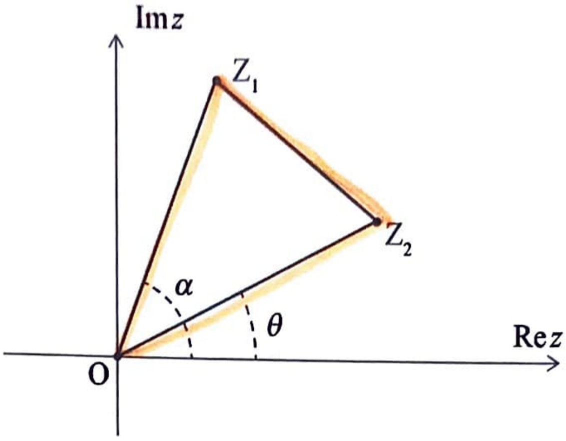
(c) Given that $(h \circ g)(a)=\frac{\pi}{4}$, find the value of $a$.

Give your answer in the form $p+\frac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

In the following Argand diagram, the points $Z_1$, O and $Z_2$ are the vertices of triangle $Z_1OZ_2$ described anticlockwise.




The point $Z_1$ represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point $Z_2$ represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles $\alpha$, $\theta$ are measured anticlockwise from the positive direction of the real axis such that $0 \le \alpha, \theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2{}^* = r_1 r_2 e^{i(\alpha - \theta)}$ where $z_2{}^*$ is the complex conjugate of $z_2$. [2]

(b) Given that $\mathrm{Re}(z_1\, z_2{}^*) = 0$, show that $Z_1OZ_2$ is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where $Z_1OZ_2$ is an $\boxed{\text{equilateral}}$ triangle.

(c) (i) Express $z_1$ in terms of $z_2$. $r_1 = r_2 = [z_1 z_2]$

(ii) Hence show that $z_1{}^2 + z_2{}^2 = z_1 z_2$. [6]

Let $z_1$ and $z_2$ be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle $Z_1OZ_2$ can be formed from the point O and the roots of this equation. [3]

References:

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4 PAGES / PÁGINAS

Candidate session number: / Numéro de session du candidat: / Número de convocatoria del alumno: 22 M TZ 2 P 1 - M A H L

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At the start of each answer to a question, write the question number in the box using your normal hand writing / Avant de répondre à une question, inscrivez son numéro à la main dans la case appropriée / Al comienzo de cada respuesta, escriba a mano el número de pregunta en la casilla.

Example Ejemplo 27 | 2 7 | Example Ejemplo 3 | 3

1 0

(a) $E(x) = 2 = p + (0.3)(2) + (q)(3) + (0.1)(4)$

$\therefore 2 = \cancel{p0}\ p + 0.6 + 3q + 0.4$

$\therefore 2 = p + 3q + 1$

$\therefore p + 3q = 1 \quad \ldots(1)$

$p + 0.3 + q + 0.1 = 1$

$\therefore p + q = 0.6 \quad \ldots(2)$

$(1) \rightarrow (2)$

$(1 - 3q) + q \cancel{2} = 0.6$

$\therefore 1 - 2q = 0.6$

$\therefore 2q = 0.4$

$\therefore \boxed{q = 0.2} \quad (= 1/5) \quad \ldots(3)$

$(3) \rightarrow (2)$

$\cancel{p0}\ p + 0.2 = 0.6$

$\therefore \boxed{p = 0.4}$

5

(b) $P(X \geq 2) = \cancel{0.3 + 0.2 + 0.1}$

$\boxed{\cancel{= 0.6}}$

$0.2 + 0.1$

$= 0.3$

2



04AX01

(c) maximum score = 20

after 3 rolls → 4

maximum score in 2 more rolls is 8

(required 6)

∴ Possibilities are :

| | |
|---|---|
| 4 | 4 |
| 4 | 3 |
| 4 | 2 |
| 2 | 4 |
| 3 | 4 |

3&3

Adding up: $P(x=4)P(x=4) + (P(x=4)P(x=3))^{\cancel{2}} \times 2$

$+ (P(X=4)P(x=2))^{\cancel{2}} \times 2$ $+P(x=3)^2$

$= (0.1)(0.1) + ((0.1)(0.2))^{\cancel{2}} \times 2$

$+ ((0.1)(0.3))^{\cancel{2}} \times 2$

$= 0.01 + (1 \times 2 \times 10^{-1} \times 10^{-1})^{\cancel{2}} \times 2$

$+ (1 \times 3 \times 10^{-1} \times 10^{-1})^{\cancel{2}} \times 2$

$= 0.01 + 2\times(0.02)^{\cancel{2}} + 2\times(0.03)^{\cancel{2}}$ FT

$= 0.01 + 0.04 + 0.06$ $+0.04$

$= 0.11$

4

(d) Yellow: 1, 2, 3, 4
Red : 1, a, a, b
1 2 2 3

Possibilities: DIE R1 DIE R2

| | |
|---|---|
| 1 | 1 |
| a | 1 |
| a | 1 |
| b | 1 |
| 1 | 2 |
| a | 2 |
| a | 2 |
| b | 2 |
| a | 2 |
| a | 2 |
| b | 2 |
| 1 | 3 |
| a | 3 |
| a | 3 |
| b | 3 |

=> Possible outcomes
$1 \times (1 \quad 1) \Rightarrow 2$
$2 \times (a \quad 1) \Rightarrow 1 + a$
$1 \times (b \quad 1) \Rightarrow 1 + b$
$2 \times (1 \quad 2) \Rightarrow 3$
$4 \times (a \quad 2) \Rightarrow 2 + a$
$2 \times (b \quad 2) \Rightarrow 2 + b$
$1 \times (1 \quad 3) \Rightarrow 4$
$2 \times (a \quad 3) \Rightarrow a + 3$
2 $1 \times (b \quad 3) \Rightarrow b + 3$

largest possible value is 8, meaning
$b + 3 = 8$
$\therefore \quad b = 5$

(e) When $a=3$, every possibility from the $P(S=s)$ distribution is accounted for (assuming $b=5$)

| | | | | | |
|---|---|---|---|---|---|
| ~~a~~ | 1 | 1 | $\Rightarrow$ | 2 | (1) |
| | a | 1 | $\Rightarrow$ | 4 | (3) |
| | b | 1 | $\Rightarrow$ | 6 | (5) |
| | 1 ~~3~~ | ~~4~~ 2 | $\Rightarrow$ | 3 | (2) |
| | a | ~~2~~ 2 | $\Rightarrow$ | 5 | (4) |
| | b | 2 | $\Rightarrow$ | 7 | (6) |
| | 1 | 3 | $\Rightarrow$ | 4 | |
| | a | 3 | $\Rightarrow$ | 6 | |
| | b | 3 | $\Rightarrow$ | 8 | (7) |

2

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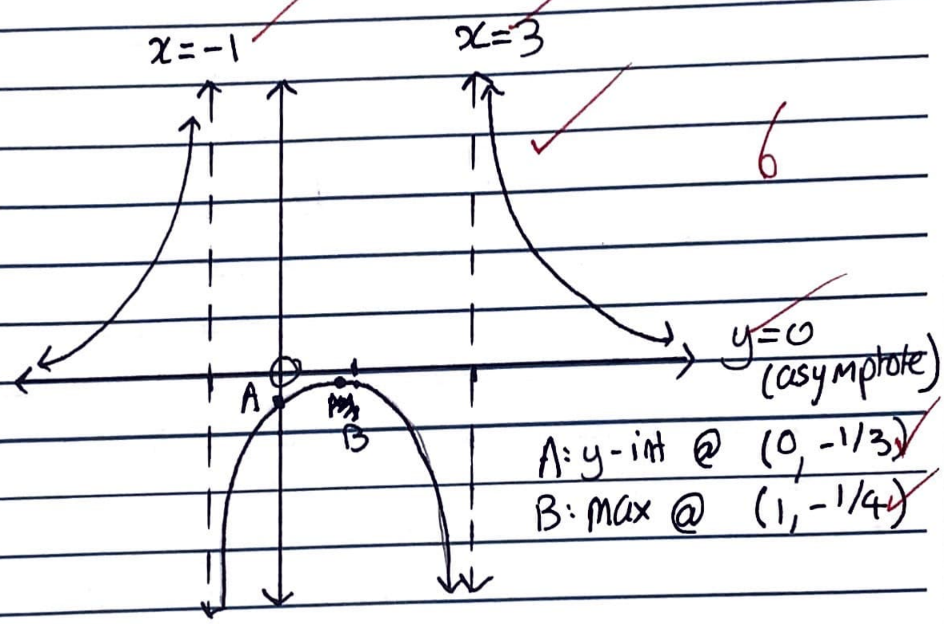
Example / Ejemplo 27 [27]   Example / Ejemplo 3 [3]

11

(a)

$f(x) = \frac{1}{x^2-2x-3}$   $x \in \mathbb{R}$   $x \neq -1,\ x \neq 3$




6

A: y-int @ $(0, -1/3)$
B: max @ $(1, -1/4)$

y-int: $y = -1/3$
horizontal asymptote: $y = 0$
x-int: does not exist.

MAXIMUM will occur @ $x = \frac{3-(+1)}{2} = 1$

$f(1) = 1/1-2-3$
$= -1/4$



04AX01

(b) (i)

$$g(x) = \frac{1}{x^2-2x-3}, \quad x > 3$$

$$x = \frac{1}{y^2-2y-3}$$

$\therefore (y^2-2y-3)x = 1$

$\therefore y^2x-2xy-3x-1 = 0$

$$\therefore y = \frac{2x \pm \sqrt{4x^2-4(x)(-3x-1)}}{2x}$$

$$= 1 \pm \frac{\sqrt{4x^2+12x^2+4x}}{2x}$$

$$= 1 \pm \frac{2\sqrt{x^2+3x^2+x}}{2x}$$

$$g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$$

$\left\{ \begin{array}{l} \cancel{\text{Only}}\ g^{-1}(x) > 3 \\ \text{thus only positive sqrt was kept} \end{array} \right\}$

(ii) $\cancel{4x^2+x \geq 0}$

$\cancel{\therefore x(4x+1) \geq 0}$

Using graph from part (a) for $x \geq 3$ to determine domain of $g^{-1}(x)$ from range of $f(x)$.

Domain: $x > 0$

6

(c) $h(x) = \arctan x/2$ $\quad x \in \mathbb{R}$

$$h(g(a)) = \pi/4$$

$$\therefore h\left(\frac{1}{a^2-2a-3}\right) = \pi/4$$

$$\therefore \arctan\left(\frac{1}{2a^2-4a-6}\right) = \pi/4$$

$$\therefore \frac{1}{2a^2-4a-6} = 1$$

$\Rightarrow$ or $a = g^{-1}(2)$

$$\therefore 2a^2-4a-6 = 1$$

$$\therefore \cancel{a^2-2a-3=}$$

$$\therefore 2a^2-4a-7 = 0$$

$$\therefore a = \frac{4 \pm \sqrt{16-4(2)(-7)}}{4}$$

$$= \frac{4 \pm \sqrt{16+56}}{4}$$

$$= \frac{4 \pm \sqrt{72}}{4}$$

$$= \frac{4 \pm 2\sqrt{18}}{4}$$

$$\therefore a = \frac{2 \pm \sqrt{18}}{2}$$

$$\therefore a = \frac{2+\sqrt{18}}{2} \quad \{a>3\}$$

$$= 1 + \frac{1}{2}\sqrt{13}$$

5

5

17

12

(a) $z_1 = r_1 e^{i\alpha}$

$z_2 = r_2 e^{i\theta}$ $\quad z_2^* = r_2 e^{-i\theta}$ ✓

$\therefore z_1 z_2^* = r_1 r_2 e^{i(\alpha-\theta)}$ ✓ 2

(b) When $z_1 O z_2$ is right angled,

$\alpha - \theta = \pi/2$ ✓

$Re(z_1 z_2^*) = r_1 r_2 \cos(\alpha-\theta) = 0$ ✓

$\therefore \cos(\alpha - \theta) = 0$

2 $\quad \therefore \alpha - \theta = \pi/2$ ✓

(c)(i) $z_1 = r_1 e^{i\alpha}$ $\quad z_2 = r_2 e^{i\theta}$

$\Rightarrow$ if equilateral : $\alpha - \theta = \pi/3$ ✓

$\therefore \theta = \alpha - \pi/3$

$\therefore \alpha = \theta + \pi/3$

$\therefore z_1 = r_1 e^{i(\theta+\pi/3)}$ ✓

$= r_1 e^{i\theta} \times e^{\pi/3 i}$

$= z_2 \times e^{\pi/3 i}$ ✓ $\{as\ r_1 = r_2\}$

3

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Example Ejemplo 27 | 2 | 7 |

Example Ejemplo 3 | | 3 |

(c)(ii)

$$z_1^2 + z_2^2 = (z_1+z_2)^2 - 2z_1z_2$$

$$= \left(z_2 + re^{\pi/3 i} + z_2\right)^2 - 2z_1z_2$$

$$= \left(2z_2 + re^{\pi/3 i}\right)^2 - 2z_1z_2$$

$$= \left(2re^{i\theta} + re^{\pi/3 i}\right)^2 - 2z_1z_2$$

$$= 4r^2e^{2i\theta} + 4r^2e^{i(\theta+\frac{\pi}{3})} + r^2e^{\frac{2\pi}{3}i}$$

$$-2re^{2i\theta}e^{i\alpha}$$

$\cancel{z_1 z_2}$

04AX01

12

(c)(ii) $z_1 z_2 = (z_2 e^{\frac{\pi}{3}i})(z_2)$

$= (z_2)^2 + (z_2)(re^{\frac{\pi}{3}i})$

$= z_2^2(e^{0+\pi/3 i})$

$= \cancel{(z_2)^2 + (z_1 - re^{\frac{\pi}{3}i})(re^{\frac{\pi}{3}i})}$

$= z_2^2 + z_2^2 e^{\frac{\pi}{3}i}$

$= \cancel{(z_2)^2 + z_1 re^{\frac{\pi}{3}i} - r^2 e^{(\frac{\pi}{3}+\frac{\pi}{3})i}}$

$= z_2^2 + z_1^2$

$= (z_2)^2 + (re^{i\theta})(re^{\frac{\pi}{3}i})$

$= (z_2)^2 + r^2 e^{i(\theta + \frac{\pi}{3})i}$

$= (z_2)^2 + r^2 e^{\alpha i}$

$= (z_2)^2 + (z_1)^2$

$= z_1^2 + z_2^2$ 1

(d) $z^2 + az + b$

$(z - z_1)(z - z_2) = z^2 + az + b$

$= z^2 - z_1 z - z_2 z + z_1 z_2$

$= z^2 - (z_1 + z_2)z + z_1 z_2$

$\therefore b = z_1 z_2 = z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2$

$a = -z_1 - z_2 = -(z_1 + z_2)$

sub a into $b = (z_1 + z_2)^2 - 2z_1 z_2$

$\therefore b = (-a)^2 - 2z_1 z_2$

$\therefore b = a^2 - 2b$

$\therefore \cancel{0 = a^2}$

$\therefore 0 = a^2 - 3b$ 5

$\therefore a^2 - 3b = 0$

(e) $z^2 + az + 12 = 0 \qquad z \in \mathbb{C} \quad a \in \mathbb{R}$

$$\therefore z_1 = \frac{-a - \sqrt{a^2 - 48}}{2} \qquad (1) \qquad 1$$

$$z_2 = \frac{-a + \sqrt{a^2 - 48}}{2} \qquad (2)$$

$\rightarrow \quad a = \pm\sqrt{36} = \pm 6$

~~Sub 6 into (1) $\rightarrow z_1 = +3 - \sqrt{36-48}$~~

for $a = -6$:

$$z_1 = \frac{6 - \sqrt{36-48}}{2}$$

$$= 3 - \sqrt{3}\,i$$

$$z_2 = 3 + \sqrt{3}\,i$$

$\Big\}\ \alpha - \theta = \frac{5\pi}{3}$

for $a = 6$: $z_1 = -3 - \sqrt{3}i$

$z_2 = -3 + \sqrt{3}i \ \Big\}\ \alpha - \theta = \pi/3$

14

5 (b)

$$T_3 = \frac{T_2 + T_4}{2}$$

$$\therefore 2 \times \frac{7!}{5!2!} x^5 = \binom{7}{1} x^6 + \binom{7}{3} x^4$$

$$\therefore \frac{7!}{5!} x^5 = 7x^6 + \frac{7!}{3!4!} x^4$$

$$\therefore (7 \times 6) x^5 = 7x^6 + \frac{7 \times 6 \times 5}{3 \times 2} x^4$$

$$\therefore 42x^5 = 7x^6 + 35x^4$$

$$\therefore 6x = x^2 + 5$$

$$\therefore x^2 - 6x + 5 = 0$$

$$\therefore (x-5)(x-1) = 0$$

5

$$\therefore x = 5, \quad x = 1$$

7

$$\int_0^{\pi/3} \sec^n x \tan x = \int u^n \tan^2 x \sec x \; du$$

1

$$= \int u^n (\sec^2 x - 1) \sec x \; du$$

2

$$= \int u^n (u^2 - 1) u \; du$$

$$= \int u^{n+1} (u^2 - 1) \; du$$

$$= \int (u^{n+2} - u^{n+1}) u \; du$$

$$= \int (u^{n+3} - u^{n+2}) du$$

$$= \left[ \frac{1}{n+4} u^{n+4} - \frac{1}{n+3} u^{n+3} \right]_0^{\pi/3}$$

$$= \left[ \frac{1}{n+4} (\sec x)^{n+4} - \frac{1}{n+3} (\sec x)^{n+3} \right]_0^{\pi/3}$$

2

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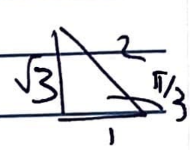
Example / Ejemplo 27 | 2 | 7 |     Example / Ejemplo 3 | | 3 |

**7**

$$= \frac{1}{n+4}\left(\sec\frac{\pi}{3}\right)^{n+4} - \frac{1}{n+3}\left(\sec\frac{\pi}{3}\right)^{n+3} - \frac{1}{n+4} + \frac{1}{n+3}$$

$$= \frac{1}{n+4}(2)^{n+4} - \frac{1}{n+3}(2)^{n+3} - \frac{1}{n+4} + \frac{1}{n+3}$$

$$= \frac{1}{n+4}\left(2^{n+4} - 1\right) + \frac{1}{n+3}\left(1 - 2^{n+3}\right)$$




**4**

$$\cos\left(\frac{x}{2} + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{x}{2} + \frac{\pi}{3} = \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$\therefore x/2 + \pi/3 = \pi/4$$

$$\therefore 6x + 4\pi = 3\pi$$

$$\therefore x = -1/6\ \pi$$

$$x/2 + \pi/3 = 7\pi/4$$

$$\therefore 6x + 4\pi = 21\pi$$

$$\therefore \boxed{x = 17/6\ \pi}$$

7

$$\int_0^{\pi/3} \sec^n x \tan x = \int_0^{\pi/3} \sec^{n-1} x \sec x \tan x \, dx$$

$u = \sec x$
$du = \sec x \tan x \, dx$
$\sec \pi/3 = 2$
$\sec 0 = 1$

$$= \int_0^{\pi/3} \sec^{n-1} x \times du$$

$$= \int_0^{\pi/3} u^{n-1} du$$

$$= \left[\frac{1}{n} u^n\right]_1^2$$

$$= \frac{1}{n}\left[\sec^n x\right]_0^{\pi/3}$$

$$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$$

$$= \frac{1}{n}\left(2^n - 1\right)$$