Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3014

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

1. [Maximum points: 24]

In this problem you will investigate the relationship between the irrationals π *and* $\sqrt{2}$.

(a) Sketch a suitable diagram to show that $\sin(\pi/4) = \cos(\pi/4) = \dfrac{\sqrt{2}}{2}$. [3]

(b) Show that [4]

(i) $2\sin(x/2)\cos(x/2) = \sin x$

(ii) $2\cos^2(x/2) - 1 = \cos x$

(c) Find the **exact** value of $\cos(\pi/8)$. [2]

(d) Hence show that [3]

$$\cos(\pi/16) = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

(e) Prove by induction that [7]

$$\sin x = 2^n \sin\left(\frac{x}{2^n}\right) \cdot \cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2^2}\right) \cdot \cos\left(\frac{x}{2^3}\right) \cdot \cdots \cdot \cos\left(\frac{x}{2^n}\right)$$

where $n \in \mathbb{Z}^+$.

(f) Use l'Hopital's rule to find $\lim_{n \to \infty} 2^n \sin\left(\dfrac{c}{2^n}\right)$ where $c \in \mathbb{R}$. [3]

(g) By considering the expression in part (e) as $n \to \infty$ show that [2]

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \cdots$$

2. [Maximum points: 31]

In this problem you will investigate the Maclaurin series of inverse functions for which an explicit inverse function does not exist.

Let $f(x) = ax + \sin x$ for $a \in \mathbb{R}$.

(a) On separate sets of axes sketch the graphs of $y = f(x)$ when $a = 1/2$ and $a = 2$ for $-4\pi \leq x \leq 4\pi$. [4]

(b) Explain which of the functions from part (a) has an inverse. [2]

(c) Show that for an inverse to exist then we must have $a \leq -1$ or $a \geq 1$. [4]

Let $a = 1$ and $y = f^{-1}(x)$.

(d) Show that $\dfrac{dy}{dx} = \dfrac{1}{1 + \cos y}$. [5]

(e) Find $\dfrac{d^2y}{dx^2}$ in terms of y. [3]

(f) Show that $\dfrac{d^3y}{dx^3} = \dfrac{3 - 2\cos y}{(1 + \cos y)^4}$. [8]

(g) Hence find the first two non-zero terms of the Maclaurin series of $f^{-1}(x)$. [5]