Mathematics: analysis and approaches

Higher level

Paper 3

ID: 3004

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

1. [Maximum points: 50]

In this problem you will determine the integral of a seemingly simple expression using two different approaches.

(a) Write $\frac{1}{x^2-1}$ using partial fractions. [4]

(b) Find an alternative expression for $\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}$ that does not contain a fraction. [2]

(c) Use integration by parts with $u=x$ and $\frac{dv}{dx}=\frac{x}{(x^2-1)^2}$ to show that [5]

$$\int \frac{x^2}{(x^2-1)^2}\,dx = -\frac{x}{2(x^2-1)}+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+C$$

where $C \in \mathbb{R}$.

(d) If $t^2=\frac{1+x^2}{x^2}$ find x^2 in terms of t^2. [2]

(e) Use the substitution $t^2=\frac{1+x^2}{x^2}$ to show that [7]

$$\int \sqrt{1+x^2}\,dx = \frac{t}{2(t^2-1)}+\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right|+D$$

where $D \in \mathbb{R}$.

(f) Hence determine $\int \sqrt{1+x^2}\,dx$. [4]

The *hyperbolic* functions $\sinh x$ and $\cosh x$ are defined as

$$\sinh x = \frac{e^x-e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x+e^{-x}}{2}$$

The inverse of these functions are $\sinh^{-1}x$ and $\cosh^{-1}x$.

(g) Show that $\sinh^{-1}x = \ln\left(x+\sqrt{1+x^2}\right)$. [5]

(h) Simplify $\cosh^2 x - \sinh^2 x$. [2]

Let $f(x) = \sinh x$ and $g(x) = \cosh x$.

(i) Show that [6]

(i) $f'(x) = g(x)$

(ii) $g'(x) = f(x)$

(iii) $\cosh 2x = 2\cosh^2 x - 1$

(j) Determine an expression for $\sinh 2x$ in terms of $\sinh x$ and $\cosh x$. [2]

(k) Show that $\sinh(2\sinh^{-1} x) = 2x\sqrt{1+x^2}$. [3]

(l) Hence use the substitution $x = \sinh u$ to determine $\int \sqrt{1+x^2}\, dx$. [8]

1. (a) We have

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1} \qquad \text{M1}$$

So

$$1=A(x-1)+B(x+1) \qquad \text{A1}$$

Giving

$$A+B=0 \quad \text{and} \quad B-A=1 \qquad \text{M1}$$

Therefore

$$A=-\frac{1}{2} \quad \text{and} \quad B=\frac{1}{2} \qquad \text{A1}$$

The expression is therefore

$$\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

(b) $$\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}=\frac{\left(\sqrt{1+x^2}+x\right)^2}{1+x^2-x^2}=\left(\sqrt{1+x^2}+x\right)^2 \qquad \text{M1A1}$$

(c) We have

$$\frac{du}{dx}=1 \quad \text{and} \quad v=-\frac{1}{2(x^2-1)} \qquad \text{A1A1}$$

So the integral becomes

$$-\frac{x}{2(x^2-1)}+\frac{1}{2}\int\frac{1}{x^2-1}\,dx=-\frac{x}{2(x^2-1)}+\frac{1}{4}\int\frac{1}{x-1}dx-\frac{1}{4}\int\frac{1}{x+1}\,dx \qquad \text{M1A1}$$

This is equal to

$$-\frac{x}{2(x^2-1)}+\frac{\ln|x-1|}{4}-\frac{\ln|x+1|}{4}+C=-\frac{x}{2(x^2-1)}+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| \qquad \text{A1}$$

(d) We have

$$x^2(t^2-1)=1 \qquad \text{M1}$$

So

$$x^2=\frac{1}{t^2-1} \qquad \text{A1}$$

(e) Use implicit differentiation M1

$$2t\frac{dt}{dx} = \frac{2x^3 - 2x(1+x^2)}{x^4} = -\frac{2}{x^3}$$ A1A1

So the integral becomes

$$-\int x^3 t\sqrt{1+x^2}\,dt = -\int \frac{x^4 t\sqrt{1+x^2}}{x}dt = -\int \frac{t^2}{(t^2-1)^2}\,dt$$ M1A1A1

By part (b) this is equal to

$$\frac{t}{2(t^2-1)} - \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| = \frac{t}{2(t^2-1)} + \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| + D$$ A1

(f) The integral is equal to

$$\frac{\sqrt{\frac{1+x^2}{x^2}}}{\frac{2(1+x^2)}{x^2}-2} + \frac{1}{4}\ln\left|\frac{\sqrt{\frac{1+x^2}{x^2}}+1}{\sqrt{\frac{1+x^2}{x^2}}-1}\right| + E$$ M1

This is equal to

$$\frac{x\sqrt{1+x^2}}{2} + \frac{1}{4}\ln\left|\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right| = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{4}\ln\left|\left(\sqrt{1+x^2}+x\right)^2\right| + E$$ A1A1

Which simplifies to

$$\frac{x\sqrt{1+x^2}}{2} + \frac{\ln\left(\sqrt{1+x^2}+x\right)}{2} + E$$ A1

(g) We have

$$x = \frac{e^{y} - e^{-y}}{2} \qquad \text{M1}$$

So

$$e^{2y} - 2x\,e^{y} - 1 = 0 \qquad \text{A1}$$

Therefore

$$e^{y} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} \qquad \text{M1}$$

The right side cannot be negative so R1

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \text{A1}$$

(h) $$\frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1 \qquad \text{M1A1}$$

(i)

(i) $$f'(x) = \frac{e^{x} - (-e^{-x})}{2} = \frac{e^{2} + e^{-x}}{2} = g(x) \qquad \text{M1A1}$$

(ii) $$g'(x) = \frac{e^{x} + (-1)e^{-x}}{2} = \frac{e^{x} - e^{-x}}{2} = f(x) \qquad \text{M1A1}$$

(iii) $$2\cosh^2 x - 1 = \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \qquad \text{M1A1}$$

(j) Differentiating the result from (i) part (iii) we have

$$4\sinh x \cosh x = 2\sinh 2x \qquad \text{M1}$$

Giving

$$\sinh 2x = 2\sinh x \cosh x \qquad \text{A1}$$

(k) We have

$$\sinh 2x = 2\sinh x\sqrt{1 + \sinh^2 x} \qquad \text{A1}$$

Replace x with $\sinh^{-1} x$ M1

$$\sinh(2\sinh^{-1} x) = 2\sinh(\sinh^{-1} x)\sqrt{1 + \left(\sinh(\sinh^{-1} x)\right)^2} = 2x\sqrt{1 + x^2} \qquad \text{A1}$$

(l) We have

$$\frac{dx}{du} = \cosh u \qquad \text{A1}$$

So the integral becomes

$$\int \sqrt{1+\sinh^2 u}\cosh u\, du = \int \cosh^2 u\, du = \frac{1}{2}\int 1 + \cosh 2u\, du \qquad \text{M1A1A1}$$

This is equal to

$$\frac{u}{2} + \frac{\sinh 2u}{4} + C = \frac{\sinh^{-1} x}{2} + \frac{\sinh(2\sinh^{-1} x)}{4} + C = \frac{\sinh^{-1} x}{2} + \frac{x\sqrt{1+x^2}}{2} + C \qquad \text{M1A1A1}$$

By part (g) this gives

$$\frac{x\sqrt{1+x^2}}{2} + \frac{\ln\left(x+\sqrt{1+x^2}\right)}{2} + C \qquad \text{A1}$$