Mathematics: analysis and approaches

Higher level

Paper 3

ID: 0005

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[57 marks]**.

Attempt 1: $\frac{35}{57} = 61\%$.

1. [Maximum points: 28]

In this problem you will investigate a Bernoulli differential equation.

Let $x\frac{dy}{dx} = y(1 - xy)$ where $y(0.2) = 0.4$.

(a) Use Euler's method with a step length of 0.1 to estimate the value of $y(0.5)$. [6]

(b) Show that $\frac{d^2y}{dx^2} = \frac{y^2(2xy - 3)}{x}$. [5]

(c) When $0 < x < 1$ and $0 < y < 1$ determine whether $\frac{d^2y}{dx^2}$ is positive or negative. [2]

(d) Hence determine whether your answer in part (a) is an overestimate or an underestimate. [2]

A *Bernoulli differential equation* is an equation of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

where $n \in \mathbb{R}$. A Bernoulli differential equation can be transformed into a linear differential equation using the substitution $u = y^{1-n}$.

(e) Show that the original differential equation used in part (a) is a Bernoulli differential equation. [2]

(f) Use the substitution $u = \frac{1}{y}$ to show that $\frac{du}{dx} + \frac{u}{x} = 1$. [3]

(g) Hence find the particular solution to the original equation. [6]

(h) Find that actual value of $y(0.5)$. [2]

2. [Maximum points: 29]

In this problem you will investigate properties of polynomials with coefficients which form a geometric sequence.

(a) Use compound angle identities to prove for $z, w \in \mathbb{C}$ then $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$. [5]

Let $f(x) = x^3 + 2x^2 + 4x + 8$.

(b) Given that $f(-2) = 0$ factorise $f(x)$. [2]

(c) Hence find all roots of $f(x)$. [2]

(d) Show that the roots of $f(x)$ form a geometric series with a complex common ratio and find the value of this ratio. [2]

(e) Follow similar steps to show that the roots of $27x^3 - 9x^2 + 3x - 1$ also form a geometric sequence with a complex common ratio. [6]

Consider the polynomial

$$g(x) = 1 + rx + r^2x^2 + \cdots + r^n x^n = \sum_{k=0}^{n} r^k x^k$$

where $r \in \mathbb{R}$ and $r \neq 0$.

(f) Find $(rx - 1)g(x)$. [2]

(g) By solving the equation $(rx - 1)g(x) = 0$ find all roots of $g(x)$. [3]

(h) Hence show that the roots of $g(x)$ form a geometric sequence with a complex common ratio and find the value of this ratio. [4]

(i) In the case when $n = 7$ and $r > 0$ illustrate the roots of $g(x)$ on an Argand diagram. [3]

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① 1.09.22

$\frac{35}{57} = 61\%$

Q1: $\frac{20}{28}$ Q2: $\frac{15}{29}$

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1 2 3 4 5 6 7 8 9 10

1

a) $x\frac{dy}{dx} = y(1-xy)$ $y(0.2) = 0.4$

$\therefore \frac{dy}{dx} = \frac{y(1-xy)}{x}$

⑥

n	x_n	y_n	
1	0.2	0.4	
2	0.3	0.584	
3	0.4	0.745	
4	0.5	0.876	

$\therefore y(0.5) \approx 0.876$

b) $x\frac{dy}{dx} = y(1-xy)$

$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = (1-xy)\frac{dy}{dx} + y\left[-\left(x\frac{dy}{dx} + y\right)\right]$

$= (1-xy)\frac{dy}{dx} - xy\frac{dy}{dx} - y^2$

$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{dx}(1-xy) - y^2 - xy\frac{dy}{dx}$

6

b)

$$x\frac{dy}{dx} = y(1-xy)$$

$$\therefore x\frac{dy}{dx} = y - xy^2$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{dx} - \left[x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right]$$

$$\therefore x\,d^2y/dx^2 = -2yx\,dy/dx - y^2$$

$$\therefore x\,d^2y/dx^2 = -2yx\left(\frac{y(1-xy)}{x}\right) - y^2$$

$$\therefore x\,d^2y/dx^2 = -2y^2(1-xy) - y^2$$

$$\therefore \quad = y^2(-2(1-xy)-1)$$

$$= y^2(-2+2xy-1)$$

$$= y^2(2xy-3)$$

$$\therefore d^2y/dx^2 = y^2(2xy-3)/x$$

5

c) take $x = 1/2$ $\quad y = 1/2$:

1

$$\frac{d^2y}{dx^2} = \left(\tfrac{1}{4}\right)\left(2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)-3\right)/(1/2)$$

$$= \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}-3\right)/(1/2)$$

$$= \left(\tfrac{1}{4}\right)(-5/2)/1/2$$

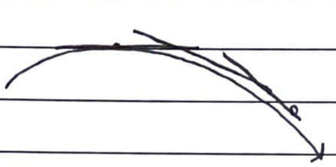
$$= -5/8 \times 2/1$$

$$= -10/8 < 0$$

need to prove for all

d) If $d^2y/dx^2 < 0$, y is concave down

2



$\therefore$ overestimate

8

e)

f) $u = 1/y$ $\quad y^{-1}$ $\quad \cancel{\to du/dx = -y^{-2}\, dy/dx}$

$\cancel{= -1/y^2\, dy/dx}$

$u = 1/y \Rightarrow y = 1/u = (u^{-1})$

∴ $\quad$ ∴ $dy/dx = -u^{-2}\, du/dx$

$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x)y^n$

∴ $dy/dx + P(x)y^{1-n+n} = Q(x)\, y^{1-n-1+2n}$ $\quad \cancel{1-n-(1-n)}$

∴ $dy/dx + P(x)y^n y^{1-n} = Qx\, y^{1-n}/y^{1-2n}$

∴ $-\frac{1}{u^2}\frac{du}{dx} y^n u$

$\frac{dy}{dx} + P(x)y = Q(x)\, y^n$

∴ $\frac{1}{y}\, dy/dx + P(x) = Q(x)y^{n-1}$

∴ $u\left(-\frac{1}{u^2}\, du/dx\right) + P(x) = Q(x)(u)(y^n)$

$$\cancel{\frac{dy}{dx} = Q(x)y^n - P(x)y}$$

$$\cancel{= y}$$

$$\cancel{u = y^{1-n}}$$

e) $x\, dy/dx = y(1 - xy)$

Right idea

$\therefore dy/dx = yx - y^2x$ ✗

$\therefore dy/dx - xy = -xy^2$, $P(x) = -x$, $Q(x) = -x$ ✓

①

f) $u = 1/y = y^{-1} \rightarrow du/dx = -y^{-2}\, dy/dx$

$= -\frac{1}{y^2}\, dy/dx$ ✓

$\therefore \frac{du}{dx} + \frac{u}{x} = -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy}$

$\therefore = -\frac{1}{y^2}\left(\frac{y - xy^2}{x}\right) + \frac{1}{xy}$ ✓

$= -\frac{1 - xy}{xy} + \frac{1}{xy}$ ✓

$= \frac{xy - 1 + 1}{xy}$ ✓

$= 1$

③

4 ✓

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9) $x \, dy/dx = y(1-xy)$

2

$\Rightarrow y = u^{-1}$

$\therefore dy/dx = -u^{-2} \, du/dx$

$= -\frac{1}{u^2} \, du/dx$

$\therefore x\left(-\frac{1}{u^2}\frac{du}{dx}\right) = \frac{1}{u}\left(1 - \frac{x}{u}\right)$

$\therefore -\frac{x}{u^2}\frac{du}{dx} = \frac{1}{u} - \frac{x}{u^2}$

$\therefore -x\frac{du}{dx} = u - x$

$\therefore dy/dx = -u/x + 1$

$= -u/x + 1$

let $u = vx \rightarrow \frac{du}{dx} = v + x \, dv/dx$

$\therefore v + x \, dv/dx = -\frac{vx}{x} + 1$

$\therefore x \, dv/dx = -v + 1 - v$

$= 1 - 2v$

$\therefore \frac{1}{1-2v}\frac{dv}{dx} = 1/x$

$\therefore \ln|1-2v| = \ln|x| + c$

$\therefore 1-2v = Ae^{x}$

$\therefore 1-2^{u}/x = Ae^{x}$

$1 - 2 \times \frac{1}{xy} = Ae^{x}$

$\therefore \frac{2}{xy} = 1 - Ae^{x}$

$\therefore xy = \frac{2}{1-Ae^{x}}$

$\therefore y = \frac{2}{x - xAe^{x}}$

2/

g) $\frac{du}{dx} + \frac{u}{x} = 1$

$\therefore I(x) = e^{\int \frac{1}{x} dx}$

$= e^{\ln x}$

$= x$

$\therefore \quad ux = \int x \, dx$

$\therefore \quad ux = \frac{1}{2}x^2 + C$

$\therefore \quad 2ux = x^2 + C$

$\therefore \quad \frac{2x}{y} = x^2 + C$

$\therefore \quad y = \frac{2x}{x^2+C}$

When $x=0.2, y=0.4.$

$\therefore \quad 0.4 = \frac{2(0.2)}{0.2^2 + C}$

$\therefore 0.4 + C = 0.4/0.4$

$\therefore \quad C = 0.96$

$\Rightarrow \quad y = \frac{2x}{x^2+0.96}$

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2

a) let $z = a+bi$, $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ ✗

let $w = c+di$, $\arg(w) = \tan^{-1}\left(\frac{d}{c}\right)$ ✗

0

b) $f(x) = x^3 + 2x^2 + 4x + 8$

Root : $x = -2$, $f(x) = (x+2)(x^2+ax+b)$

$= x^3 + ax^2 + bx$

$+ 2x^2 + 2ax + 2b$ ✓

$= x^3 + (a+2)x^2 + (b+2a)x + 2b$

$\therefore b = 4$, $a = 0$

$\therefore f(x) = (x+2)(x^2+4)$ ✓

~~$= (x+2)$~~

2

2/

c) $x^2 + 4 = 0$

$\therefore \quad x^2 = -4$

$= 4\ cis(\pi + 2k\pi)$

$\therefore \quad x = \sqrt{4}\ cis\left(\frac{\pi + 2k\pi}{2}\right)$

$= \cancel{\sqrt{2}}\ 2\ cis\left(\frac{\pi + 2k\pi}{2}\right)$

$\therefore \quad x = 2cis(\pi/2),\ 2cis(3\pi/2),\ 2cis(5\pi/2)$

$= -2,\ 2(0 - i),\ 2(i)$ ✓

$= -2,\ -2i,\ 2i$ ✓ 2

d) $u_1 = -2$

$u_2 = -2i$

$u_3 = 2i$

2

$\therefore \quad r = \frac{u_2}{u_1} = -\frac{2i}{-2} = i$

$= \frac{u_3}{u_2} = \frac{2i}{-2i} = -1$

$u_1 = -2$

$u_2 = -2i$

$u_3 = 2i$

$\therefore \quad r = i$ ✓

$u_2 = -2$

$u_1 = 2i$

$u_3 = -2i$

$\therefore \quad \frac{u_3}{u_2} = \frac{-2i}{-2} = i$

$\frac{u_2}{u_1} = -\frac{2}{2i} = -\frac{1}{i} = i$

4

e) G.D.C: $\text{cPolyRoots}(27 \cdot x^3 - 9 \cdot x^2 + 3 \cdot x - 1, x)$

$\left\{-\frac{1}{3} \cdot i, \frac{1}{3} \cdot i, \frac{1}{3}\right\}$

~~let $u_1 = 1/3$~~

~~$u_2 = 1/3\, i$~~

~~$u_3 = -1/3\, i$~~

~~$\therefore r = \frac{1/3\, i}{1/3} = i$~~

~~$\Rightarrow$ test for $\frac{-1/3\, i}{1/3} = -i$~~

6

f) $g(x) = 1 + rx + r^2x^2 + \cdots + r^nx^n = \sum_{k=0}^{n} r^k x^k$

2

$\therefore (rx-1)g(x) = (rx-1)(1 + rx + r^2x^2 + \cdots + r^nx^n)$
$= rx + r^2x^2 + \cdots + r^{n+1}x^{n+1} - (1 + rx + r^2x^2 + \cdots + r^nx^n)$
$= -1 + r^{n+1}x^{n+1}$

g) $-1 + r^{n+1}x^{n+1} = 0$

1

$r^{n+1}x^{n+1} = 1$

$\therefore x^{n+1} = \cancel{\frac{1}{r^{n+1}}}$ cos...

$\therefore x = \cancel{\frac{1}{(r^{n+1})^{1/n+1}}}$ cos...

$\therefore x = \cancel{\frac{1}{r}}$

$\therefore x = 1/r$ cos...

9