Mathematics: analysis and approaches
Higher level
Paper 2

2 hours

Candidate session number

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all the questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

At a café, the waiting time between ordering and receiving a cup of coffee is dependent upon the number of customers who have already ordered their coffee and are waiting to receive it.

Sarah, a regular customer, visited the café on five consecutive days. The following table shows the number of customers, x, ahead of Sarah who have already ordered and are waiting to receive their coffee and Sarah's waiting time, y minutes.

Number of customers (x)	3	9	11	10	5
Sarah's waiting time (y)	6	10	12	11	6

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$.

a) (i) Find the value of a and the value of b.

(ii) Write down the value of Pearson's product-moment correlation coefficient, r. [3]

b) Interpret, in context, the value of a found in part (a)(i). [1]

On another day, Sarah visits the café to order a coffee. Seven customers have already ordered their coffee and are waiting to receive it.

c) Use the result from part (a)(i) to estimate Sarah's waiting time to receive her coffee. [2]

a) using the calculator

i) $a = 0.805$, $b = 2.88$

A1 A1

ii) $r = 0.978$ → A1

b) a: the wait time per customer ⇒ A1

c) $\underbrace{y = 0.807 \times 7 + 2.88}_{M1} = 8.529 = \underbrace{8.53}_{A1}$

2. [Maximum mark: 5]

An arithmetic sequence has first term 60 and common difference -2.5 .

(a) Given that the kth term of the sequence is zero, find the value of k . [2]

Let S_n denote the sum of the first n terms of the sequence.

(b) Find the maximum value of S_n . [3]

$u_1 = 60$, $d = -2.5$

a) $u_n = u_1 + (n-1)d$

$0 = 60 + (k-1) \times -2.5$ ⇒ M1

$k - 1 = \dfrac{60}{2.5}$

$k = 24 + 1 = 25$ ⇒ A1

b) $\underbrace{60,\ 57.5\ \ldots\ 0}_{\text{Max}},\ -2.5$ → sum will start to decrease as adding negative numbers

M1 / R1 (recognise that only 25 terms)

$S_n = \dfrac{n}{2}(u_1 + u_n)$

$= \underbrace{\dfrac{25}{2}(60+0)}_{M_1} = 75$ ⇒ A1

3. [Maximum mark: 8]

At a school, 70% of the students play a sport and 20% of the students are involved in theatre. 18% of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school 48% of the students are girls, and 25% of the girls are involved in theatre.

A student is selected at random. Let G be the event “the student is a girl” and let T be the event “the student is involved in theatre”.

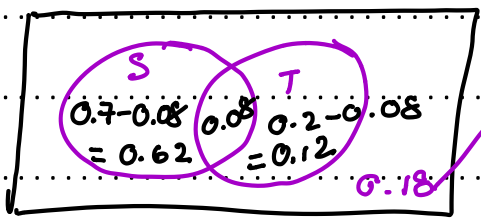
(c) Find $P(G \cap T)$. [2]

(d) Determine if the events G and T are independent. Justify your answer. [2]

S: 70 % = 0.7 T: 20 % = 0.2 Neither: 18% = 0.18

Using Venn diagram

a)



$S+T = 1-0.18 = 0.82$

$\Rightarrow S \cap T = 0.7+0.2-0.82$
$= 0.08$

→ M1

P(Sport and theatre) = 0.08 → A1

b) $\underbrace{P(T \cap S') = 0.2 - 0.08}_{M1} = \underbrace{0.12}_{A1}$ or just using venn diagram

c) G: 0.48 → 0.25 of which in Theatre ↳ $P(T|G)$

$P(G \cap T) = P(G) \times P(T|G)$ → M1

$= 0.48 \times 0.25 = 0.12$ → A1

d) $P(G) \times P(T) = 0.48 \times 0.2 = 0.096 \neq P(G \cap T)$ M1

∴ not independent ⇒ A1

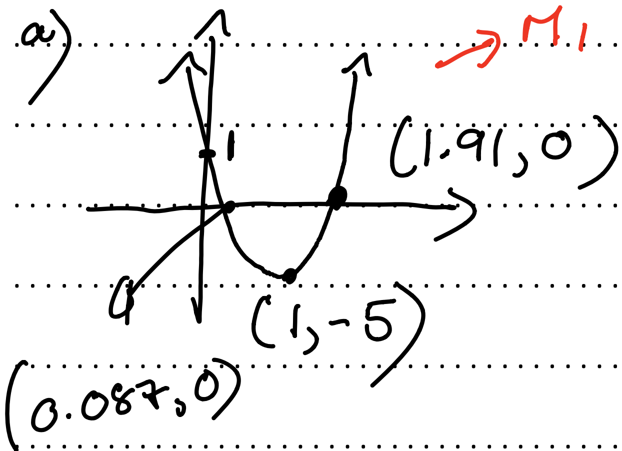
4. [Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

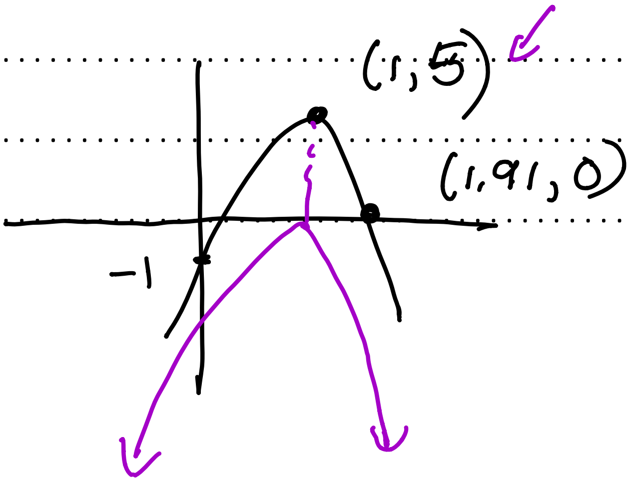
a) → M1



Range: $x \geqslant -5$ → A1

b) $(g \circ f)(x) = g(f(x))$

$= -6x^2 + 12x - 1 + c \leq 0$



shift the graph up/down

$\Rightarrow$ if $(g \circ f)(x) \leq 0$

Turning point must b at $(1, 0)$ or below

$\Rightarrow c \leq -5$.

5. [Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

(a) when $t = 0$, $A = 100$

$100 = A_0 e^{-0k}$

$100 = A_0$ → A1

(b) $\frac{1}{2}$ of $A_0 = 50$, $t = 5730$

When $A = 50$ ⇒ $50 = 100e^{-k \times 5730}$ M1

$\frac{1}{2} = e^{-5730k}$

$\ln\left(\frac{1}{2}\right) = -5730k$

$\ln(1) - \ln 2 = -5730k$

$-\ln 2 = -5730k$ M1

$\boxed{k = \frac{\ln 2}{5730}}$ → A1

c) 25% of 100 = 75

when $A = 75$

M1 $75 = 100e^{-t \times \left(\frac{\ln 2}{5730}\right)}$

M1 $\ln\left(\frac{3}{4}\right) = -t\left(\frac{\ln 2}{5730}\right)$

$t = 2378.16$

$= 2380$ years → A1

6. [Maximum mark: 6]

A continuous random variable X has the probability density function f_n given by

$$f_n(x) = \begin{cases} (n+1)x^n, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

where $n \in \mathbb{R}$, $n \ge 0$.

(a) Show that $E(X) = \frac{n+1}{n+2}$. [2]

(b) Show that $Var(X) = \frac{n+1}{(n+2)^2(n+3)}$. [4]

a) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^1 x^n (n+1)\, x\, dx$

M1

$= \int_0^1 (n+1)\, x^{n+1}\, dx$

$= \left[\frac{n+1}{n+2} x^{n+2}\right]_0^1 = \frac{n+1}{n+2}\left(1^{n+2} - 0^{n+2}\right) = \frac{n+1}{n+2}$

A1

b) $Var(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 = \int_0^1 x^2 x^n (n+1)\, dx - \left(\frac{n+1}{n+2}\right)^2$ M1

$= \int_0^1 x^{2+n} (n+1)\, dx - \left(\frac{n+1}{n+2}\right)^2$

$= \frac{n+1}{n+3}\left[x^{n+3}\right]_0^1 - \left(\frac{n+1}{n+2}\right)^2$ M1

$= \frac{n+1}{n+3}$ M1 $- \frac{(n+1)^2}{(n+2)^2} = \frac{(n+1)(n+2)^2 - (n+1)^2(n+3)}{(n+3)(n+2)^2}$

$= \frac{(n+1)\left[n^2+4n+4 - (n+1)(n+3)\right]}{(n+3)(n+2)^2} = \frac{(n+1)(n^2+4n+4-n^2-4n-3)}{(n+2)^2(n+3)}$

A1

7. [Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

a) Jack finish immediately after Andrea

$\Rightarrow$ Jack and Andrea count as 1 person

$\Rightarrow$ $7! = 5040$

M1 A1

b) Any position after Andrea

(1) Andrea finished 1^{st} $\Rightarrow$ Jack: any out of 7: ${}^7C_1$

$\rightarrow$ the other 6: $6!$

(2) Andrea: 2^{nd} $\Rightarrow$ J: ${}^6C_1$ $\Rightarrow$ 6 places

Other: $6!$

(3) Andrea 3^{rd} $\Rightarrow$ J: ${}^5C_1$

Others: $6!$

R1 M1

⋮

Andea: 7^{th} $\Rightarrow$ Jack ${}^1C_1$

others: $6!$

$\Rightarrow {}^7C_1\, 6! + {}^6C_1\, 6! + {}^5C_1\, 6! + {}^4C_1\, 6! + {}^3C_1\, 6! + {}^2C_1\, 6! + {}^1C_1\, 6!$ M1

$= 6!\,(7+6+5+4+3+2+1) = 20160$

A1

8. [Maximum mark: 5]

Consider $z = \cos\theta + i\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $Re\left(\frac{1+z}{1-z}\right) = 0$.

$$\frac{1+z}{1-z} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta} \quad M_1$$

$$= \frac{[(1+\cos\theta)+i\sin\theta][(1-\cos\theta)+i\sin\theta]}{[(1-\cos\theta)-i\sin\theta][(1-\cos\theta)+i\sin\theta]}$$

$$= \frac{(1-\cos^2\theta)+i\sin\theta+\cancel{i\sin\theta\cos\theta}+i\sin\theta-\cancel{i\sin\theta\cos\theta}+i^2\sin^2\theta}{(1-\cos\theta)^2-i^2\sin^2\theta} \quad M_1$$

$$= \frac{1-\cos^2\theta+2i\sin\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta} \quad M_1$$

$$= \frac{1+2i\sin\theta-(\cos^2\theta+\sin^2\theta)}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{1-1+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \frac{0+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} \quad M_1$$

$$Re\left(\frac{1+z}{1-z}\right) = \frac{0}{(1-\cos\theta)^2+\sin^2\theta} = 0 \quad A_1$$

9. [Maximum mark: 8]

$n \in \mathbb{Q}$

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $arc\tan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x\, arc\tan 2x}{\sec x - 1}\right)$. [3]

a) $(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 \dots\right)$

$(1+t)^{-1} = 1^{-1}\left(1+(-1)t+\frac{(-1)(-2)}{2!}t^2\right)$

$= 1-t+t^2 \dots \rightarrow A_1$

b) $\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}\dots$

$\cos x - 1 = -\frac{x^2}{2}+\frac{x^4}{24} \rightarrow \sec x = (\cos x)^{-1}$

$\sec x = \left(1+(\cos -1)\right)^{-1} \rightarrow M_1$

$(1+t)^{-1} \Rightarrow \left(1+\overbrace{(\cos -1)}^{t}\right)^{-1} = \sec x$

$= 1-\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2$ M_1

$= \underbrace{1+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^4}{4}} - \frac{x^8}{24}+\frac{x^8}{24^2}$ M_1

$\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{24} \rightarrow$ up to and including x^4. R_1/A_1

c) $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}\dots \rightarrow \arctan 2x = 2x-\frac{2^3x^3}{3}\dots$ M_1

$\lim_{x\to 0}\left(\frac{x\arctan 2x}{\sec x - 1}\right) = \lim_{x\to 0}\left[\frac{2\cancel{x^2}-\frac{2^3 x^{\cancel{4}\,2}}{3}+\frac{2^5 x^{\cancel{6}\,4}}{5}\dots}{\frac{\cancel{x^2}}{2}+\frac{5x^{\cancel{4}\,2}}{24}\dots}\right]$ M_1

cancelation of x^2.

$= \frac{2}{1/2} = 4. \rightarrow A_1$

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

0.02

(a) Given that 2% of the flight times are longer than 82 minutes, find the value of σ. [3]

(b) Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. [2]

(c) Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. [4]

On a particular day, there are 64 flights scheduled between these two cities.

(d) Find the expected number of flights that will have a flight time of more than 80 minutes. [3]

(e) Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. [3]

$\rightarrow M_1$

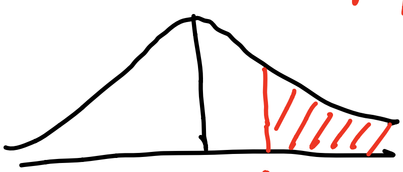
(a) $\mu = 75$, $x = 82$, $z = \text{invnorm}(0.98) = 2.05375$

$z = \dfrac{x-\mu}{\sigma} \Rightarrow 2.05375 = \dfrac{82-75}{\sigma} \Rightarrow \sigma = \dfrac{7}{2.05375}$ M_1

$= 3.41 \rightarrow A_1$

(b) $P(T > 80) = \text{normCdf}(80, 1000, 75, 3.41) = 0.0713 \rightarrow A_1$

M_1



75 80

(c) Given $T > 80$, find $P(T < 82) \Rightarrow$ M_1 $P(T < 82 \mid T > 80) = \dfrac{P(80 < T < 82)}{P(T > 80)}$ M_1

M_1 $= \dfrac{0.05124}{0.07128} = 0.719$ A_1

(d) $P(T > 80) = 0.0713 \Rightarrow Y \sim B(64, 0.0713) \Rightarrow E(Y) = 64 \times 0.0713$ M_1 M_1

binomal dis

A_1 $= 4.56$

(e) $P(Y > 6) = P(Y \geq 7) \rightarrow$ binomial cdf: $n: 64$, $p = 0.0713$ M_1 M_1

$= 0.17$ A_1

Do **not** write solutions on this page.

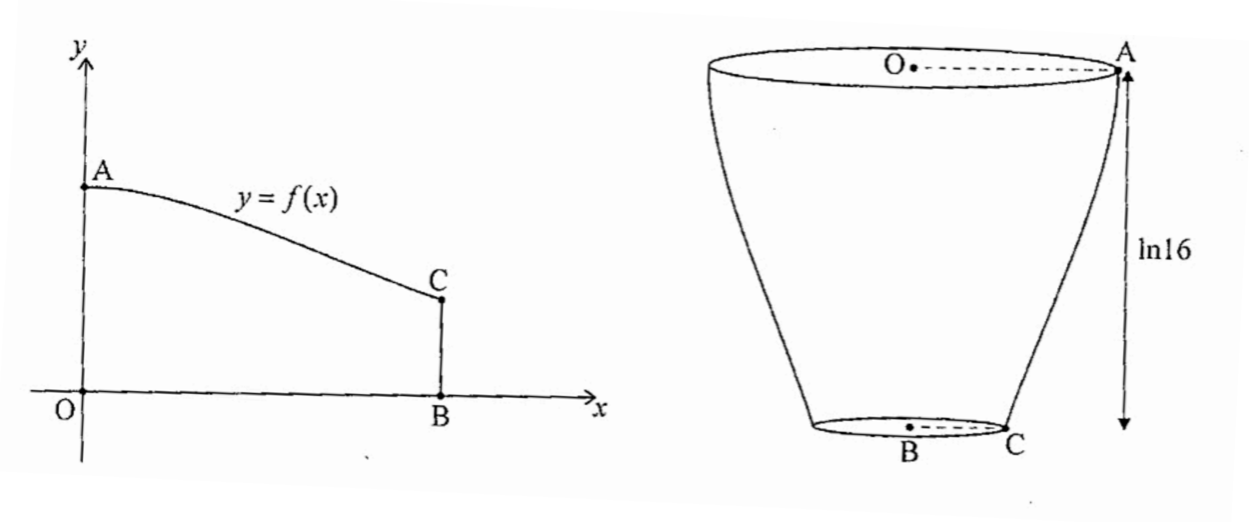
11. [Maximum mark: 18]

A function f is defined by $f(x) = \frac{ke^{\frac{x}{2}}}{1+e^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis and the line $x = \ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of 300 cm^3
Pedro's design is shown in the following diagrams.



The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(iii) State the cross-sectional radius of the bowl at this point. [6]

M1 M1

a) $V = \pi \int_0^{\ln 16} \left(\frac{ke^{x/2}}{1+e^{x}} \right)^2 dx = k^2\pi \int_0^{\ln 16} \frac{e^{x}}{(1+e^{x})^2} dx$

M1 M1

$u = 1+e^{x} \Rightarrow x = \ln 16 \Rightarrow u = 17 \Rightarrow V = k^2\pi \int_2^{17} u^{-2} du$

$x = 0 \Rightarrow u = 2$

$du = e^{x} dx$

$= -k^2\pi \left[u^{-1}\right]_2^{17}$ M1

$= -k^2\pi \left(\frac{1}{17} - \frac{1}{2}\right) = \frac{15k^2\pi}{34}$

Do **not** write solutions on this page.

12. [Maximum mark: 21]

A function f is defined by $f(x) = arc\sin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2(x^2+1)}}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = arc\sin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1} . [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

M1

(a) $f(-x) = \arcsin\left(\frac{(-x)^2-1}{(-x)^2+1}\right) = \arcsin\left(\frac{x^2-1}{x^2+1}\right) = f(x)$

$\therefore f(x)$ is an even function.

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \arcsin\left(\frac{x^2-1}{x^2+1}\right)$

$= \lim_{x\to\infty} \arcsin\left(\frac{x^2/x^2 - 1/x^2}{x^2/x^2 + 1/x^2}\right)$ M1

$= \lim_{x\to\infty} \arcsin\left(\frac{1 - 1/x^2}{1 + 1/x^2}\right)$

$= \arcsin(1) = \frac{\pi}{2}$.

horizontal asymptote : $y = \frac{\pi}{2}$ A1 c)

i) $f(x) = \arcsin\left[\frac{x^2-1}{x^2+1}\right]$

$$u = \frac{x^2-1}{x^2+1} \Rightarrow u' = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

M1

$$= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

M1

$$f'(x) = \frac{u'}{\sqrt{1-u^2}} = \frac{4x}{(x^2+1)^2} \times \frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}$$

$$= \frac{4x}{(x^2+1)^2} \times \frac{1}{\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)^2}}}$$ M1

$$= \frac{4x}{(x^2+1)^2} \times \frac{1}{\sqrt{\frac{(x^2+1-x^2+1)(x^2+1+x^2-1)}{(x^2+1)^2}}}$$

$$= \frac{4x}{(x^2+1)^2} \times \frac{1}{\sqrt{\frac{4x^2}{(x^2+1)^2}}}$$ M1

$$= \frac{4x}{(x^2+1)^2} \times \frac{x^2+1}{2\sqrt{x^2}}$$

$$= \frac{2x}{\sqrt{x^2}\,(x^2+1)^2}$$ $\Rightarrow$ A1

ii) $f(x)$ is decreasing if

$f'(x) < 0$

$$f'(x) = \frac{2x}{\sqrt{x^2}\,(x^2+1)^2} < 0$$

$$\frac{2x}{|x|\,(x^2+1)} < 0$$

if $x < 0$ R1

$\Rightarrow |x| > 0$

$x^2+1 > 0$

but $2x < 0$

$\therefore f(x)$ is decreasing A1

if $x < 0$

d) $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}, x \geq 0$

$$y = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$$

$$\Rightarrow x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$$ M1

$$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$$

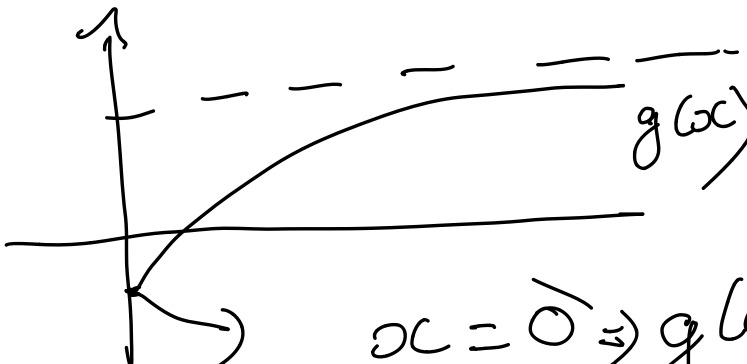
$$y^2(\sin x - 1) = -1 - \sin x$$

M1

$$y^2 = \frac{-1-\sin x}{\sin x - 1} = \frac{1+\sin x}{1-\sin x} \quad M1$$

$$y = \sqrt{\frac{1+\sin x}{1-\sin x}} = g^{-1}(x), \quad \underbrace{\sin x \neq 1}_{M1} \quad A1$$

e) Domain $g^{-1}(x)$ = Range $g(x)$



$x = 0 \Rightarrow g(0) = \arcsin(-1)$

$\Rightarrow \arcsin(-1) = y$

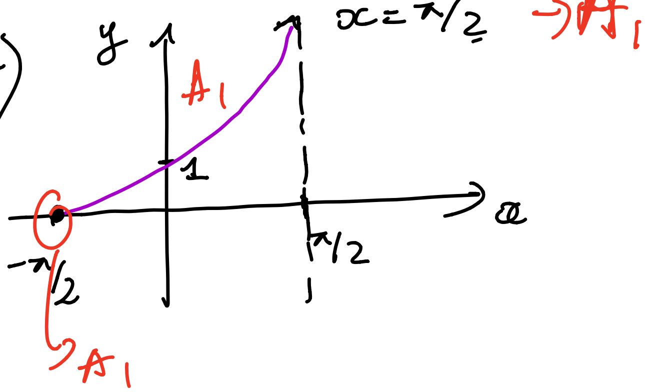
$\sin y = -1$

$\therefore y = -\frac{\pi}{2}$

Domain $g^{-1}(x)$ $\leftarrow$

$-\frac{\pi}{2} \le x < \frac{\pi}{2}$ A1

f)



Question 11: (continued

b) $\frac{15k^3\pi}{34} = 300$ M1

$k = \sqrt[3]{\frac{300 \times 34}{15\pi}} = 14.7\text{cm}$ A1

c) i) at A : $x = 0$

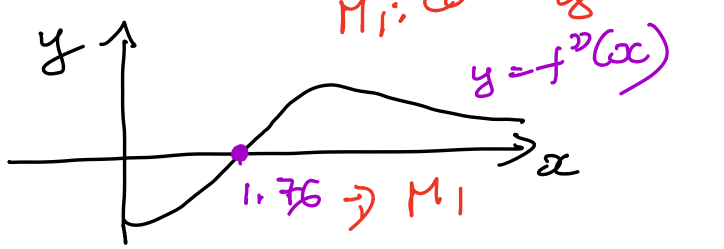
$f(0) = \frac{14.7e^{0/2}}{1+e^0}$ M1 $= \frac{14.7}{2} = 7.35$ cm

$OA = 7.35\text{cm}$ A1

at B: $x = \ln 16 \rightarrow$ M1

$f(\ln 16) = \frac{14.7e^{\ln 16/2}}{1+e^{\ln 16}} = 3.46\text{ cm}$ A1

d) i) $f''(x) = 0 \rightarrow$ M1



Radius decreasing most rapidly when $x = 1.76$ A1

ii) $f(1.76) = \dfrac{14.7e^{1.76/2}}{1+e^{1.76}}$ M1

$= 5.2$

Radius: 5.2 cm A1