Solutions.

Mathematics: Analysis and Approaches
Higher level
2022 Semester 2 Examinations
Paper 2

ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Monday, August 29th (morning)

2 hours

Candidate number

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all the questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider a geometric sequence with a first term of 4 and a fourth term of −2.916.

(a) Find the common ratio of this sequence. [3]

(b) Find the sum to infinity of this sequence. [2]

(a) $u_4 = u_1 r^3 = -2.916$

$4r^3 = -2.916$

$r = 0.9$

(b) $S_\infty = \frac{u_1}{1-r}$

$= \frac{4}{1-0.9}$

$= 40$

2. [Maximum mark: 10]

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where a and b are positive integers.

(a) Given that $x^2 - 1$ is a factor of $f(x)$ find the value of a and the value of b. [4]

(b) Factorize $f(x)$ into a product of linear factors. [3]

(c) Using your graph state the range of values of c for which $f(x) = c$ has exactly two distinct real roots. [3]

(a) $x^2-1=(x+1)(x-1)$ $\therefore \pm 1$ are factors

$f(1)=a+b-8=0$

$a+b=8$ ①

$f(-1)=-a+b+6=0$

$-a+b=-6$ ②

solve ① & ② $a=7$

$b=1$

(b)

$$\begin{array}{r} 3x^2+7x+4 \\ x^2+0x-1\overline{)\,3x^4+7x^3+x^2-7x-4} \\ \underline{3x^4+0x^3-3x^2} \\ 7x^3+4x^2-7x \\ \underline{7x^3+0x^2-7x} \\ 4x^2+0-4 \\ \underline{4x\quad -4} \\ 0 \end{array}$$

$f(x)=(x^2-1)(3x^2+7x+4)$

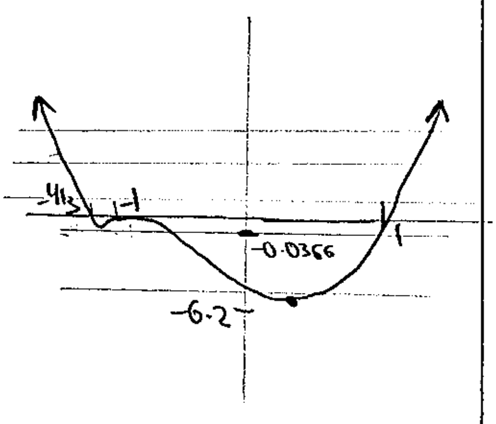
$=(x-1)(x+1)(x+1)(3x+4)$

$=(x-1)(x+1)^2(3x+4)$

(d)

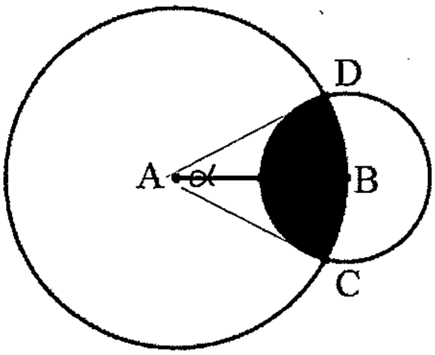
$c>0$ and

$-6.2<c<-0.0366$



3. [Maximum mark: 8]

The diagram shows two circles with centres at the points A and B and radii $2r$ and r, respectively. The point B lies on the circle with centre A. The circles intersect at the points C and D.



Let α be the measure of the angle CAD and θ be the measure of the angle CBD in radians.

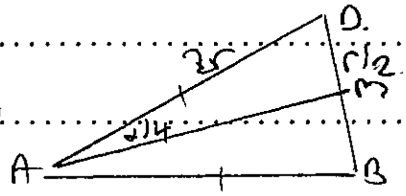
(a) Find an expression for the shaded area in terms of α, θ and r. [3]

(b) Show that $\alpha = 4\arcsin\frac{1}{4}$. [2]

(c) Hence find the value of r given that the shaded area is equal to 4. [3]

(a) $A = 2(\alpha - \sin\alpha)r^2 + \frac{1}{2}(\theta - \sin\theta)r^2$

(b) ΔABD



M is midpoint BD

As ΔABD is isosceles $AM \perp BD$

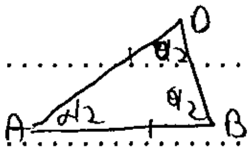
$\sin\left(\frac{\alpha}{4}\right) = \frac{r/2}{2r}$

$\sin\frac{\alpha}{4} = 1/4$

$\frac{\alpha}{4} = \arcsin 1/4$

$\alpha = 4\arcsin 1/4$

(c)



$\alpha/2 + \theta/2 + \theta/2 = \pi$

$\theta = \pi - \alpha/2$

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Question 3 (writing space continued)

$\therefore$ From (a)

$$4 = (\alpha - \sin\alpha) r^2 + \tfrac{1}{2}\left(\left(\pi - \tfrac{\alpha}{2}\right) - \sin\left(\pi - \tfrac{\alpha}{2}\right)\right) r^2 \quad (1)$$

From (b), $\alpha = 4\arcsin \tfrac{1}{4}$

$$= 1.01$$

$$\pi - \tfrac{\alpha}{2} = 2.63$$

into (1)

$$r^2 = \frac{4}{(1.01 - \sin 1.01) + \tfrac{1}{2}(2.63 - \sin(2.63))}$$

$$r = 1.69$$

4. [Maximum mark: 6]

(a) Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. [3]

(b) Hence find the least value of n for which $\binom{3n+1}{3n-2} > 10^6$. [3]

(a) $\binom{3n+1}{3n-2} = \frac{(3n+1)!}{(3n-2)!\,3!}$

$= \frac{(3n+1)3n(3n-1)}{3!}$

$= \frac{3n(9n^2-1)}{6}$

$= \frac{27n^3-3n}{6}$

(b) $\frac{27n^3-3n}{6} > 10^6$

n solve $n > 60.57$

$\therefore n = 61$

5. [Maximum mark: 7]

The curve C has equation $e^{2y} = x^3 + y$.

(a) Show that $\frac{dy}{dx} = \frac{3x^2}{2e^{2y}-1}$. [3]

(b) The tangent to C at the point P is parallel to the y-axis.

Find the x-coordinate of P. [4]

a) $e^{2y} = x^3 + y$

$2e^{2y}\frac{dy}{dx} = 3x^2 + \frac{dy}{dx}$

$(2e^{2y}-1)\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{(2e^{2y}-1)}$

b) parallel to y-axis $\Rightarrow \frac{dy}{dx}$ undefined.

$\therefore$ at $2e^{2y}-1=0$

$e^{2y} = \frac{1}{2}$

$\ln\frac{1}{2} = 2y$

$y = \frac{1}{2}\ln\frac{1}{2}$

$= -0.347\ldots$

$e^{2y} = x^3 + y$

$x = \sqrt[3]{e^{2y}-y}$ sub $y = \frac{1}{2}\ln\frac{1}{2}$

$x = 0.946$ (3 sf)

6. [Maximum mark: 7]

By using the substitution $x^2 = 2\sec\theta$, show that $\int \frac{\mathrm{d}x}{x\sqrt{x^4-4}} = \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + c$.

$x^2 = 2\sec\theta$ $\quad\quad x = \sqrt{2}(\sec\theta)^{1/2}$

$2x\frac{dx}{d\theta} = 2\sec\theta\tan\theta$

$dx = \frac{2\sec\theta\tan\theta}{2x}\, d\theta$

$dx = \frac{\sec\theta\tan\theta}{x}\, d\theta$

$\int \frac{dx}{x\sqrt{x^4-4}} = \int \frac{\sec\theta\tan\theta\, d\theta}{x^2\sqrt{4\sec^2\theta-4}}$

$= \int \frac{\cancel{\sec\theta}\tan\theta}{2\cancel{\sec\theta}\sqrt{4\sec^2\theta-4}} d\theta$

$= \frac{1}{2}\int \frac{\tan\theta}{\sqrt{4(\tan^2\theta)}} d\theta$

$= \frac{1}{2}\int \frac{\tan\theta}{2\tan\theta} d\theta$

$= \frac{1}{2}\int \frac{1}{2} d\theta$

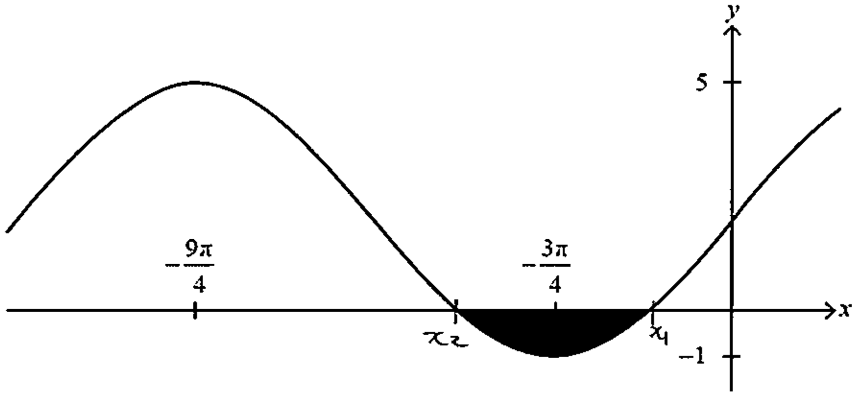
$= \frac{1}{4}\theta + C$ $\quad\quad$ Now $x^2 = 2\sec\theta$

$= \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + C$ $\quad\quad \cos\theta = \frac{2}{x^2}$

$\theta = \arccos\left(\frac{2}{x^2}\right)$

7. [Maximum mark: 8]

The following diagram shows part of the graph of $y = p + q\sin(rx)$. The graph has a local maximum point at $\left(-\frac{9\pi}{4}, 5\right)$ and a local minimum point at $\left(-\frac{3\pi}{4}, -1\right)$.



(a) Determine the values of p, q and r. [4]

(b) Hence find the area of the shaded region. [4]

(a) axis $= \frac{5+(-1)}{2}$ $= 2$ $\therefore p = 2$

amp $= \frac{5-(-1)}{2}$ $= 3$ $q = 3$

Period: $\frac{2\pi}{r} = 3\pi$ $r = \frac{2}{3}$

$y = 2 + 3\sin\left(\frac{2x}{3}\right)$

(b) GDC $x_1 = -1.09459\ldots$ $x_2 = -3.617797\ldots$

$$A = \int_{x_2}^{x_1} \left(2 + 3\sin\left(\frac{2x}{3}\right)\right)dx.$$

$= |-1.66\ldots|$

$= 1.66$ (3sf).

8. [Maximum mark: 7]

There are eight boys and five girls who attend the Mathematics Club. How many ways can the teacher select a group of 6 students form the club to represent the school in a Mathematics competition if:

(a) There are no gender restrictions [2]

(b) The team is to be made up of three girls and three boys [2]

(c) At least two of each gender are included in the team [3]

a) ${}^{13}C_6 = 1716$

b) ${}^{8}C_3 \times {}^{5}C_3 = 560$

c) $(2b, 4g) + (3b, 3g) + (4b, 2g)$

$= {}^{8}C_2 \times {}^{5}C_4 + {}^{8}C_3 \times {}^{5}C_3 + {}^{8}C_4 \times {}^{5}C_2 = 1400$

9 $v = e^{-3t}\sin 6t \quad 0 < t < \pi/2$

(a) rest: $v = 0$

$e^{-3t}\sin 6t = 0$

$e^{-3t} = 0 \qquad \sin 6t = 0$

NS $\qquad 6t = 0, \pi, 2\pi, 3\pi$

$t = 0, \pi/6, \pi/3, \pi/2 \ldots$

$0 < t < \pi/2 \therefore \underline{t = \pi/6, \pi/3}$

(b) $s = \int e^{-3t}\sin 6t \qquad u = \sin 6t \quad dv = e^{-3t}$

$du = 6\cos 6t \quad v = -1/3\, e^{-3t}.$

$s = -1/3\, e^{-3t}\sin 6t - \int -1/3\, e^{-3t}\, 6\cos 6t \, dt.$

$s = -1/3\, e^{-3t}\sin 6t + 2\int e^{-3t}\cos 6t \, dt. \qquad u = \cos 6t \quad dv = e^{-3t}$

$du = -6\sin 6t \quad v = -1/3\, e^{-3t}$

$s = -1/3\, e^{-3t}\sin 6t + 2\left[-1/3\, e^{-3t}\cos 6t - \int -1/3\, e^{-3t}(-6\sin 6t)\, dt\right.$

$s = -1/3\, e^{-3t}\sin 6t - 2/3\, e^{-3t}\cos 6t - 4\int e^{-3t}\sin 6t \, dt$

$$s = \frac{-e^{-3t}\sin 6t - 2e^{-3t}\cos 6t}{3} - 4s$$

$$5s = \frac{-e^{-3t}\sin 6t - 2e^{-3t}\cos 6t}{3}.$$

$$s = \frac{-e^{-3t}\sin 6t - 2e^{-3t}\cos 6t}{15} + C$$

$s(0) = 0 \qquad 0 = \dfrac{0 - 2}{15} + C$

$C = 2/15.$

$$s = \frac{2}{15} - \frac{e^{-3t}\sin 6t + 2e^{-3t}\cos 6t}{15}$$

(c) max when $t = \pi/6$ (see (a))

$$S = \frac{2}{15} - \frac{\left(e^{-\pi/2}\sin\pi + 2e^{-\pi/2}\cos\pi\right)}{15}$$

OR use GDC or $\int$

$$= \frac{2}{15} - \frac{\left(0 - 2e^{-\pi/2}\right)}{15}$$

$$= 0.161\text{m} \ (3\text{sf})$$

(d) $d = \int_0^{1.5} \left| e^{-3t}\sin 6t \right| dt$

$= 0.201\text{ m. } (3\text{sf})$

(e) $v = e^{-3t}\sin 6t$

$\frac{dv}{dt} = 6\cos 6t\, e^{-3t} + -3e^{-3t}\sin 6t = 0$

$3e^{-3t}(2\cos 6t - \sin 6t) = 0$

$\sin 6t = 2\cos 6t.$

$\frac{\sin 6t}{\cos 6t} = 2.$

$\tan 6t = 2$

(f)

attempt to evaluate $t_1,\ t_2,\ t_3$ in exact form ***M1***

$$6t_1 = \arctan 2 \left(\Rightarrow t_1 = \frac{1}{6}\arctan 2\right)$$

$$6t_2 = \pi + \arctan 2 \left(\Rightarrow t_2 = \frac{\pi}{6} + \frac{1}{6}\arctan 2\right)$$

$6t_3 = 2\pi + \arctan 2 \left(\Rightarrow t_3 = \frac{\pi}{3} + \frac{1}{6}\arctan 2\right)$ ***A1***

Note: The ***A1*** is for any two consecutive correct, or showing that $6t_2 = \pi + 6t_1$ or $6t_3 = \pi + 6t_2$.

showing that $\sin 6t_{n+1} = -\sin 6t_n$

eg $\tan 6t = 2 \Rightarrow \sin 6t = \pm\frac{2}{\sqrt{5}}$ ***M1A1***

showing that $\frac{e^{-3t_{n+1}}}{e^{-3t_n}} = e^{-\frac{\pi}{2}}$ ***M1***

eg $e^{-3\left(\frac{\pi}{6}+k\right)} \div e^{-3k} = e^{-\frac{\pi}{2}}$

9(f) con't

Note: Award the *A1* for any two consecutive terms.

$\frac{v_3}{v_2} = \frac{v_2}{v_1} = -e^{-\frac{\pi}{2}}$ *AG*

[5 marks]

10. (a) (i) $\lim_{x\to\infty} \frac{x^3}{e^x} = \frac{\infty}{\infty}$ $\therefore$ appropriate to use LH's rule 2

(ii) $\therefore \lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} \Rightarrow = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{6x}{e^x} \Rightarrow = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{6}{e^x}$ 3

$= 0$

(b) $\int_0^\infty x^3 e^{-x}\,dx = \lim_{R\to\infty} \int_0^R x^3 e^{-x}\,dx$ $u = x^3$ $dv = e^{-x}$ $du = 3x^2$ $v = -e^{-x}$

$= \lim_{R\to\infty} \left[\left[-e^{-x}x^3\right]_0^R + 3\int_0^R e^{-x} x^2\,dx \right]$ $u = x^2$ $du = 2x$

$= \lim_{R\to\infty} \left[-e^{-R}R^3 + 3\left(\left[-e^{-x}x^2\right]_0^R + 2\int_0^R e^{-x} x\,dx \right) \right]$ $u = x$ $du = 1$ $dv = e^{-x}$ $v = -e^{-x}$

$= \lim_{R\to\infty} \left[-e^{-R}R^3 + 3\left(-e^{-R}R^2 + 2\left(\left[-xe^{-x}\right]_0^R + \int_0^R e^{-x}\,dx\right)\right) \right]$

$= \lim_{R\to\infty} \left[-e^{-R}R^3 + 3\left(e^{-R}R^2 + 2\left(-Re^{-R} + e^{-R} + 1\right)\right) \right]$

$= \lim_{R\to\infty} \left[e^{-R}R^3 - 3e^{-R}R^3 - 6e^{-R}R - 6e^{-R} + 6 \right]$ 9

$= 6$

11 (a)

METHOD 1

for example

$\vec{PQ} = \begin{pmatrix} -1 \\ -5 \\ 8 \end{pmatrix}, \vec{PR} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$ ***A1A1***

$\vec{PQ} \times \vec{PR} = 33\boldsymbol{i} + 11\boldsymbol{j} + 11\boldsymbol{k}$ ***(M1)A1***

r.n = a.n

$33x + 11y + 11z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ 11 \\ 11 \end{pmatrix} = 22$ ***(M1)***

$\Rightarrow 3x + y + z = 2$ or equivalent ***A1***

METHOD 2

assume plane can be written as $ax + by + cz = 1$ ***M1***

substituting each set of coordinates gives the system of equations:

$$a + 6b - 7c = 1$$

$$0a + b + c = 1$$

$2a + 0b - 4c = 1$ ***A1***

solving by GDC ***(M1)***

$a = \frac{3}{2}, b = \frac{1}{2}, C = \frac{1}{2}$ ***A1A1A1***

$\Rightarrow \frac{3}{2}x + \frac{1}{2}y + \frac{1}{2}z = 1$ or equivalent

(b)

METHOD 1

substitution of equation of line into both equations of planes ***M1***

$3\left(\frac{5}{4} + \frac{\lambda}{2}\right) + \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 2$ ***A1***

$\left(\frac{5}{4} + \frac{\lambda}{2}\right) - 3\lambda - \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 3$ ***A1***

METHOD 2

adding Π_1 and Π_2 gives $4x - 2y = 5$ ***M1***

given $y = \lambda \Rightarrow x = \frac{5}{4} + \frac{\lambda}{2}$ ***A1***

$z = 2 - y - 3x = -\frac{7}{4} - \frac{5\lambda}{2}$ ***A1***

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$ ***AG***

(c)

normal to Π_3 is perpendicular to direction of L

$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 0$ ***A1***

$\Rightarrow a + 2b - 5c = 0$ ***AG***

[1 mark]

(d)

substituting $\begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix}$ into Π_3: ***M1***

$\frac{5a}{4} - \frac{7c}{4} = 1$ ***A1***

$5a - 7c = 4$ ***AG***

[2 marks]

(e)

attempt to find scalar products for Π_1 and Π_3, Π_2 and Π_3.

and equating ***M1***

$\frac{3a+b+C}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = \frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***M1***

Note: Accept $3a + b + c = a - 3b - c$.

$\Rightarrow a + 2b + c = 0$ ***A1***

attempt to solve $a + 2b + c = 0, a + 2b - 5c = 0, 5a - 7c = 4$ ***M1***

$\Rightarrow a = \frac{4}{5},\ b = -\frac{2}{5},\ c = 0$ ***A1***

hence equation is $\frac{4x}{5} - \frac{2y}{5} = 1$

for second equation:

$\frac{3a+b+C}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = -\frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***(M1)***

$\Rightarrow 2a - b = 0$

attempt to solve $2a - b = 0, a + 2b - 5c = 0, 5a - 7c = 4$ ***M1***

$\Rightarrow a = -2, b = -4, c = -2$ ***A1***

hence equation is $-2x - 4y - 2z = 1$

[7 marks]