**Mathematics: analysis and approaches**
**Higher level**
**2022 Semester 2 Examinations**
**Paper 3**

ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Friday, September 2nd (morning)

1 hour

Candidate number

| 0 | 0 | 3 | 3 | 7 | 6 | 0 | 0 | 4 | 1 |
|---|---|---|---|---|---|---|---|---|---|

JAMES SULLIVAN

**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**. 50

8 + 13 = 21 / 50

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you could sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 25]

This question asks you to investigate definite integrals of powers of $(1+x^2)^{-1}$

(a) On the same set of axes, sketch and label $y=(1+x^2)^{-1}$, $y=(1+x^2)^{-2}$ and $y=(1+x^2)^{-3}$, for $0<x<1$. [2]

(b) Find the exact value of $\int_0^1 (1+x^2)^{-1}dx$. [3]

(c) By substituting $x=\tan\theta$, or otherwise, find the exact value of $\int_0^1 (1+x^2)^{-2}dx$. [7]

Let $I_n = \int_0^1 (1+x^2)^{-n}dx$.

(d) (i) By expressing $(1+x^2)^{-n}$ as $(1+x^2)^{-n}(1)$, or otherwise, show that

$$I_{n+1} = \left(1-\frac{1}{2n}\right)I_n + \frac{2^{-n-1}}{n}, \text{ for } n\geq 1.$$

(ii) Find the exact value of $I_n = \int_0^1 (1+x^2)^{-3}dx$ [9]

(e) Find the exact value of $\int_0^1 (x^2-2x+2)^{-3}dx$ [4]

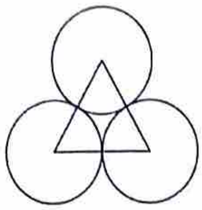
**2.** [Maximum mark: 30]

This question asks you to investigate the packing density of circles and spheres.

(a) An infinitely large table is covered by non-overlapping circular disks of equal radii.

Show the maximum proportion of the table that is covered is $\dfrac{\pi}{2\sqrt{3}}$ [4]

Hint:



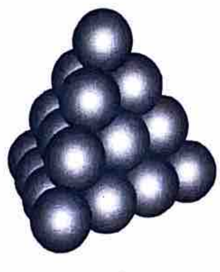
(b) Show by mathematical induction that $1+2+3+4+....+k=\dfrac{k(k+1)}{2}$ [5]

(c) (i) Expand $(k+1)^3$

(ii) By summing each side of (b) (i) from $k=1$ to $k=n$, show that

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$ [6]

The diagram below shows spherical balls of diameter 1 arranged in a triangular pyramid with $n$ layers



(d) (i) Find the number of balls in the $k$ th layer from the top, simplifying your answer.

(ii) Show that the number of balls in the pyramid is $\dfrac{1}{6}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{3}n$. [5]

Let *A, B, C* be the centres of the balls at the vertices of the pyramid.

(e) Show that the volume of the tetrahedron *ABCD* is $\dfrac{\sqrt{2}}{12}(n-1)^3$. [6]

(f) Find the exact value of the proportion of the tetrahedron *ABCD* that is occupied by the balls as $n$ approaches infinity. [4]

**End of paper 3**

$\dfrac{\pi r^2}{\sqrt{3} r^2}$

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**ANSWER BOOKLET**

**4 PAGES** 1

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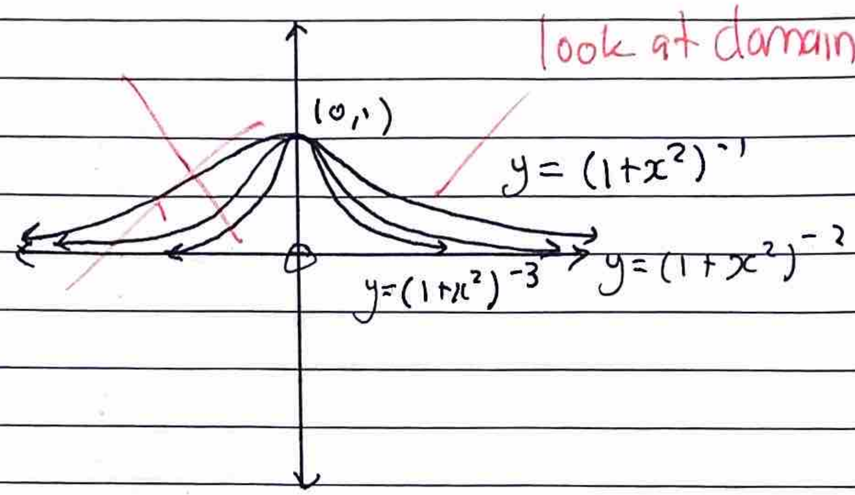
At the start of each answer to a question, write the question number in the box using your normal handwriting

Example 27 [2][7] Example 3 [ ][3]

[ ][1]

(a)

look at domain.




1.

b) $\cancel{\int_0^1 (1+x^2)^{-1}dx = \int_0^1 (1)(1+x^2)^{-1}dx}$

$\cancel{u = (1+x^2)^{-1}}$ $\quad du = -(1+x^2)^{-2}(2x)$

$\cancel{du = 1}$ $\quad \cancel{v = x}$

$= \cancel{\int [x(1+x^2)^{-1}]}$

$= \cancel{[x(1+x^2)^{-1}]_0^1 - \int -x(1+x^2)^{-2}2x}$

$= \cancel{[x(1+x^2)^{-1}]_0^1 + \int \frac{2x^2}{(1+x^2)^{-2}}}$

(b) $\int_0^1 \frac{1}{1+x^2}\,dx = [\arctan x]_0^1$

$= \arctan(1) - \arctan(0)$ 3

$= \pi/4$

(c) $\int_0^1 (1+x^2)^{-2}\,dx = \int \frac{1}{1+x^2}$

$= \int_0^1 \frac{1}{(1+x^2)^2}\,dx$

let $x = \tan\theta$
$\therefore dx = \sec^2\theta\,d\theta$

$= \int_0^1 \frac{1}{(1+x^2)^2}\,dx$

$= \int_0^1 \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta$

Change limits

$= \int_0^1 \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta$

$= \int_0^1 \frac{1}{\sec^2\theta}\,d\theta$ be careful.

$x = \tan\theta$
$\to 0 = \tan\theta, \theta = 0$
$1 = \tan\theta, \theta = \pi/4$

$= [\tan\theta]_0^1$ M1

4.

$= \tan(1) - \tan(0)$

$= \tan(1)$

$\to \int_0^{\pi/4} \cos^2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{\cos 2\theta}{2} + \theta\right) + C$

$= \int_0^{\pi/4} \frac{\cos 2\theta + 1}{2}\,d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/4}$

$= \frac{1}{2}\frac{\pi}{2} + 1/4$

(d) $I_n = \int_0^1 (1+x^2)^{-n} \, dx$

(i) $I_{n+1} = \int_0^1 (1+x^2)^{-(n+1)} (1) \, dx$

$\therefore \int_0^1 (1+x^2)^{n+1} \, dx = \int_0^1 (1) \, d I_{n+1}$

$\therefore \cancel{\oint \frac{1}{(n+2)(1+x^2)^{n+2}}}$

$\therefore \frac{1}{n+2} (1+x^2)^{n+2} \times \frac{1}{2x} = I_{n+1}$

$\therefore I_{n+1} = \frac{1}{n+2} \times (1+x^2)^{n+1+1} \times \frac{1}{2x}$

$= \frac{(1+x^2)}{n+2} \times (1+x^2)^{n+1} \times \frac{1}{2x}$

$I_n = \int_0^1 (1+x^2)^{-n} \, dx$

$= \int_0^1 (1)(1+x^2)^{-n}$

$=$

$\begin{cases} u = (1+x^2)^{-n} \rightarrow du = -n(1+x^2)^{-n-1} \\ dv = 1 \rightarrow v = x \end{cases}$

$\therefore I_n = x(1+x^2)^{-n} - \int_0^1 -xn(1+x^2)^{-(n+1)} \, dx$

$= \left[x(1+x^2)^{-n}\right]_0^1 + n \int_0^1 x(1+x^2)^{-(n+1)} \, dx$

$= \left[x(1+x^2)^{-n}\right]_0^1 + n \cdot \frac{1}{2} \int_0^1 u^{-(n+1)} \, du$

$= 2^{-n} - 0 + \frac{n}{2} \left[-(n+1) u^{-n}\right]_0^1$

heart mind life

(ii) $I_n = \int_0^1 (1+x^2)^{-n} dx$

$\therefore I_{n+1} = \int_0^1 (1+x^2)^{-n-1} (1) dx$

$\frac{dy}{dx} = \int (1+x^2)^{-n-1} (+)$

$\therefore \frac{1}{1}\frac{dy}{dx} =$

let $u = (1+x^2)^{-n-1}$ $\quad du = (-n-1)(1+x^2)^{-n-2}$

$dv = 1$ $\quad v = x$

$\therefore I_{n+1} = \left[x(1+x^2)^{-n-1}\right]_0^1 - \int_0^1 x(-n-1)(1+x^2)^{-n-2} dx$

$= \cancel{2^{-n-1} - 0 - \int -nx(1+x^2)^{-n-2}}$

$= 2^{-n-1} + \int_0^1 x(+n+1)(1+x^2)^{-(n+2)} dx$

$= 2^{-n-1} + \int_0^1 x(n)(1+x^2)^{-n} dx + \int_0^1 x(n+1)(1+x^2)^{-2} dx$

$= 2^{-n-1} + \int_0^1 x(n+1) I_n dx + \int_0^1 x(n+1)(1+x^2)^{-2} dx$

(e) $\int_0^1 (x^2-2x+2)^{-3} dx = 0.544524$ (GDC)

$= \left[\frac{-\frac{1}{2}(x^2-2x+2)^{-2}}{2x-2}\right]_0^1$

$= -\frac{1}{2(}$

0

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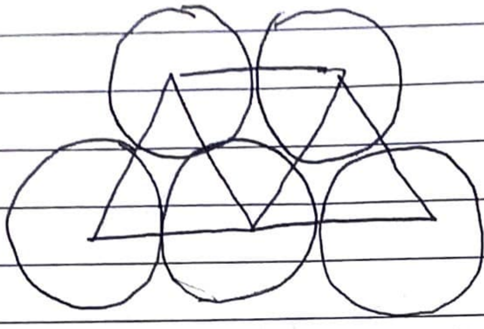
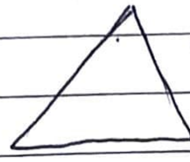
Example 27 [2][7]   Example 3 [ ][3]

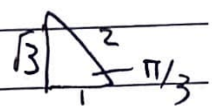
[ ][2]

(a)






sectors in the triangle.




$$\therefore \quad A_{circle} = \cancel{2\pi r^2} \quad \frac{1}{2}\pi r^2$$

$$A_{triangle} = \frac{1}{2}(2r)^2 \sin 60$$

$$= 2r^2 \times \sqrt{3}/2$$

$$= \sqrt{3}\, r^2$$

$$\cancel{\therefore \quad \frac{A_{triangle}}{A_{circle}} = \frac{\sqrt{3} r^2}{\pi r^2} = \sqrt{3}}$$

4.

$$\therefore \quad \frac{A_{circle}}{A_{triangle}} = \frac{\pi r^2}{2\sqrt{3}\, r^2}$$

$$= \frac{\pi}{2\sqrt{3}}$$

(b) $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

Step 1: Prove for $k=1$:

$LHS = 1$

$RHS = 1(1+1)/2$

$= 1$

$= LHS$

$\therefore$ true for $k=1$

Step 2: assume true for $k=n$

$\therefore 1+2+3+4+\cdots+n = n(n+1)/2$

Step 3: $1+2+3+4+\cdots+n+1 = \frac{(n+1)(n+2)}{2}$

$LHS = \frac{n(n+1)}{2} + (n+1)$ {by assumption}

$= \frac{n(n+1)+2(n+1)}{2}$

$= \frac{(n+1)(n+2)}{2}$

$= RHS$

$\therefore$ true for ~~$n=k+1$~~ $k=n+1$ whenever $n=k$

Step 4.

As it is true for $k=1$, and also true for $k=n+1$ whenever $k=n$ is true, it is true for all $k \in \mathbb{Z}^+$ by mathematical induction.

5

(c) (i) $(k+1)^3 = k^3+3k^2+3k+1$ $/2$

(ii) $$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}(k^3+3k^2+3k+1)$$

$$\therefore \sum_{k=1}^{n}(k^2) = \sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}(3k^2+3k+1)$$

$$= \sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}$$ 1

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}(k^3+3k^2+3k+1)$$

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$\therefore \sum_{k=1}^{n}k^2 = \left(\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}k^3 - 3\sum_{k=1}^{n}k + n\right)\times\frac{1}{3}$$

$$= \left(-1+(n+1)^3 - \frac{3n(n+1)}{2} + n\right)\times\frac{1}{3}$$

$$= \frac{n^3+3n^2+3n+1}{3} - \frac{1}{3} - \frac{n^2+n}{2} + \frac{n}{3}$$

$$= \frac{1}{3}n^3 + \left(\frac{3}{3}-\frac{1}{2}\right)n^2 + \left(\frac{3}{3}-\frac{1}{2}+\frac{1}{3}\right)n$$

$$= \frac{1}{3}n^3 - \frac{1}{2}n^2$$

heart
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(d) (i)

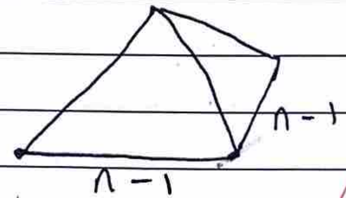
| | | |
|---|---|---|
| $k=1$ | 1 | |
| $k=2$ | 3 | 2 |
| $k=3$ | 6 | 3 |
| $k=4$ | 10 | 4 |

$\therefore I_k = k^3$

X

(d)(ii)

(e) $V_{BALL} = \frac{4}{3}\pi r^3$




$V = \frac{4}{3}\pi r^3\left(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right) \times \frac{2\sqrt{3}}{\pi}$

$= \frac{8\sqrt{3}}{3} r^3\left(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right)$

$= \frac{8\sqrt{3}}{3}\left(\frac{1}{6}\right) r^3\left(n^3 + 3n^2 + 2n\right)$

$=$

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Example 27 | 2 | 7 |   Example 3 | | 3 |

2

(f) $V = \frac{\sqrt{2}}{12}(n-1)^3$

$$\therefore V_\infty = \frac{\sqrt{2}}{12}\lim_{n\to\infty}\left((n-1)^3\right)$$

$$= \frac{\sqrt{2}}{12}\lim_{n\to\infty}\left(n^3 + 3n^2(-1) + 3n(-1)^2 + (-1)^3\right)$$

$$= \frac{\sqrt{2}}{12}\lim_{n\to\infty}\left(n^3 - 3n^2 + 3n - 1\right)$$

$$= \frac{\sqrt{2}}{12}\lim_{n\to\infty}\left(1 - 3/n + 3/n^2 - 1/n^3\right)$$

$$= \frac{\sqrt{2}}{12} \text{ units}^3$$

$$V_{BALLS} = \lim_{n\to\infty}\left(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{6} + \frac{1}{2n} + \frac{1}{3n^2}\right)$$

$$= 1/6 \text{ units}^3$$

$$\therefore \frac{V_{BALLS}}{V_\infty} = \frac{\sqrt{2}}{12} \times \frac{6}{1}$$

$$= \frac{\sqrt{2}}{2}$$