Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 2

Tuesday 14 May 2019 (morning) 1:47:00

2 hours ~~1:52:00~~

Candidate session number

19	M	TZ	1	P	2	M	A	H	L

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is ~~**[100 marks]**~~.

89 marks

$\frac{76}{89} = 85.4\%$

12 pages

2219–7204

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Let l be the tangent to the curve $y = xe^{2x}$ at the point $(1, e^2)$.

Find the coordinates of the point where l meets the x-axis.

$y = xe^{2x}$

$\therefore dy/dx = e^{2x} + 2xe^{2x}$

$= e^{2x}[1 + 2x]$

$= m$

COMPLETED IN AB.

$e^{2x} + e^{2}$

$= (e^{x})^{2x} + e^{2}$

$\Rightarrow \therefore y - y_0 = m(x - x_0)$

$\therefore y - e^2 = e^{2x}(1+2x)(x-1)$

$\Rightarrow$ meets x axis at $y=0$:

$-e^2 = (e^{2x} + 2xe^{2x})(x-1)$

$= xe^{2x} + 2x^2e^{2x} - e^{2x} - 2xe^{2x}$

$= e^{2x}(x + 2x^2 - 1 - 2x)$

$\therefore -e^2 = e^{2x}(2x^2 - x - 1)$

$\therefore 0 =$

$\therefore y - e^2 = e^{2x}(x - 1 + 2x^2 - 2x)$

$= e^{2x}(2x^2 - x - 1)$

$\therefore y = e^{2x}(2x^2 - x - 1) + e^2 = 0$

$\therefore 2x^2 - x - 1 = -\dfrac{e^2}{e^{2x}}$

$= -e^{2-2x}$

COMPLETED IN AB

2. [Maximum mark: 5]

Solve $z^2 = 4e^{\frac{\pi}{2}i}$, giving your answers in the form

(a) $re^{i\theta}$ where $r, \theta \in \mathbb{R}, r > 0$; [3]

(b) $a + ib$ where $a, b \in \mathbb{R}$. [2]

a) $z^2 = 4e^{\pi/2\, i}$

$= 4\,\text{cis}\left(\pi/2 + 2k\pi\right)$

$= 4\,\text{cis}\left(\frac{\pi + 4k\pi}{2}\right)$

$\therefore z = \cancel{4\,\text{cis}}\ 2\,\text{cis}\left(\frac{\pi + 4k\pi}{4}\right)$

$= 2\,\text{cis}\left(-\frac{3\pi}{4}\right),\ 2\,\text{cis}\left(\pi/4\right),\ \cancel{2\,\text{cis}\left(5\pi/4\right)}$

$= 2e^{-\frac{3\pi}{4}i},\ 2e^{\frac{\pi}{4}i},\ \cancel{2e^{-\frac{5\pi}{4}i}}$

b) $2\,\text{cis}\left(-\frac{3\pi}{4}\right) = \cancel{2\,\text{cis}}$ Q3

$= \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \times 2$ 1

$2\,\text{cis}\left(\frac{\pi}{4}\right) =$ Q1

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \times 2$

$\cancel{2\,\text{cis}\left(\frac{5\pi}{4}\right) = Q}$

4

3. [~~Maximum mark: 7~~]

The marks achieved by eight students in a class test are given in the following list.

8	4	7	6	10	9	7	3

(a) Find

(i) the mean;

(ii) the standard deviation. [2]

(b) The teacher increases all the marks by 2. Write down the new value for

(i) the mean;

(ii) the standard deviation. [2]

A ninth student also takes the test.

(c) Explain why the median is unchanged. [3]

4. [Maximum mark: 5]

The function f is defined by $f(x) = \sec x + 2,\ 0 \le x < \frac{\pi}{2}$.

(a) Write down the range of f. [1]

(b) Find $f^{-1}(x)$, stating its domain. [4]

$f(x) = \sec x + 2$

a) $\{y \mid y \geq 3,\ \cancel{y \leq 1}\}$ → not within domain

b) $f^{-1}(x)$ occurs at $x = \sec y + 2$

$\therefore x = \frac{1}{\cos y} + 2$

$\therefore x\cos y = 1 + 2\cos y$

$\therefore \cos y(x-2) = 1$

$\therefore \cos y = \frac{1}{x-2}$

$\therefore y = \arccos\left(\frac{1}{x-2}\right)$

$\therefore f^{-1}(x) = \arccos\left(\frac{1}{x-2}\right)$

Domain = Range of $f(x)$

$= \{x \mid x \geq 3,\ x \leq 1\}$ ECF

4

5. [Maximum mark: 6]

Use integration by parts to find $\int (\ln x)^2 \, dx$.

$\int (\ln x)^2 dx = \int 1.(\ln x)^2 dx$ $\quad u = (\ln x)^2 \quad du = \frac{2\ln x}{x}$

$= x(\ln x)^2 - \int 2\ln x \, dx$ $\quad dv = 1 \quad v = x$

$= x(\ln x)^2 - 2\int \ln x \, dx$

$= x(\ln x)^2 - 2\left[x\ln x - \int 1 \, dx\right]$ $\quad u = \ln x \quad du = 1/x$ $\quad dv = 1 \quad v = x$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

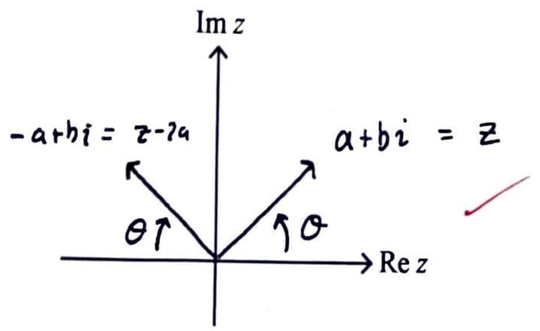
$\cancel{= \ln x(x\ln x - 2x) + 2x}$

6

6. [Maximum mark: 7]

Let $z = a + bi$, $a, b \in \mathbb{R}^+$ and let $\arg z = \theta$.

(a) Show the points represented by z and $z - 2a$ on the following Argand diagram. [1]



(b) Find an expression in terms of θ for

(i) $\arg(z - 2a)$;

(ii) $\arg\left(\dfrac{z}{z-2a}\right)$. [3]

(c) Hence or otherwise find the value of θ for which $\text{Re}\left(\dfrac{z}{z-2a}\right) = 0$. [3]

bi) $\arg(z) = \theta$ $\quad \arg(z-2a) = 180 - \theta$
$= \pi - \theta$

bii) $\arg\left(\dfrac{z}{z-2a}\right) = \arg(z) - \arg(z-2a)$
$= \theta - 180 + \theta$
$= 2\theta - 180 = 2\theta - \pi$ {rad}

c) $\dfrac{z}{z-2a} = \text{cis}(2\theta - 180) \times \left|\dfrac{z}{z-2a}\right|$

$\therefore \text{Re}\left(\dfrac{z}{z-2a}\right) = \left|\dfrac{z}{z-2a}\right| \cos(2\theta - 180) = 0$

$\therefore \cos(2\theta - 180) = 0$

$\therefore 2\theta - 180 = -90$

$\therefore 2\theta = 270$

$\therefore \theta = 135°$ ECF

$= 3\pi/4$

! stay in radians

keep in mind the domain restriction

$\frac{\pi}{2} + \pi = \frac{3\pi}{2}$

5/

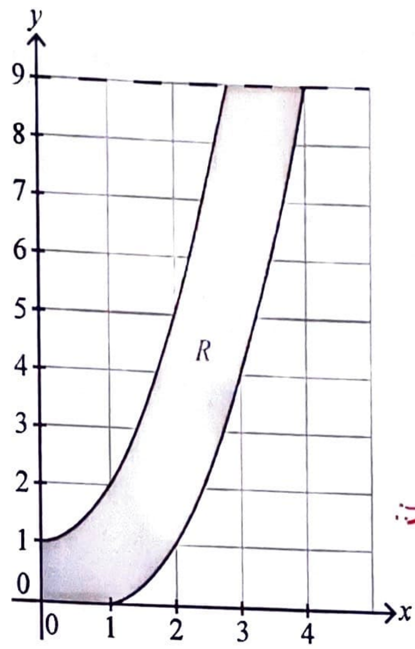
7. [Maximum mark: 7]

The function f is defined by $f(x) = (x-1)^2, x \geq 1$ and the function g is defined by $g(x) = x^2 + 1, x \geq 0$.
The region R is bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $y = 0$, $x = 0$ and $y = 9$ as shown on the following diagram.

$V = \int_0^9 (\sqrt{y}+1)^2 dy - \pi \int_1^9 (\sqrt{y-1})^2 dy$

$=$ {using GDC}

$V = 168.1$ units^3



find inverse:

$f(x) = (x-1)^2$

$\therefore y^{1/2} = x - 1$

$\therefore x = y^{1/2} + 1$

$g(x) = (x^2+1)$

$y - 1 = x^2$

$\therefore \cancel{y}\, x = \sqrt{y-1}$

$\cancel{V = \pi \int_0^9}$

The shape of a clay vase can be modelled by rotating the region R through 360° about the y-axis.

Find the volume of clay used to make the vase.

M1 M1

$V = \pi \int_0^4 \left[g(x)^2 - f(x)^2 \right] dx$

$= \pi \int_0^4 \left[(x^2+1)^2 - (x^2 - 2x + 1)^2 \right] dx$

$= \pi \int_0^4 \left[x^4 + 2x^2 + 1 - (x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1) \right] dx$

$\cancel{= \pi \int_0^4 [2x^2 - x^2 - 4x^2 - x^2 + 2x^3 + 2x^3}$

$= \pi \int_0^4 (\cancel{x^4} + 2x^2 + \cancel{1} - \cancel{x^4} + 4x^3 - 6x^2 + 4x - \cancel{1}) dx$

$= \pi \int_0^4 (6x - 6x^2 + 4x^3) dx$

$= \pi \left[3x^2 - \frac{6}{3}x^3 + x^4 \right]_0^4$

$= \pi (3 \cdot 4^2 - 2 \cdot 4^3 + 4^4)$

$\approx 552.920\ldots$ units^3

≈ 553 units^3

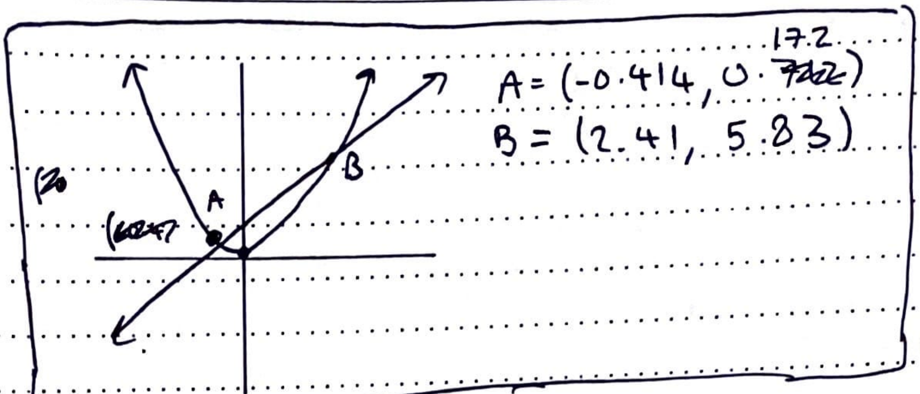
2/

8. [Maximum mark: 9]

(a) Solve the inequality $x^2 > 2x + 1$. [2]

(b) Use mathematical induction to prove that $2^{n+1} > n^2$ for $n \in \mathbb{Z}, n \geq 3$. [7]

a)



$A = (-0.414, 0.172)$

$B = (2.41, 5.83)$

GDC

$\therefore x^2 > 2x + 1$

$\downarrow$

$x < -0.414$, $x > 2.41$

b) $2^{n+1} > n^2$ $n \in Z$, $n \geq 3$

step 1: prove for $n = 3$:

$2^4 > 2^2$

$\therefore$ true for $n = 3$

step 2: assume true for $n = k$:

$2^{k+1} > k^2$

step 3: prove ~~true~~ for $n = k + 1$

$\therefore 2^{k+2} > (k+1)^2$

$\therefore 2^{(k+1)+1} > k^2 + 2k + 1$

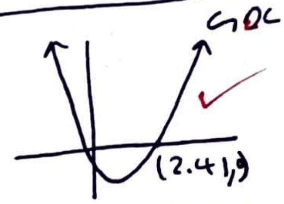
$\therefore 2 \cdot k^2 > k^2 + 2k + 1$ {from assumption}

$\therefore$ ~~$k^2 - 2k - 1 > 0$~~ $\therefore k^2 - 2k - 1 > 0$

$\therefore$ continued in second column

COLUMN 2

GDC



$\therefore$ ~~true~~ $k+1$ is true whenever k is true as $k \geq 3$ {only false $k < 2.41$}

step 4: as true for $n = 3$, and true for $n = k+1$ whenever ~~the~~ $n = k$ is true, then true for all $k \geq 3$, $k \in Z$ by mathematical induction.

9

Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 13]

A café serves sandwiches and cakes. Each customer will choose one of the following three options; buy only a sandwich, buy only a cake or buy both a sandwich and a cake.
The probability that a customer buys a sandwich is 0.72 and the probability that a customer buys a cake is 0.45.

(a) Find the probability that a customer chosen at random will buy

(i) both a sandwich and a cake;

(ii) only a sandwich. [4]

On a typical day 200 customers come to the café.

(b) Find

(i) the expected number of cakes sold on a typical day;

(ii) the probability that more than 100 cakes will be sold on a typical day. [4]

It is known that 46% of the customers who come to the café are male, and that 80% of these buy a sandwich.

(c) (i) A customer is selected at random. Find the probability that the customer is male and buys a sandwich.

(ii) A female customer is selected at random. Find the probability that she buys a sandwich. [5]

Do **not** write solutions on this page.

10. [Maximum mark: 20]

The voltage v in a circuit is given by the equation

$v(t) = 3\sin(100\pi t)$, $t \geq 0$ where t is measured in seconds.

(a) Write down the maximum and minimum value of v. [2]

The current i in this circuit is given by the equation

$$i(t) = 2\sin(100\pi(t + 0.003)).$$

(b) Write down two transformations that will transform the graph of $y = v(t)$ onto the graph of $y = i(t)$. [2]

The power p in this circuit is given by $p(t) = v(t) \times i(t)$.

(c) Sketch the graph of $y = p(t)$ for $0 \leq t \leq 0.02$, showing clearly the coordinates of the first maximum and the first minimum. [3]

(d) Find the total time in the interval $0 \leq t \leq 0.02$ for which $p(t) \geq 3$. [3]

The average power p_{av} in this circuit from $t = 0$ to $t = T$ is given by the equation

$$p_{av}(T) = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t, \text{ where } T > 0.$$

(e) Find $p_{av}(0.007)$. [2]

(f) With reference to your graph of $y = p(t)$ explain why $p_{av}(T) > 0$ for all $T > 0$. [2]

(g) Given that $p(t)$ can be written as $p(t) = a\sin(b(t - c)) + d$ where $a, b, c, d > 0$, use your graph to find the values of a, b, c and d. [6]

Do **not** write solutions on this page.

11. [Maximum mark: 17]

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.
The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.
The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

(a) Show that $abc = 8$. [5]

The values a, b, and c are consecutive terms in a geometric sequence.

(b) Show that one of the real roots is equal to 1. [3]

(c) Given that $q = 8d^2$, find the other two real roots. [9]

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Please write question numbers in the following format: / Veuillez numéroter les questions en utilisant la présentation suivante: / Sírvase escribir los números de las preguntas en el siguiente formato:

1 2 3 4 5 6 7 8 9 10

9

ai) ~~$P(A \cap B) = P(A) + P$~~

$P(S \cap C) = P(S) + P(C) - 1$

$= 0.72 + 0.45 - 1$

$= 0.170$

aii) $P(S \text{ only}) = P(S) - P(S \cap C)$

$= 0.72 - 0.17$

$= 0.550$

4

~~b~~ ci) $P(M) = 0.46$ $\therefore P(F) = 0.54$

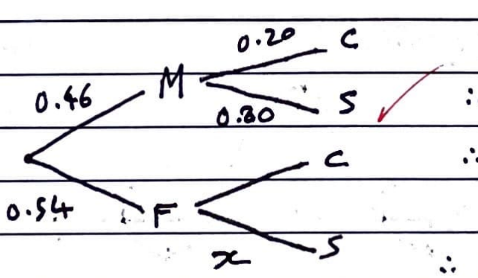
$P(S|M) = 0.800$ $\therefore$

$\therefore P(M \cap S) = P(M) \times P(S|M)$

$= 0.46 \times 0.80$

$= 0.368$

cii)



$P(S) = P(MS) + P(FS)$

$\therefore 0.72 = 0.368 + P(FS)$

$\therefore P(FS) = 0.352$

$\therefore 0.352 = 0.54 \times x$

$\therefore x = 0.652$

$\therefore P(S|F) = 0.652$

5

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10

a) $v(t) = 3\sin(100\pi t)$

$\therefore v'(t) = 3\cos(100\pi t)(100\pi)$

$= 300\pi\cos(100\pi t)$

max/min : $300\pi\cos(100\pi t) = 0$

$\therefore \cos(100\pi t) = 0$

$\therefore 100\pi t = \frac{\pi}{2} + k\pi$

$\therefore 100\pi t = \frac{\pi + 2k\pi}{2}$

$\therefore t = \frac{\pi + 2k\pi}{200\pi}$

$\therefore t = \frac{1+2k}{200}$ ✗

Read the question: $y_{max} = 3$

$y_{min} = -3$

b) $v(t) = 3\sin(100\pi t)$

$i(t) = 2\sin(100\pi(t+0.003))$

$\therefore$ $3\sin(100\pi t)$

$\downarrow$ - vertical stretch by scale factor $2/3$

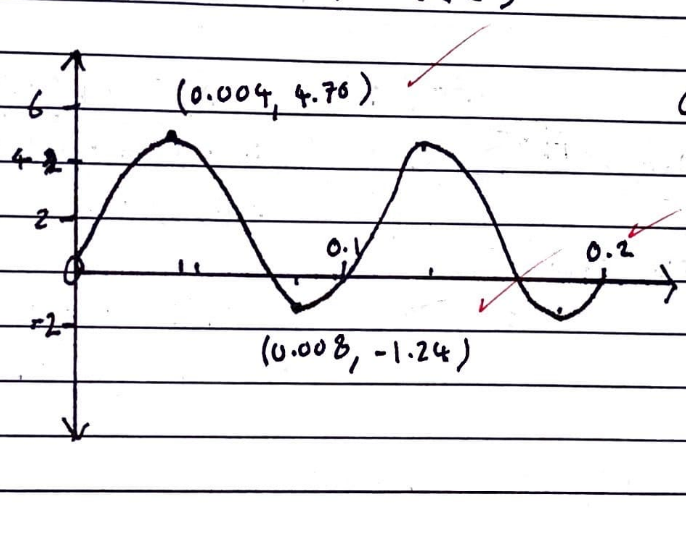
$2\sin(100\pi t)$

$\downarrow$ - translation to the left by 0.003

$2\sin(100\pi(t+0.003))$

2

c) $P(t) = v(t) \times i(t)$



G.D.C.

3

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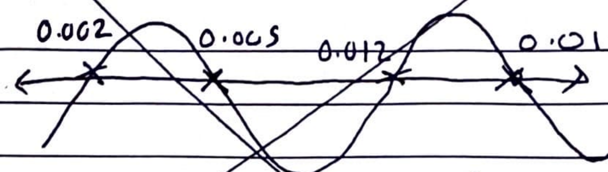
d) ~~x-intercepts: 0.007~~
~~0.010~~ ~~negative~~
~~0.017~~

~~$\therefore$ Graphing $P(x) - 3$ on the GDC, finding zeroes:~~

~~GDC:~~



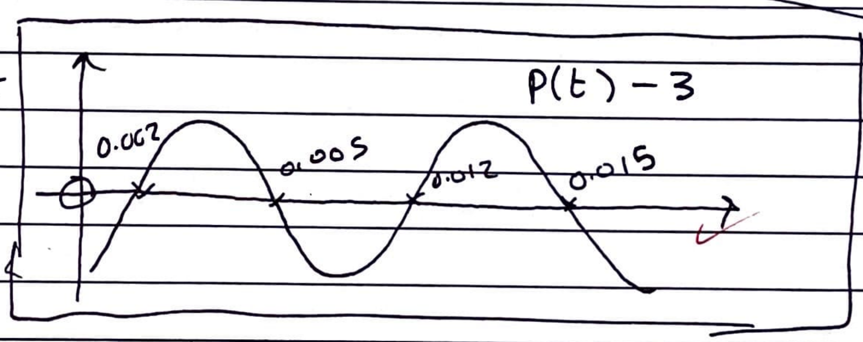
~~GDC.~~

~~$\therefore$ Total power $= \int_{0.002}^{0.005} (P(x)-3)\,dx + \int_{0.012}^{0.015} (P(x)-3)\,dx$~~

~~$= 0.004016 + 0.004016$~~
~~$= 0.008031$~~
~~≈ 0.00803~~

~~Total time $= 0.005 - 0.002 + 0.015 - 0.012$~~
~~$= 0.006$ seconds~~

GDC



$\therefore P(t) > 3$ for $0.002 \to 0.005$
$0.012 \to 0.015$

$\therefore t_{total} = 0.006$ seconds

3

3/

e) $P_{AV}(T) = \frac{1}{T}\int_0^T p(t)\,dt$

$\therefore P_{AV}(0.007) = \frac{1}{0.007}\int_0^{0.007} p(t)\,dt$

$= 2.867 \text{ W}$ {GDC}

2.87

2

f) ~~Fare~~ $\frac{\text{max} - \text{min}}{2} > 0$

$\Rightarrow$ this corresponds to an average that is > 0.

2

g) $p(t) = a\sin(b(t-c)) + d$

$d = \frac{\text{max}-\text{min}}{2} = \frac{4.76-(-1.24)}{2}$

$\therefore d = 3$ ✗

$c = 0$ ✗

$a = \text{Amplitude} = \text{Max} - d$

$= 4.76 - 3$

$\therefore a = 1.76$ ✗

$b = \text{frequency} = 2\pi / \text{period}$

$= 1 / 0.01$

$\therefore b = 100$ ✗

4

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1 2 3 4 5 6 7 8 9 10

1 1

$x^5 - 3x^4 + mx^3 + \cancel{n} nx^2 + px + q = 0$

roots: ~~$\log_2 \alpha$~~ $\log_2 a$, $\log_2 b$, $\log_2 c$, di, $-di$ } 5 distinct roots

a) ~~the~~ ~~sum of roots $= -q = \log_2 a + \log_2 b + \log_2 c$~~
~~$\therefore \log_2 abc = -q$~~
~~$\therefore abc =$~~

sum of roots:
$\log_2 a + \log_2 b + \log_2 c + di - di = 3/1$
$\therefore \log_2 abc = 3$
$\therefore abc = 2^3$
$\therefore abc = 8$

{SUM}

5

5/

b) let r be the common ratio

$\therefore a = b/r \qquad c = br \qquad \{GP\}$

$\because abc = 8$

$\therefore (\frac{b}{r})(b)(br) = 8$

$\therefore b^3 = 8$

$\therefore b = 2$

If a root is $\log_2 b$, then

$\log_2 b = \log_2 2$

$= 1$

$\therefore$ root of equation $= 1$.

3

c) Product :

$-q = (\log_2 a)(\log_2 b)(\log_2 c)(di)(-di)$

$= (\log_2 a)(\log_2 c)(d^2)(+1)$

$= (\log_2 a)(\log_2 c)d^2$

$\therefore q = -(\log_2 a)(\log_2 c)d^2$

$\Rightarrow$ If $q = 8d^2$, then

$-(\log_2 a)(\log_2 c) = 8$

$\therefore -(\log_2(b/r))(\log_2(br)) = 8$

$\therefore -(\log_2 b - \log_2 r)(\log_2 b + \log_2 r) = 8$

$\therefore -(1-\log_2 r)(1+\log_2 r) = 8$

$\therefore -(1-(\log_2 r)^2) = 8$

3/

$\therefore$ ~~$1 = \log_2 r$~~

$\therefore -1 + (\log_2 r)^2 = 8$

$\therefore$ ~~$-7 - (\log_2 r)^2 = \log 0$~~

$\therefore$ ~~$7 + (\log_2 r)^2 = 0$~~

$\therefore$ ~~$(\log_2 r)^2 = -7$~~

$\therefore \log_2 r = 3$

$\therefore r = 8$

Using r to find the other two roots

1) $\log_2 a = \log_2 (b/r)$
$= \log_2 b - \log_2 r$
$= 1 - 3$
$= -2$

2) $\log_2 b = \log_2 (br)$
$= \log_2 b + \log_2 r$
$= 1 + 3$
$= 4$

9

Alternative method.

Product: $r_1 \times r_2 \times 1 \times -di \times di = -8d^2$
$\therefore r_1 \times r_2 \times d^2 = -8d^2$
$\therefore r_1 \times r_2 = -8$

Sum: $r_1 + r_2 + 1 + di - di = 3$
$\therefore r_1 + r_2 = 2$

Solve simultaneously

$r_1 = -2, r_2 = 4$

9

$y = xe^{2x}$

$\therefore dy/dx = e^{2x} + 2xe^{2x}$

$= e^{2x}[1+2x]$

at $(1, e^2)$, $\frac{dy}{dx} = e^2 (3)$

$= 3e^2$

$= M$

$\therefore y - e^2 = 3e^2(x-1)$

$\therefore y = \cancel{\ } 3e^2(x-1) + e^2$

$\therefore$ x-intercept at $y=0$

$\left\{ \text{GDC: nSolve}(3 \cdot e^2 \cdot (x-1) + e^2 = 0, x) \quad 0.666667 \right\}$

$\therefore$ l meets x axis at $(\frac{2}{3}, 0)$

4