ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Diploma Programme

Candidate session number

Solutions.

Mathematics

Higher level

Paper 1

Trial Examination 2020

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[100 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines if necessary.

1. [Maximum mark: 9]

Given the function $f(x) = \ln x - \ln(1-x)$,

(a) Find:

(i) the domain

(ii) the range

(iii)the inverse function $f^{-1}(x)$ [5]

(b) Sketch $y = f(x)$ and $y = f^{-1}(x)$, labelling any intercepts and asymptotes. [4]

(a) (i) $x > 0 \quad 1-x > 0$

$x < 1$

$\therefore D: \ 0 < x < 1$

(ii) $y \in \mathbb{R}$

(iii) $x = \ln y - \ln(1-y)$

$x = \ln\left(\frac{y}{1-y}\right)$

$\frac{y}{1-y} = e^x$

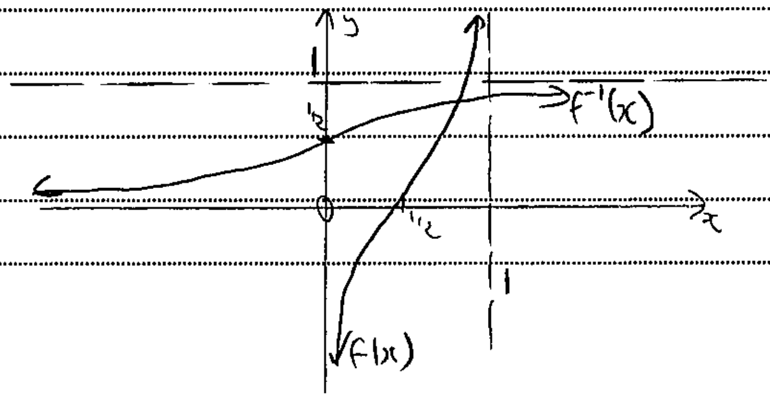
$y = e^x - e^x y$

$y + ye^x = e^x$

$y(1+e^x) = e^x$

$\therefore f^{-1}(x): \ y = \frac{e^x}{1+e^x}$

(b)



2. [Maximum mark: 6]

(a) Prove the identity

$$\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}=2\sec\theta.$$ [2]

(b) Solve the equation $\sec^2 x+2\tan x=0,\quad -\pi\le x\le\pi$. [4]

a) LHS $= \dfrac{\cos^2\theta + 1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$

$= \dfrac{2+2\sin\theta}{(1+\sin\theta)\cos\theta}$ $\sin^2\theta+\cos^2\theta=1$

$= \dfrac{2}{\cos\theta}$

$= 2\sec\theta = \text{RHS}$

b) $\sec^2 x+2\tan x=0$

$(\tan^2 x+1)+2\tan x=0$

$\tan^2 x+2\tan x+1=0$

$(\tan x+1)^2=0$

$\tan x=-1$

$\therefore x=-\pi/4, \frac{3\pi}{4}$

3. [Maximum mark: 7]

(a) Write the first three derivatives of $f(x) = x^2 e^x$. [3]

(b) Use mathematical induction to prove that

$$f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$$

where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative. [4]

(a) $f(x) = x^2 e^x$

$f'(x) = 2xe^x + x^2e^x$

$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x$

$f'''(x) = 2e^x + 2e^x + 2xe^x + 2e^x + 2xe^x + 2xe^x + x^2e^x$

$= 6e^x + 6xe^x + x^2e^x$

$= e^x[x^2 + 6x + 6]$

(b) Prove true for $n=1$ ie $f(x) = e^x[x^2 + 2x + 0]$

True (see part (a))

Assume true for $n=k$ ie $f^{(k)}x = e^x[x^2 + 2kx + k(k-1)]$

Prove true for $n=k+1$ ie $f^{(k+1)}x = e^x[x^2 + 2(k+1)x + (k+1)k]$

From assumption $f^{(k)}x = e^x x^2 + 2ke^x x + k(k-1)e^x$

$\therefore f^{(k+1)}x = 2xe^x + x^2e^x + 2ke^x + 2ke^x x + k(k-1)e^x$

$= x^2e^x + (2+2k)xe^x + (2k + k(k-1))e^x$

$= x^2e^x + 2(k+1)xe^x + (k+k^2)e^x$

$= x^2e^x + 2(k+1)xe^x + k(k+1)e^x$

$= e^x[x^2 + 2(k+1)x + k(k+1)]$

$\therefore$ True for $n=k+1$

If true for $n=1$ and $n=k+1$, then it is true for $n=1+1=2$ and $n=2+1=3$ etc for all $n \in \mathbb{Z}^+$

4. [Maximum mark: 8]

(a) Factorise $2x^2 - 3x - 5$. [2]

(b) Hence, or otherwise, find the coefficient of x^{23} in the expansion of $(2x^2 - 3x - 5)^{12}$, writing your answer in the form $k \times 2^m$ where $k, m \in \mathbb{Z}$. [6]

(a) $(2x-5)(x+1)$

(b) $(2x^2-3x-5)^{12} = (2x-5)^{12}(x+1)^{12}$

$U_{n+1} = {}^{12}C_n (2x)^{12-n}(-5)^n \times {}^{12}C_k x^k$

$= {}^{12}C_n \times {}^{12}C_k \, x^{12-n+k} \, 2^{12-n} (-5)^n$

$\therefore$ Coefficient x^{23} when $12-n+k=23$

$-n+k=11 \quad \therefore \; n=1 \quad k=12 \quad$ ①

$\underline{\text{and}} \quad n=0 \quad k=11 \quad$ ②

from ① $\quad {}^{12}C_1 \times {}^{12}C_{12} \; 2^{11}(-5)^1 = -60 \times 2^{11}$

from ② $\quad {}^{12}C_0 \times {}^{12}C_{11} \times 2^{12}(-5)^0 = 12 \times 2^{12}$

$\therefore$ Coefficient $= -60 \times 2^{11} + 12 \times 2^{12}$

$= -60 \times 2^{11} + 24 \times 2^{11}$

$= -36 \times 2^{11}$

$= -9 \times 4 \times 2^{11}$

$= -9 \times 2^{13}$

$k=-9, \; m=13$

5. [Maximum mark: 6]

(a) Find $\int x^2 \sin x \, dx$. [4]

(b) Evaluate $\int_{-1}^{1} x^2 \sin x \, dx$. [2]

(a) $\int x^2 \sin x \, dx$ let $u = x^2$ $dv = \sin x$

$du = 2x$ $v = -\cos x$

$= -x^2 \cos x - \int 2x(-\cos x) dx$

$= -x^2 \cos x + \int 2x \cos x \, dx$ let $u = 2x$ $dv = \cos x$

$du = 2$ $v = \sin x$

$= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$

(b) $\int_{-1}^{1} x^2 \sin x \, dx$

$= \left[-x^2 \cos x + 2x \sin x + 2\cos x \right]_{-1}^{1}$

$= 0$

6. [Maximum mark: 9]

Let the probability that it rains on any one day be p and the weather on any day is independent of the weather on any other day.

(a) Using $p = 0.5$, find the probability that during a period of one week:

(i) it will rain on at least five ;

(ii) it will rain on the last day;

(iii) raining and non-raining days will alternate. [5]

(b) Find p, if during a full week period, it is equally likely that there will be five raining days as there will be six raining days. [4]

(a) (i) $p = 0.5 \therefore q = 0.5$

$P(x \geq 5) = P(x=5) + P(x=6) + P(x=7)$

$= {}^7C_5 (1/2)^5 (1/2)^2 + {}^7C_6 (1/2)^6 (1/2) + {}^7C_1 (1/2)^7$

$= {}^7C_5 (1/2)^7 + {}^7C_6 (1/2)^7 + (1/2)^7$

$= (1/2)^7 (21 + 7 + 1)$

$= \frac{1}{2^7} \times 29$

$= 29/2^7$

$= 29/128$

(ii) $p = 1/2$

(iii) $RR'RR'RR'R + R'RR'RR'RR'$

$= pq\,pq\,pqp + qp\,qp\,qpq$

$= p^7 + p^7$

$= 2p^7$

$= 1/64$

(iv) $P(x=5) = P(x=6)$

${}^7C_5 p^5 (1-p)^2 = {}^7C_6 p^6 (1-p)$

$21(1-p) = 7p$

$3 - 3p = p$

$4p = 3$

$p = 3/4$

7. [Maximum mark: 5]

Prove that $1 + \text{cis}\,\theta + \text{cis}\,2\theta + \ldots + \text{cis}\,n\theta = \frac{1-\text{cis}(n+1)\theta}{1-\text{cis}\theta}$.

Prove true for $n=1$

$LHS = 1+\text{cis}\,\theta \qquad RHS = \frac{1-\text{cis}\,2\theta}{1-\text{cis}\,\theta}$

$$= \frac{1-(\cos 2\theta + i\sin 2\theta)}{1-\text{cis}\,\theta}$$

$$= \frac{1-(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)}{1-\text{cis}\,\theta}$$

$$= \frac{1-(\cos\theta + i\sin\theta)^2}{1-\text{cis}\,\theta}$$

$$= \frac{(1-\text{cis}\,\theta)(1+\text{cis}\,\theta)}{1-\text{cis}\,\theta}$$

$= 1+\text{cis}\,\theta = LHS \quad \therefore$ True.

Assume true for $n=k+1$

i.e. $1+\text{cis}\,\theta+\text{cis}\,2\theta+\ldots+\text{cis}\,k\theta = \frac{1-\text{cis}(k+1)\theta}{1-\text{cis}\,\theta}$.

Prove true for $n=k+1$

ie $\underline{1+\text{cis}\,\theta+\text{cis}\,2\theta+\ldots+\text{cis}\,k\theta} + \text{cis}(k+1)\theta = \frac{1-\text{cis}(k+2)\theta}{1-\text{cis}\,\theta}$

$LHS = \frac{1-\text{cis}(k+1)\theta}{1-\text{cis}\,\theta} + \text{cis}(k+1)\theta$ From assumption.

$$= \frac{1-\text{cis}(k+1)\theta + (\text{cis}(k+1)\theta)(1-\text{cis}\,\theta)}{1-\text{cis}\,\theta}$$

$$= \frac{1-\text{cis}(k+1)\theta + \text{cis}(k+1)\theta - \text{cis}(k+1)\theta \times \text{cis}\,\theta}{1-\text{cis}\,\theta}$$

$$= \frac{1-\text{cis}(k+2)\theta}{1-\text{cis}\,\theta} = RHS.$$

$\therefore$ True for $n=k+1$

Section B

8 (a) By the triangle inequality

$$2x+3 < x+1+2x+1$$

$$\underline{1 < x}$$

(b) $(x+1)^2 + (2x+1)^2 = (2x+3)^2$

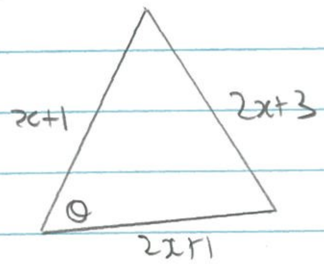
$$x^2+2x+1+4x^2+4x+1 = 4x^2+12x+9$$

$$x^2-6x-7=0$$

$$(x-7)(x+1)=0$$

$$\underline{x=7} \quad \text{as } x > 1.$$

(c) (i) largest angle is opposite largest side $2x+3$.



$$\cos\theta = \frac{(x+1)^2+(2x+1)^2-(2x+3)^2}{2(x+1)(2x+1)}$$

$$= \frac{x^2-6x-7}{2(x+1)(2x+1)}$$

$$= \frac{(x-7)(x+1)}{2(x+1)(2x+1)}$$

$$\underline{\cos\theta = \frac{x-7}{2(2x+1)}}$$

(ii) $\cos 120 = \dfrac{x-7}{2(2x+1)}$ as $120°$ must be the largest angle.

$$-\frac{1}{2} = \frac{x-7}{2(2x+1)}.$$

$$4x+2 = 14-2x$$

$$6x = 12$$

$$\underline{x=2}$$

(d) on next page

8 (d) 60° must be the middle angle

$\therefore$ opposite $2x+1$.

$$\cos 60 = \frac{(2x+3)^2 + (x+1)^2 - (2x+1)^2}{2(2x+3)(x+1)}.$$

$$1/2 = \frac{4x^2 + 12x + 9 + x^2 + 2x + 1 - 4x^2 - 4x - 1}{2(2x+3)(x+1)}$$

$$1/2 = \frac{x^2 + 10x + 9}{2(2x+3)(x+1)}$$

$$\frac{1}{2} = \frac{(x+9)(x+1)}{2(2x+3)(x+1)}$$

$$\frac{1}{2} = \frac{x+9}{2(2x+3}.$$

$$4x+6 = 2x+18$$

$$2x = 12$$

$$x = 6$$

9. (a) $y = x^3 - 3x^2 - 2x - 6$

$\frac{dy}{dx} = 3x^2 - 6x - 2 = 0$ for stat. points.

$x = \frac{6 \pm \sqrt{60}}{6}$

$= \frac{3 \pm \sqrt{15}}{3}$

(b) $\frac{d^2y}{dx^2} = 6x - 6 = 0$ for pt of inf

$x = 1$.

mean of stat points $= \frac{\frac{3+\sqrt{15}}{3} + \frac{3-\sqrt{15}}{3}}{2}$

$= 1$.

$\therefore$ true

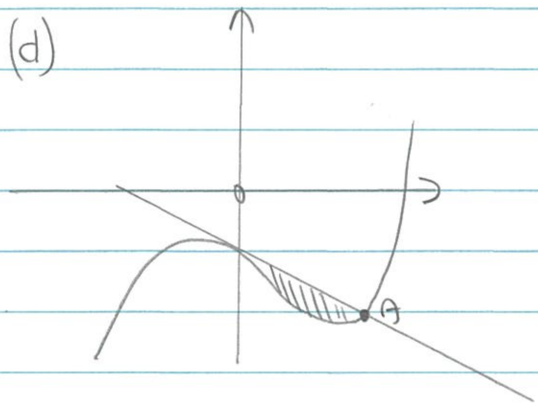
(c) $x = 0,\ y = -6$.

$x = 0 \quad \frac{dy}{dx} = -2$.

$y - (-6) = -2(x - 0)$.

$y = -2x - 6$

(d)



A: $y = x^3 - 3x^2 - 2x - 6$

$y = -2x - 6$

$x^3 - 3x^2 - 2x - 6 = -2x - 6$.

$x^3 - 3x^2 = 0$

$x^2(x-3) = 0$

$x = 0 \quad x = 3$.

$A = \int_0^3 [(-2x-6) - (x^3 - 3x^2 - 2x - 6)]\,dx$

$= \int_0^3 (-x^3 + 3x^2)\,dx$

$= \left[-\frac{x^4}{4} + x^3\right]_0^3$

$= -\frac{3^4}{4} + 27$

$= 6\,{}^3\!/_4\ u^3$

Section B

Q 10)

a) (i) $\frac{x-1}{2} = \frac{y-(-2)}{1} = \frac{z-(-1)}{3}$

direction vector $\begin{pmatrix}2\\1\\3\end{pmatrix}$

point A on the line l $\begin{pmatrix}1\\-2\\-1\end{pmatrix}$

line m : dir. vector $\begin{pmatrix}3\\-2\\1\end{pmatrix}$

point B on line m $\begin{pmatrix}2\\2\\4\end{pmatrix}$

$x_l = x_m \Rightarrow \quad 1+2\mu = 2+3\lambda \longrightarrow \lambda = 1$

$y_l = y_m \Rightarrow \quad -2+\mu = 2-2\lambda \rightarrow \mu = 4-2\lambda \rightarrow \mu = 2$

$z_l = z_m \Rightarrow \quad -1+3\mu = 4+\lambda \longrightarrow$ consistent with $\begin{pmatrix}\lambda=1\\\mu=2\end{pmatrix}$

l & m intersect at point C

(ii) $x_c = 5$ $\quad C\begin{pmatrix}5\\0\\5\end{pmatrix}$

$y_c = 0$

$z_c = 5$

b) plane Π has direction vector $\vec{n}$ $\perp$ to both (l) & m

$\vec{n} = \begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\2 & 1 & 3\\3 & -2 & 1\end{pmatrix} = \hat{i}\,(1+6) - \hat{j}\,(2-9) + k\,(-4-3)$

$= 7\hat{i} + 7\hat{j} - 7k$

$\vec{n} = \begin{pmatrix}7\\7\\-7\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\begin{pmatrix}x-5\\y-0\\z-5\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix} = 0 = x-5+y-z+5 \Rightarrow \boxed{x+y-z=0}$

Q.10 Continued

c) $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\hat{n} = \frac{\vec{n}}{\sqrt{1^2+1^2+(-1)^2}} = \frac{\vec{n}}{\sqrt{3}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k})$

d) line K: $\vec{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\frac{x_{P_1}+x_{P_2}}{2} = x_C$ $x_{P_1} = 5+t_1$; $x_{P_2} = 5+t_2 \Rightarrow t_1+t_2=0$

$t_2 = -t_1$

$\frac{y_{P_2}+y_{P_2}}{2} = y_C$ $y_{P_1} = t_1$; $y_{P_2} = t_2$

$\frac{z_{P_1}+z_{P_2}}{2} = z_C$ $z_{P_1} = 5-t_1$; $z_{P_2} = 5-t_2$

$P_1C = 5 \Rightarrow \sqrt{(\cancel{5+t}\ t-5)^2+(t-0)^2+(5-t-5)^2} = \sqrt{3t^2} = 5$

$t = \frac{5}{\sqrt{3}} \Rightarrow P_1 : x_{P_1} = 5+\frac{5}{\sqrt{3}}$; $y_{P_1} = \frac{5}{\sqrt{3}}$; $z_{P_1} = 5-\frac{5}{\sqrt{3}}$

$P_2 \left(t_2 = -\frac{5}{\sqrt{3}}\right) : x_{P_2} = 5-\frac{5}{\sqrt{3}}$; $y_{P_2} = -\frac{5}{\sqrt{3}}$; $z_{P_2} = 5+\frac{5}{\sqrt{3}}$

e) $P_1 : \begin{pmatrix} 5+\frac{5}{\sqrt{3}} \\ \frac{5}{\sqrt{3}} \\ 5-\frac{5}{\sqrt{3}} \end{pmatrix}$ The plane is $\parallel$ to $S_1 \Rightarrow$ they share the normal vector $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x-5-\frac{5}{\sqrt{3}} \\ y-\frac{5}{\sqrt{3}} \\ z-5+\frac{5}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow x-\cancel{5}-\frac{5}{\sqrt{3}}+y-\frac{5}{\sqrt{3}}-z+\cancel{5}-\frac{5}{\sqrt{3}} = 0$

$$\boxed{x+y-z = 5\sqrt{3}}$$