

Mathematics: analysis and approaches
Higher level
Paper 3

~~EZ0~~ TZ1

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.



Diploma Programme
Programme du diplôme
Programa del Diploma

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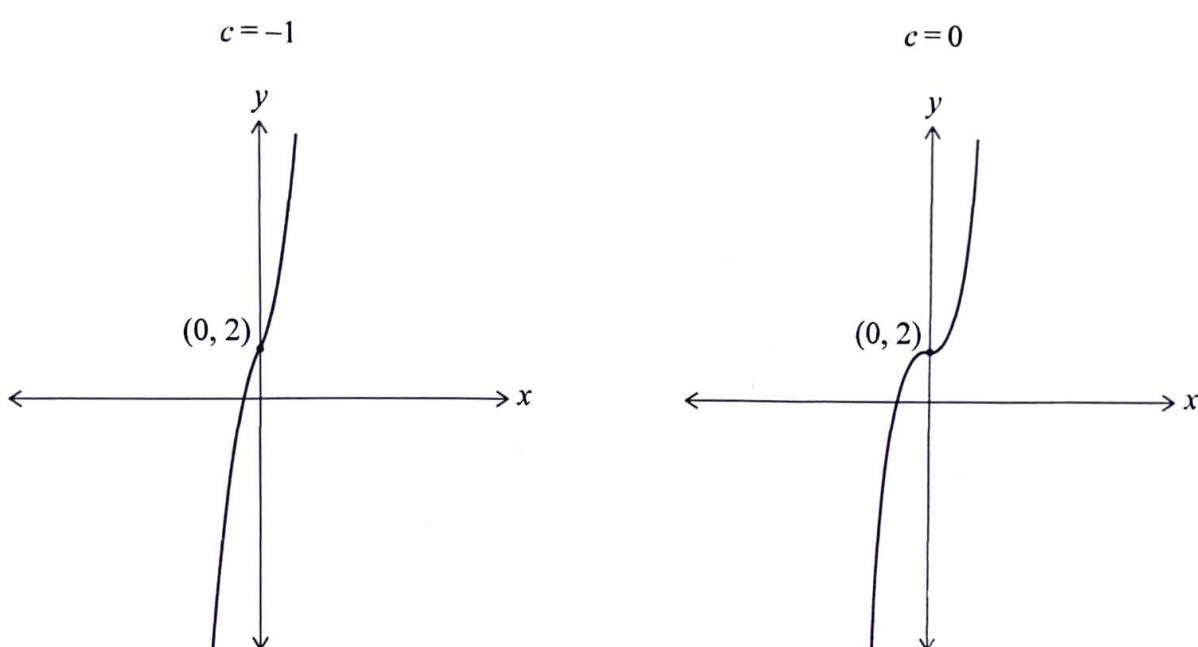
Answer all questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



- (a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y -intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

- (b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has
- (i) a point of inflexion with zero gradient; [1]
 - (ii) one local maximum point and one local minimum point; [2]
 - (iii) no points where the gradient is equal to zero. [1]
- (d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that
- (i) the y -coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]
 - (ii) the y -coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]
- (e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has
- (i) exactly one x -axis intercept; [2]
 - (ii) exactly two x -axis intercepts; [2]
 - (iii) exactly three x -axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c, d \in \mathbb{R}$.

- (f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x -axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

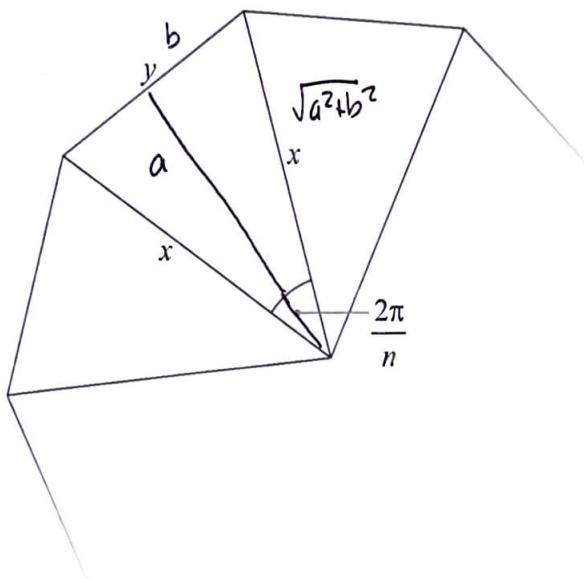
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P .

- (a) Find the side length, s , where $s > 0$, of a square such that $A = P$. [3]

An n -sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n -sided regular polygon is shown in the following diagram.



- (b) Write down, in terms of x and n , an expression for the area, A_T , of one of these isosceles triangles. [1]

- (c) Show that $y = 2x \sin \frac{\pi}{n}$. [2]

Consider a n -sided regular polygon such that $A = P$.

- (d) Use the results from parts (b) and (c) to show that $A = P = 4n \tan \frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

- (e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right)$. [3]
- (ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2 + b^2}$, where $a \geq b$, such that $A = P$.

- (f) Show that $a = \frac{8}{b-4} + 4$. [7]
- (g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a, b, A, P \in \mathbb{Z}$. [3]
- (ii) Determine the area and perimeter of these two right-angled triangles. [1]
-

References:

ANSWER BOOKLET
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4 PAGES / PÁGINAS

①
44
 $\frac{44}{55} = 0.8$



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Q1: $\frac{24}{27}$

Q2: $\frac{30}{28}$

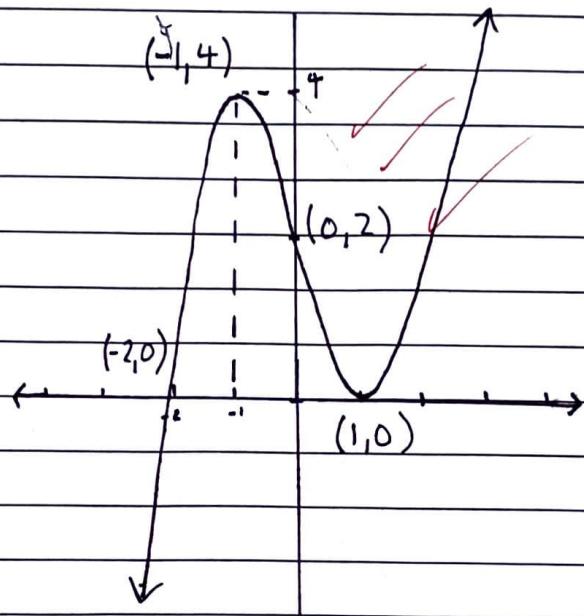
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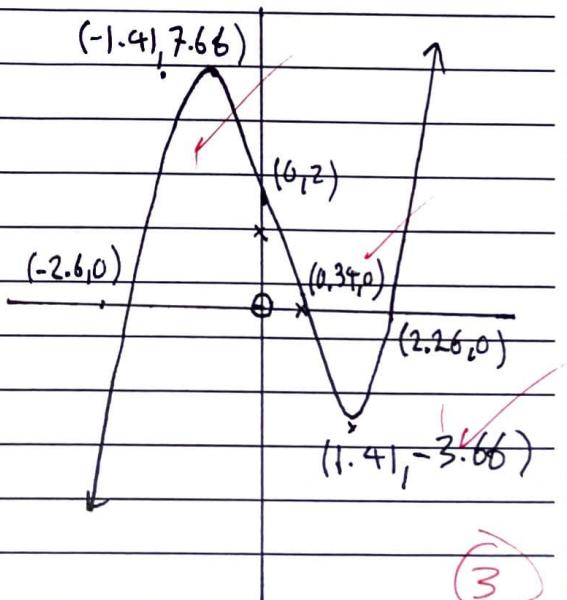
1 2 3 4 5 6 7 8 9 10

a) i) $f(x) = x^3 - 3x + 2$



③

ii) $f(x) = x^3 - 6x + 2$



6



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b) $f(x) = x^3 - 3cx + 2$ (1)

$$\therefore f'(x) = 3x^2 - 3c$$

c) i) $f'(x) = 0 = 3x^2 - c$

$\therefore x^2 = \frac{c}{3}$

$\therefore x = \pm\sqrt{\frac{c}{3}}$

$\therefore c = x^2$

$c \rightarrow 0 \quad c=0$ (1)

ii) $c > 0$ (1) $3x^2 - 3c = 0$

$c > 0$

iii) $c < 0$ (1)

d) i) $f'(x) = 3x^2 - 3c = 0$

$\therefore x^2 = c$

$\therefore x = \pm\sqrt{c}$ (1)

When $x = -\sqrt{c}$, $f(-\sqrt{c}) = (-\sqrt{c})^3 + 3(-\sqrt{c})c + 2$

$$= -\sqrt{c}^3 + 3c\sqrt{c} + 2$$
$$= -c^{3/2} + 3c^{3/2} + 2$$
$$\therefore y = 2c^{3/2} + 2$$

{NOTE: THIS IS MAX DETERMINED BY G.D.C}

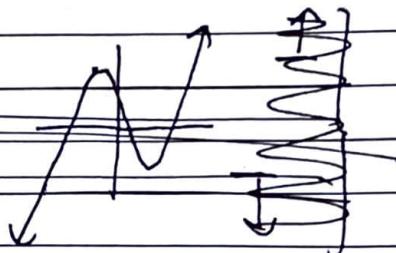
ii) When $x = \sqrt{c}$, $f(\sqrt{c}) = (\sqrt{c})^3 - 3(\sqrt{c})c + 2$

$$= c^{3/2} - 3c^{3/2} + 2$$
$$= -2c^{3/2} + 2$$

{NOTE: MIN determined graphically}

e) When $c > 0$, there is one local min and one local max.

i) ~~1 intercept~~ :



$$f(x) = x^3 - 3cx + 2$$

e) i) one intercept : local min ≥ 0

$$\therefore -2c^{3/2} + 2 \geq 0$$

$$\therefore 2c^{3/2} \leq 2$$

$$\therefore c^{3/2} \leq 1$$

$$\therefore c \leq 1$$

$$\therefore 0 < c \leq 1$$

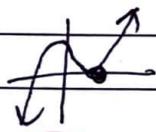
{note: this includes touching, but not crossing the x-axis}

ii) two intercepts : local min = 0

$$\therefore -2c^{3/2} + 2 = 0$$

$$\therefore 2c^{3/2} = 2$$

$$\therefore c = 1$$



iii) three intercepts : local min < 0

$$\therefore c > 1$$



f) $g(x) = x^3 - 3cx + d$ $x \in \mathbb{R}$
 $c, d \in \mathbb{R}$

$\Rightarrow g'(x) = 3x^2 - 3c$ {same as from part b}

\Rightarrow Finding y-coord of local min.

~~$y =$~~ $\Rightarrow x = \sqrt[3]{c}$

$$\begin{aligned}\therefore g(\sqrt[3]{c}) &= (\sqrt[3]{c})^3 - 3\sqrt[3]{c}(\cancel{x}) + d \\ &= c^{3/2} - 3c^{3/2} + d \\ &= -2c^{3/2} + d\end{aligned}$$

\Rightarrow one x-intercept occurs when $g(\sqrt[3]{c}) > 0$

(3)

$\therefore -2c^{3/2} + d > 0$

(4)

$$\therefore 2c^{3/2} < d$$

$$\therefore d > 2c^{3/2}, c > 0$$

OR $c \leq 0$ for all d.

4

4 PAGES / PÁGINAS

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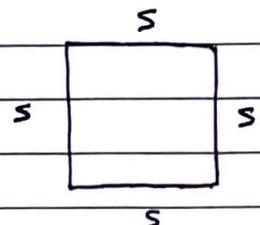
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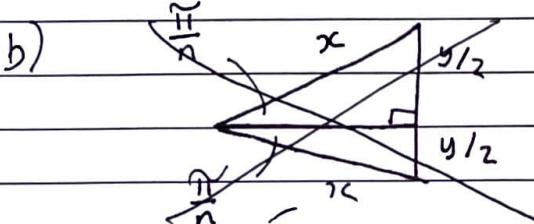
1 2 3 4 5 6 7 8 9 10

a)



$$\begin{aligned} \therefore 4s &= s^2 \\ \therefore s^2 - 4s &= 0 \\ \therefore s(s-4) &= 0 \\ \therefore s &\neq 0, s = 4 \\ \therefore s &= 4 \quad \{s > 0\} \end{aligned}$$

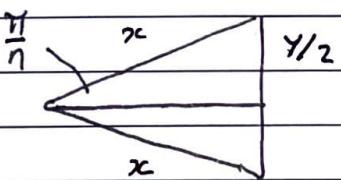
(3)



$$\begin{aligned} A &= \frac{1}{2}(x)(x) \sin(2\pi/n) \\ \therefore A_T &= \frac{1}{2}x^2 \sin(2\pi/n) \end{aligned}$$

(1)

c)

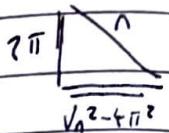


$$\begin{aligned} \sin \frac{\pi}{n} &= \frac{y/2}{x} \\ \therefore y/2 &= x \sin \pi/n \\ \therefore y &= 2x \sin \pi/n \end{aligned}$$

(2)

8 ✓

d)



$$A = n A_T \quad (1)$$

$$= n \times \frac{1}{2} x^2 \sin\left(\frac{2\pi}{n}\right) \quad (1)$$

$$P = n \times y \quad \checkmark$$

$$= 2nr \sin\left(\frac{\pi}{n}\right) \quad (2)$$

If $A = P$, then

$$\frac{1}{2} n x^2 \sin\left(\frac{2\pi}{n}\right) = 2nr \sin\left(\frac{\pi}{n}\right)$$

$$\therefore x \sin\left(\frac{2\pi}{n}\right) = 4 \sin\left(\frac{\pi}{n}\right)$$

$$\therefore x = 4 \frac{\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\therefore A = \frac{1}{2} n \left(4 \frac{\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)} \right)^2 \sin\left(\frac{2\pi}{n}\right)$$

$$= \frac{1}{2} \times 16n \times \frac{\sin^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{2\pi}{n}\right)} \sin\left(\frac{2\pi}{n}\right)$$

$$= -8n \frac{\sin^2\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)}$$

$$x = 4 \frac{\sin\left(\frac{\pi}{n}\right)}{2 \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)}$$

$$= 2 \tan\left(\frac{\pi}{n}\right) \times \frac{1}{\sin\left(\frac{\pi}{n}\right)} \quad (3)$$

$$(3) \rightarrow (2) : P = 2n \left(\frac{2 \tan\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} \right) \sin\left(\frac{\pi}{n}\right)$$

$$= 4n \tan\left(\frac{\pi}{n}\right)$$

$$\therefore A = P = 4n \tan\left(\frac{\pi}{n}\right)$$

7

7

e) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

i)
$$\begin{aligned} 4n \tan \frac{\pi}{n} &= 4n \left[\frac{\pi}{n} + \frac{(\pi/n)^3}{3} + \frac{2(\pi/n)^5}{15} \right] \\ &= 4n \left(\frac{\pi}{n} + \frac{(\pi/n)^3}{3} + \frac{2(\pi/n)^5}{15} + \dots \right) \\ &= 4\pi + \frac{4\pi^3}{3n^2} + \frac{28\pi^5}{15n^4} + \dots \\ \dots \lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right) &= \lim_{n \rightarrow \infty} \left(4\pi + \frac{4\pi^3}{3n^2} + \frac{28\pi^5}{15n^4} + \dots \right) \\ &= 4\pi \end{aligned}$$

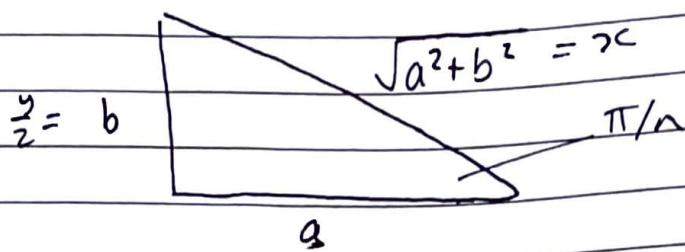
(3)

ii) As $n \rightarrow \infty, y \rightarrow 0$

↳ the perimeter / area converges to 4π
↳ the polygon approaches a circle
with circumference 4π units
and area $4\pi \text{ units}^2$

(1)

f)



$$\Rightarrow P = a + b + \sqrt{a^2 + b^2}$$

$$\Rightarrow A = \frac{1}{2}ab$$

If $P = A$, then

② ③

$$\cancel{\frac{1}{2}ab} = a + b + \sqrt{a^2 + b^2}$$

$$\therefore \cancel{\frac{1}{4}a^2b^2} = \cancel{a^2 + b^2} + \cancel{a^2 + b^2}$$

$$\therefore \cancel{a^2b^2} = 4a^2 + 4a^2 + 4b^2 + 4b^2$$

$$\cancel{\frac{1}{2}ab} \leq$$

$$P = x + \frac{y}{2} + a$$

$$a + b + \sqrt{a^2 + b^2} = \frac{1}{2}ab$$

$$\therefore a^2 + b^2 = \left(\frac{1}{2}ab - (a+b)\right)$$

$$\therefore a^2 + b^2 = \frac{1}{4}a^2b^2 - ab(a+b) + (a+b)^2$$

$$\therefore a^2 + b^2 = \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + 2ab + b^2$$

$$\therefore 0 = \frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab$$

$$= ab\left(\frac{1}{4}ab - a - b + 2\right) = 0$$

$$\therefore a\left(\frac{1}{4}b - 1\right) = b - 2$$

$$\therefore a(b-4) = 4b-8$$

$$a = \frac{4b-8}{b-4}$$

$$= \frac{4b-16}{b-4} + \frac{8}{b-4}$$

3/

4 PAGES / PÁGINAS

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1 2 3 4 5 6 7 8 9 10

g) i)

$$a = \frac{8}{b-a} + 4$$

$$\therefore ab - 4a = 8 + 4$$

$$\therefore ab - 4a = 12$$

$$\therefore 4a = ab - 12 \quad \text{X}$$

$$\therefore 4 = b - 12/a$$

∴

$$\therefore b = 8, a = 3 \quad \dots \text{ (1)} \sqrt{b^2 + a^2} = \sqrt{73}$$

OR

$$b = 7, a = 4 \quad \dots \text{ (2)} \sqrt{a^2 + b^2} = \sqrt{65}$$

g) ii)

$$P = a + b + \sqrt{a^2 + b^2}$$

$$A = \frac{1}{2}ab$$

$$\text{for (1)} : P = 3 + 8 + \sqrt{3^2 + 8^2}$$

$$= 11 + \sqrt{73} \text{ units}$$

$$A = \frac{1}{2}(3)(8)$$

$$= 12 \text{ units}^2 \quad \text{X}$$

$$\text{for (2)} : P = 4 + 7 + \sqrt{4^2 + 7^2} \quad \text{X}$$

$$= 11 + \sqrt{65} \text{ units}$$

$$A = \frac{1}{2}(4)(7)$$

$$= 14 \text{ units}^2$$