Markscheme

Additional Practice

Contradiction & Counterexample (Non-Calculator)

ID: 4002

Mathematics: analysis and approaches

Higher level

1. (a) This is true. A1

Proof

Let $g(a) = 2$ then

$$(f \circ g)(a) = f(2) = 4$$ M1A1

(b) Use disproof by counterexample. M1

e.g. Let $g(x) = x$. We have

$$(f \circ g)(x) = x^2$$ A1

The solutions to $x^2 = 4$ are $x = \pm 2$. A1

The solution to $x = 2$ is $x = 2$. A1

The statement is therefore false. A1

2. If the events are mutually exclusive then $P(A \cap B) = 0$. A1

If the events are independent then $P(A \cap B) = P(A)P(B)$. A1

If the events are both mutually exclusive and independent then we have

$$0 = P(A)P(B) \qquad \text{A1}$$

This is a contradiction since both $P(A)$ and $P(B) > 0$. A1

3. Assume that $\log_4 10$ is rational and use proof by contradiction. M1

This means it can be written in the form

$$\log_4 10 = \frac{a}{b}$$

where a and b are integers. A1

Rewrite in exponential form. M1

$$4^{a/b} = 10$$

Therefore

$$4^a = 10^b$$ A1

However, the left side of this has a final digit of 4 or 6 and the right side has a final digit of zero. R1

So our original assumption must be incorrect, and therefore $\log_4 10$ must be irrational. A1

4. Assume that $2^{1/3}$ is rational. This means it can be written in the form $2^{1/3} = \frac{a}{b}$ where a and b are integers in the lowest terms. M1

We therefore have

$$2 = \frac{a^3}{b^3}$$ M1

So

$$a^3 = 2b^3$$

This means that a^3, and therefore a, is even. A1

Let $a = 2k$ where $k \in \mathbb{N}$. We have

$$8k^3 = 2b^3$$ M1

So

$$b^3 = 4k^3$$ A1

This means that b^3, and therefore b, is even. This is a contradiction since a/b is in the lowest terms.

So the original assumption must be false and $2^{1/3}$ must be irrational. A1

5. (a) Assume that

$$\frac{x^2+1}{x} < 2 \qquad \text{M1}$$

So

$$x^2 - 2x + 1 < 0 \qquad \text{A1}$$

Which gives

$$(x-1)^2 < 0 \qquad \text{M1}$$

However, this is a contradiction since the left side can never be negative. So the original assumption must be false and therefore R1

$$\frac{x^2+1}{x} \geq 2$$

(b) $2 \times 2 = 4$ M1A1

6. (a) $\dfrac{x^n - 1}{x - 1}$ A1

(b) $x^n - 1 = (x-1)S_n$ so $f(x) = x - 1$ A1

(c) Replace x with 2^a. We have M1

$$(2^a)^b - 1 = (2^a - 1)(1 + 2^a + (2^a)^2 + \cdots + (2^a)^{b-2} + (2^a)^{b-1}) \quad \text{A1}$$

So

$$2^{ab} - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{a(b-2)} + 2^{a(b-1)}) \quad \text{A1}$$

(d) If $2^n - 1$ is prime assume that n is composite. This means $n = ab$ where $1 < a,b < n$. M1

However, by part (c) this means that $2^a - 1$ is a factor of $2^n - 1$ which is a contradiction. R1

So the assumption that n is composite must be false. So n must be prime. A1

7. (a) We have

$$4N = 4 + \frac{4}{2} + \frac{4}{3} + \frac{4}{4} + \frac{4}{5} + \frac{4}{6} + \frac{4}{7} + \frac{4}{8} + \frac{4}{9} \qquad \text{M1}$$

Simplify

$$4N = 4 + 2 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2} + \frac{4}{9} \qquad \text{A1}$$

Rearrange to the given equation

$$-\frac{1}{2} = 4 + 2 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{4}{9} - 4N$$

(b) Since every denominator in the expression is odd when written as a single fraction (in the lowest terms) it will also be odd. R1 A1

(c) The denominator on the left side is even and the denominator on the right side is odd. This is a contradiction so the original assumption that N is an integer cannot be true. A1 A1

(d) Multiple the value of M by 8. When the fractions are simplified the only fraction with an even denominator will be $8/16 = 1/2$ and the rest of the proof is the same. A1A1

8. (a) We have

$$(f(-x))^2 = (-f(x))^2 = (f(x))^2$$ M1A1

The statement is therefore true. A1

(b) We have

$$f(-x) \cdot g(-x) = -f(x) \cdot g(x)$$ M1

The statement is therefore true. A1

(c) Use disproof by counterexample. M1

e.g. Let $f(x) = x$ and $g(x) = x^2$. A1

We then have $f(-x) + g(-x) = -x + x^2$ which is not equal to $f(x)$ nor $-f(x)$. A1

The statement is therefore false. A1

9. (a) $a = 243, \ b = 3125$ A1

$c = 49, d = 729$ A1

(b) The final digit follows the pattern 7, 9, 3, 1. R1

Since $165 = 41 \times 4 + 1$ the final digit will be the first digit in the pattern. M1

So the final digit is 7. A1

(c) Assume that $\sqrt{5}$ is rational. This means it can be written in the form $\sqrt{5} = a/b$ where $a, b \in \mathbb{N}$ and are in their lowest terms. M1

This gives

$$a^2 = 5b^2$$ A1

So a^2 is a multiple of 5. This means its final digit must be 5. R1

Only the square of a multiple of 5 has a final digit of 5 so a must also be a multiple of 5. R1

Let $a = 5m$ where $m \in \mathbb{N}$. This gives

$$25m^2 = 5b^2$$ M1

So $b^2 = 5m^2$ which means b^2, and therefore b, must be a multiple of 5. A1

This is a contradiction since a and b should be in their lowest terms. So the original assumption that $\sqrt{5}$ is rational must be false. So $\sqrt{5}$ is irrational. R1

10. (a) The integer x can be written as the product of k prime factors. So let

$$x = p_1 p_2 \ldots p_k$$ A1

This means that

$$x^2 = {p_1}^2 {p_2}^2 \ldots {p_k}^2$$ A1

If p divides x^2 then because p is prime it must be equal to exactly one of p_1, or p_2, or ..., or p_k. A1

Therefore p must also divide x. A1

(b) Assume that $\sqrt{11}$ is rational. M1

This means it can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and a and b are in the lowest terms. A1

This means that $11b^2 = a^2$. A1

So a^2 must be a multiple of 11. A1

If a^2 is a multiple of 11 then a must also be a multiple of 11. Let $a = 11n$ for $n \in \mathbb{Z}$. A1

This means that $11b^2 = 121n^2$ so $b^2 = 11n^2$. A1

This must mean that b^2 and therefore b is also a multiple of 11. A1

However, this contradicts the fact that a and b are in the lowest terms, so our claim that $\sqrt{11}$ is rational cannot be true. A1

11. (a) The integer x can be written as the product of k prime factors. So let

$$x = p_1 p_2 \ldots p_k$$ A1

This means that

$$x^2 = p_1{}^2 p_2{}^2 \ldots p_k{}^2$$ A1

If p divides x^2 then because p is prime it must be equal to exactly one of p_1, or p_2, or ..., or p_k. A1

Therefore p must also divide x. A1

(b) Assume that $\sqrt{7}$ is rational. M1

This means it can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and a and b are in the lowest terms. A1

This means that $7b^2 = a^2$. A1

So a^2 must be a multiple of 7. A1

If a^2 is a multiple of 7 then a must also be a multiple of 7. Let $a = 7n$ for $n \in \mathbb{Z}$. A1

This means that $7b^2 = 49n^2$ so $b^2 = 7n^2$. A1

This must mean that b^2 and therefore b is also a multiple of 7. A1

However, this contradicts the fact that a and b are in the lowest terms, so our claim that $\sqrt{7}$ is rational cannot be true. A1

12. (a) $2 \times 2 \times 2 \times 3 \times 5$ A1

(b) 2, 3, 5 A1

(c)

(i) $2 + 1 = 3$ is prime A1

(ii) $2 \times 3 + 1 = 7$ is prime A1

(iii) $2 \times 3 \times 5 + 1 = 31$ is prime. A1

(d) For example

$$\frac{p_1 p_2 \cdots p_n + 1}{p_1} = p_2 p_3 \cdots p_n + \frac{1}{p_1}$$ M1

So p_1 doesn't divide P. A1

Using a similar method we can show none of the p_i divide P. R1

(e) We have shown that none of the p_i divide P so P must be a new prime number not in the list. A1

This is a contradiction so the original assumption that there are a finite number of primes must be false. R1

(f) This is false since $3 \times 5 + 1 = 16$ is not prime. A1R1

13. (a) Use the product rule M1

$$f'(x) = \cos x \tan x + \frac{\sin x}{\cos^2 x} = \sin x + \frac{\sin x}{\cos^2 x}$$ A1A1

(b) Use the quotient rule M1

$$f''(x) = \cos x + \frac{\cos^3 x + 2\sin^2 x \cos x}{\cos^4 x} = \frac{\cos^4 x + \cos^2 x + 2\sin^2 x}{\cos^3 x}$$ A1A1

This is equal to

$$\frac{\cos^4 x - \cos^2 x + 2(1-\cos^2 x)}{\cos^3 x} = \frac{\cos^4 x - \cos^2 x + 2}{\cos^3 x}$$ M1A1

(c) We have

$$\cos x = 0$$ M1

So

$$x = -3\pi/2 \quad x = -\pi/2 \quad x = \pi/2 \quad x = 3\pi/2$$ A1A1

(d) We have

$$\sin x + \frac{\sin x}{\cos^2 x} = 0$$

So

$$\sin x(\cos^2 x + 1) = 0$$ M1

Giving

$$x = -\pi \quad x = 0 \quad x = \pi$$ A1

The corresponding y-coordinates are all 0. A1

Use $f''(x)$ to classify the points. M1

Since $f''(-\pi) < 0$ there is a maximum point at $(-\pi, 0)$.

Since $f''(0) > 0$ there is a minimum point at $(0, 0)$.

Since $f''(\pi) < 0$ there is a maximum point at $(\pi, 0)$. A1

(e) Assume the graph has a point of inflection. So

$$\cos^4 x - \cos^2 x + 2 = 0$$ M1

The discriminant is equal to $(-1)^2 - 4(1)(2) = -7$. Since the discriminant is negative there are no real solutions. So the assumption that the graph has a point of inflection is false. A1 R1

(f) The asymptotes are consistent with part (d) and the turning points are consistent with part (e). A1

The domain is correct. A1

The shape is approximately correct. A1