Practice Set C: Paper 3 Mark scheme

1 a ( )( ) (( )) (( )(A2

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[2 marks]

b i 16 A1

ii 8 A1

iii 12 870 A1

${}^{2n}\mathrm{C}_n$ A1

[4 marks]

c i ( )( ) (( )) A1

ii ((( ))) ( )(( )) ( )( )( ) (( )) ( ) (( )( )) M1

So $B_3 = 5$ A1

[3 marks]

d $\frac{B_1}{A_1} = \frac{1}{2}$ $\frac{B_2}{A_2} = \frac{1}{3}$ $\frac{B_3}{A_3} = \frac{5}{20} = \frac{1}{4}$ $\frac{B_8}{A_8} = \frac{1}{9}$ (M2)

This suggests $\mathrm{f}(n) = \frac{1}{n+1}$ A1

$B_n = \frac{1}{n+1}\,{}^{2n}\mathrm{C}_n$

[3 marks]

e When $n = 1$ M1

$B_1 = \frac{1}{2} \times {}^{2}\mathrm{C}_1 = \frac{1}{2} \times 2 = 1$

So the conjecture is true when $n = 1$ A1

Assume that it is true when $n = k$ M1

$B_1 = \frac{1}{k+1}\,{}^{2k}\mathrm{C}_k = \frac{1}{k+1}\,\frac{(2k)!}{k!k!}$ A1

Then using the given recursion relation:

$B_{k+1} = \frac{4k+2}{k+2} \times \frac{1}{k+1}\,\frac{(2k)!}{k!k!}$ M1

$= \frac{2(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!}$

$= \frac{2(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!} \times \frac{(2k+1)(2k+2)}{(2k+1)(2k+2)}$ M1

$= \frac{(2k+1)}{(k+2)(k+1)} \times \frac{(2k)!}{k!k!} \times \frac{(2k+1)(2k+2)}{(2k+1)(k+1)}$

$= \frac{1}{k+2} \times \frac{2(k+1)!}{(k+1)!(k+1)!}$ A1

$= \frac{1}{(k+1)+1}\,{}^{2(k+1)}\mathrm{C}_{k+1}$

So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$ therefore it works for all $n \in \mathbb{Z}^+$ A1

[8 marks]

Tip: You might wonder where the given recursion relation comes from. The most natural way is from the triangulation of a polygon interpretation of Catalan numbers.

f $\frac{B_{20}}{A_{20}} = \mathrm{f}(20) = \frac{1}{21}$ M1A1

[2 marks]

g Let (be equivalent to a vote for Elsa and) be equivalent to a vote for Asher M1

Then the total number of ways of ending in a draw is A_{50} and the number where Asher is never ahead is B_{50} M1

The probability is then $\frac{B_{50}}{A_{50}} = \frac{1}{51}$ A1

[3 marks]

Total [25 marks]

2 **a** $\left|e^{\frac{2\pi i}{3}}-1\right|=\left|\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}-1\right|$ M1

$=\left|-\frac{1}{2}+\frac{\sqrt{3}}{2}i-1\right|$ A1

$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$=\sqrt{3}$ A1

[3 marks]

b Bill: $e^{\frac{2\pi i}{3}}$ A1

Charlotte: $e^{\frac{4\pi i}{3}}$ A1

[2 marks]

c Using part **a**: $\sqrt{3}$ units in $\sqrt{3}$ seconds A1A1

[2 marks]

d The direction from z_A to z_B is z_B-z_A A1

The distance travelled per unit time is one, so this is $\frac{z_B-z_A}{|z_B-z_A|}$ A1

[2 marks]

e $z_B=e^{\frac{2\pi i}{3}}z_A$ A1

[1 mark]

f $\frac{dz_A}{dt}=\frac{dr}{dt}e^{i\theta}+ire^{i\theta}\frac{d\theta}{dt}$ M1A1

[2 marks]

g $\frac{dr}{dt}e^{i\theta}+ire^{i\theta}\frac{d\theta}{dt}=\frac{e^{\frac{2\pi i}{3}}z_A-z_A}{\left|e^{\frac{2\pi i}{3}}z_A-z_A\right|}=\frac{z_A\left(e^{\frac{2\pi i}{3}}-1\right)}{|z_A|\left|e^{\frac{2\pi i}{3}}-1\right|}$ M1A1

$=\frac{re^{i\theta}\left(e^{\frac{2\pi i}{3}}-1\right)}{r\left|e^{\frac{2\pi i}{3}}-1\right|}$ M1A1

$=\frac{e^{i\theta}}{\sqrt{3}}\left(-\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)$ A1

$=e^{i\theta}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

Dividing through by $e^{i\theta}$:

1. $\frac{dr}{dt}+ir\frac{d\theta}{dt}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

Comparing real and imaginary parts:

$\frac{dr}{dt}=-\frac{\sqrt{3}}{2}$ A1

$r\frac{d\theta}{dt}=\frac{1}{2}$ A1

[7 marks]

h $r=-\frac{\sqrt{3}}{2}t+c$ M1

When $t=0$, $r=1$ so $c=1$ M1

$r=1-\frac{\sqrt{3}}{2}t$ A1

$\frac{d\theta}{dt}=\frac{1}{2\left(1-\frac{\sqrt{3}}{2}t\right)}=\frac{1}{2-\sqrt{3}t}$ M1

$\theta=-\frac{1}{\sqrt{3}}\ln(2-\sqrt{3}t)+c$ A1

When $t=0$, $\theta=0$ so $c=\frac{1}{\sqrt{3}}\ln 2$ M1

$\theta=\frac{1}{\sqrt{3}}\ln\left(\frac{2}{2-\sqrt{3}t}\right)$ A1

[7 marks]

i Meet when $r=0$ M1

This happens when $1-\frac{\sqrt{3}}{2}t=0$

So $t=\frac{2}{\sqrt{3}}$ A1

Since $v=1$ the distance travelled is $\frac{2}{\sqrt{3}}$ units A1

As $t\to\frac{2}{\sqrt{3}}$, $\theta\to\infty$ so the snails make an infinite number of rotations A1

[4 marks]

Total [30 marks]