Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2018

Mathematics

Higher level

Paper 2

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking May 2018"**. It is essential that you read this document before you start marking. In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Note: Accept answers that round to the correct 2sf unless otherwise stated in the markscheme.

Section A

1. (a) $z = \frac{(2+7\text{i})}{(6+2\text{i})} \times \frac{(6-2\text{i})}{(6-2\text{i})}$ ***(M1)***

$= \frac{26+38\text{i}}{40} \left(= \frac{13+19\text{i}}{20} = 0.65 + 0.95\text{i} \right)$ ***A1***

[2 marks]

(b) attempt to use $|z| = \sqrt{a^2+b^2}$ ***(M1)***

$|z| = \sqrt{\frac{53}{40}} \left(= \frac{\sqrt{530}}{20} \right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

(c) **EITHER**

$\arg z = \arg(2+7\text{i}) - \arg(6+2\text{i})$ ***(M1)***

OR

$\arg z = \arctan\left(\frac{19}{13}\right)$ ***(M1)***

THEN

$\arg z = 0.9707$ (radians) ($= 55.6197$ degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

Total [6 marks]

2. **METHOD 1**

substitute each of $x = 1, 2$ and 3 into the quartic and equate to zero ***(M1)***

$p+q+r = -7$

$4p+2q+r = -11$ or equivalent ***(A2)***

$9p+3q+r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations ***(M1)***

$p = -7$, $q = 17$, $r = -17$ ***A1***

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

continued…

Question 2 continued

METHOD 2

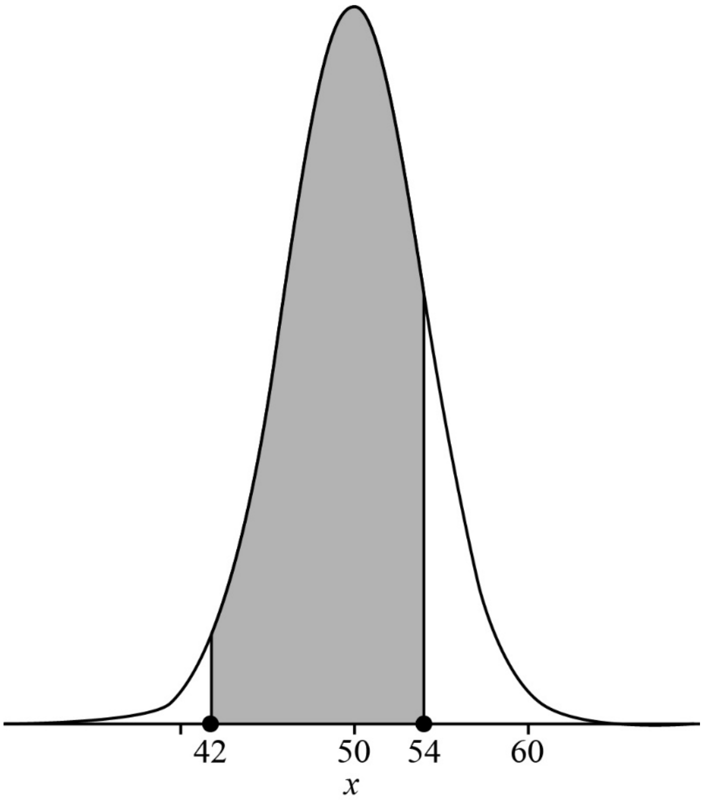
attempt to find fourth factor **(M1)**
$(x-1)$ **A1**
attempt to expand $(x-1)^2(x-2)(x-3)$ **M1**
$= x^4 - 7x^3 + 17x^2 - 17x + 6 \left(p = -7,\ q = 17,\ r = -17\right)$ **A2**

Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

[5 marks]

3. (a)



normal curve centred on 50 **A1**
vertical lines at $x = 42$ and $x = 54$, with shading in between **A1**

[2 marks]

(b) $\mathrm{P}(42 < X < 54) \left(= \mathrm{P}(-2 < Z < 1)\right)$ **(M1)**
$= 0.819$ **A1**

[2 marks]

continued…

Question 3 continued

(c) $P(\mu - k\sigma < X < \mu + k\sigma) = 0.5 \Rightarrow P(X < \mu + k\sigma) = 0.75$ ***(M1)***

$k = 0.674$ ***A1***

Note: Award ***M1A0*** for $k = -0.674$.

[2 marks]

Total [6 marks]

4. (a) (i) **METHOD 1**

$\text{PC} = \frac{\sqrt{3}}{2}$ or 0.8660 ***(M1)***

$\text{PM} = \frac{1}{2}\text{PC} = \frac{\sqrt{3}}{4}$ or 0.4330 ***(A1)***

$\text{AM} = \sqrt{\frac{1}{4} + \frac{3}{16}}$

$= \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

Note: Award ***M1*** for attempting to solve triangle AMP.

METHOD 2

using the cosine rule

$\text{AM}^2 = 1^2 + \left(\frac{\sqrt{3}}{4}\right)^2 - 2 \times \frac{\sqrt{3}}{4} \times \cos(30°)$ ***M1A1***

$\text{AM} = \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

(ii) $\tan\left(\text{A}\hat{\text{M}}\text{P}\right) = \frac{2}{\sqrt{3}}$ or equivalent ***(M1)***

$= 0.857$ ***A1***

[5 marks]

continued…

Question 4 continued

(b) **EITHER**

$\frac{1}{2}\text{AM}^2\left(2\text{A}\hat{\text{M}}\text{P} - \sin(2\text{A}\hat{\text{M}}\text{P})\right)$ ***(M1)A1***

OR

$\frac{1}{2}\text{AM}^2 \times 2\text{A}\hat{\text{M}}\text{P} - \frac{\sqrt{3}}{8}$ ***(M1)A1***

$= 0.158(\text{m}^2)$ ***A1***

Note: Award ***M1*** for attempting to calculate area of a sector minus area of a triangle.

[3 marks]

Total [8 marks]

5. (a) $\binom{3n+1}{3n-2} = \frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$= \frac{(3n+1)3n(3n-1)}{3!}$ ***A1***

$= \frac{9}{2}n^3 - \frac{1}{2}n$ or equivalent ***A1***

[3 marks]

(b) attempt to solve $\frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

Total [6 marks]

6. let P_n be the statement: $(1-a)^n > 1-na$ for some $n \in \mathbb{Z}^+, n \geq 2,$ where $0 < a < 1$

consider the case $n=2$: $(1-a)^2 = 1-2a+a^2$ ***M1***

$> 1-2a$ because $a^2 > 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1-ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$> 1-(k+1)a + ka^2$ ***A1***

$> 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1-(k+1)a + ka^2$ ***A1***

$(1-a)^{k+1} > 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

THEN

P_2 is true and P_k is true $\Rightarrow P_{k+1}$ is true so P_n true for all $n \geq 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

7. (a) attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441\text{(s)}$ ***A1***

[2 marks]

(b) (i) $a(t) = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{e}^{-t} - 16t\mathrm{e}^{-2t} + 16t^2\mathrm{e}^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

(ii) $a(t_1) = -2.28\left(\mathrm{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [5 marks]

8. (a) $np = 3.5$ ***(A1)***

$p \leq 1 \Rightarrow$ least $n = 4$ ***A1***

[2 marks]

continued…

Question 8 continued

(b) $(1-p)^n + np(1-p)^{n-1} = 0.09478$ ***M1A1***

attempt to solve above equation with $np = 3.5$ ***(M1)***

$n = 12,\ p = \frac{7}{24}\ (= 0.292)$ ***A1A1***

Note: Do not accept n as a decimal.

[5 marks]

Total [7 marks]

Section B

9. (a) (i) $X \sim \text{Po}(5.3)$

$\text{P}(X = 4) = \text{e}^{-5.3}\dfrac{5.3^4}{4!}$ ***(M1)***

$= 0.164$ ***A1***

(ii) **METHOD 1**

listing probabilities (table or graph) ***M1***
mode $X = 5$ (with probability 0.174) ***A1***

Note: Award ***M0A0*** for 5 (taxis) or mode $= 5$ with no justification.

METHOD 2

mode is the integer part of mean ***R1***
$E(X) = 5.3 \Rightarrow \text{mode} = 5$ ***A1***

Note: Do not allow ***R0A1***.

(iii) attempt at conditional probability ***(M1)***

$\dfrac{\text{P}(X = 7)}{\text{P}(X \geq 6)}$ or equivalent $\left(= \dfrac{0.1163...}{0.4365...}\right)$ ***A1***

$= 0.267$ ***A1***

[7 marks]

(b) **METHOD 1**

the possible arrivals are $(2,0)$, $(1,1)$, $(0,2)$ ***(A1)***
$Y \sim \text{Po}(0.65)$ ***A1***
attempt to compute, using sum and product rule, ***(M1)***
$0.070106\ldots \times 0.52204\ldots + 0.026455\ldots \times 0.33932\ldots + 0.0049916\ldots \times 0.11028\ldots$

(A1)(A1)

Note: Award ***A1*** for one correct product and ***A1*** for two other correct products.

$= 0.0461$ ***A1***

[6 marks]

continued…

Question 9 continued

METHOD 2

recognising a sum of 2 independent Poisson variables *eg* $Z = X + Y$ ***R1***

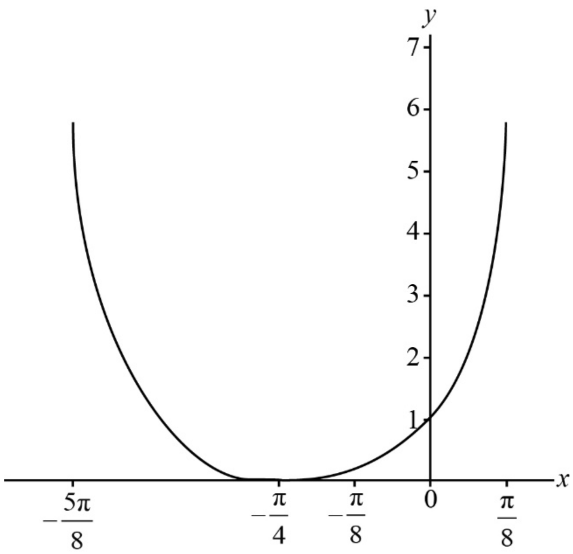
$\lambda = 5.3 + \frac{1.3}{2}$ ***A1***

$\mathrm{P}(Z = 2) = 0.0461$ ***(M1)A3***

[6 marks]

Total [13 marks]

10. (a) (i)



A1A1

A1 for correct concavity, many to one graph, symmetrical about the midpoint of the domain and with two axes intercepts.

Note: Axes intercepts and scales not required.

A1 for correct domain

(ii) for each value of x there is a unique value of $f(x)$ ***A1***

Note: Accept "passes the vertical line test" or equivalent.

(iii) no inverse because the function fails the horizontal line test or equivalent ***R1***

Note: No **FT** if the graph is in degrees (one-to-one).

(iv) the expression is not valid at either of $x = \frac{\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$ ***R1***

[5 marks]

continued…

Question 10 continued

(b) **METHOD 1**

$$f(x) = \frac{\tan\left(x + \frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$ ***M1***

$$= \frac{\dfrac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}}{\dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}}$$ ***M1A1***

$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

METHOD 2

$$f(x) = \tan\left(x + \frac{\pi}{4}\right)\tan\left(\frac{\pi}{2} - \frac{\pi}{4} + x\right)$$ ***(M1)***

$$= \tan^2\left(x + \frac{\pi}{4}\right)$$ ***A1***

$$g(t) = \left(\frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}\right)^2$$ ***A1***

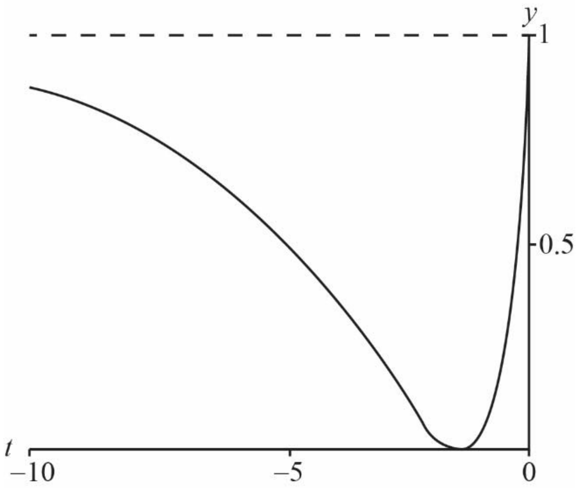
$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

[3 marks]

continued…

Question 10 continued

(c)



for $t \le 0,$ correct concavity with two axes intercepts and with asymptote $y = 1$ ***A1***

t intercept at $(-1, 0)$ ***A1***

y intercept at $(0, 1)$ ***A1***

[3 marks]

(d) (i) **METHOD 1**

α, β satisfy $\frac{(1+t)^2}{(1-t)^2} = k$ ***M1***

$1 + t^2 + 2t = k(1 + t^2 - 2t)$ ***A1***

$(k-1)t^2 - 2(k+1)t + (k-1) = 0$ ***A1***

attempt at using quadratic formula ***M1***

$\alpha, \beta = \frac{k + 1 \pm 2\sqrt{k}}{k-1}$ or equivalent ***A1***

METHOD 2

α, β satisfy $\frac{1+t}{1-t} = (\pm)\sqrt{k}$ ***M1***

$t + \sqrt{k}t = \sqrt{k} - 1$ ***M1***

$t = \frac{\sqrt{k}-1}{\sqrt{k}+1}$ (or equivalent) ***A1***

$t - \sqrt{k}t = -\left(\sqrt{k}+1\right)$ ***M1***

$t = \frac{\sqrt{k}+1}{\sqrt{k}-1}$ (or equivalent) ***A1***

so for *eg*, $\alpha = \frac{\sqrt{k}-1}{\sqrt{k}+1}$, $\beta = \frac{\sqrt{k}+1}{\sqrt{k}-1}$

continued…

Question 10 continued

(ii) $\alpha + \beta = 2\frac{(k+1)}{(k-1)}\left(= -2\frac{(1+k)}{(1-k)}\right)$ ***A1***

since $1 + k > 1 - k$ ***R1***

$\alpha + \beta < -2$ ***AG***

Note: Accept a valid graphical reasoning.

[7 marks]

Total [18 marks]

11. (a) attempt at implicit differentiation ***M1***

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent ***A1***

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ ***AG***

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$P\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), Q\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ ***A1***

continued…

Question 11 continued

(ii) $m_1 = -\left(\dfrac{1-\sqrt{\dfrac{\pi}{2}} \times -1}{1+\sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***M1A1***

$m_2 = -\left(\dfrac{1+\sqrt{\dfrac{\pi}{2}} \times -1}{1-\sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 ***M1***

$(y - x)\sin(xy) = 0$ ***(A1)***

$y = x$, $\sin(xy) = 0$ ***R1***

in the first case, attempt to solve $2x = \cos\left(x^2\right)$ ***M1***

$(0.486, 0.486)$ ***A1***

in the second case, $\sin(xy) = 0 \Rightarrow xy = 0$ and $x + y = 1$ ***(M1)***

$(0,1)$, $(1,0)$ ***A1***

[7 marks]

Total [19 marks]