Mathematics: analysis and approaches
Higher level
Paper 3 Practice Set B (Hodder)

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: analysis and approaches formula book is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

$\frac{34}{55}$

17/10/22

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$\frac{34}{55}$ $= 61.8\%$

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(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$\therefore 1+2+\cdots+n = \frac{n(n+1)}{2}$

M1

Step 1: place for $n=1$:

A1

$LHS = 1$

$RHS = \frac{1(2)}{2}$

$= 1$

$= LHS$

$\therefore$ true for $n=1$

M1

Step 2: assume true for $n=k$: $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ A1

Step 3: consider $n=k+1$

$\therefore 1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$ M1

$\therefore LHS = 1+2+3+\cdots+k+(k+1)$

$= \frac{k(k+1)}{2} + k+1$ {by assumption}

$= \frac{k(k+1)+2k+2}{2}$

$= \frac{k^2+3k+2}{2}$

04AX01

$= \frac{k^2+2k+k+2}{2}$

$= \frac{k(k+2)+(k+2)}{2}$

$= \frac{(k+1)(k+2)}{2}$ ✓A1

$=$ RHS

$\therefore$ true for $n=k+1$ ✓A1

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Step 4: as true for $n=1$ and true for $n=k+1$ whenever the assumption of $n=k$ holds true, then true for all $n \in \mathbb{Z}^+$ by the process of mathematical induction.

(b) $(n+1)^3-n^3 = \cancel{n^3}+3n^2+3n+1-\cancel{n^3}$

$= 3n^2+3n+1$ ✓✓M1A1

2

(c) $\sum_1^n (r+1)^3-r^3 = \sum_1^n 3r^2+3r+1$

$\therefore \sum_1^n (r+1)^3 - \sum_1^n r^3 = 3\sum_1^n r^2 + 3\sum_1^n r + 1$

$\therefore 3\sum_1^n r^2 = \sum_1^n (r+1)^3 - \sum_1^n r^3 - 3\sum_1^n r - 1$

$= (n+1)^3 - 1^3 - 3\frac{n(n+1)}{2} - 1$

$= n^3+3n^2+3n-\frac{3}{2}n^2-\frac{3}{2}n - 1$

$\therefore \sum_1^n r^2 = \frac{1}{3}n^3+n^2+n-\frac{1}{2}n^2-\frac{1}{2}n-1$

$= \frac{2n^3+6n^2+6n-3n^2-3n-6}{6}$

$= \frac{2n^3+3n^2+3n-6}{6}$

$= \frac{n(2n^2+3n+3)-6}{6}$

$= n(2n^2+$

(c) $\sum_1^n (r+1)^3-r^3 = \sum_1^n (3r^2+3r+1)$

$\therefore \sum_1^n (r+1)^3 - \sum_1^n r^3 = 3\sum_1^n r^2 + 3\sum_1^n r + n$

$\therefore 3\sum_1^n r^2 = \sum_1^n (r+1)^3 - \sum_1^n r^3 - 3\sum_1^n r - n$

$= (n+1)^3 - 1^3 - 3\frac{n(n+1)}{2} - n$ ✓M1A1 ✓M1A1

$= n^3+3n^2+3n+1-1-3\frac{n^2+n}{2} - n$

$\therefore 6\sum_1^n r^2 = 2n^3+6n^2+6n-3n^2-3n-2n$

$= 2n^3+3n^2+n$ ✓✓M1A1

$\therefore \sum_1^n r^2 = \frac{n(2n^2+3n+1)}{6}$

$= \frac{n(2n^2+2n+n+1)}{6}$

$\therefore \sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$ ✓A1

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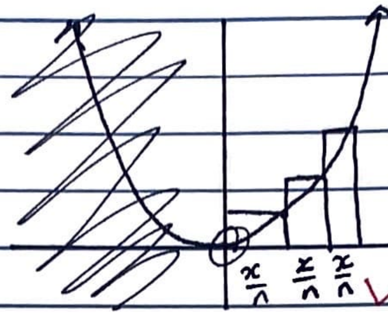
(d)



The boxes protrude over the line $y=x^2$. Hence, the summation of their area,

$$w \times h = \frac{x}{n}\sum_{r=1}^{n}\left(\frac{rx}{n}\right)^2,$$

will be greater than the actual area under the curve, $\int_0^x t^2\,dt$

2

$$\cancel{\frac{x}{n}\sum_{r=1}^{n}\left(\frac{rx}{n}\right)^2 = \frac{\left(\frac{rx}{n}\right)\left(\frac{rx}{n}+1\right)\left(\frac{2rx}{n}+1\right)}{6}\left(\frac{x}{n}\right)}$$

$$\cancel{= \frac{\left(\left(\frac{rx}{n}\right)^2+\frac{rx}{n}\right)\left(2\frac{rx}{n}+1\right)}{6}\left(\frac{x}{n}\right)}$$

$$\cancel{= \frac{2\left(\frac{rx}{n}\right)^3+\left(\frac{rx}{n}\right)^2+2\left(\frac{rx}{n}\right)^2+\left(\frac{rx}{n}\right)}{6}\left(\frac{x}{n}\right)}$$

$$\frac{x}{n}\left(\frac{x}{n}\right)^2\sum_{r=1}^{n} r^2 = \frac{x^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{x^3}{n^2}\,\frac{(n+1)(2n+1)}{6}$$

$$= \frac{x^3(2n^2+3n+1)}{6n^2}$$

$$= \frac{x^3(2n^2)}{6n^2}+\frac{x^3 3n}{6n^2}+\frac{x^3}{6n^2}$$

$$= \frac{x^3}{3}+\frac{x^3}{2n}+\frac{1}{6n^2}$$

As $\int_0^x t^2\,dt = \frac{1}{3}x^2$, $\int_0^x t^2\,dt \leq \frac{x}{n}\sum_{r=1}^{n}\left(\frac{rx}{n}\right)^2$

$$\therefore \frac{1}{3}x^2 \leq \frac{1}{3}x^2+\frac{1}{2n}x^3+\frac{1}{6n^2}(x^3)$$

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Example / Ejemplo 27 | 2 7 |

Example / Ejemplo 3 | 3 |

1

(e)

$$\frac{x}{n}\sum_{r=0}^{n-1}\left(\frac{rx}{n+1}\right)^2 \leq \int_0^x t^2\,dt.$$ A1

1

(f) $$\int_0^x t^2\,dt = \lim_{n\to\infty}\frac{x}{n}\sum_{r=1}^{n}\left(\frac{rx}{n}\right)^2$$

$$= \lim_{n\to\infty}\frac{x^3}{n^3}\sum_{r=1}^{n} r^2$$ M1

$$= \lim_{n\to\infty}\left(\frac{x^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$$ A1

$$= \lim_{n\to\infty}\left(\frac{x^3}{n^2}\right)\left(\frac{\cancel{3}\,2n^2+3n+1}{6}\right)$$

$$= \lim_{n\to\infty}\left(\cancel{\frac{x^3}{3}}\frac{2n^2x^3+3nx^3+x^3}{6n^2}\right)$$ M1 A1

$$= \lim_{n\to\infty}\left(\frac{2x^3+3x^3/n+x^3/n^2}{6}\right)$$ M1

$$= \frac{2x^3}{6}$$

$$= x^3/3$$ A1

6

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04AX01

(a) $\bar{X} = \frac{\mu_1 + \mu_2}{2}$

$= \frac{E(X_1) + E(X_2)}{2}$

$\therefore \bar{X} = \frac{E(X_1 + X_2)}{2}$

(b)

(a) $\bar{X} \sim N(\mu, \sigma^2)$

(b) $E(\bar{X}) = \mu$ {part (a)}

$Var(\bar{X}) = \sigma^2$ {part (a)}

(c)(i)

2 (a) X_1 and X_2 are both observations.

$\therefore \bar{X} = \frac{X_1 + X_2}{2}$ ✓A1 1

(b) $E(\bar{X}) = E\left(\frac{X_1}{2} + \frac{X_2}{2}\right)$ ✓M1

$= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2)$

$= \frac{E(X_1) + E(X_2)}{2}$ A0

$= \mu$

1

$Var(\bar{X}) = Var\left(\frac{X_1}{2} + \frac{X_2}{2}\right)$

~~$= \frac{1}{2}Var(X_1) + \frac{1}{2}Var(X_2)$~~ ×

$= \frac{Var(X_1) + Var(X_2)}{2}$ $= \frac{1}{2}\sigma^2$

(c)(i) $S^2 = \frac{X_1^2 + X_2^2}{2} - \bar{X}^2$

$E(X^2) = Var(X) + E(X)^2$ ✓A1 1

(ii) $E(S^2) = Var(S) + E(S)^2$ ✓M1

$= \frac{X_1^2 + X_2^2}{2} - \bar{X}^2 + \bar{X}^2$

$= \frac{X_1^2 + X_2^2}{2}$ 1

$= \frac{1}{2}\sigma^2$

(d) (i)

$$M = \frac{2X_1 + 3X_2}{5}$$

$$\therefore E(M) = E\left(\frac{2}{5}X_1 + \frac{3}{5}X_2\right)$$ ✓M1

$$= \frac{2}{5}E(X_1) + \frac{3}{5}E(X_2)$$

$$= \frac{2}{5}\mu + \frac{3}{5}\mu$$ ✓A1

$$= \mu$$

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$\therefore E(M) = \mu$

(ii) NA

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| 2 | (e)(i)

Y_1 : ☐ ☐ ☐ ☐T

$a=4$

Y_2 : ☐ ☐ ☐ ☐ ☐ ☐T

$b=6$

let $Y_1 \sim B(a, p)$ and $Y_2 \sim B(b, p)$

$$L = \binom{a}{p}(1-p)^{a-1}(p) \times \binom{b}{p}(1-p)^{b-1}(p)$$

$$= \binom{a}{p}\binom{b}{p}(p^2)\frac{(1-p)^{a+b}}{(1-p)^2}$$

$$P(Y_1 = a) = \binom{a}{a-1}(1-p)^{a-(a+1)}(p)$$

$$= \frac{a!}{(a-1)!(a-a+1)!}(1-p)^1 p$$ × M1

$$= \frac{a(a-1)!}{(a-1)!}(p-p^2)$$

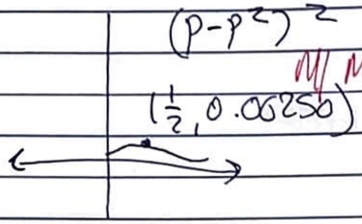
$$= a(p-p^2)$$

$$P(Y_2 = b) = b(p-p^2)$$ ✓ M1

$\therefore L = P(Y_1 = a)\,P(Y_2 = b)$ M1

$= ab(p-p^2)^2$. 1

(ii)



M1 M1 2

$\therefore p = 1/2$ will maximise L for any value $a, b \in Z^+$

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