Mathematics: analysis and approaches

Higher level

Additional Practice

Series and Sequences (Non-Calculator)

ID: 4004

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[140 marks]**.

1. [Maximum points: 4]

Find the value of $\sum_{k=1}^{\infty} \frac{4}{(-3)^k}$.

2. [Maximum points: 6]

Calculate the value of each infinite geometric series, if it exists.

(a) $1000 + 100 + 10 + 1 + \cdots$ [2]

(b) $64 - 16 + 4 - 1 + \cdots$ [2]

(c) $400 - 600 + 900 - 1350 +$ [2]

3. [Maximum points: 6]

Let $x = 0.\dot{5}\dot{2}$.

(a) If x is written as an infinite geometric series find, as a fraction in the lowest terms, the value of [3]

(i) the first term

(ii) the common ratio

(b) Hence show that x is rational. [3]

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4. [Maximum points: 6]

By writing $0.\dot{8}\dot{5}$ (= 0.8585858585...) as an infinite geometric series, show that it is rational.

5. [Maximum points: 7]

Consider the geometric sequence 4, 8, 16 and 32.

(a) Write down [2]

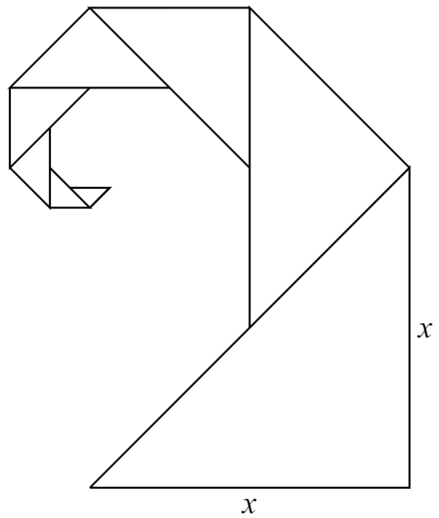
(i) the common ratio

(ii) the sum of the sequence

(b) Show that there are no other geometric sequences with 4 terms and first term equal to 4 whose sum is equal to the value from (a) part (ii). [5]

6. [Maximum points: 8]

The diagram below shows a series of similar right-isosceles triangles where the largest triangle has a base and height each of length x. All other triangles have one vertex on the midpoint of the hypotenuse of the next largest triangle.



(a) Find the length of the base and height of the second largest triangle writing your answer in the form kx where $k \in \mathbb{Q}'$. [2]

(b) Show that the length of the base and height of the third largest triangle is $x/2$. [2]

If the pattern is continued to infinity the total area of all of the triangles is equal to 100.

(c) Find the value of x. [4]

7. [Maximum points: 10]

Prove by induction that the sum S_n of the first n terms of a geometric sequence with first term t_1 and common ratio r is equal to

$$S_n = \frac{t_1(1-r^n)}{1-r}$$

8. [Maximum points: 10]

(a) For $|x| < 1$ write down the value of $\sum_{n=0}^{\infty}(-x)^n$ in terms of x. [2]

(b) Write $\frac{2x+3}{(x+1)(x+2)}$ using partial fractions. [2]

(c) Hence find the first three terms of the binomial expansion of $\frac{2x+3}{(x+1)(x+2)}$. [4]

(d) Determine the restrictions on the value of x for the binomial expansion in part (c) to converge. [2]

9. [Maximum points: 13]

Let $f(x) = \sum_{k=1}^{n} (3 + 2k)x^{k-1}$ where $x \neq 1$.

(a) Write down the first 4 terms of [2]

(i) $f(x)$

(ii) $x f(x)$

(b) Show that [3]

$$f(x) - x f(x) = 5 - (3 + 2n)x^n + \frac{2x(1 - x^{n-1})}{1 - x}$$

(c) Hence find an expression for $f(x)$ that does not use sigma notation. [2]

(d) If $|x| < 1$ show that [3]

$$\sum_{k=1}^{\infty} (3 + 2k)x^{k-1} = \frac{5}{1 - x} + \frac{2x}{1 - x}$$

(e) Hence evaluate $\sum_{k=1}^{\infty} \frac{3 + 2k}{2^{k-1}}$. [3]

10. [Maximum points: 16]

Consider the infinite series

$$1 - x^2 + x^4 - x^6 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

(a) If $|x| < 1$ show that the value of the series is equal to $\dfrac{1}{1+x^2}$. [3]

(b) Show that $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$. [7]

(c) Hence determine the Maclaurin series for $\arctan x$. [2]

(d) By considering the value of $\arctan\left(\dfrac{\sqrt{3}}{3}\right)$ show that [4]

$$\pi = 2\sqrt{3} \times \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k \times (2k+1)}$$

11. [Maximum points: 17]

A mathematics teacher is trying to make a problem for her students. The wording of the problem is as follows:

Jason and Maria play a game where Jason thinks of an integer between 1 *and* x. *Maria then tries to guess the number in as few guesses as possible. After each guess Jason states whether the guess is correct, too big or too small. Maria always guesses the median value of the available numbers.*

The teacher tests some values of x in order to find values of x so that after each guess if the correct integer has not been found then there is always an odd number of values remaining from which to guess (an odd number of values means that the median will always be one of those values).

(a) Show that one possible value of x is 7. [2]

(b) Find the next three possible values of x greater than 7. [6]

(c) If the possible values of x are of the form $f(n)$ where $n \in \mathbb{Z}^+$ hypothesise the function $f(n)$. [2]

(d) Prove your answer to (c) by induction. [7]

12. [Maximum points: 18]

Let $f(x) = \sum_{k=0}^{\infty} x^k$ where $|x| < 1$.

(a) Write down the first four non-zero terms of [3]

(i) $f(x)$

(ii) $f'(x)$

(b) Explain why $f(x) = \dfrac{1}{1-x}$. [2]

(c) Hence find an expression for $f'(x)$ writing your answer as a single fraction. [3]

A regular six-sided die is repeatedly rolled. Let the random variable X represent the number of rolls needed until a 6 appears.

(d) The table below shows the values of $P(X = x)$ for the first four values of x. [3]

x	1	2	3	4
$P(X=x)$	a	b	$\frac{25}{216}$	$\frac{125}{1296}$

Find the values of a and b.

(e) Verify that $\sum_{x=1}^{\infty} P(X = x) = 1$. [3]

(f) Find $E(X)$. [4]

13. [Maximum points: 19]

Let $f(x) = 1 + x + x^2 + \cdots = \sum_{r=0}^{\infty} x^r$ where $-1 < x < 1$.

(a) Determine an expression for $f(x)$ that does not use sigma notation. [2]

(b) Use your answer to part (a) to find [4]

(i) $f'(x)$

(ii) $f''(x)$

(c) Hence show that $\sum_{r=1}^{\infty} r^2 x^{r-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$. [6]

A regular six sided die is rolled until a five appears. Let X represent the total number of rolls.

(d) Write down an expression for the probability that the dice is rolled exactly n times. [1]

(e) Determine the value of [6]

(i) $E(X)$

(ii) $\mathrm{Var}(X)$