Practice Set B: Paper 3 Mark scheme

1 **a** Check that the statement is true for $n = 1$: M1

LHS = 1 RHS = $\frac{1 \times 2}{2} = 1$ A1

Assume true for $n = k$ M1

$\sum_{r=1}^{r=k} r = \frac{k(k+1)}{2}$ A1

Then

$\sum_{r=1}^{r=k+1} r = \sum_{r=1}^{r=k} r + (k+1) = \frac{k(k+1)}{2} + (k+1)$ M1

$= (k+1)\left(\frac{k}{2} + 1\right)$

$= \frac{(k+1)(k+2)}{2}$ A1

So if the statement works for $n = k$ then it works for $n = k + 1$ and it works for $n = 1$, therefore it works for all $n \in \mathbb{Z}^+$. A1

[7 marks]

b $3n^2 + 3n + 1$ M1A1

[2 marks]

c $\sum_{r=1}^{n} (r+1)^3 - r^3$

$= [(n+1)^3 - n^3] + [n^3 - (n-1)^3] \ldots + [3^3 - 2^3] + [2^3 - 1^3]$ M1

$= (n+1)^3 - 1 = n^3 + 3n^2 + 3n$ A1

Also:

$\sum_{r=1}^{n} (r+1)^3 - r^3 = \sum_{r=1}^{n} 3r^2 + 3r + 1$

$= 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$ M1

$= 3\sum_{r=1}^{n} r^2 + \frac{3n(n+1)}{2} + n$ A1A1

Therefore:

$3\sum_{r=1}^{n} r^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$ M1

$= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$

$= \frac{1}{2}n(2n^2 + 3n + 1)$

$= \frac{1}{2}n(n+1)(2n+1)$ A1

Therefore $\sum_{r=1}^{n} r^2 + \frac{n(n+1)(2n+1)}{6}$ AG

[7 marks]

d The coordinate of the bottom right hand corner of the rth rectangle is $\frac{rx}{n}$. M1

The height of the rectangle is $\left(\frac{rx}{n}\right)^2$ A1

So the area of each rectangle is $\frac{x}{n}\left(\frac{rx}{n}\right)^2$ A1

The total area is $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{rx}{n}\right)^2$

Each rectangle has a portion above the curve, so the total area is an overestimate of the true area under the curve. A1

Tip: A diagram would be a great way to form and illustrate this argument!

[4 marks]

e The coordinate of the bottom left hand corner of the rth rectangle is $\frac{(r-1)x}{n}$ M1

The height of the rectangles with top left corner on the curve is $\left(\frac{(r-1)x}{n}\right)^2$ A1

The total area is $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{(r-1)x}{n}\right)^2$

This is less than the area under the curve, so M1

$\frac{x}{n}\sum_{r=1}^{n}\left(\frac{(r-1)x}{n}\right)^2 \leqslant \int_0^x t^2 \mathrm{d}t$ A1

[4 marks]

f $\sum_{r=1}^{n} \frac{x}{n}\left(\frac{rx}{n}\right)^2 = \frac{x^3}{n^3}\sum_{r=1}^{n} r^2 = \frac{x^3}{n^3}\frac{n(n+1)(2n+1)}{6}$ M1

$= x^3 \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$ A1

$\frac{x}{n}\sum_{r=1}^{n}\left(\frac{(r-1)x}{n}\right)^2 = \frac{x}{n}\sum_{r=1}^{n-1}\left(\frac{rx}{n}\right)^2$ M1A1

$= \frac{x^3}{n^3}\frac{(n-1)(n)(2n-1)}{6}$

$= x^3 \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}$

Taking the limit

$\lim_{n\to\infty} x^3 \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6} \leqslant \int_0^x t^2\,dt \leqslant \lim_{n\to\infty} x^3 \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$ M1

$\frac{x^3}{3} \leqslant \int_0^x x^2\,dx \leqslant \frac{x^3}{3}$

Since $\int_0^x t^2\,dt$ is sandwiched between two quantities tending towards $\frac{x^3}{3}$,
it must also tend towards $\frac{x^3}{3}$. A1

[6 marks]

Total [30 marks]

2 **a** $\overline{X} = \frac{X_1 + X_2}{2}$ A1

[1 mark]

b $E(\overline{X}) = E\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2)$ M1

$= \frac{1}{2}\mu + \frac{1}{2}\mu$ A1

$= \mu$ AG

$\text{Var}(\overline{X}) = \text{Var}\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{4}\text{Var}(X_1) + \frac{1}{4}\text{Var}(X_2)$ M1

$\frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2$

$= \frac{1}{2}\sigma^2$ A1

[4 marks]

c **i** $E(X^2) = \text{Var}(X) + E(X)^2$ A1

ii $E(S^2) = E\left(\frac{X_1^2 + X_2^2}{2} - \overline{X}^2\right) = \frac{1}{2}E(X_1^2) + \frac{1}{2}E(X_2^2) - E(\overline{X}^2)$ M1

$= \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2) + \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2)$ M1

$-(\text{Var}(\overline{X}) + E(\overline{X})^2)$

$= \frac{1}{2}(\sigma^2 + \mu^2) + \frac{1}{2}(\sigma^2 + \mu^2) - \left(\frac{1}{2}\sigma^2 + \mu^2\right)$ A1

$= \frac{1}{2}\sigma^2$ AG

[4 marks]

d **i** $E(M) = \frac{2}{5}E(X_1) + \frac{3}{5}E(X_2)$ M1

$= \frac{2}{5}\mu + \frac{3}{5}\mu$ A1

$= \mu$ AG

ii $\text{Var}(M) = \frac{4}{25}\text{Var}(X_1) + \frac{9}{25}\text{Var}(X_2)$ M1

$= \frac{13}{25}\sigma^2$ A1

$> \frac{1}{2}\sigma^2$ therefore $\overline{X}$ is a more efficient estimator A1

[5 marks]

e **i** $L = P(Y = a)P(Y = b)$ M1

$= p(1-p)^{a-1} \times p(1-p)^{b-1}$ A1

ii $L = p^2(1-p)^{a+b-2}$

$\dfrac{dL}{dp} = 2p(1-p)^{a+b-2} - (a+b-2)p^2(1-p)^{a+b-3}$ M1A1

At a max, $\dfrac{dL}{dp} = 0$ M1

$p(1-p)^{a+b-3}(2(1-p) - (a+b-2)p) = 0$ M1

Since $p \neq 0$ and $p \neq 1$ at the maximum value of L A1

$2 - 2p = ap + bp - 2p$

$2 = ap + bp$

$p = \dfrac{2}{a+b}$ A1

[8 marks]

f **i** $S^2 = \dfrac{4^2 + 8^2}{2} - 6^2 = 4$ M1

Unbiased estimate of $\sigma^2 = 2S^2 = 8$ A1

ii $p = \dfrac{2}{4+8} = \dfrac{1}{6}$ A1

[3 marks]

Total [25 marks]