ST ANDREW'S CATHEDRAL SCHOOL
FOUNDED 1885

Diploma Programme

Candidate session number

Solutions

Mathematics
Higher Level
Paper 2

August 2020

2 hours

Instructions to candidates

- Write your session number in the boxes above
- Do not open this examination paper until instructed to do so.
- A graphics display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklets provided. Start each question in a new booklet
- A clean copy of the ***mathematics HL and further mathematics HL formula booklet*** is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correctly to three significant figures.
- The maximum mark for this examination paper is **[100 marks].**

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.

1. [Maximum mark: 6]

Two dice are rolled four times and the sums of the scores were: 2, 5, 6, 9. The dice are then rolled twice more. If the new mean score sum was 7 and the standard deviation of the score sum was 3. Find the values of the last two score sums.

Let the new scores be a and b

mean: $\frac{2+5+6+9+a+b}{6} = 7$

$22 + a + b = 42$

$a + b = 20$ ①

so: $3^2 = \frac{(2-7)^2 + (5-7)^2 + (6-7)^2 + (9-7)^2 + (a-7)^2 + (b-7)^2}{6}$

$54 = 34 + (a-7)^2 + (b-7)^2$

$20 = (a-7)^2 + (b-7)^2$ ②

① ⇒ $a = 20 - b$

⇒ ② $20 = (13-b)^2 + (b-7)^2$

$20 = 169 - 26b + b^2 \quad b^2 - 14b + 49$

$0 = 2b^2 - 40b + 198$

$0 = b^2 - 20b + 99$

$b = 9, 11$

$\therefore a = 11$ or 9.

∴ the two numbers are 9 and 11

2. [Maximum mark: 4]

Find the values of m such that this system of equations has no unique solution

$$mx + 2y = 1 \quad ①$$
$$4x + (m+2)y = 4 \quad ②$$

No solutions when parallel ie. equal gradients

① $\Rightarrow$ $y = -\frac{m}{2}x + \frac{1}{2}$

② $\Rightarrow$ $y = -\frac{4}{m+2}x + \frac{4}{m+2}$

$\therefore$ no solution when $-\frac{m}{2} = -\frac{4}{m+2}$

$m^2 + 2m = 8$

$m^2 + 2m - 8 = 0$

$m = -4$ or 2

3. [Maximum mark: 6]

Let $p(z) = z^4 + az^3 + bz^2 + cz + d$ be a polynomial with real coefficients. Given that $z = 3 + i$ and $z = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$ are two complex zeros of $p(z)$, find the values of the real coefficients.

$\sqrt{2}$ $1-i$

M1 A1

$z_1 = 3+i$ Sum $= 6$
M1 $z_2 = 3-i$ product $= 10$ $\Rightarrow z^2 - 6z + 10$ A1
$z_3 = 1-i$ Sum $= 2$
$z_4 = 1+i$ product $= 2$ $\Rightarrow z^2 - 2z + 2$

$p(z) = (z^2-2z+2)(z^2-6z+10)$ M1

$= z^4 - 6z^3 + 10z^2$
$\quad -2z^3 + 12z^2 - 20z$
$\quad + 2z^2 - 12z + 20$

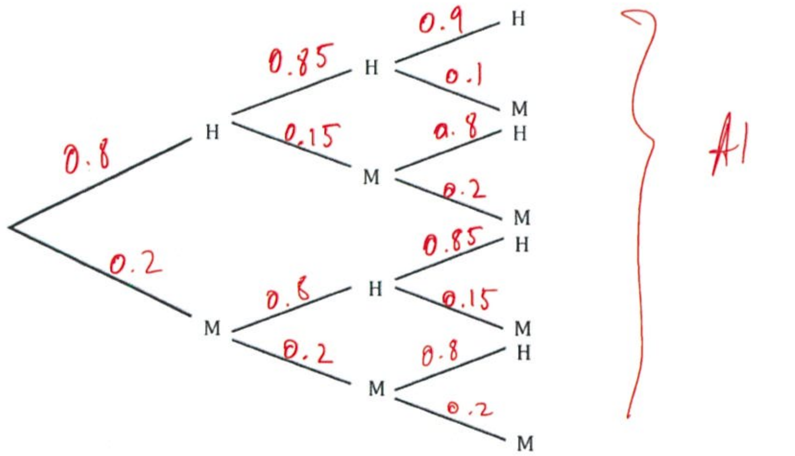
$= z^4 - 8z^3 + 24z^2 - 32z + 20$

$a = -8$
$b = 24$
$c = -32$
$d = 20$
A1

4. [Maximum mark: 7]

Klay attempts three free throws in a basketball game. The probability that he hits (H) his first free throw is 0.8. Whenever he hits a free throw, his confidence increases so that the probability of him making the next free throw increases by 0.05. Whenever he misses (M) a free throw the probability of him making the next one is 0.8.

a) Complete the given probability tree diagram for Klay's three attempts, labelling each branch with the correct probability.



b) Calculate the probability that he hits at least two free throws.

c) Calculate the probability that he hits his last (third) free throw, given that he hit exactly one previous free throw.

b). Let X = no. of free throws hit

$P(X \geq 2) = P(X=2) + P(X=3)$ M1

$= 0.8 \times 0.85 \times 0.1 + 0.8 \times 0.15 \times 0.8 + 0.2 \times 0.8 \times 0.85$ A1

$+ 0.8 \times 0.85 \times 0.9$

$= \mathbf{0.912}$ A1

c) $P(H_3 \mid H_1 \cup H_2) = \dfrac{0.8 \times 0.15 \times 0.8 + 0.2 \times 0.8 \times 0.85}{0.8 \times 0.15 + 0.2 \times 0.8}$ M1 A1

$= 0.829$ A1

M

5. [Maximum mark: ~~6~~ 5]

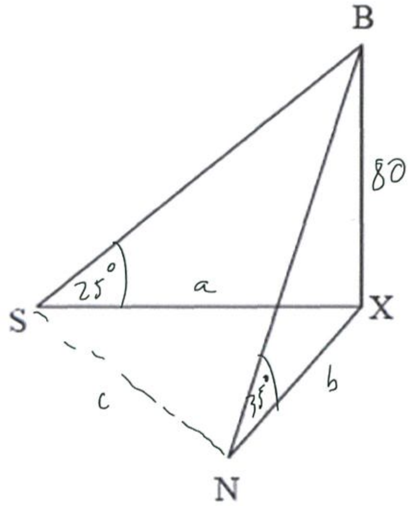
Brett is at the top of a cliff, standing 80 m above sea level, and observes two yachts in the sea.

"*Seaview*" (S) is at an angle of depression of 25°.

"*Nauti Buoy*" (N) is at an angle of depression of 35°.

The following three dimensional diagram shows Brett and the two yachts at S and N.

X lies at the foot of the cliff and angle $\text{SXN} = 70°$.



Find, to 3 significant figures, the distance between the two yachts.

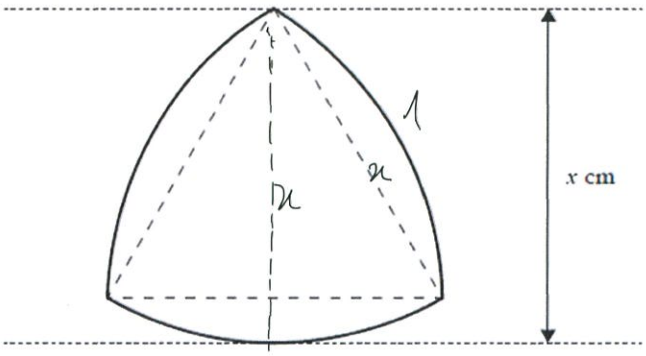
M1 $c = \sqrt{\left(\frac{80}{\tan 25}\right)^2 + \left(\frac{80}{\tan 35°}\right)^2 - 2\left(\frac{80}{\tan 25}\right)\left(\frac{80}{\tan 35°}\right)(\cos 70°)}$ A1 A1 A1

$c \simeq$ 171 m A1

6. [Maximum mark: 7]

The following shape consists of three arcs of a circle, each with centre at the opposite vertex of an equilateral triangle as shown in the diagram.

diagram not to scale



to

Given that the perimeter and area have the same value (in cm and cm^2 respecitively), find x. [7]

M1 A1

$l = r\theta = x\left(\frac{\pi}{3}\right)$ $P = 3l = \cancel{3}x\left(\frac{\pi}{\cancel{3}}\right) = \pi x$ cm

$A = \frac{1}{2}x^2\left(\frac{\pi}{3}\times 3 - 2\sin\left(\frac{\pi}{3}\right)\right)$ M1

$= \frac{1}{2}x^2\left(\pi - \sqrt{3}\right)$ A1

$= \frac{1}{2}x^{\cancel{2}}\left(\pi - \sqrt{3}\right) = \pi \cancel{x}$ M1

$x = \dfrac{2\pi}{(\pi - \sqrt{3})}$ A1

$x \simeq 4.4576$ A1

7. [Maximum mark: 7]

Consider the sequence $\ln b, \ln\sqrt{b}, \ln\sqrt[4]{b}, \ln\sqrt[8]{b}, \ldots$ where $b > 1$.

a) Show that the series $\ln b + \ln\sqrt{b} + \ln\sqrt[4]{b} + \ln\sqrt[8]{b} + \ldots$ converges and find its sum.

b) A data set consists of the first n terms of the sequence. Find an expression for the mean of the data set in terms of n and $\ln b$.

c) Hence find the least value of n for which the mean is less than 5% of the first term of the sequence.

a) $\frac{\ln\sqrt{b}}{\ln b} = \frac{\ln\sqrt[4]{b}}{\ln\sqrt{b}} = \frac{1}{2}$ M1 A1

$S_\infty = \frac{\ln b}{1-\frac{1}{2}} = 2\ln b$ A1

b) mean $= \frac{\text{sum}}{n} = \frac{\ln b\left(1-\left(\frac{1}{2}\right)^n\right)/\left(1-\frac{1}{2}\right)}{n}$ M1

$= \frac{2\ln b\left(1-\left(\frac{1}{2}\right)^n\right)}{n}$ A1

c) least n such that

mean $< 0.05\ln b$

$\frac{2\cancel{\ln b}\left(1-\left(\frac{1}{2}\right)^n\right)}{n} < 0.05\cancel{\ln b}$ M1

$n = 41$ (from GDC) A1

8. [Maximum mark: ~~7~~] 8

a) Given that the coefficients of x^{r-1}, x^r, x^{r+1} in the expansion of $(1+x)^n$ are in an arithmetic sequence, show that, $n^2 + 4r^2 - 2 - n(4r+1) = 0$. [4]

b) Hence find three consecutive coefficients of the expansion of $(1+x)^{14}$ which form an arithmetic sequence. [3]

a) $\binom{n}{r} - \binom{n}{r-1} = \binom{n}{r+1} - \binom{n}{r}$ M1

$$\frac{n!}{r!(n-r)!} - \frac{n!}{(r-1)!(n-(r-1))!} = \frac{n!}{(r+1)!(n-(r+1))!} - \frac{n!}{r!(n-r)!}$$

$$\frac{2n!}{r!(n-r)!} - \frac{n!}{(r-1)!(n-r+1)!} - \frac{n!}{(r+1)!(n-r-1)!} = 0 \quad \text{A1}$$

$[\quad '' \qquad '' \qquad '' \quad] \times (n-r-1)!\,(r-1)!$ M1

$$\left(\frac{2}{r(n-r)} - \frac{1}{(n-r+1)(n-r)} - \frac{1}{r(r+1)} = 0\right) \times r(n-r)(n-r+1)(r+1) \quad \text{A1 M1}$$

$$2(n-r+1)(r+1) - r(r+1) - (n-r)(n-r+1) = 0$$

$$2(nr + n - r^2 \cancel{-r} \cancel{+r} + 1) - r^2 \cancel{-r} - (n^2 - nr + n - nr + r^2 \cancel{-r}) = 0$$

$$4nr + n - 4r^2 + 2 - n^2 = 0$$

$$n^2 + 4r^2 - 2 - n(4r+1) = 0 \quad \text{AG}$$

b) $n = 14 \Rightarrow 14^2 + 4r^2 - 2 - 14(4r+1) = 0 \Rightarrow r = 9,\ r = 5$ M1

Coefficients: $\binom{14}{8}, \binom{14}{9}, \binom{14}{10}$ OR $\binom{14}{6}, \binom{14}{5}, \binom{14}{4}$

: 3003, 2002, 1001 A1

End of Section A

Do **not** write solutions on the following pages.

Section B

Answer all questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

A survey is conducted in a large office building. It is found that 28% of the office workers weigh less than 70 kg and that 15% of the office workers weigh more than ~~92.5~~ 98 kg.

The weights of the office workers may be modelled by a normal distribution with mean μ and standard deviation σ.

a) (i) Determine two simultaneous linear equations satisfied by μ and σ. [3]

(ii) Find the values of μ and σ. [3]

b) Find the probability that an office worker weighs more than 100 kg. [1]

There are elevators in the office building that take the office workers to their offices.

c) Given that there are 10 workers in a particular elevator, find the probability that at least four of the workers weigh more than 100 kg. [2]

d) Given that there are 10 workers in an elevator and at least one weighs more than 100 kg, find the probability that there are fewer than four workers exceeding 100 kg. [3]

The arrival of the elevators at the ground floor between $08{:}00$ and $09{:}00$ can be modelled by a Poisson distribution. Elevators arrive on average every 36 seconds.

e) Find the probability that in any half hour period between $08{:}00$ and $09{:}00$ more than 60 elevators arrive at the ground floor. [3]

An elevator can take a maximum of 10 workers. Given that 400 workers arrive in a half hour period independently of each other,

f) find the probability that there are sufficient elevators to take them to their offices. [3]

a) i) $P\left(Z<\frac{70-\mu}{\sigma}\right)=0.28$; $P\left(Z>\frac{\cancel{98}\ 92.5-\mu}{\sigma}\right)=0.15$ M1

$\Rightarrow 70-\mu=-0.5828\sigma$ A1 $\Rightarrow \cancel{98}\ 92.5-\mu=1.036\sigma$ A1

ii) Solve (GDC) M1 $\mu=\cancel{80.08}\ 78.1$ A1 ; $\sigma=\cancel{17.3}0\ 13.9$ A1

b) $P(X>100)=\cancel{0.125}\ 0.0576$ A1

c) $X\sim B(10,0.125)$ (M1) d) $P(X<4\,|\,X\geq 1)$ (M1)

$P(X\geq 4)=\cancel{0.0275}\ 0.00174$ A1 $=\frac{P(1\leq X\leq 3)}{\ldots}$

$P(X \geq 4) = 0.0275$ A1

$= \dfrac{P(1 \leq X \leq 3)}{P(X \geq 1)}$

$= \dfrac{0.7095}{0.7369}$ A1

$= \underline{0.963}$ A1

e) $\dfrac{1800 \text{ secs}}{36 \text{ secs/elev}} = 50$ elev. A1

$\therefore E \sim P(50)$ (M1)

$P(E > 60) = 0.0722$ A1

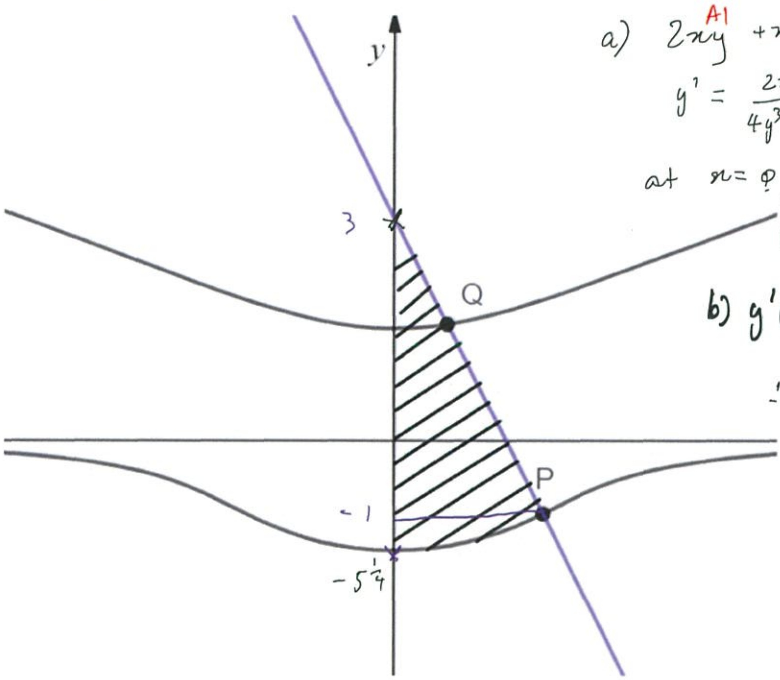
f) 400 workers need at least 40 elevators A1

$P(E \geq 40) = 0.935$ (M1) A1

Do **not** write solutions on the following pages.

11. [Maximum mark: 22]

The following graph shows the two parts of the curve defined by the equation $\frac{d}{dx}\left(x^2y = y^4 - 5\right)$, and the normal to the curve at the point P(2, -1). M1



a) $2xy + x^2y' = 4y^3y'$ A1 A1 A1

$y' = \frac{2xy}{4y^3 - x^2}$ A1

at $x = 0$, $y' = 0$ $(y \neq 0)$ R1

(i.e 2 y-intercepts) R1

b) $y'(2,-1) = \frac{-4}{-4-4} = \frac{1}{2}$ (M1) A1

$\therefore m_n = -2$ A1

$y + 1 = -2(x-2)$ M1

$y + 1 = -2x + 4$

$y = -2x + 3$ A1

(a) Show that there are exactly two points on the curve where the gradient is zero. [7]

(b) Find the equation of the normal to the curve at the point P. [5]

(c) The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

(d) The shaded region is rotated by about the y-axis. Find the volume of the solid formed. [7]

c) $y = -2x + 3$ -① , $x^2y = y^4 - 5$ -② } ① into ② and solve (GDC) M1 A1 (equation in x)

$x_Q = 0.724$ A1

d) $V = \left[\frac{1}{3}\pi r^2 \cdot h\right] + \pi\int_{-5\frac{1}{4}}^{-1} \frac{y^4 - 5}{y}\,dy$ M1 A1 A1 M1 (for volume form)

$= \frac{1}{3} \cdot \pi \cdot 2^2 \cdot 4 + ''$ A1 A1

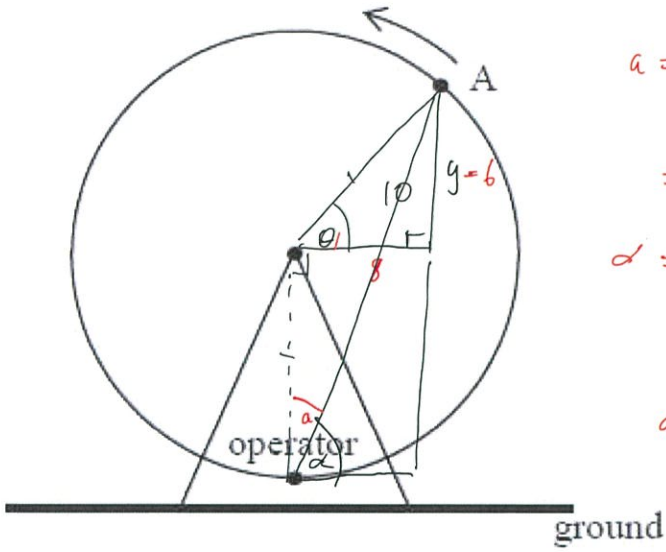
$= 19.9\ u^3$ A1

turn over.

Do **not** write solutions on the following pages.

12. [Maximum mark: 13]

Below is a sketch of a Ferris wheel, an amusement park ride carrying passengers around the rim of the wheel.



$a = \frac{\pi - \left(\frac{\pi}{2} + \theta\right)}{2}$

$= \frac{\pi}{4} - \frac{\theta}{2}$

$\alpha = \frac{\pi}{2} - a$

$= \frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$\alpha = \frac{\pi}{4} + \frac{\theta}{2}$

a) The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising. [7]

b) The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle α with the horizontal. Find the rate of change of α at point A. [3]

End of Paper Two

a) $\frac{d\theta}{dt} = 3$ rad/min A1

$y = 10\sin\theta$ A1

$\frac{dy}{d\theta} = 10\cos\theta$ m/rad M1

$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} = 10\cos\theta \cdot 3 = 30\cos\theta$ M1

at $y = 6$, $\cos\theta = 8/10 \Rightarrow \frac{dy}{dt} = 30\left(\frac{8}{10}\right)$ M1 A1

$= 24$ m/min A1

b) $\alpha = \frac{\pi}{4} + \frac{\theta}{2}$ A1

$\frac{d\alpha}{dt} = \frac{1}{2}\frac{d\theta}{dt}$ M1

$= 1.5$ rad/min A1