BANGABANDHU SHEIKH MUJIBUR RAHMAN SCIENCE AND TECHNOLOGY UNIVERSITY

GOPALGANJ-8100

Assignment on

Complex Integration and Cauchy's Theorem

Course Title: Complex Variables and Laplace

Course Code: MAT255

Submitted by	Submitted to
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4.32. Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along.

a) the curve $y = x^2+1$,

b) the stright line joining (0,1) and (2,5)

c) the straight lines from (0,1) to (0,5) and then from (0,5) to (2,5),

d) the straight lines from (0,1) to (2,1) and then from (2,1) to (2,5)

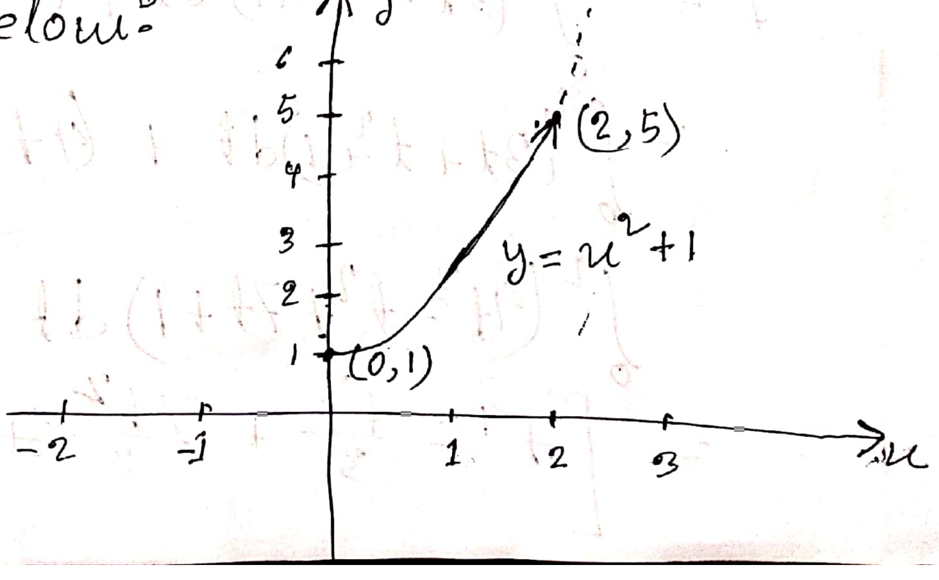
Solve:

Consider the line integral to evalute given below:

$$I = \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy \quad \text{----} \ (i)$$

(a) Along the curve $y = x^2+1$.

Consider the graph of the curve $y=x^2+1$ given below:



The parametric equation for $y=x^2+1$ are given by.

$$x = t$$

$$\therefore y = t^2+1 \;,\; 0 \le t \le 2$$

The points $(0,1)$ and $(2,5)$ on the curve $y=x^2+1$ correspond to $t=0$ and $t=2$ respectively.

Differentiate $x=t$, $y=t^2+1$ with respect to t, this gives,

$$dx = dt$$

$$dy = 2t\,dt$$

Substitute $x=t$, $y=t^2+1$, $dx=dt$, and $dy=2t\,dt$ into (i),

$$I = \int_0^2 (3t+t^2+1)\,dt + (2(t^2+1)-t)\,2t\,dt$$

$$= \int_0^2 (3t+t^2+1)\,dt + (4t^3-2t^2+4t)\,dt$$

$$= \int_0^2 (4t^3-t^2+7t+1)\,dt$$

$$= \left[t^4 - \frac{t^3}{3} + \frac{7t^2}{2} + t\right]_0^2$$

$$= 2^4 - \frac{2^3}{3} + \frac{7(2)^2}{2} + 2$$

$$= \frac{88}{3} \text{ (Ans)}$$

(b) The equation of the straight line joining (0,1) and (2,5) is given by,

$$\frac{y-1}{1-5} = \frac{x-0}{0-2}$$

$$\Rightarrow \frac{y-1}{-4} = \frac{x}{-2}$$

$$\Rightarrow y-1 = 2x$$

$$\Rightarrow y = 2x+1$$

The parametric equ^n fo $y=2x+1$ are given by,

$x = t$

$\therefore y = 2t+1, \; 0 \leq t \leq 2$

The points (0,1) and (2,5) on the straight line $y = 2x+1$ correspond to $t=0$ and $t=2$ respectively.

So, $\frac{d}{dt} x = \frac{d}{dt} t$

$\Rightarrow dx = dt$

and, $\frac{d}{dt} y = \frac{d}{dt}(2t+1)$

$\Rightarrow dy = 2dt.$

Substitute $x=t$, $y=2t+1$, $dx=dt$, and $dy=2dt$ into eqn (i)

$$\therefore I = \int_0^2 (3t+2t+1)\,dt + (2(2t+1)-t)2dt$$

$$= \int_0^2 (5t+1)\,dt + (6t+4)\,dt$$

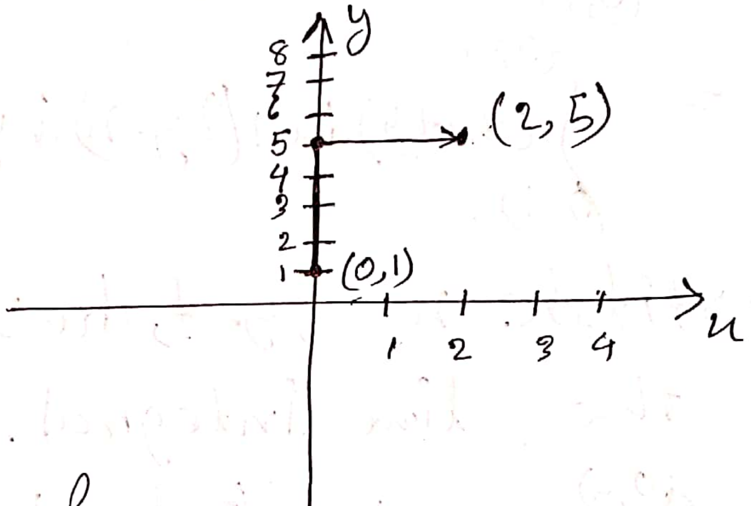
$$= \int_0^2 (11t+5)\,dt$$

$$= \left[\frac{11t^2}{2} + 5t\right]_0^2$$

$$= \frac{11(2)^2}{2} + 5(2)$$

$$= 32 \quad \text{(Ans)}$$

(c) Consider the graph of straight lines from (0, 1) and (0, 5) and then from (0, 5) to (2, 5) given below.



Equation for (0, 1) to (0, 5) is

$x = 0$

$y = t, \ 1 \leq t \leq 5$

Differentiate $x = 0$ with respect to t,

$dx = 0$

$dy = dt$

and Equation for (0, 5) to (2, 5) is

$y = 5$

$x = t \ [0 \leq t \leq 2]$

Differentiate $y = 5$ with respect to t

$dx = dt$

$dy = 0$

The line integral.

$$I = \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_{(0,1)}^{(0,5)} (3x+y)\,dx + (2y-x)\,dy + \int_{(0,5)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

Substitute $x=0$, $y=t$, $dx=0$ and $dy=dt$ into the line integral.

$$\int_{(0,1)}^{(0,5)} (3x+y)\,dx + (2y-x)\,dy$$

and $x=t$, $y=5$, $dx=dt$, and $dy=0$ into the line integral.

$$\int_{(0,5)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$\text{So, } I = \int_0^5 (3(0)+t)(0) + (2t-0)\,dt + \int_0^2 (3t+5)$$

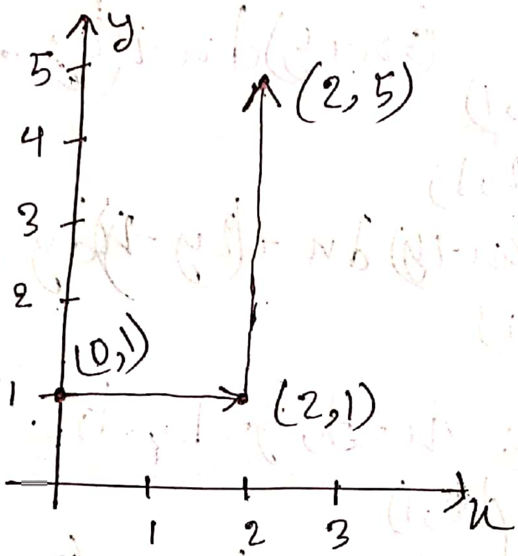
$$= 2\int_1^5 t\,dt + \int_0^2 (3t+5)\,dt$$

$$= \left[t^2\right]_1^5 + \left[\frac{3t^2}{2} + 5t\right]_0^2$$

$$= 24 + 6 + 10$$

$$= 40 \quad (\text{Ans})$$

(d) Consider the graph of straight line from (0,1) to (2,1) and then from (2,1) to (2,5) given below:



Equation for (0,1) to (2,1) is.

$y = 1$

$x = t \quad [0 \leq t \leq 2]$

Differentiate $x = t$ and $y = 1$ with respect to t,

$dx = dt$

$dy = 0$

and Equation for (2,1) to (2,5) is

$x = 2$

$y = t, \quad 1 \leq t \leq 5$ →

Differentiate $x=2$ and $y=t$ with respect to t,

$dx=0$

$dy=dt$

So, $I=\int_{(0,1)}^{(2,5)} (3x+y)dx+(2y-x)dy$

$$=\int_{(0,1)}^{(2,1)} (3x+y)dx+(2y-x)dy+\int_{(2,1)}^{(2,5)} (3x+y)dx+(2y-x)dy$$

Substitute $x=t$, $y=1$, $dx=dt$ and $dy=0$ into

$$\int_{(0,1)}^{(2,1)} (3x+y)dx+(2y-x)dy$$

and $x=2$, $y=t$, $dx=0$ and $dy=dt$

into

$$\int_{(2,1)}^{(2,5)} (3x+y)dx+(2y-x)dy$$

Then, $I=\int_0^2 (3t+1)\,dt+(2-t)(0)+$

$$\int_1^5 (6+t)(0)+(2t-2)dt$$

$$=\int_0^2 (3t+1)\,dt+\int_1^5 (2t-2)\,dt$$

$$=\left[\frac{3t^2}{2}+t\right]_0^2+\left[t^2-2t\right]_1^5$$

$=6+2+25-10-1+2$

$=24$ (Ans)

(8)

4.34. Evaluate $\int_c (x^2 - iy^2)\,dz$ along

a) the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$,

b) the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$,

c) the straight line from $(1,1)$ to $(2,8)$

Solve:

a)

$$I = \int_c (x^2 - iy^2)\,dz \quad \dots \text{(i)}$$

Let $z = x + iy \quad \dots \text{(ii)}$

Differentiate eqn (ii)

$$dz = dx + i\,dy$$

Substitute $dz = dx + i\,dy$ into $I = \int_c (x^2 - iy^2)\,dz$

this gives.

$$I = \int_c (x^2 - iy^2)(dx + i\,dy)$$

$$= \int_c (x^2 dx + ix^2 dy - iy^2 dx + y^2 dy$$

$$= \int_c x^2 dx + y^2 dy + i\int_c x^2 dy - y^2 dx$$

The parametric equations for $y=2x^2$

are given by,

$x=t$

$y=2t^2, 1\leq t\leq 2$

Differentiate $x=t$, $y=2t^2$ with respect

to t, this gives.

$dx=dt$

$dy=4t\,dt$

So, $I=\int_c x^2dx+y^2dy+i\int_c x^2dy-y^2dx$

$=\int_1^2 (t^2dt+4t^4(4t)dt+i\int_1^2 t^2(4t)dt-4t^4dt$

$=\int_1^2 (t^2+16t^5)dt+i\int_1^2 (4t^3-4t^4)dt$

$=\left[\frac{t^3}{3}+\frac{16t^6}{6}\right]_1^2+i\left[t^4-\frac{4t^5}{5}\right]_1^2$

$=\left[8/3+\frac{8\times64}{3}-1/3-8/3\right]+i\left[16-\frac{4\times32}{5}-1+4/5\right]$

$=\frac{511}{3}-\frac{49}{5}i$ (Ans)

b) The equation of the straight line from $(1,1)$ to $(1,8)$ is given by,

$x = 1$

$y = t, \quad 1 \leq t \leq 8$

Differentiate $x = 1$ with respect to t, this gives,

$dx = 0$

$dy = dt$

and the Equation for $(1,8)$ to $(2,8)$ is.

$y = 8$

$x = t, \quad 1 \leq t \leq 2$

Differentiate $y = 8$ and $x = t$ with respect to t.

$dx = dt$

$dy = 0$

Substitute $x = 1, y = t$ $dx = 0, dy = dt$

$$I_1 = \int_1^8 (0) + t^2 dt + i \int_1^8 dt - t^2(0)$$

$$= \int_1^8 t^2 dt + i \int_1^8 dt$$

$$= \left[\frac{t^3}{3}\right]_1^8 + i\left[t\right]_1^8$$

$$= \frac{(8)^3}{3} - \frac{1}{3} + 7i$$

$$= \frac{511}{3} + 7i$$

Substitute $x = t, y = 8, dx = dt, dy = 0$ into I_2.

$$I_2 = \int_1^2 t^2 dt + 64(0) + i\int_{1}^{2} t^2(0) - 8^2 dt$$

$$= \int_1^2 t^2 dt - i\int_1^2 (8)^2 dt$$

$$= \left[\frac{t^3}{3}\right]_1^2 - 64i\left[t\right]_1^2$$

$$= \frac{8}{3} + \frac{1}{3} - 64i$$

$$= \frac{7}{3} + 64i \quad \text{(Ans)}$$

So, $I = I_1 + I_2$

$$= \frac{511}{3} + 7i + \frac{7}{3} - 64i$$

$$= \frac{518}{3} - 57i \quad \text{(Ans)}$$

c) The equation of the straight line from $(\overset{x_1}{1},\overset{y_1}{1})$ to $(\overset{x_2}{2},\overset{y_2}{8})$ is

$$\frac{y-1}{1-8} = \frac{x-1}{1-2}$$

$$\Rightarrow \frac{y-1}{7} = x-1$$

$$\Rightarrow y = 7x-6 \quad \cdots \text{(ii)}$$

The parametric equations for (i) are

$x = t$

$y = 7t-6, \quad 1 \le t \le 2$

Differentiate $x=t$, $y=7t-6$ with t,

$dx = dt$

$dy = 7dt$

Substitute $x=t$, $y=7t-6$, $dx=dt$, $dy=7dt$

$$I = \int_1^2 t^2 dt + 7(7t-6)^2 dt + i\int_1^2 7t^2 dt - (7t-6)^2 dt$$

$$= \int_1^2 (344t^2 - 588t + 252)\, dt + i\int_1^2 (-42t^2 + 84t - 36)\, dt$$

$$= \left[\frac{344t^3}{3} - 294t^2 + 252t\right]_1^2 + i\left[-14t^3 + 42t^2 - 36t\right]_1^2$$

$$= \left[\frac{344(2)^3}{3} - 294(2)^2 + 252(2) - \frac{344}{3} + 294 - 252 \right]$$

$$+ i \left[-14(2)^3 + 42(2)^2 - 36(2) + 14 - 42 + 36 \right]$$

$$= \frac{518}{3} - 8i$$

(Ans)

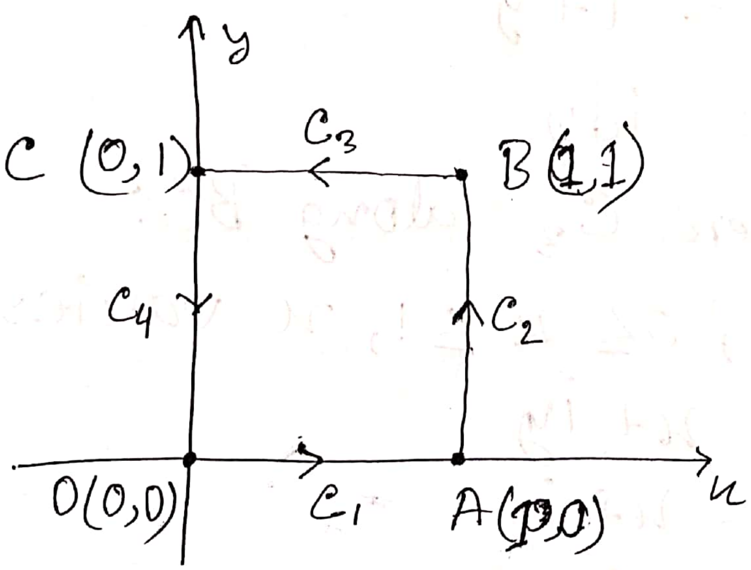
4.35. Evaluate $\oint_C |z|^2 dz$ around the square with vertices at (0,0), (1,0), (1,1), (0,1).

Solve:

Consider the line integral to evaluate around the square with vertices at (0,0), (1,0), (1,1), (0,1) given below:

$$I = \oint_C |z|^2 dz$$

Consider the square OABC with vertices at (0,0), (1,0), (1,1), (0,1) in the Z-plan given below.



Let $z = x+iy$ and hence $|z|^2 = x^2+y^2$

Consider C_1 along OA:

$y = 0, \ 0 \leq x \leq 1$

$z = x + iy$

$z = x$

$|z|^2 = x^2$

$dz = dx$

Consider C_2 along AB:

$x = 1, \ 0 \leq y \leq 1$

$z = x + iy$

$|z|^2 = 1 + y^2$

$dz = i\,dy$

Consider C_3 along BC:

$y = 1, \ 0 \leq x \leq 1$, x varies from 1 to 0

$z = x + iy$

$z = x + i$

$|z|^2 = x^2 + 1$

$dz = dx$

Consider C_4 along CO:

$x=0$, $0 \le y \le 1$, y varies from 1 to 0

$z = x+iy$

$z = iy$

$|z|^2 = y^2$

$dz = i\,dy$

The line integral $\oint_{C} |z|^2 dz$ is given by

$$\oint_C |z|^2 dz = \oint_{C_1} |z|^2 dz + \oint_{C_2} |z|^2 dz + \oint_{C_3} |z|^2 dz + \oint_{C_4} |z|^2 dz$$

$$= \int_0^1 x^2 dx + i\int_0^1 (1+y^2)\,dy + \int_1^0 (1+x^2)\,dx + i\int_1^0 y^2 dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[x+\frac{x^3}{3}\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= 1/3 + 4/3\,i + \left[0-1-1/3\right] + i\left[0-1/3\right]$$

$$= 1/3 - 4/3 + (4/3 - 1/3)\,i$$

$$= -1 + i$$

(Ans)

4.39. Evaluate $\oint_C \bar{z}^2 dz$ around the circles

a) $|z| = 1$,

b) $|z-1| = 1$.

Solve:

Consider the line integral

$$I = \oint_C \bar{z}^2 dz \quad \cdots\cdots (i)$$

a) Along the circle $|z| = 1$.

The radius of the circle $|z| = 1$ is $r = 1$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ and hence,

$$dz = ie^{i\theta} d\theta \text{ and } \bar{z} = e^{-i\theta}$$

Substitute $dz = ie^{i\theta}$ and $\bar{z} = e^{-i\theta}$ into (i)

$$I = \oint_C \bar{z}^2 dz$$

$$= \int_0^{2\pi} (e^{-i\theta})^2 \, ie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta} d\theta$$

But, $e^{-i\theta} = \cos\theta - i\sin\theta$

So, $I = i \int_0^{2\pi} (\cos\theta - i\sin\theta) d\theta$

$$I = i\left[\sin\theta + i\cos\theta\right]_0^{2\pi}$$

$$= i\left[\sin 2\pi + i\cos 2\pi - \sin\theta - i\cos\theta\right]$$

$$= 0$$

(Ans)

b) The radius of the circle $|z-1|=1$ is $r=1$.

Let $z = 1+e^{i\theta}$, $0 \le \theta \le 2\pi$ and hence,

$dz = ie^{i\theta}\,d\theta$ and $\bar{z} = -1+e^{-i\theta}$

Substitute $dz = ie^{i\theta}$ and $\bar{z} = -1+e^{-i\theta}$ into (i),

$$I = \oint_C \bar{z}^2\,dz$$

$$= \oint_0^{2\pi} \left(1+e^{-i\theta}\right)^2 ie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \left(1+2e^{-i\theta}+e^{-i2\theta}\right)e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \left(e^{i\theta}+2+e^{-i\theta}\right)d\theta$$

But $e^{-i\theta} = \cos\theta - i\sin\theta$,

$e^{i\theta} = \cos\theta + i\sin\theta$,

so,
$$I = i\int_0^{2\pi} (\cos\theta + i\sin\theta + 2 + \cos\theta - i\sin\theta)\, d\theta$$

$$= 2i\int_0^{2\pi} (\cos\theta - 1)\, d\theta$$

$$= 2i\left[\sin\theta + \theta\right]_0^{2\pi}$$

$$= 2i\left[\sin 2\pi + 2\pi - \sin 0 - 0\right]$$

$$= 4\pi i$$

(Ans)

4.40. Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around

a) the circle $|z| = 1$,

b) the square with verticees at $(0,0), (1,0), (1,1), (0,1)$.

c) The curve consisting of the parabolas $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$.

Solve:

a) Consider the line integral to evaluate given below:

$$I = \oint_C (5z^4 - z^3 + 2)\,dz \quad \cdots \text{(i)}$$

The radius of the circle $|z| = 1$ is $r = 1$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ and $dz = ie^{i\theta}\,d\theta$

Substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}\,d\theta$ into the equation (i)

$$I = \oint_C (5z^4 - z^3 + 2)\,dz$$

$$= \int_0^{2\pi} \left(5(e^{i\theta})^4 - (e^{i\theta})^3 + 2\right) ie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \left(5e^{i5\theta} - e^{i4\theta} + 2e^{i\theta}\right) d\theta$$

$$= i\left[\frac{e^{i5\theta}}{i} - \frac{e^{i4\theta}}{4i} + \frac{2e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= \left[e^{i5\theta} - \frac{e^{i4\theta}}{4} + 2e^{i\theta}\right]_0^{2\pi}$$

$$= \Big[\cos 5\theta + i\sin 5\theta - \cos 4\theta - i\sin 4\theta + 2\cos\theta + 2i\sin\theta\Big]_0^{2\pi}$$

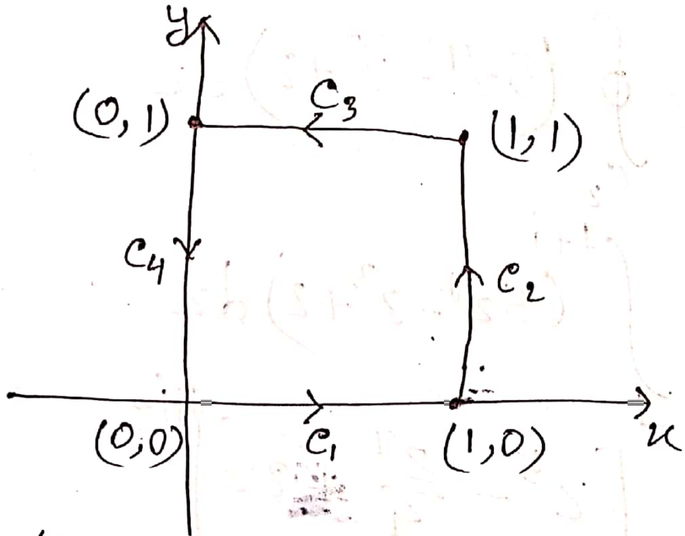
$$= \Big[\cos 10\pi + i\sin 10\pi - \cos 8\pi - i\sin 8\pi + 2\cos 2\pi + 2i\sin 2\pi - \cos 0 - i\sin 0 + \cos 0 + i\sin 0 - 2\cos 0 - 2i\sin 0\Big]$$

$$= 1 - 1 + 2 - 1 + 1 - 2$$

$$= 0$$

(Ans)

b) Consider the square with vertices at (0, 0), (1, 0), (1,1) and (0, 1) given below:



Along the line C_1, the points (0, 0) to (1,0) correspond to $z=0$ to $z=1$

So, Along the line C_1,

$$I_1 = \oint_{C_1} (5z^4 - z^3 + 2)\, dz$$

$$= \int_0^1 (5z^4 - z^3 + 2)\, dz$$

$$= \left[\frac{5z^5}{5} - \frac{z^4}{4} + 2z\right]_0^1$$

$$= 1 - \frac{1}{4} + 2$$

$$= \frac{11}{4}$$

Along the line C_2, the points $(1,0)$ to $(1,1)$ correspond to $z=1$ to $z=1+i$.

$$I_2 = \oint_{C_2} (5z^4 - z^3 + 2)\, dz$$

$$= \int_1^{1+i} (5z^4 - z^3 + 2)\, dz$$

$$= \left[z^5 - \frac{z^4}{4} + 2z \right]_1^{1+i}$$

$$= (1+i)^5 - \frac{(1+i)^4}{4} + 2(1+i) - 1 + \frac{1}{4} - 2$$

$$= -3/4 + 2i + 1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5$$

$$- \frac{1+4i-6-4i+1}{4}$$

$$= -3/4 + 2i + 1 + 5i - 10 - 10i + 5 + i + 1$$

$$= -\frac{15}{4} - 2i$$

Along the line C_3, the points $(1,1)$ to $(0,1)$ corresponds to $z=1+i$ to $z=i$.

$$I_3 = \oint_{C_3} (5z^4 - z^3 + 2)\, dz$$

$$= \int_{1+i}^{i} (5z^4 - z^3 + 2)\, dz$$

$$= \left[z^5 - \frac{z^4}{4} + 2z \right]_{1+i}^{i}$$

$$= i^5 - \frac{i^4}{5} + 2i - (1+i)^5 + \frac{(1+i)^4}{4} - 2(1+i)$$

$$= i - \frac{1}{4} + 2i - (1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5)$$
$$+ \frac{(1 + 4i + 6i^2 + 4i^3 + i^4)}{4} - 2 - 2i$$

$$= -1/4 - 2 + i + 2i - 2i - (-4 - 4i) - 1$$

$$= (-1/4 - 2 + 4 - 1) + (3i - 2i + 4i)$$

$$= 3/4 + 5i$$

~~Ans~~) (P.T.O)

Along the line C_4, the points $(0,1)$ to $(0,0)$ corresponds to $z=i$ to $z=0$.

$$I = \oint_{C_4} (5z^4 - z^3 + 2)\, dz$$

$$= \int_i^0 (5z^4 - z^3 + 2)\, dz$$

$$= \left[z^5 - \frac{z^4}{4} + 2z \right]_i^0$$

$$= 0 - i^5 + \frac{i^4}{4} - 2i$$

$$= \frac{1}{4} - 3i$$

Add $I_1 = \frac{11}{4}$, $I_2 = -\frac{15}{4} - 2i$, $I_3 = \frac{3}{4} + 5i$, $I_4 = \frac{1}{4} - 3i$

$$I = I_1 + I_2 + I_3 + I_4$$

$$= 11/4 - 15/4 - 2i + 3/4 + 5i + 1/4 - 3i$$

$$= 11/4 - 15/4 + 3/4 + 1/4 - 2i + 5i - 3i$$

$$= 0$$

(Ans)

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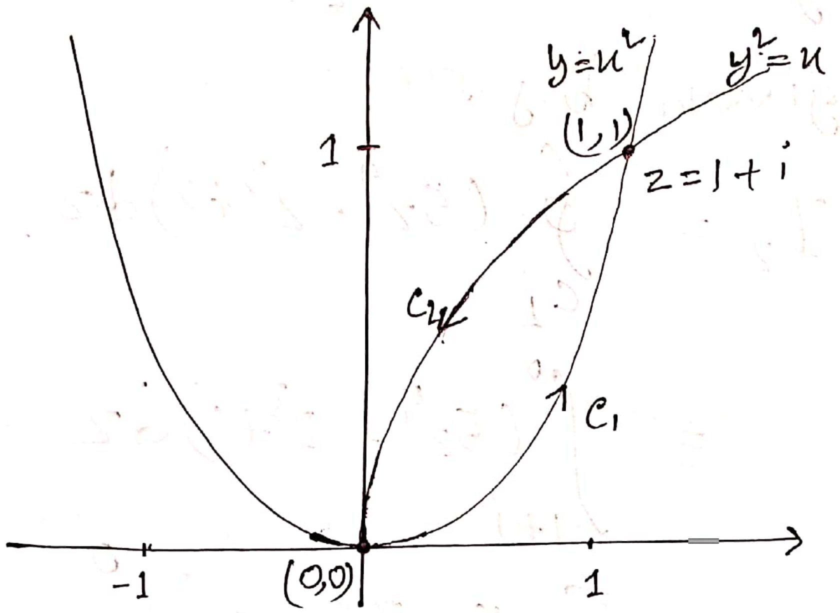
c) Consider the graph of the curve consisting of the parabolas $y=x^2$ from (1,1) to (0,0) given below.



Along the line C_1, the points (0,0) to (1,1) correspond to $z=0$ to $z=1+i$

$$I_1 = \oint_{C_1} (5z^4 - z^3 + 2)\, dz$$

$$= \int_0^{1+i} (5z^4 - z^3 + 2)\, dz$$

$$= \left[z^5 - \frac{z^4}{4} + 2z \right]_0^{1+i}$$

$$= (1+i)^5 - \frac{(1+i)^4}{4} + 2(1+i)$$

Along the line C_2, the points $(1,1)$ to $(0,0)$ correspond to $z = 1+i$ to $z=0$.

Along the line C_2, the line integral is given by.

$$I_2 = \oint_{C_2} (5z^4 - z^3 + 2)\,dz$$

$$= \int_{1+i}^{0} (5z^4 - z^3 + 2)\,dz$$

$$= \left[z^5 - \frac{z^4}{4} + 2z \right]_{1+i}^{0}$$

$$= -(1+i)^5 + \frac{(1+i)^4}{4} - 2(1+i)$$

Add I_1 and I_2

$$I = I_1 + I_2$$

$$= (1+i)^5 - \frac{(1+i)^4}{4} + 2(1+i)$$

$$-(1+i)^5 + \frac{(1+i)^4}{4} - 2(1+i)$$

$$= 0$$

(Ans)

4.43. Evaluate $\oint_C \frac{dz}{z-2}$ around

a) the circle $|z-2|=4$,

b) the circle $|z-1|=5$,

c) the square with vertices at

$3 \pm 3i, -3 \pm 3i$.

Solve:

Consider the line integral to evaluate given below:

$$I = \oint_C \frac{dz}{z-2}$$

a) The radius of the circle $|z-2|=4$ is $r=4$.

Suppose $a=2$ is inside C and let Γ be the circle of radius $r=4$ with center at $z=a$, so that Γ is inside C.

Thus, the line integral is given by.

$$\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$$

Let $z-2=4e^{i\theta}$, $[0 \leq \theta \leq 2\pi]$

$$dz = 4ie^{i\theta}\,d\theta$$

Substitute $z-2=4e^{i\theta}$, $dz=4ie^{i\theta}\,d\theta$ into the right side of $\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$,

The line integral is given by.

$$\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$$

$$= \int_0^{2\pi} \frac{4ie^{i\theta}}{4e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} \theta\,d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

(Ans)

b) The radius of the circle $|z-1| = 5$ is $r=5$

Suppose $a=2$ is inside C and let Γ be a circle of radius $r=5$ with center at $z=a$ so that Γ is inside C.

$$\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$$

Let $z-2 = 5e^{i\theta}$, $0 \leq \theta \leq 2\pi$ and there

$$dz = 5ie^{i\theta}\, d\theta$$

Substitute $z-2 = 5e^{i\theta}$, $dz = 5ie^{i\theta} d\theta$ into the right side of $\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$,

$$= \int_0^{2\pi} \frac{5ie^{i\theta}}{5e^{i\theta}}\, d\theta$$

$$= i\int_0^{2\pi} \theta\, d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

(Ans)

c) Assume the circle $|z-2|=1$ is inside the square with vertices at $3\pm3i, -3i\pm3i$.

The radius of the circle $|z-2|=1$

Suppose $a=2$ is inside C and let Γ be a circle of radius $r=1$ with center at $z=a$ so that Γ is inside C.

$$\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$$

Let $z-2=e^{i\theta}$, $0\leq\theta\leq 2\pi$

and $dz = ie^{i\theta}\,d\theta$

Substitute $z-2=e^{i\theta}$, $dz=ie^{i\theta}$ into the right side of $\oint_C \frac{dz}{z-2} = \oint_\Gamma \frac{dz}{z-2}$

$$= \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta$$

$$= i\,[\theta]_0^{2\pi}$$

$$= 2\pi i$$

(Ans)

4.45. Verify Green's theorem in the plane for $\oint_c (x^2-2xy)\,dx + (y^2-x^3y)\,dy$ where c is a square with vertices at (0,0),(2,0),(2,2), and (0,2).

Solve:

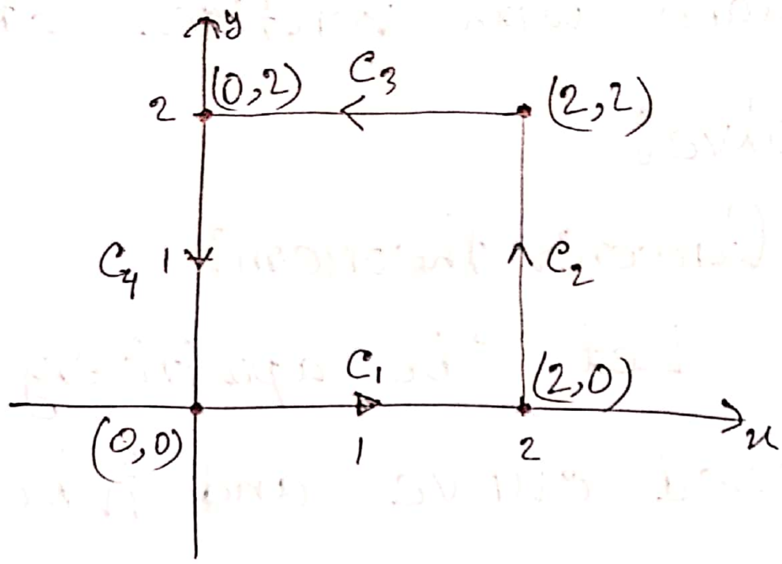
Green's theorem:

Let C be a positively oriented closed curve and R be the region enclosed by the curve. If P and Q have continious first order partial derivatives then,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

[P.T.O]

Consider the graph of the square with vertices at (0,0), (2,0), (2,2), (0,2) is given below.



The equation for C_1 are given below.

$y = 0$

Along the line C_1, the line integral is

$$I_1 = \oint_{C_1} (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$$

But $y = 0$, x varies from 0 to 2

$$\therefore I_1 = \oint_{C_1}^{2} (x^2 - 2x(0))\,dx + ((0)^2 - x^3(0))\,dy$$

$$= \int_0^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{3}$$

The equation for C_2 is given by

$x = 2$

$dx = 0$

Along the line C_2, the line integral is

$$I_2 = \oint_{C_2} (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$$

But $x = 2$, $dx = 0$, y varies from 0 to 2

$$I_2 = \oint_{C_2} ((2)^2 - 4y)(0) + (y^2 - (2)^3 y)\,dy$$

$$= \int_0^2 (y^2 - 8y)\,dy$$

$$= \left[\frac{y^3}{3} - 4y^2\right]_0^2$$

$$= 8/3 - 16$$

$$= -40/3$$

The equation for C_3 are given by.

$y = 2$

$dy = 0$

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Along the line C_3, the line integral is

$$I_3 = \oint_{C_3} (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$$

But $y = 2$, $dy = 0$, x varies from 2 to 0

$$I_3 = \oint_{C_3} (x^2 - 4x)\,dx + (4 - 2x^3)(0)$$

$$= \int_2^0 (x^2 - 4x)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2\right]_2^0$$

$$= 0 - 8/3 + 8$$

$$= 16/3$$

The equation for C_4 are given by,

$y = 2$

$dy = 0$

Along the line C_4, the line integral is

$$I_4 = \oint_{C_4} (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$$

But $x = 0$ and y varies from 2 to 0

$$I_4 = \oint_{C_4} (0-0)\,dx + (y^2-0)\,dy$$

$$= \int_2^0 y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_2^0$$

$$= 0 - 8/3$$

$$= -8/3$$

Add $I_1 = 8/3$, $I_2 = -\frac{40}{3}$, $I_3 = 16/3$ and $I_4 = -8/3$, the line integral is given by,

$$\oint_C (x^2-2xy)\,dx + (y^2-x^2y)\,dy = I_1+I_2+I_3+I_4$$

$$= \frac{8}{3} - \frac{40}{3} + \frac{16}{3} - \frac{8}{3}$$

$$= -8$$

Compare $\oint_C (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$ with the left side of Green's theorem,

this gives,

$$P = x^2 - 2xy$$

$$Q = y^2 - x^3 y$$

Take the partial derivative of $P = x^2 - 2xy$ with respect to y and $Q = y^2 - x^3 y$ with respect to x, this gives

$$\frac{\partial P}{\partial y} = -2x$$

$$\frac{\partial Q}{\partial x} = -3x^2 y$$

Substitute $\frac{\partial Q}{\partial x} = -3x^2 y$ and $\frac{\partial P}{\partial y} = -2x$ into the right side of Green's theorem, this gives.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left((-3x^2y) - (-2x)\right) dx\,dy$$

$$= \int_0^2\int_0^2 (2x - 3x^2y)\, dx\,dy$$

$$= \int_0^2 \left[2xy - \frac{3x^2y^2}{2}\right]_0^2 dx$$

$$= \int_0^2 \left[4x - 6x^2\right] dx$$

$$= \left[2x^2 - 2x^3\right]_0^2$$

$$= -8$$

Hence, the Green's theorem is verified fo $\oint_C (x^2 - 2xy)\,dx + (y^2 - x^3y)\,dy$, Here C is a square with vertices al $(0,2)$, $(2,2)$, $(0,0)$, $(2,0)$

(Ans)

4.46. Evaluate $\oint_C (3x+6y-3)\,dx + (3x-4y+2)\,dy$ around a triangle in the xy plane with vertices at (0,0), (4,0), and (4,3).

Solve:

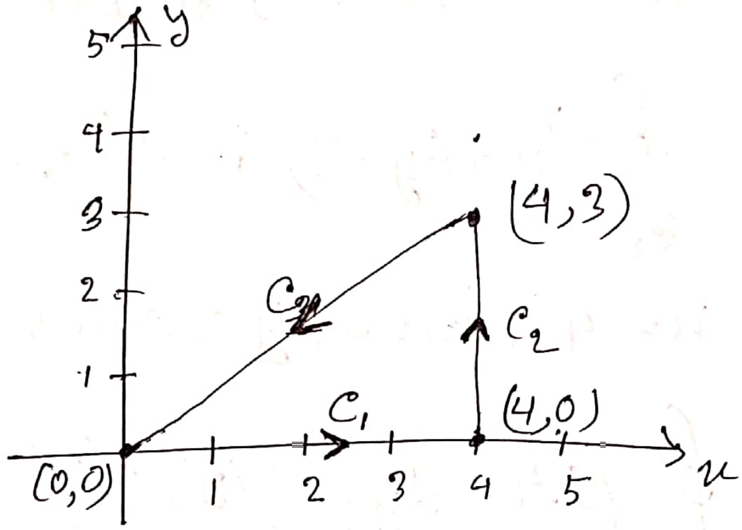
Green's theorem:

Let C be a positively oriented, closed curve and R be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on R then,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$\boxed{\text{P.T.O.}}$

Consider the graph of the triangle in the xy-plane withe verticees at (0,0), (4,0), and (4,3) given below



The equation for C_1 are given by.

$y = 0$
$dy = 0$

Along the line C_1, the line integral is

$$I_1 = \oint_{C_1} (5x + 6y - 3)dx + (3x - 4y + 2) dy$$

But $y = 0$, $dy = 0$, x varies from 0 to 4

$$I_1 = \oint_{c} (5x + 6(0) - 3)dx + (3x - 4(0) + 2)(0)$$

$$= \int_0^4 5x - 3 \, dx$$

$$= \left[\frac{5x^2}{2} - 3x\right]_0^4 = 28$$

(41)

The equation for C_2 are given by

$$x = 4$$
$$dx = 0$$

Along the line C_2, the line integral is

$$I_2 = \oint_{C_2} (3x + 6y - 3)\,dx + (3x - 4y + 2)\,dy$$

But $x = 4$, $dx = 0$, y varies from 0 to 3

$$I_2 = \oint_{C_2} (3.4 + 6y - 3)(0) + (3.4 - 4y + 2)\,dy$$

$$= \int_0^3 (14 - 4y)\,dy$$

$$= \left[14y - 2y^2\right]_0^3$$

$$= 24$$

The equation for C_3 are given by,

$$\frac{y - 0}{0 - 3} = \frac{x - 0}{0 - 4}$$

$$\Rightarrow y = \frac{3}{4}x$$

So, $dy = \frac{3}{4}\,dx$

Along the line C_3, the line integral is

$$I_3 = \oint_{C_3} (5x+6y-3)\,dx + (3x-4y+2)\,dy$$

But $y = \frac{3}{4}x$, $dy = \frac{3}{4}dx$, x varies from 4 to 0

$$I_3 = \oint_C \left(\frac{19}{2}x - 3\right)dx + \frac{3}{2}dx$$

$$= \int_4^0 \left(\frac{19}{2}x - \frac{3}{2}\right)dx$$

$$= \cancel{\left[\frac{19x^2}{4} + \frac{3(4)}{2}\right]}$$

$$= \left[\frac{19x^2}{4} - \frac{3x}{2}\right]_4^0$$

$$= 0 - \frac{19(4)^2}{4} + \frac{3.4}{2}$$

$$= -70$$

Add $I_1 = 28$, $I_2 = 24$ and $I_3 = -70$, the line integral is.

$$\oint_C (5x+6y-3)\,dx + (3x-4y+2)\,dy = I_1 + I_2 + I_3$$
$$= 28 + 24 - 70$$
$$= -18$$

Compare $\oint_C (5x+6y-3)\,dx + (3x-4y+2)\,dy$ with the left side of $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

So,

$$P = 5x + 6y - 3$$

$$\Rightarrow \frac{\partial P}{\partial y} = 6 \quad \text{[Partial derivative of P with respect to y]}$$

$$Q = 3x - 4y + 2$$

$$\Rightarrow \frac{\partial Q}{\partial x} = 3 \quad \text{[Partial derivative of Q with respect to x]}$$

Substitute $\frac{\partial Q}{\partial x} = 3$ and $\frac{\partial P}{\partial y} = 6$ into the right side of Green's theorem.

this gives

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R (3-6)\,dx\,dy$$

$$= -3 \int_0^4 \int_0^{y=\frac{3}{4}x} dy\,dx$$

$$= -3 \int_0^4 [y]_0^{\frac{3}{4}x}\,dx$$

$$= -\frac{9}{4} \int_0^4 x\,dx$$

$$= -\frac{9}{8}\left[x^2\right]_0^4$$

$$= -\frac{9}{8}[16]$$

$$= -18$$

Hence the Green's theorem is varified for $\oint_C (5x+6y-3)dx + (3x-4y+2)dy$

Here C is a triangle with vertices at (0,0), (4,0), and (4,3)

(Ans)

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