Hence

$$P(C) = \frac{.22}{P(D)} = \frac{.22}{.36} = 0.61$$

Independence of More than Two Events

The multiplication rule for independent events extends very simply to three or more independent events. For three events we have the following rule:

> If A, B, C are all independent of each other (i.e.the occurrence of any one is not affected by the occurrence of any combination of the others), then $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$

This result can naturally be extended to n events $A_1, A_2, \ldots\ldots, A_n$. We state this result as saying that n events are independent provided the probability of their intersections is equal to the product of their individual probabilities: That is

$$P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n) = P(A_1) \times P(A_2) \times \ldots\ldots \times P(A_n) \qquad (7.1)$$

But what about the independence of the pairs of events, such as (A_1 and A_2) or (A_2 and A_3) or (A_4 and A_5)? It is possible that the events A_1, A_2, A_3.. are not independent (i.e. equation (7.1) is not satisfied), even though all possible combinations of the events are independent. For example, we might have $P(A_1 \cap A_2) = P(A_1) \times P(A_2)$ or $P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$ or any other combination of the events $A_1, A_2, A_3, \ldots, A_n$., satisfying this relation but not 7.1. In such instances, we raise the question of **complete independence**, which we define below:

***Definition* 7.12**: *The n events are said to be completely independent if and only if every combination of these events, taken any number at a time, is independent.*

If every combination other than the one in (7.1) is independent, then we say that the events are **pairwise** independent but not **completely independent.**

For three events A_1, A_2, A_3, the complete independence ensures that the following equations are satisfied:

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

If first three of the above relations are satisfied but not the last one, we say that the events A_1, A_2, and A_3 are pairwise independent but not completely independent.

It is important to note that any combination of the events formed by replacing one or two or three..... by their complements, will also make the above results valid.

Example 7.49: Two coins are tossed. If A is the event "head on the first coin", B is the event "head on the second coin" and C is the event "coins fall alike", then show that the events A, B, and C are pairwise independent but not completely independent.

Solution: The sample space is $S=\{HH, HT, TH, TT\}$ and the events A, B, C are as follows

$$A=\{HH, HT\},\ B=\{HH, TH\} \text{ and } C=\{HH, TT\}.$$

and

$$A\cap B=\{HH\},\ A\cap C=\{HH\},\ B\cap C=\{HH\},\ A\cap B\cap C=\{HH\}.$$

And the associated probabilities are

$$P(A)=P(B)=P(C)=\frac{1}{2}, \text{ and } P(A\cap B)=P(A\cap C)=P(B\cap C)=\frac{1}{4}$$

which shows that

$$P(A)\times P(B)=P(A\cap B),\ P(A)\times P(C)=P(A\cap C) \text{ and } P(B)\times P(C)=P(B\cap C)$$

Hence the events are pairwise independent. But

$$P(A\cap B\cap C)=\frac{1}{4} \text{ and } P(A)\times P(B)\times P(C)=\frac{1}{8}.$$

owing that

$$P(A\cap B\cap C)\neq P(A)\times P(B)\times P(C)$$

nce they are not independent taken altogether, in other words, not completely independent.

7.13 SELECTED THEOREMS ON PROBABILITY

On the basis of the axioms stated earlier, we state and prove the following theorems:

***Theorem* 7.2:** *For any event A,* $P(\overline{A})=1-P(A)$

Proof: Since A and $\overline{A}$ are exhaustive, their union constitute the sample space S. That is $A\cup\overline{A}=S$. Consequently

$$P(A \cup \overline{A}) = P(S) = 1 \qquad \text{[by Axiom 2]}$$

Since A and $\overline{A}$ are disjoint

$$P(A \cup \overline{A}) = P(A) + P(\overline{A}) = 1$$

Hence

$$P(\overline{A}) = 1 - P(A)$$

***Theorem* 7.3**: *If* ϕ *is an empty set,* $P(\phi) = 0$.

Proof: Since S and $\bar{S}$ are exhaustive, $S \cup \overline{S} = S$, we have

$$P(S \cup \overline{S}) = P(S)$$

Also S and $\overline{S}$ are mutually exclusive, so that

$$P(S) + P(\overline{S}) = P(S)$$

Hence it follows that

$$P(\phi) = 0$$

***Theorem* 7.4**: $P(A) \le 1$.

Proof: Since by Axiom 1, $P(\overline{A}) \ge 0$, and by Axiom, $P(A) + P(\overline{A}) = 1$, it follows that

$$P(A) \le 1.$$

***Theorem* 7.5**: If $A \cap B \neq \phi$, *then* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

This theorem is variously known as the **addition theorem** (law), **additive law** or **law of total probability,**

Proof: Look at the following Venn diagram.

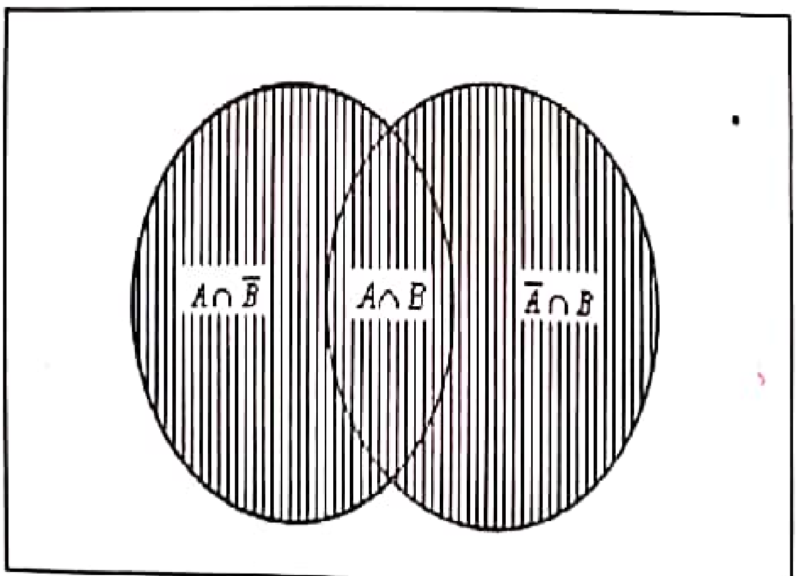

Figure 7.9: Union of A & B : $A \cup B$

The union of A and B can be partitioned as follows:

$$A \cup B = (A \cap \overline{B}) \cup B.$$

Thus by Axiom 3

$$P(A \cup B) = P(A \cap \overline{B}) + P(B)$$
$$= P(A) - P(A \cap B) + P(B)$$

Hence the proof

$$= P(A) + P(B) - P(A \cap B)$$

We can also write $(A \cup B)$ as follows:

$$(A \cup B) = (\overline{A} \cap B) \cup A$$

and proceed to prove the theorem as above

Example 7.50: Two coins are tossed. A is the event 'getting two heads' and B is the event 'second coin shows head'. Evaluate $P(A \cup B)$.

Solution: The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

and the events A, B and $A \cap B$ are

$$A = \{HH\},\ B = \{HH, TH\},\ A \cap B = \{HH\}$$

and the associated probabilities are

$$P(A) = \frac{1}{4},\ P(B) = \frac{2}{4},\ P(A \cap B) = \frac{1}{4}$$

and hence

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{4} + \frac{2}{4} - \frac{1}{4} = \frac{1}{2}$$

...orem 7.6. *For any three events, A, B, C, which are not mutually ...xclusive,*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \text{... (a)}$$

Proof: Consider the following **Venn diagram**, which shows how the union of 3 events can be partitioned into seven mutually exclusive events $a_1, a_2, a_3, \ldots a_7$.

We shall denote the probabilities of these events by the values $P(a_1)$, $P(a_2), P(a_3)$, ... $P(a_7)$ respectively as indicated in the diagram.

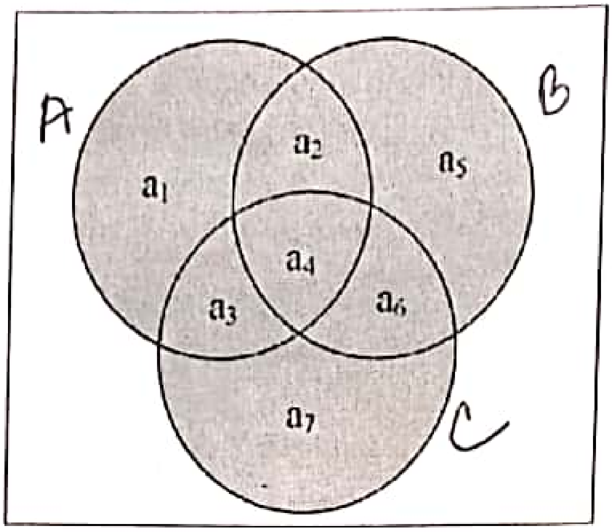

Figure 7.10: Union of A, B and C: $A \cup B \cup C$

Then

$$P(A \cup B \cup C) = P(a_1) + P(a_2) .... + P(a_7) = \sum_{i=1}^{7} P(a_i)$$

And we must show that the right-hand side of (1) is also equal to $\Sigma P(a_i)$

Since

$$A = a_1 \cup a_2 \cup a_3 \cup a_4,\ B = a_2 \cup a_4 \cup a_5 \cup a_6 \text{ and}$$
$$C = a_3 \cup a_4 \cup a_6 \cup a_7.$$

$$P(A) = P(a_1 \cup a_2 \cup a_3 \cup a_4) = P(a_1) + P(a_2) + P(a_3) + P(a_4)$$
$$P(B) = P(a_2 \cup a_4 \cup a_5 \cup a_6) = P(a_2) + P(a_4) + P(a_5) + P(a_6)$$
$$P(C) = P(a_3 \cup a_4 \cup a_6 \cup a_7) = P(a_3) + P(a_4) + P(a_6) + P(a_7)$$

Adding

$$\begin{aligned} P(A)+P(B)+P(C) &= P(a_1)+2P(a_2)+2P(a_3)+3P(a_4)+P(a_5) \\ &\quad +2P(a_6)+P(a_7) \\ &= [P(a_1)+P(a_2)+ \dots\dots\dots + P(a_7)] \\ &\quad +[P(a_2)+P(a_4)]+[P(a_3)+P(a_4)] \\ &\quad +[P(a_4)+P(a_6)] - P(a_4). \\ &= \sum_{i=1}^{7} p(a_i) + [P(a_2)+P(a_4)]+[P(a_3)+P(a_4)] \\ &\quad + [P(a_4)+P(a_6)] - P(a_4) \quad \dots\dots\dots\dots\dots\dots (2) \end{aligned}$$

But an examination of the Venn diagram shows that

$$\sum_{i=1}^{7} P(a_i) = P(A\cup B\cup C)$$

$$P(a_4)=P(A\cap B\cap C),\ P(a_2)+P(a_4)=P(A\cap B),\ P(a_3)+P(a_4)=P(A\cap C)$$
$$P(a_4)+P(a_6)=P(B\cap C)$$

Substituting these values in (2) above and transposing

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)$$
$$-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

An alternative way of proving the above theorem is as follows:

$$P(A\cup B\cup C)=P[(A\cup B)\cup C)]$$
$$=P(A\cup B)+P(C)-P[(A\cup B)\cap C]$$
$$=P(A)+P(B)-P(A\cap B)+P(C)-P[(A\cap C)\cup(B\cap C)]$$

The last term of the above expression can be written as

$$P[(A\cap C)\cup(B\cap C)]=P(A\cap C)+P(B\cap C)-P[(A\cap C)\cap(B\cap C)]$$
$$=P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)$$

On substitution, the result follows.

Note: *If A, B, C are mutually exclusive, the above relation reduces to*

$$P(A\cup B\cup C)=P(A)+P(B)+P(C$$

For four events A, B, C and D, which are not mutually exclusive, the formula can be extended as follows:

$$P(A\cup B\cup C\cup D)=P(A)+P(B)+P(C)+P(D)-P(A\cap B)-P(A\cap C)$$
$$-P(A\cap D-P(B\cap C)-P(B\cap D)-P(C\cap D)$$
$$+P(A\cap B\cap C)+P(A\cap B\cap D)+P(A\cap C\cap D)$$
$$+P(B\cap C\cap D)-P(A\cap B\cap C\cap D)$$

Note: For n events, $A_1, A_2, \ldots A_n$, which are not mutually exclusive

$$P(A_1\cup A_2\cup\ldots\cup A_n)=\sum_{i=1}^{n}P(A_1)-\sum_{i<j}^{n}P(A_i\cap A_j)+\sum_{i<j<k}^{n}P(A_i\cap A_j\cap A_k)$$
$$-\sum_{i<j<k<l}^{n}P(A_i\cap A_j\cap A_k\cap A_l)+\ldots$$
$$+(-1)^{n+1}P((A_i\cap A_j\cap A_k\cap\ldots\cap A_n)$$

From the above expression, it follows that

ex

Pr

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a_1,

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) \le P(A_1) + P(A_2) + \ldots + P(A_k)$$

Occasionally, this is referred to as **Boole's inequality.**

When $A_1, A_2, \ldots A_n$ are all mutually exclusive,

$$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Theorem **7.7:** *For any event A, prove that*

$$0 \le P(A) \le 1.$$

Proof: It is known from Axiom 1 that $P(A) \ge 0$. If $P(A) > 1$, then it follows from Theorem 1 that $P(\overline{A}) < 0$. Since these results contradict Axiom 1 (which states that the probability of every event must be non-negative), it must also be true that $P(A) \le 1$. Hence the proof.

Theorem **7.8**: *If* $A \subset B$, *then* $P(A) \le P(B)$

The theorem states that if A is contained in B, then probability of A happening can not exceed the probability of happening of B.

Proof: As illustrated in figure below, the event B may be treated as the union of two disjoint events A and $B \cap \overline{A}$ $\overline{A}$

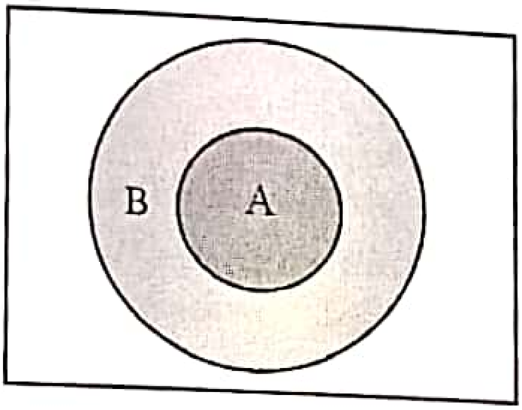

Figure 7.11: Venn diagram displaying $A \subset B$

Therefore

$$P(B) = P(A) + P(B \cap \overline{A})\overline{A})$$

Since $P(B \cap \overline{A}) \ge 0$, it follows that

$$P(A) \le P(B)$$

Theorem **7.9** *For two events A and B, prove that*

$$P[(\overline{A} \cap B) \cup (A \cap \overline{B})] = P(A \cup B) - P(A \cap B)$$

Proof: Look at the Venn diagram. Since the events $\bar{A} \cap B$ and $A \cap \bar{B}$ are mutually exclusive.

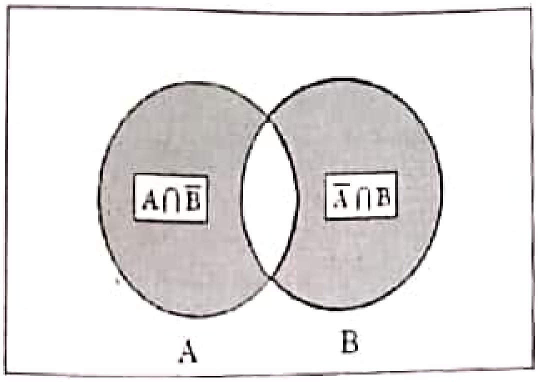

Figure 7.12: Venn diagrams displaying $\bar{A} \cap B$ and $A \cap \bar{B}$

$$\begin{aligned} P[(\bar{A} \cap B) \cup (A \cap \bar{B})] &= P(\bar{A} \cap B) + P(A \cap \bar{B}) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A \cup B) - P(A \cap B) \end{aligned}$$

Hence the proof.

Theorem 7.10: *If the events A and B are independent, so are $\bar{A}$ and $\bar{B}$, A and $\bar{B}$, $\bar{A}$ and B*

Proof: Since A and B are independent, $P(A \cap B) = P(A)\,P(B)$ and

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A \cap B) = P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] = P(A).P(\bar{B}) \end{aligned}$$

Hence A and $\bar{B}$ are independent.

To prove the independence of $\bar{A}$ and B, we proceed as follows:

$$\begin{aligned} P(\bar{A} \cap B) &= P(B) - P(A \cap B) = P(B) - P(A)P(B) \\ &= P(B)[1 - P(A)] = P(\bar{A}).P(B) \end{aligned}$$

To prove that $\bar{A}$ and $\bar{B}$ are independent, we find that

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= P(\bar{A}) - P(\bar{A} \cap B) = P(\bar{A}) - P(\bar{A})P(B) \\ &= P(\bar{A})[1 - P(B)] = P(\bar{A}).P(B) \end{aligned}$$

Hence $\bar{A}$ and $\bar{B}$ are independent.

We wish to clearly emphasize that independence of events should never be confused with disjoint or mutually exclusive events. If two events, each with non-zero probability, are mutually exclusive, they are obviously dependent, since the occurrence of one will automatically preclude the occurrence of the other. Similarly, if A and B are independent and $P(A)>0$, $P(B)>0$, the A and B cannot be mutually exclusive.

Corollary: If A and B are independent events, then

$$P(A \mid B) = P(A) \qquad \text{if } P(B) > 0$$

and

$$P(B \mid A) = P(A) \qquad \text{if } P(A) > 0$$

Example 7.51: A certain retail shop accepts either the American Express or the VISA credit card. A total of 25 percent of its customers carry an American Express card, 60 percent carry VISA credit card and 15 percent carry both. What is the probability that a randomly chosen customer will have at least one of these cards? What is the probability that the customer has neither an American Express nor a VISA card? Are the events 'accepting an American card' and accepting a VISA card' independent?

Solution: Let A be the event that the customer has an American card and B be the event that he has a VISA card. Then

$$P(A) = 0.25 \quad P(B) = 0.60 \quad P(A \cap B) = 0.15$$

Using the addition law of probability, the desired probability $P(A \cup B)$ is

$$P(A \cup B) = 0.25 + 0.60 - 0.15 = 0.70$$

We conclude that 70 percent of the customers carry at least one of the cards that it will accept.

The probability of having neither is $P(\overline{A} \cap \overline{B})$, which equals $P(\overline{A \cup B})$:

$$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.70 = 0.30$$

Example 7.52: Of the total students of a women's college, 60% wear neither a ring nor a necklace, 20% wear a ring, and 30% wear a necklace. If one of the women is randomly chosen, find the probability that she is wearing (a) A ring or a necklace (b) Both.

Solution: Let R and N respectively denote the events that a woman wears a ring and a necklace. We are given that

$$P(R) = 0.20, \; P(N) = 0.30 \text{ and } P(\overline{A} \cap \overline{N}) = 0.60$$

The probability that she is wearing a ring or a necklace is

$$P(R\cup N)=1-P(\overline{R\cup N})=1-P(\overline{R}\cap\overline{N})=0.40$$

The probability that she wears both is

$$P(R\cap N)=P(R)+P(N)-P(R\cup N)=0.20+0.30-0.40=0.10$$

Example 7.53: A newly married couple is planning to have two children and suppose that each child is equally likely to be a boy or a girl. Construct a sample space and find the probability that the couple will have (a) two boys, (b) one boy and one girl and (c) at least one girl.

Solution: We let B denote that the child is a boy and G denote that the child is a girl. The 'double' BG, for instance, represents the outcome 'first child is boy and the second is a girl'. Then one possible sample space is

$$S=\{BB,BG,GB,GG\}$$

where each of the outcomes is equally likely so that

$$P(BB)=P(BG)=P(GB)=P(GG)=\frac{1}{4}$$

(a) The probability that the couple will have two boys is

$$P(BB)=\frac{1}{4}$$

he probability that the couple will have one boy and one girl

$$P(BG)+P(GB)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

(c) The probability that the couple will have at least one boy

$$P(BB)+P(GB)+P(BG)=\frac{3}{4}$$

Example 7.54: A school teacher has received five complementary tickets to watch a final football match to be played between Abahoni Krira Chakra and Mohamedan Sporting Club. He decides to give these tickets to five of his best students: Ariq (A), Bashir (B), Chandon (C), Delwar (D) and Erfan (E). What is the probability that (i) both Ariq and Bashir are chosen, (ii) both Chandon and Erfan are chosen, (iii) Bashir, Chandon, and Delwar are chosen?

Solution: There are ${}^5C_3$ or 10 possible selections of the 5 persons taken 3 at a time, so that a sample space is

$$S=\{ABC,ABD,ABE,ACD,ACE,ADE,BCD,BCE,BDE,CDE\}$$

Let A_1, A_2, and A_3 be the events of interest, so that

7.14 BAYES' THEOREM

Very often we begin our probability analysis with initial or **prior** probability estimates for specific event of interest. Then, from sources, such as a sample, a special report or document, we obtain some additional information about the events. Given this new information, we want to revise and up-date the prior probability values. The new and revised probabilities for the events are referred to as **posterior** probabilities. **Bayes' theorem**, which we will deal with here, provides a means of computing these revised probabilities. We cite an example below to illustrate the use of Bayes' theorem.

7.14.1 Understanding Bayes' Rule

Suppose an Electric Company has two machines A and B for manufacturing electric bulbs. The authority is seriously concerned with the quality of bulbs manufactured by these two machines. The company receives complaints from the dealers that they frequently receive defective bulbs. Once a dealer reports to the authority with a defective product. The manager now wants to make a probabilistic guess as to which machine could have produced this bulb. For this purpose he needs some prior information. Office record shows that, of the total output, 60 percent are produced by machine A and the remaining 40 percent by machine B. Thus we can reasonably assume that the probability that a bulb taken at random from the stock is produced by machine A is 0.60 and that it is produced by machine B is 0.40. These, in our probabilistic terminology, are referred to as the **prior probabilities.** These are prior probabilities, because we know these probabilities before it is known that the selected item is defective. We designate these probabilities by $P(A)$ and $P(B)$. But these prior information are not enough to come to a valid conclusion about the problem in hand. Which machine is to be blamed for manufacturing defective bulbs? The manager wants to make some educated guess on this. Further scrutinizing of the office records reveals that 3 percent of the bulbs produced by machine A in the recent past were defectives, while this proportion is 5 percent for machine B. If D stands for the event that the bulb is defective, then $P(D|A)$ will be the conditional probability of getting a defective bulb, if it is known in advance that the bulb was produced by machine A. This in the present instance is .03. Similarly $P(D|B)$ =.05. By definition, $P(D)=P(D|A)+P(D|B)$, because the defective bulb could be produced by either of the machines. Given the defective bulb (D) in hand, the manager now wants to know: what is the chance that this bulb was produced by machine A? By machine B? Symbolically, we want to evaluate $P(A|D)$ and $P(B|D)$. These probabilities are the modified probabilities of $P(A)$ and $P(B)$ respectively and are referred to as the **posterior probabilities**, which the Bayes' theorem deals with. $P(A|D)$ and $P(B|D)$ are posterior probabilities, because these are the probabilities of the events after it is known that the selected item is defective. Note that the events of interest are mutually exclusive and collectively exhaustive. Thus $P(A \cap B)=0$ and the sum of their probabilities $P(A)+P(B)=1$. Also since the defective items are either produced by A or B, so that $P(A|D)+P(B|D)=1$. In all applications of Bayes' theorem, the events of interest will be mutually exclusive and collectively exhaustive.

7.14.2 Toward Developing Bayes' Theorem

Let us now see how the formula for Bayes' theorem is developed. For this purpose, we make use of the foregoing example.

Since the probability $P(A|D)$ that we are seeking for, is a conditional probability, we can start with the definition of conditional probability expressed as follows:

$$P(A|D) = \frac{P(A \cap D)}{P(D)}$$

Using the multiplication rule, we can replace $P(A \cap D)$ by $P(A) \times P(D|A)$ so that

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} \quad \dots (7.6)$$

nilar formula can be presented for $P(B|D)$, the conditional probability B given D

$$P(B|D) = \frac{P(B)P(D|B)}{P(D)} \quad \dots (7.7)$$

The above equations are the simplest forms of Bayes' theorem involving two events A and B.

The numerators of (7.6) and (7.7) are easy to compute since $P(A)$, $P(B)$, $P(D|A)$ and $P(D|B)$ are all known. But how can we compute $P(D)$, the probability that the bulb is defective? To compute $P(D)$, consider the following partitioning of the sample space.

Clearly $$D = (A \cap D) \cup (B \cap D)$$

and hence

$$\begin{aligned} P(D) &= P(A \cap D) + P(B \cap D) \\ &= P(A)P(D \mid A) + P(B)P(D \mid B) \end{aligned} \quad \text{... (7.8)}$$

Substituting (7.8) in (7.6) and (7.7)

$$P(A \mid D) = \frac{P(A)\,P(D \mid A)}{P(A)P(D \mid A) + P(B)P(D \mid B)} \quad \text{... (7.9)}$$

and

$$P(B \mid D) = \frac{P(B)P(D \mid B)}{P(A)P(D \mid A) + P(B)P(D \mid B)} \quad \text{... (7.10)}$$

which are the alternative forms of Bayes' rule for two events.

For three events A, B and C, the Bayes'formulae will assume the following form:

$$P(A \mid D) = \frac{P(A)\,P(D \mid A)}{P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C)}$$

and so on.

For $\underline{k}$ events $A_1, A_2, \ldots, A_k$

$$P(A \mid D) = \frac{P(A_i)\,P(D \mid A_i)}{\sum_{i=1}^{k} P(A_i)P(D \mid A_i)} \quad \text{... (7.11)}$$

where

$P(A_i)$ = prior probability of event A_i, $i=1, 2, \ldots\ldots\ldots, k$.
$P(D|A_i)$ = conditional probability of D given A_i
$P(A_i|D)$ = posterior probability of A_i given D

We now provide a general proof of the Bayes' theorem.

***Theorem* 7.11:** *Let $A_1, A_2, \ldots, A_k$ be k mutually exclusive events forming partitions of the sample space S of an experiment.. Let B be any event of S such that P(B)>0, for i=1,2,..,k. Then*

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}$$

Proof: The left-hand side of the above equation can be written as

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

Since the events $A_1, A_2, \ldots A_k$ are mutually exclusive, their union is the sample space S. This implies that $A_1 \cup A_2 \cup \ldots \cup A_k = S$. Further, if B is any other event in S, then the events $A_1 \cap B, A_2 \cap B, \ldots, A_k \cap B$ will form partitions of B, as illustrated in Figure 7.9 below. Hence we can write

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B). \quad \ldots (*)$$

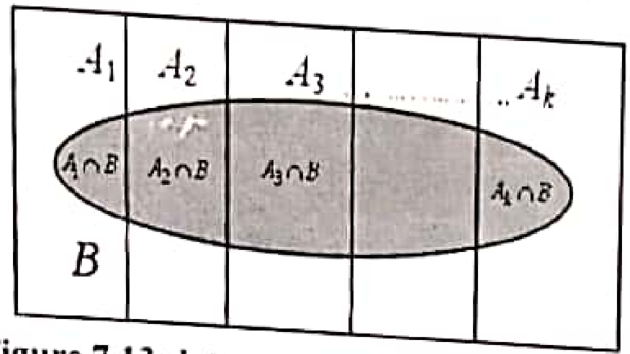

Figure 7.13: Intersection of B with $A_1, A_2 \ldots A_k$

Furthermore, since the events on the right-hand side of the equation (*) are disjoint,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_k \cap B)$$
$$= \sum_{i=1}^{k} P(A_i \cap B)$$

But for $P(A_i) > 0$

$$P(A_i \cap B) = P(A_i)P(B \mid A_i),$$

so that

$$P(B) = \sum P(A_i)P(B \mid A_i)$$

Hence

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}$$

This proves the theorem.

Example 7.63: An opinion poll in Dhaka city shows that 45 percent of the city dwellers support Awami League (A), 40 percent support the Bangladesh Nationalist Party (B) and the remaining 15 percent support Communist and other parties (C). Previous records reveal that in city elections, 65 percent of the Awami League, 80 percent of the Bangladesh Nationalist Party and 50 percent of the Communist or other party supporters turned up to cast their votes. A person in the City is chosen at random and it is learned that he did not cast his vote in the last election. What is the probability that he is a supporter of A? B? C?

Solution: The prior probabilities are $P(A)=0.45$, $P(B)=0.40$ and $P(C)=0.15$. If V stands for the event: Casting vote, then the conditional probabilities are

$$P(V|A)=0.65,\ P(V|B)=0.80 \text{ and } P(V|C)=0.50$$

and the conditional probabilities for not-voting will be

$$P(\bar{V}\mid A))=0.35,\ P(\bar{V}\mid B)=0.20 \text{ and } P(\bar{V}\mid C)=0.50$$

Hence the probability that a person chosen at random did not cast his vote, denoted by $P(\bar{V})$ is

$$\begin{aligned} P(\bar{V}) &= P(A\cap\bar{V})+P(B\cap\bar{V})+P(C\cap\bar{V}) \\ &= P(A)P(\bar{V}\mid A)+P(B)P(\bar{V}\mid B)+P(C)P(\bar{V}\mid C) \\ &= 0.45\times0.35+0.40\times0.20+0.15\times0.50=0.3125 \end{aligned}$$

We need now to obtain $P(A|\bar{V})$, $P(B|\bar{V})$ and $P(C|\bar{V})$.

Using Bayes' rule

$$P(A\mid\bar{V})=\frac{P(A\cap\bar{V})}{P(\bar{V})}=\frac{P(A)P(\bar{V}\mid A)}{P(\bar{V})}=\frac{(0.45)(0.35)}{0.3125}=0.504$$

Similarly, the probabilities of the other two conditional events can be computed, which appear below:

$$P(B\mid\bar{V})=\frac{P(B\cap\bar{V})}{P(\bar{V})}=\frac{P(B)P(\bar{V}\mid A)}{P(\bar{V})}=\frac{(0.40)(0.20)}{0.3125}=0.256$$

$$P(C\mid\bar{V})=\frac{P(C\cap\bar{V})}{P(\bar{V})}=\frac{P(C)P(\bar{V}\mid C)}{P(\bar{V})}=\frac{(0.15)(0.50)}{0.3125}=0.240$$

Example 7.64: The percentages of students favoring a 4-year honors course in three different universities were as follows: Dhaka University: 21 percent, Rajshahi University: 45 percent and Chittagong University: 75 percent. If a university is chosen at random and a student is selected from this university also at random, what is the probability that the student so selected will be in favor of introducing a 4-year honors course in the university? Given that the student is in favor of a 4-year course, what is the probability that he comes from Dhaka University? From Chittagong University? From Rajshahi University?

Solution: If D, C and R stand respectively for the events that a student is selected from Dhaka, Chittagong and Rajshahi Universities, then

$$P(D)=P(C)=P(R)=\frac{1}{3}$$

Denoting the event 'favoring' by F, we have the following conditional probabilities:

$$P(F|D)=0.21,\ P(F|C)=0.45 \text{ and } P(F|R)=0.75$$

where for example, $P(F|D)$ is read as: 'probability that the student so selected, will favor the issue given that he/she belongs to Dhaka University'. Thus, if $P(F)$ denotes the probability that a student chosen at random is in favor of a 4-year course,

$$\begin{aligned}P(F)&=P(D\cap F)+P(C\cap F)+P(R\cap F)\\&=P(D)P(F\mid D)+P(C)P(F\mid C)+P(R)P(F\mid R)\\&=\frac{1}{3}(0.21+0.45+0.75)=0.47\end{aligned}$$

We can now use Bayes' theorem to compute the probability that a student chosen at random will belong to the Dhaka University, given that he/she is in favor of a 4-year degree course. If it is denoted by $P(D|F)$,

$$P(D\mid F)=\frac{P(D\cap F)}{P(F)}=\frac{P(D)P(F\mid D)}{P(F)}=\frac{\frac{1}{3}(0.21)}{0.47}=0.15$$

Similarly, $P(C|F)$ can be computed as

$$P(C\mid F)=\frac{P(C\cap F)}{P(F)}=\frac{P(C)P(F\mid C)}{P(F)}=\frac{\frac{1}{3}(0.45)}{0.47}=0.32$$

and finally

$$P(R\mid F)=\frac{P(R\cap F)}{P(F)}=\frac{P(R)P(F\mid R)}{P(F)}=\frac{\frac{1}{3}(0.75)}{0.47}=0.53$$

Since the events are mutually exclusive and exhaustive, you can also evaluate $P(R|F)$ from the relation $P(R|F)+P(C|F)+P(D|F)=1$ from which $P(R|F)=1-[P(C|F)+P(D|F)]=0.53$

Example 7.65: An insurance company believes that people can be classified into two classes–those who are prone to have accident and those who are not. The data indicate that an accident prone person will meet an accident in a one-year period is 0.1; and the probability to all others is 0.05. Suppose that the probability is 0.2 that a new policyholder is accident-prone.

a) What is the probability that a new policyholder will have an accident in the first year?

b) If a new policyholder has an accident in the first year, what is the probability that he/she is accident-prone?

Solution: Let E be the event that the new policyholder is accident-prone, and A denote the event that he/she has an accident in the first year. Then we can compute $P(A)$ by conditioning on whether the person is accident-prone:

$$\begin{aligned}P(A)&=P(E\cap A)+P(\overline{E}\cap A)\\&=P(E)P(A\mid E)+P(\overline{E})P(A\mid \overline{E})\\&=(0.2)(0.1)+(0.80)(0.05)=0.06\end{aligned}$$

This value implies that there is a 6 percent chance that a new policyholder will meet an accident in one- year time.

In the second case, the conditional probability is

$$P(E\mid A)=\frac{P(E\cap A)}{P(A)}=\frac{P(E)P(A\mid E)}{P(A)}=\frac{(0.2)(0.1)}{0.06}=0.33$$

Example 7.66: A blood test is 90 percent effective in detecting a certain disease when the disease is present. However, the test also yields a **false-positive** result for 5 percent of the healthy patients tested. (That is, if a healthy person is tested, then with probability 0.05 the test will say that this person has the disease.) Suppose 1 percent of the population has the disease. Find the conditional probability that a rand