① Chapter name : Random variable and it's Probability Distribution.

Variable :- A variable is a characteristic whose values are different from one to another.

Random variable :- A random variable is a variable whose values cannot be predicted in advance.

Types of random variable :- There are two types of random variable.

① Discrete random variable :- A random variable defined over a discrete sample space (i.e. that may only take on a finite or countable number of different is referred to as a discrete random variable.

Example :-

- Number of telephone calls received in a telephone booth.
- Number of correct answers in 100-MCQ type questions.
- Number of defective bulbs in a factory.
- Number of under-five children in a family.

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Continuous random variable:- A random variable defined over a continuous sample space (i.e. which may take on any value in a certain interval) is referred to as a continuous random variable.

Example:-

- Time taken to serve a customer in a bank counter.
- weight of a six month-old baby
- longevity of an electric bulb.

Probability Distribution:-

Probability distribution is a mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment.

Discrete probability Distribution :-

If a coin is tossed three times the sample space will consist of eight possible outcomes as follows

$$S = \{HHH, HHT, HTT, HTH, THT, THH, TTH, TTT\}$$

If x denote the number of heads, then X, by definition, is a discrete random variable.

The possible values of x of the random variable X and their associated probabilities can be presented in a tabular form as follows

values of $X : x$	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

X হল random variable এবং x হল সেই random variable -এর যেকোনো value.

Note that, in the above table, each value of X is paired with its probability, and thus the table represents a probability distribution. Since X is discrete, it is a discrete probability distribution.

Frequently, it is convenient to represent all the probabilities of a random variable X by a mathematical formula. We can write

$$f(x) = P(X=x)$$

It is easy to very that $f(x) = P(X=x)$ in the above table is expressible in the form

$$f(x) = {}^{n}C_{x}\left(\frac{1}{2}\right)^{x}\left(1-\frac{1}{2}\right)^{n-x} \text{; for } x = 0, 1, 2, 3 \ldots n$$

$$= {}^{3}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x}$$

Here the coin is thrown three times that is why $n = 3$

Say, we will try for $X = 0$ that means there is no head in the three trials. Then

$$f(0) = P(X=0) = {}^{3}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{3-0}.$$

$$= 1 \times 1 \times \frac{1}{8}.$$

$$= \frac{1}{8}.$$

We also see from the table that

$$P(X=x) = P(X=0) = \frac{1}{8}.$$

We can now try another one. Say $X = 2$.

$$f(2) = P(X=2) = {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2}$$

$$= 3 \cdot \frac{1}{4} \cdot \frac{1}{2}$$

$$= \frac{3}{8}.$$

we also see from the table that

$$P(X=x) = P(X=2) = \frac{3}{8}.$$

From the above example, we are clear that when we put the values of X in the function above we find some probabilities.

$\boxed{\text{All the discussions above are for your under-standing}}$

Now, the official definition of discrete Probability distribution:

The Probability distribution for the discrete random variable X is defined as the function f such that for any real number x,

$$f(x) = P(X=x).$$

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Probability distribution of discrete random variable are often called Probability mass function (pmf).

Pmf has some properties

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$
3. $P(X=x) = f(x)$.

<u>Example</u> :- Verify that the following functions are probability mass functions.

a) $f(x) = \frac{2x-1}{8}$; $x = 0, 1, 2, 3$.

b) $f(x) = \frac{x+1}{16}$; $x = 0, 1, 2, 3$.

c) $f(x) = \frac{3x+6}{21}$, $x = 1, 2$.

Solution : a)

Summing the function over the entire range of X

$$\sum_{x=0}^{3} f(x) = f(0) + f(1) + f(2) + f(3)$$

$$= \frac{2.0-1}{8} + \frac{2.1-1}{8} + \frac{2.2-1}{8} + \frac{2.3-1}{8}$$

We will test all the problems just using property 2.

be a function of the numerical values x, that we shall denote by $f(x)$, $g(x)$, $h(x)$ and so forth. Hence we write $f(x)=P(X=x)$. In particular $f(3)=P(X=3)$. It is easy to verify that $f(x)$ in the above table is expressible in the form

$$f(x)={}^{3}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x}, \text{ for } x = 0, 1, 2, 3$$

In the foregoing example, the set of ordered pairs, each of the form

{Number of heads (X), probability of X} or $\{x, P(X=x)\}$

is called the **probability distribution** or the **probability mass function** (*pmf*) of the discrete random variable X.

Definition 8.6: *If a random variable X has a discrete distribution, the probability distribution of X is defined as the function f such that for any real number x, $f(x)=P(X=x)$*

It is important to emphasize that all functions are not probability mass functions. The function $f(x)$ defined above must satisfy the following conditions in order to be a probability mass function:

1. $f(x) \geq 0$
2. $\sum_{x} f(x) = 1$
3. $P(X = x) = f(x)$

Example 8.3: Verify if the following functions are probability mass functions.

(a) $f(x) = \dfrac{2x-1}{8}$, $x = 0, 1, 2, 3$. P[illegible]

(b) $f(x) = \dfrac{x+1}{16}$, $x = 0, 1, 2, 3$.

(c) $f(x) = \dfrac{3x+6}{21}$, $x = 1, 2$.

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Solution: (a) Summing the function over the entire range of X,

$$\sum_{x=0}^{3} f(x) = f(0) + f(1) + f(2) + f(3)$$

$$= -\frac{1}{8} + \frac{1}{8} + \frac{3}{8} + \frac{5}{8} = 1$$

Putting the values of x in $f(x)$ we get

$f(0) = \dfrac{2 \cdot 0 - 1}{8} = -\dfrac{1}{8}$

i.e. it does not satisfies $f(x) \geq 0$

So it is not a pmf

$$= -\frac{1}{8}+\frac{1}{8}+\frac{3}{8}+\frac{5}{8}$$

$$= 1$$

It is verified that the function saties the property of pmf. that's why it is a pmf.

(b) & (c) Try yourself.

Example :- A bag contains 10 balls of which 4 are black. If three balls are drawn at random without replacement, obtain the probability distribution for the number of black balls.

Solution :- If x denote the number of black balls drawn, then clearly x can assume values 0, 1, 2 & 3. To obtain the probabilities a function can be written in the form

$$f(x) = P(X=x) = \frac{{}^4C_x \; {}^6C_{3-x}}{{}^{10}C_3} \; ; \; x = 0, 1, 2, 3$$

Now, we need to compute probabilities associated with 0, 1, 2, 3

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$f(0) = P(X=0) = \frac{{}^4C_0 \times {}^6C_3}{{}^{10}C_3} = \frac{20}{120} = \frac{1}{6}$

$f(1) = P(X=1) = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{60}{120} = \frac{1}{2}.$

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So, in tabular form.

x	0	1	2	3
$f(x)$	1/6	1/2	3/10	1/30

H.W A department store has 6 television sets of which 2 are defective. A purchaser makes a random choice of three sets. If X is the number of defective sets purchased, find the probability distribution of X.

Continuous Probability Distribution :-

Continuous Probability distribution are often called Probability density function or Pdf.

Pdf has some properties;

1. $f(x) \geq 0$.

2. $\int_{-\alpha}^{\alpha} f(x)dx = 1$

3. $P(a < x < b) = \int_{a}^{b} f(x)\,dx$

Example :- A random variable x has the following functional form;

$$f(x) = kx \quad , \quad 0 < x < 4.$$
$$= 0 \quad , \quad \text{elsewhere}.$$

i) Determine k for which f(x) is a density function.

ii) Find $P(1 < X < 2)$ and $P(X > 2)$.

Solution :- i) Since f(x) be a density function

$$\int_{-\alpha}^{\alpha} f(x)dx = 1.$$

$$\Rightarrow \int_{0}^{4} kx\,dx = 1.$$

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$$\Rightarrow k \int_0^4 x\,dx = 1.$$

$$\Rightarrow k \left[\frac{x^2}{2}\right]_0^4 = 1.$$

$$\Rightarrow k \left[\frac{4^2}{2}\right] = 1.$$

$$\Rightarrow 8k = 1.$$

$$\Rightarrow k = \frac{1}{8}.$$

ii) $P(1 < x < 2) = \int_1^2 kx\,dx$

$$= \int_1^2 \frac{1}{8} x\,dx.$$

$$= \frac{1}{8} \int_1^2 x\,dx$$

$$= \frac{1}{8} \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{8} \cdot \frac{1}{2} [2^2 - 1].$$

$$= \frac{1}{16} \cdot 3 = \frac{3}{16}.$$

$P(x > 2) = \int_2^4 \ldots$

(11)

H.W A continuous random variable X has the following density function

$$f(x) = \frac{2}{27}(1+x) \quad ; \quad 2 < x < 5$$

$$= 0 \quad , \text{ elsewhere}$$

a) Verify that it satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$

b) Find $P(X < 4)$.

c) Find $P(3 < X < 4)$.

Mathematical Expectation

*. If x is a discrete random variable with probability function $f(x)$, then the expected value or mathematical expectation of x is $E(X)$ which is defined as

$$E(X) = \sum_{x} x f(x)$$

If x is continuous having a density function $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Theorem:- Let x be a discrete random variable with probability function $f(x)$ and c be a constant. Then $E(c) = c$.

Proof:- $E(c) = \sum_{x} c f(x) = c \sum f(x)$

But $\sum f(x) = 1$ and hence

$$E(c) = c \cdot 1 = c \quad \text{(Proved)}$$

*. Variance, $V(X) = E[X^2] - [E(X)]^2$

* If X be a r. v. with probability distribution $f(x)$. The expected value of the function $w(X)$ of the random variable X is

$$E[w(X)] = \sum_{x} w(x) f(x) \quad ; \text{ if } X \text{ is discrete}.$$

$$= \int_{-\infty}^{\infty} w(x) f(x)\, dx \quad ; \text{ if } X \text{ is continuous}$$

* the variance of a r.v. X is defined to be the expected value of the square of the differences between the values of X and their mean μ. That is

$$V(X) = E(X-\mu)^2$$

If mean is denoted by μ

Bernoulli Trial

If a random variable (r.v.) has only two outcomes success and failure then the r.v. is called Bernoulli r.v. And the trial is called Bernoulli trial. when Bernoulli trial is repeated n times then the experiment is called Binomial experiment.

Bernoulli Experiment $\longrightarrow$ Binomial Experiment

Binomial Distribution :-

When an experiment has two possible outcomes, success and failure and the experiment is repeated n times independently, and the probability p of success of any given trial remains constant from trial to trial, the experiment is known as Binomial experiment.

Binomial Distribution has the following properties

(1) The overall experiment can be described in terms of a sequence of n identical experiments, which are called trial.

(2) Each trial result in an outcome that may be classified as success or failure.

(3) The probability of success on a single trial is equal to some value p which remains constant from one trial to the next. the probability of failure is equal to $q = 1 - p$.

(4) The repeated trials are independent.

(5) The random variable of interest is X, the number of success observed in n trials.

Examples of Binomial Experiment : Book

~~Derivation of Binomial Experiment~~ :-

*. Note : The distribution associated with Binomial Experiment is called Binomial distribution.

Derivation of Binomial Distribution :-

If a Bernoulli trial can result in a success with probability p and failure with probability $q = 1-p$, then the probability distribution of binomial r.v. X, where X is the number of success in n independent trials is

$$b(x, n, p) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$$; x = 0, 1, \cdots n$$

Proof:- Let us suppose that a sequence of n Bernoulli trials results in x successes and $n-x$ failures. If we denote a success by S and a failure by F, then one possible outcome of the sequence is of the form

$$\underbrace{S\,S\,S \ldots S}_{x \text{ terms}}\ \underbrace{F\,F \ldots F}_{n-x \text{ terms}}$$

Note: It may also be the case that there is one S and $n-1$ F. ~~then~~ x Success and $n-x$ Failure is just any possibility

Since we have assumed that trials are independent. we can multiply the probabilities corresponding to different outcomes.

i.e. ~~$P(SS\ldots S\,FF\ldots F) = P(S)+P(S)$~~

$+ \cdots + P(S)+P(F) + \cdots + P(F)$

~~= P.P~~

Note: ~~$P(A \cap B) = P(AB)$~~

~~$= P(A)+P(B)$~~ if A & B are independent.

i.e. $P(SS\cdots SFF\cdots F) = P(S)\,P(S)\cdots P(S)\,P(F)\cdots P(F)$

Note :- $P(A\cap B) = P(AB)$
$= P(A)\cdot P(B)$ if A & B
are independent

$= p.p\cdots p.\,q\,q\cdots q$

$= p^{x}\,q^{n-x}$

This is one possible arrangement of x successes and $n-x$ failures in n trials. The total number of different arrangements is equal to $^{n}c_{x}$. And probability of each such arrangement is $p^{x}\,q^{n-x}$. Because these arrangements are mutally exclusive the probability that there are x success and $n-x$ failures is obtained by adding the probabilities of

Note :- when we toss a coin three times and x is the number of heads then

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

One possible arrangement is HHH and it is present once.

i.e. $^{3}c_{3} = 1$.

all the different arrangements are multiplying $p^x q^{n-x}$ by ${}^nc_x$.

In other words, the probability of x success and $n-x$ failures for a binomial experiment of n trials with probability p as success and q as failure in a single trial is

$$b(x; n, p) = {}^nc_x\, p^x\, q^{n-x}$$

$$; x = 0, 1, 2, \cdots n$$

$$; p + q = 1.$$

Mean of Binomial Distribution :-

$$E(X) = \sum_x x\, b(x; n, p)$$

$$= \sum_{x=0}^{n} x \cdot {}^nc_x\, p^x q^{n-x}$$

$$= 0 + 1 \cdot {}^nc_1\, p^1 q^{n-1} + 2 \cdot {}^nc_2\, p^2 q^{n-2} + \cdots + n \cdot {}^nc_n\, p^n q^0$$

We know that $E(X) = \sum_x x f(x)$

here $f(x) = b(x; n, p)$

$$= {}^nc_1\, p q^{n-1} + 2\, {}^nc_2\, p^2 q^{n-2} + \cdots + n\, p^n$$

$$= npq^{n-1} + 2 \cdot \frac{n!}{2!(n-2)!} p^2 q^{n-2} + \cdots + np^n$$

$$= npq^{n-1} + \frac{n(n-1)(n-2)!}{(n-2)!} p^2 q^{n-2} + \cdots + np^n$$

$$= npq^{n-1} + n(n-1) p^2 q^{n-2} + \cdots + np^n$$

$$= np\left[q^{n-1} + (n-1) p q^{n-2} + \cdots + p^{n-1}\right]$$

Formula mathematical

$$= np(p+q)^{n-1}$$

$$= np \quad [\because p+q=1]$$

$$x^{n-1} + (\cancel{n}-1)(1-x)x^{n-2} + \cdots \cancel{+} (1-x)^{n-1}$$

$=$

Variance of Binomial :-

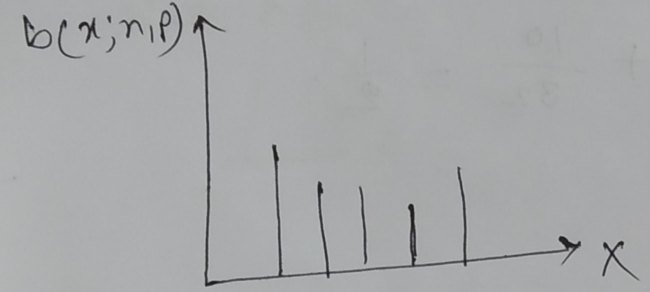
$$np(1-p)$$

$$= npq.$$

Binomial Distribution is a discrete distribution. Since the values of x are isolated i-e.

$x = 0, 1, 2 \ldots n$.

The curve of Binomial distribution be like



<u>Problem:-</u> A fair coin is tossed five times. Find the probability of obtaining (i) Exactly 4 heads (ii) fewer than 3 heads.

<u>Solution:-</u> Since the experiment has two outcomes that is head and tail. The experiment is a Binomial experiment. Let x is the number of heads.

i) $P(X=4) = b(x; n, 0.5)$

$= b(4; 5, 0.5)$

$p = 0.5 = \frac{1}{2}$

$q = 0.5 = \frac{1}{2}$

$n = 5$

$$= {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4}$$

$$= \frac{5}{32}.$$

1/2 or 0.5 you can write any of them

ii) $P(X<3) = \sum_{x=0}^{2} b(x; 5, \frac{1}{2})$

$$= b(0;5,\tfrac{1}{2}) + b(1;5,\tfrac{1}{2}) + b(2;5,\tfrac{1}{2})$$

$$= {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} + {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1} + {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2}$$

$$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{1}{2}.$$

Problem: A trafic control officer reports that 75% of the trucks passing through a check post are from within Dhaka city. What is the probability that at least three of the next five trucks are from out of Dhaka city?

Hint:- In the coin toss problem our interest is in number of heads. So the probability of head is the success probability (p) and probabity of tail is faire probability (q).

For this math success is the truck from out side Dhaka city. And this probability can be calculated from the information given.