Poisson Distribution :- Poisson distribution has been found applicable to many processes that involve an observation falling in a given time interval or in a specified region or space.
Experiments yielding numerical values of a random variable X within the given interval is often called poisson experiment.

Example :-

1. The number of telephone calls received at a switchboard per minute.

2. The number of customers arriving at a bank counter per 5 minute period.

Probability Distribution of poisson variable :-

Let X be a poisson variable, the distribution of poisson variable is denoted by $f(x, \mu)$ where μ is the average number of success occuring in a given time interval or specifed region. then

$$f(x, \mu) = \frac{e^{-\mu} \mu^{x}}{x!} \; ; \; x = 0, 1, \cdots \infty$$

verify that

$$f(x,\mu) = \frac{e^{-\mu}\,\mu^{x}}{x!}, \quad x = 0, 1, 2, \ldots \infty$$

is a pmf.

Proof:- It is clear that $f(x,\mu) \geq 0$ for each value of x. In order to verify that the function $f(x,\mu)$ satisfies the requirements of every probability function, it must be shown that

$$\sum_{x=0}^{\infty} f(x,\mu) = 1$$

It is known from the elementary algebra that for any real number μ,

$$e^{\mu} = \sum_{x=0}^{\infty} \frac{\mu^{x}}{x!}$$

~~Therefore,~~ $\cancel{\sum_{x=0}^{\infty} f(x,\mu)} = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^{x}}{x!}$

$\cancel{= e^{-\mu}\, e^{\mu} = 1}$.

Therefore,
$$\sum_{x=0}^{\infty} f(x,\mu) = \sum_{x=0}^{\infty} e^{-\mu} \frac{\mu^{x}}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^{x}}{x!}$$

$$= e^{-\mu}\, e^{\mu} = \underline{1}.$$

(proved)

Example:- Telephone calls arrive at a switch board at a mean reate of 0.5 calls per minute. Calculate the probability that two calls will arrive in a particular five minute period.

Solution:- Since a particular time period is given the problem fits the poisson distribution with average number of calls per minute $= 0.5$

So, the average number of calls per five minute, $\mu = 5 \times 0.5 = 2.5$.

$$\therefore \mu = 2.5$$

If x is the random variable. we want to compute $f(x;\mu) = f(2, 2.5)$

$$= \frac{e^{-2.5} \cdot (2.5)^2}{2!}$$

$$= 0.257$$

H.W The average number of calls received by a telephone operator during a time interval of 10 minutes during 5 P.M. to 5.10 P.M daily is 3.

What is the probability that the operator will receive tomorrow during the same time interval

i) No call?

ii) Exactly one call?

iii) At least two calls

Soln:- i) $P(X=0)$

ii) $P(X=1)$

iii) $P(X \geq 2) = \cancel{1-P(X \leq 1)}$

$= 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

Properties of poisson distribution :-

Mean, $E(X) = \sum_{x=0}^{\infty} x f(x, \mu)$

$= \sum_{x=0}^{\infty} x \frac{e^{-\mu} \mu^{x}}{x!}$

$= \mu \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^{x-1}}{(x-1)!}$

the first term is 0. that's why we put $\sum_{x=1}^{\infty}$

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Properties of poisson distribution :-

Mean, $E(X) = \sum_{x=0}^{\infty} x f(x, \mu)$

$= \sum_{x=1}^{\infty} x f(x, \mu)$

$= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\mu} \mu^{x}}{x!}$

Since putting $x=0$ will make the whole term zero. That's why we omit the term 0 and start from $\sum_{x=1}^{\infty}$

$= \sum_{x=1}^{\infty} \frac{x \cdot e^{-\mu} \cdot \mu \mu^{x-1}}{x \cdot (x-1)!}$

$= \mu \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^{x-1}}{(x-1)!}$

$$= \mu \sum_{x-1=0}^{\infty} \frac{e^{-\mu} \mu^{x-1}}{(x-1)!}$$

Let $y = x-1$

Then $$E(X) = \mu \sum_{y=0}^{\infty} \frac{e^{-\mu} \mu^{y}}{y!}$$

$$= \mu e^{-\mu} \sum_{y=0}^{\infty} \frac{\mu^{y}}{y!}$$

$$= \mu e^{-\mu} e^{\mu} = \mu.$$

$\therefore \; E(X) = \mu.$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= E(X^2) - \mu^2.$$

Now, $E(X^2) = \mu^2 + \mu.$

$$V(X) = \mu^2 + \mu - \mu^2$$

$$= \mu.$$

$\therefore$ Mean and variance of Poisson distribution are equal.

Problem Suppose that there are, on the average four vehicle accidents per day on the Asian highway running from Dhaka to Manikganj. What is the probability that on a given day

i) there is no vehicle accident ?

ii) there are three or fewer accidents?

iii) there are three or more accidents?