Jacob Miller

Training with Topology

Using the Euler Characteristic Transform

December 13, 2024

Outline

What is topology?

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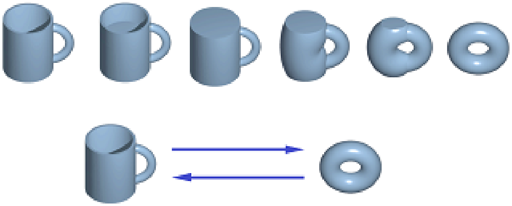

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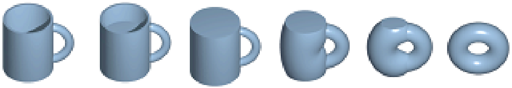

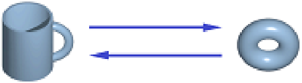

Remember that, as topologists, we cannot:

1. cut
2. glue

Topological Data Analysis:

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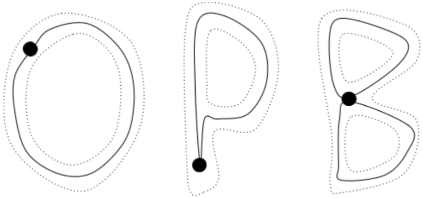

- Ignore this issue for now...

Simplices: Building Blocks

Simplicial complexes are a common way to describe shapes in topology

Definition (Simplex)

A *k-simplex* is the convex hull of $k+1$ affinely independent points:

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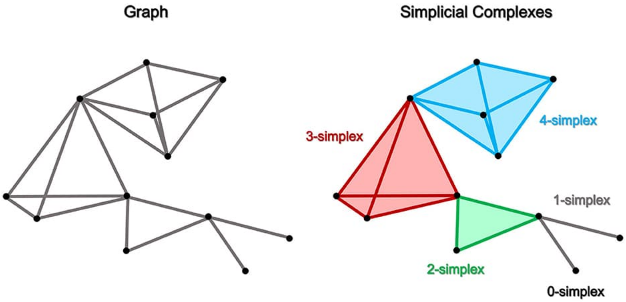

Simplicial Complexes

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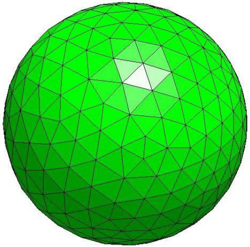

Euler Characteristic for Simplicial Complexes

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The Euler characteristic $\chi(\Delta)$ of a finite simplicial complex Δ is:

$$\chi(\Delta) = \sum_{i=0}^{\infty} (-1)^i f_i$$

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Example

A filled in triangle (a.k.a. 2-simplex):

- $f_0 = 3$ (vertices), $f_1 = 3$ (edges), $f_2 = 1$ (faces)
- $\chi(\text{triangle}) = 3 - 3 + 1 = 1$

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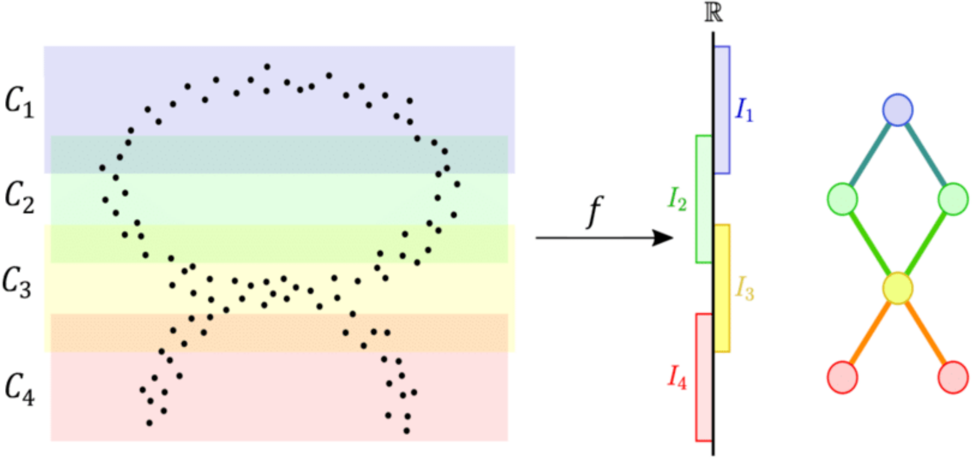
Output: A simplicial complex – a topological approximation of the manifold M.

Diagram of Mapper



Euler Characteristic Transform

Input: An embedded simplicial complex Δ – normalized to live in a unit ball in $\mathbb{R}^n$.

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Output: A function

$$ECT : S^{n-1} \times [-1, 1] \to \mathbb{Z}$$

ECT Setup

Given an embedded simplicial complex Δ and a direction $\vec{w} \in S^{n-1} \subseteq \mathbb{R}^n$ we can define a function $F_{\vec{w}} : \Delta \to \mathbb{R}$ inductively:

$$F_{\vec{w}}(v) := v \cdot \vec{w} \qquad \text{For any vertex } v \in \Delta$$

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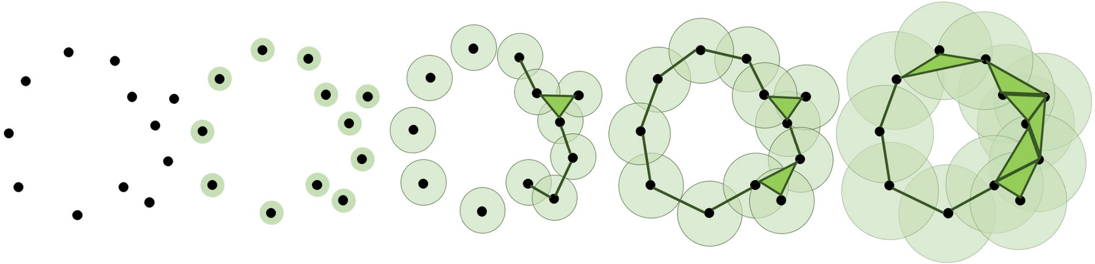
$$F_{\vec{w}}(\sigma) := \max \{ F_{\vec{w}}(v_i) \,|\, 1 \leq i \leq k \}$$

For any threshold value $t \in [-1, 1]$

$$F_{\vec{w}}^{-1}[[-1, t)] \text{ is a subcomplex of } \Delta$$

An example of a filtration

Warning: This is a picture of a Vietoris-Rips filtration



K_1 K_2 K_3 K_4 K_5

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If we pick a direction and a threshold value, we get a simplicial complex.

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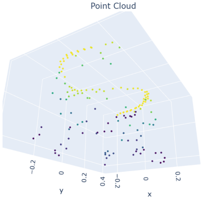
IDEA: For each direction and threshold value, calculate the Euler characteristic!

ECT Definition

The Euler Characteristic Transform ECT is given by

$$ECT(\vec{w}, t) := \chi\left(F_{\vec{w}}^{-1}[(-\infty, t)]\right)$$

ECT Results



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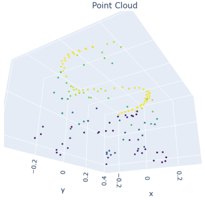
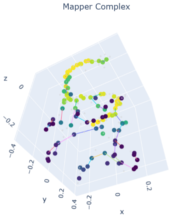
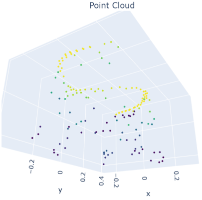

Point Cloud
−0.2
0
0.2
0.4
y
−0.2
0
0.2
x


Mapper Complex
z
0
−0.2
−0.4
−0.2
0
0.2
0.4
y
−0.2
0
0.2
x

ECT Results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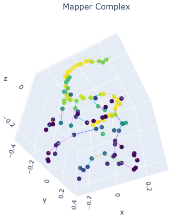
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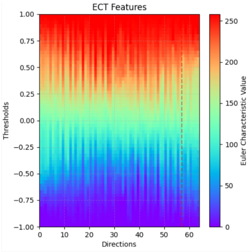
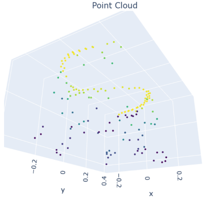
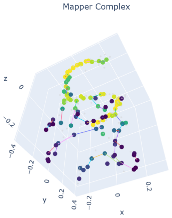
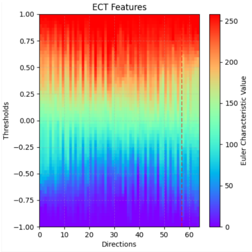

ECT Features
Thresholds
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
10
20
30
40
50
60
Directions
Euler Characteristic Value
250
200
150
100
50
0

ECT Results







Theorem (Turner et al., 2014)

The ECT is ***injective*** *on the space of constructible* sets in* $\mathbb{R}^d$. *So:*

$$\text{If } ECT(K) \neq ECT(K'), \text{ then } K \neq K'.$$

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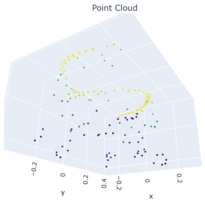

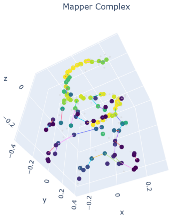

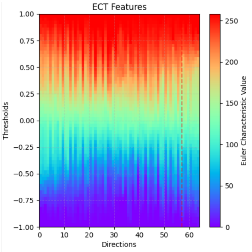

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So the above theorem is true for simplicial complexes in $\mathbb{R}^d$.

Preprocessing

- Load image from MNIST dataset
- Filter out empty pixels
- Normalize pixel and grayscale values to live in the unit 3-ball
- Translate by the mean to standardize
- Apply some Gaussian smoothing (to look good)

Processing

- Load data cloud
- Compute mapper complex using zen-mapper
  - filter function: Projection to the line $y + x = 0$.
  - Covering Scheme: Width Balanced with 10 elements, 40% overlap
  - Clusterer: Affinity Propagation with 0.9 damping (random preferences)
- Embed into $\mathbb{R}^3$
- Approximate ECT
  - 64 directions (sampled via Fibonacci spiral)
  - 64 thresholds (uniformly distributed in $[-1, 1]$)

ECTNet Architecture

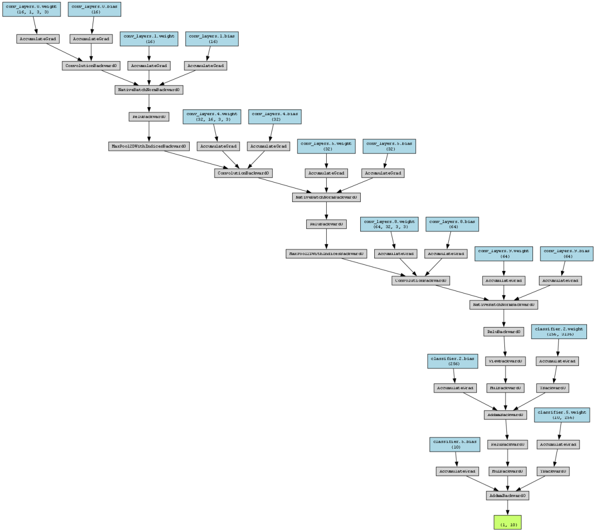
Designing your own CNN - Sanjay Dutta
Creating a CNN from Scratch using Pytorch - Abhishek

- **Convolutional Layers:**
  - Conv1: 1 $\rightarrow$ 16 channels (3$\times$3)
  - BatchNorm + ReLU + MaxPool(2$\times$2)
  - Conv2: 16 $\rightarrow$ 32 channels (3$\times$3)
  - BatchNorm + ReLU + MaxPool(2$\times$2)
  - Conv3: 32 $\rightarrow$ 64 channels (3$\times$3)
  - BatchNorm + ReLU
- **Classifier Head (Make Decision):**
  - Flatten
  - Dropout (0.3) – randomly deactivates neurons
  - Linear: input_dim $\rightarrow$ 256
  - ReLU + Dropout(0.3)
  - Linear: 256 $\rightarrow$ 10

Simplified:

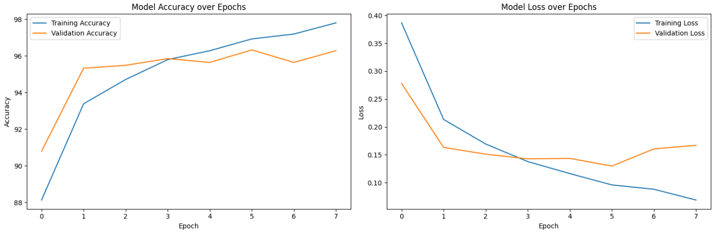
- Two main components:
  - Feature extraction (conv layers)
  - Classification (dense layers)
- Convolutional section:
  - Progressively expands features
  - Spatial reduction via max pooling (takes maximum value from the input region)
- Classifier section:
  - Flattens features
  - Final 10-class prediction

Torchviz Diagram:

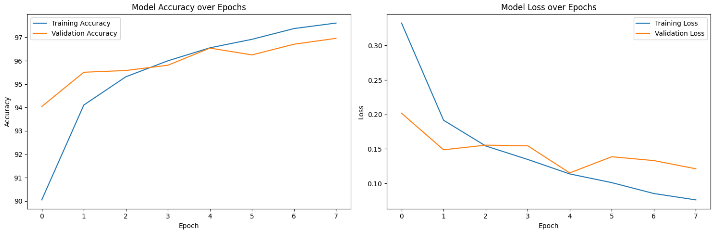


Training Results:

Mapper Complex Training Results: 97.8% accuracy



Point Complex Training Results: 97.6% accuracy



IOWA

Thank you!

Thanks for listening! If you want to see the implementation checkout the GitHub:

https://github.com/Jamiller137/ect-mnist

There is an interactive app that will let you draw a number and see what the model is doing.

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