Jacob Miller

Training with Topology

Using the Euler Characteristic Transform

December 11, 2024

Outline

What is topology?

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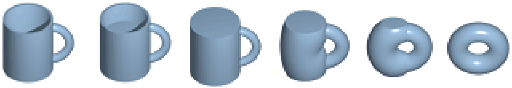

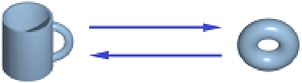

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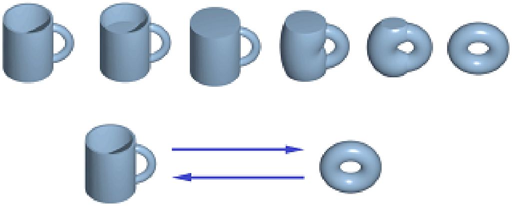

Remember that, as topologists, we cannot:

1. cut
2. glue

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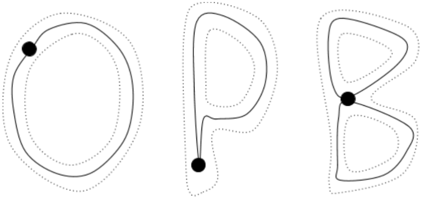

- Ignore this issue for now...

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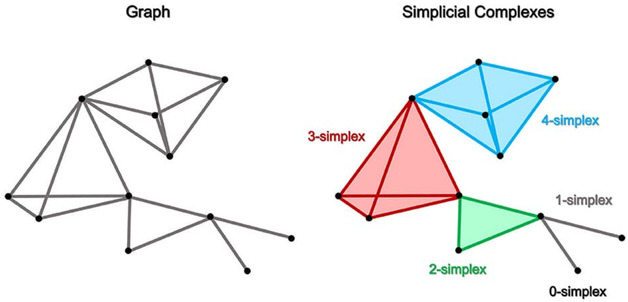

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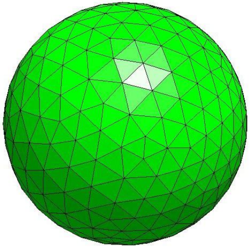

Euler Characteristic for Simplicial Complexes

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Example

A tetrahedron (3-simplex):

- $f_0 = 4$ (vertices), $f_1 = 6$ (edges), $f_2 = 4$ (faces), $f_3 = 1$ (tetrahedron)
- $\chi(\text{tetrahedron}) = 4 - 6 + 4 - 1 = 1$

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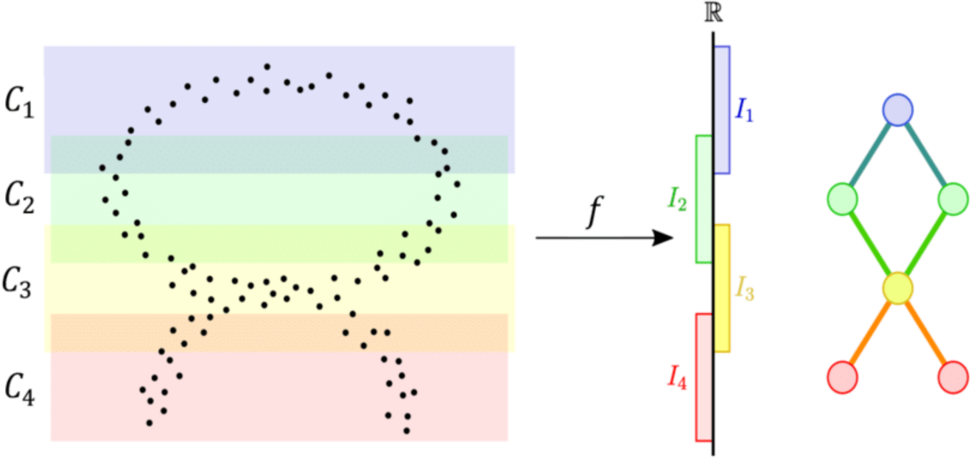
Output: A simplicial complex – a topological approximation of the manifold M.

Diagram of Mapper



Euler Characteristic Transform

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Output: A function

$$ECT : S^{n-1} \times [-1, 1] \to \mathbb{Z}$$

ECT Setup

Given an embedded simplicial complex Δ and a direction $\vec{w} \in S^{n-1} \subseteq \mathbb{R}^n$ we can define a function $F_{\vec{w}} : \Delta \to \mathbb{R}$ inductively:

$$F_{\vec{w}}(v) := v \cdot \vec{w} \qquad \text{For any vertex } v \in \Delta$$

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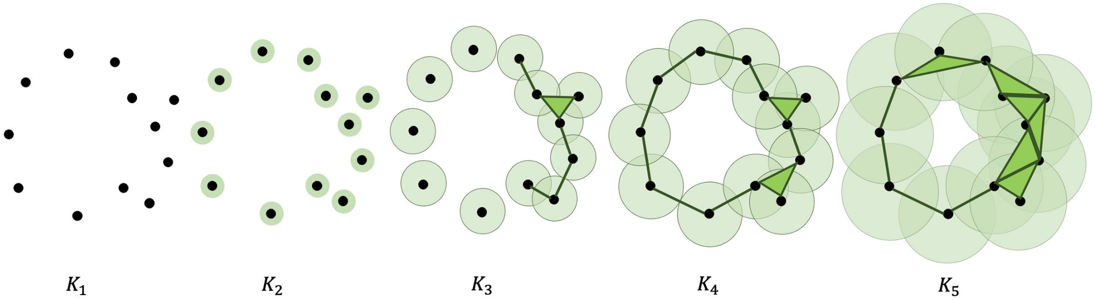
For any threshold value $t \in \mathbb{R}$

$$F_{\vec{w}}^{-1}[(-\infty, t)] \text{ is a subcomplex of } \Delta$$

This is called a filtration!

An example of a filtration



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IDEA: For each direction and threshold value, calculate the Euler characteristic!

ECT Definition

The Euler Characteristic Transform ECT is given by

$$ECT(\vec{w}, t) := \chi\left(F_{\vec{w}}^{-1}\left[(-\infty, t)\right]\right)$$

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Theorem

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The above theorem was proved for simplicial complexes embedded in dimension $d \leq 3$. And is also true in 'nice subsets' of any dimension.

Preprocessing

- Load image from MNIST dataset
- Filter out empty pixels
- Normalize pixel and grayscale values to live in the unit 3-ball
- Translate by the mean to standardize
- Apply some Gaussian smoothing (to look good)

Processing

- Load data cloud
- Compute mapper complex using zen-mapper
  - filter function: Projection to the line $y + x = 0$.
  - Covering Scheme: Width Balanced
  - Clusterer: Affinity Propagation – to avoid having to manually set parameters
- Embed into $\mathbb{R}^3$
- Approximate ECT with 64 directions and 64 threshold samples

ECTNet Architecture

- **Convolutional Layers (Look for Patterns):**
  - Conv1: 1 $\rightarrow$ 16 channels (3$\times$3)
  - BatchNorm + ReLU + MaxPool(2$\times$2)
  - Conv2: 16 $\rightarrow$ 32 channels (3$\times$3)
  - BatchNorm + ReLU + MaxPool(2$\times$2)
  - Conv3: 32 $\rightarrow$ 64 channels (3$\times$3)
  - BatchNorm + ReLU
- **Classifier Head (Make Decision):**
  - Flatten
  - Dropout(0.3)
  - Linear: input_dim $\rightarrow$ 256
  - ReLU + Dropout(0.3)
  - Linear: 256 $\rightarrow$ 10

Simplified:

- Two main components:
  - Feature extraction (conv layers)
  - Classification (dense layers)
- Convolutional section:
  - Progressively expands features
  - Each block includes normalization
  - Spatial reduction via pooling
- Classifier section:
  - Flattens spatial features
  - Uses dropout for regularization
  - Final 10-class prediction

Placeholder