Supplement | Modular framework for the estimation of varying parameters in dynamical systems

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December 16, 2024

1 Discrete Time Problem Formulation

For a Discrete Time (DT) system, there is no need to differentitate between a continuously varying parameter and a discretely switching one.

Definition 1. ***Parameter Switches in Discrete Time Systems*** *For a discrete-time system,* $X(\tau+1) = f(X(\tau), p(\tau))$*, with piece-wise constant parameters* $p(\tau) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n_p}$ *given by*

$$p(\tau) = \begin{cases} p_1 & \text{if} \quad \tau \in \{k_0, ..., k_1\} \\ p_2 & \text{if} \quad \tau \in \{k_2, ..., k_3\} \\ & \vdots \\ p_n & \text{if} \quad \tau \in \{k_{n-1}, ..., k_n\} \end{cases} \tag{1}$$

where $n \in \mathbb{N}$*. Then we have parameter vectors,* $p_i \in \mathbb{R}^{n_p}$ *for each* i*-th ordered set where* $i \in \{1, ..., n\}$*. If* $p(\tau) \neq p(\tau+1)$ *for a given* τ*, then we say that a* ***parameter switch*** *has occured at* τ*. If the system experiences* $N_s := n - 1$ *parameter switches resulting in* n *partitioned ordered sets of* $\mathbb{Z}_{\geq 0}$*.*

1.1 General Problem Formulation (DT)

Assume that we have data denoted $X_{data} \in \mathbb{R}^{m \times N}$ from a parameter-varying system denoted

$$X(\tau + 1) = f(X(\tau), p(\tau)) \tag{2}$$

with $m \in \mathbb{N}$ states, $N \in \mathbb{N}$ data points, vector field $f : \mathbb{R}^m \to \mathbb{R}^m$, and $n_p \in \mathbb{N}$ parameters $p(\tau) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n_p}$. Further, assume we know the form of the system's model, $f(X, p(\tau))$ but we do not know the varying parameters $p(\tau)$.

Problem: Identify $p(\tau)$ given X_{data}, i.e. find the minimizing parameters such that

$$p(\tau) = \mathbf{argmin} \|X_{data} - X_{model}(p(\tau))\|_2 \quad . \tag{3}$$

1.2 Framework for DT Systems

We provided a jupyter notebook example of our framework on a DT nonlinear parameter varying system in the github repository. Note that for chaotic systems, perturbations to system parameters lead to higly differentiated dynamics, and thus, slight errors in the estimation of these parameters would lead to unreliably reconstructed trajectories (moreso than in the non-chaotic instances).

Algorithm Outline for Parametric Curves with Sparse Regression

1. **Instantiate: model, known static parameters,** $X_{data} \in \mathbb{R}^{m \times N}$

   $X_{model}(\tau + 1) = f(X(\tau), p(\tau), p_{static})$

2. **Detect parameter switches and segment data**

   $X_{data} = [\ X_1 \ | \ X_2 \ | \ \dots \ | \ X_i \ | \ \dots \ X_n \ ]$

   $p = [\ p_1 \ | \ p_2 \ | \ \dots \ | \ p_i \ | \ \dots \ p_n \ ]$

3. **Minimize objective function to obtain parameters for** X_1

   $\longrightarrow \{p_1, p_{static}\} = \mathbf{argmin} \left\| X_{data_1} - X_{model_1} \right\|_1$

4. **Repeat step 3 for the** $i = 2, \dots, n$ **data intervals**

   $\circlearrowleft \{p_i\} = \mathbf{argmin} \left\| X_{data_i} - X_{model_i} \right\|_2$

5. **Sparse regression on** p

   $\longrightarrow \{\theta, w\} = \mathbf{argmin} \left\| p - \mathcal{D}_\theta w \right\|_2 + \lambda \|w\|_0$

Figure 1: Outline of framework with sparse regression for the estimation of DT parameters.